Rotation Matrix Reconstruction in Quaternions with an Optimal Observer: A Navigation Algorithm Based on Nonlinear Complementary Observers

Ariel Iporre Rivas* and Mauricio Améstegui Moreno†
*†Electronic Engineering - Faculty of Engineering - Universidad Mayor de San Andrés
La Paz, Bolivia

***Abstract*—This work proposes a variation of an earlier navigation algorithm in the literature composed by SO(3) complementary observers. Such modification means the inclusion of a quaternion optimal observer to determine the rotation matrix instead using the original vectorial reconstruction approach. In particular, this proposal involves an experimental comparison between the original and modified method. The results show about 40% increase for estimation quality while 21% more complexity using this new approach.**

***Index Terms*—Inertial navigation, Kalman filters, Quaternions, INS, Navigation Systems, Optimal Observers, Nonlinear Filters**

I. Introduction

The navigation problem can be summarized in two simple questions: where am I and where am I going, which mathematically refers to the identification of the position and the velocity. In general, this problem is solved by interpreting measurements fixed to the internal states of the body in motion, somewhat similar to the observer problem from where derives the navigation algorithm, hereby navigation can be defined in terms of the observer problem. Most recent trends for designing this kind of observers explode tools from control theory, namely nonlinear filters developed in the framework of the Lyapunov theory e.g. [1]–[3]. Some authors worked to reduce the complexity involved in navigation equations by dividing the process into separated observers [3], [4]. We pay particular attention to the work of [4] who combines two nonlinear complementary filters to determine rotation and position in two separate processes, mainly based on [5]. Based on this initial idea, in this work, we propose the pre-calculation of the rotational matrix in quaternions using an optimal observer by modifying the initial navigation algorithm proposed in [4] and [5], where we identified that the performance could be enhanced.

One of the first formal approaches dealing the navigation problem had been addressed as an application of the popular Kalman Filter [6] extended through the linear perturbation equation to solve the trajectory estimation for a nonlinear model of motion in navigation computer of the Apollo project [7]. Since that milestone event, many other authors have issued an other of ideas diverging from Kalman's core paper in the field of navigation algorithms e.g. [8]–[10]. Some versions of the Kalman Filter work with representations of motion based on untypical algebraic structures like quaternions. Among them, we remark the work of [11] who pre-calculates the quaternion using a Gauss-Newton iteration algorithm to find the best quaternion that relates the measured accelerations and earth magnetic field in the body coordinate frame, which is a very similar approach to what we did in this work.

Besides, navigation algorithms are also connected with the Luenberger Observers [12], who develops a theoretical foundation through the use of modern control applied to the observer problem, which triggers the engagement of non-linear control theory e.g. [13] in this field. Therefore, tendencies in this approach implement nonlinear observers when the system's dynamic entails nonlinear terms as the navigation equations. Hence this approach has been used to develop navigation algorithms founded on the design of Lyapunov's exponential stability; again many of these algorithms grip to the strong, rugged representation of rotation using quaternions, e.g. [14], [15]. Besides, a unique approach is extended by [16] which proposes a general geometrical framework for the design of observers on finite-dimensional Lie groups. One particular case investigated in [16] explains the general form of nonlinear observers on SO(3) where the dynamics and the correction terms are expressed on a tangential Lie Algebras $\mathfrak{so}(3)$ and the orientation is determined from two vectorial measurements depending on the rotational transformation on the Lie group. Such structure identifies the nonlinear observer proposed in [5] who presents an extensive analysis of non-linear attitude observers on SO(3) regarding the Lyapunov stability and the reconstruction of the Rotation matrix also based on vectorial measurements. Also, they extend a globally stable observer dynamics, where global asymptotic stability is proved under the assumption that the observer's error trajectory does not intersect a forward invariant set of instability conditions. Further, [5] implicate a sub-optimal solution of the algebraic reconstruction of the rotation equation which we believe can be improved, the main topic of this work.

What we proposed is to extend the idea of non-linear complementary observers from [4] and [5] named *Mahony-Scandaroli's navigation algorithm* by pre-computing the rotation matrix with an optimal observer expressed in quaternions, which we expected to reduce the prediction errors. Two main points were studied: first, the modification of the original approach of *Mahony-Scandaroli's navigation algorithm of nonlinear observers on $SO(3)$* by including an optimal ob-

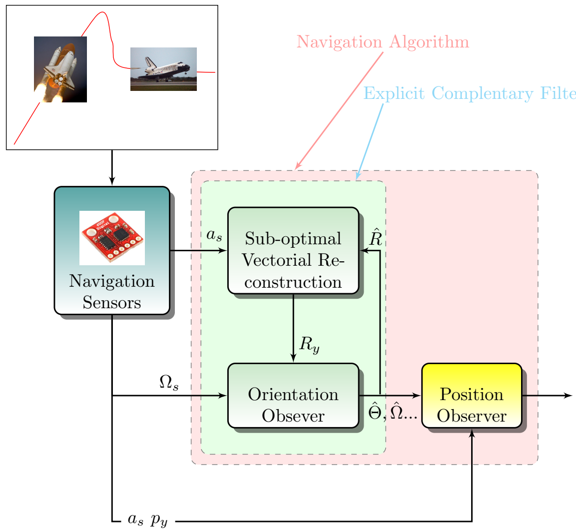

Fig. 1. Simplified scheme of the original navigation algorithm of Mahoni-Scandaroli.

server, which optimally calculates the rotation matrix based on vectorial measurements; and second, the behavior of our approach compared with the original method in two different experiments.

II. Navigation problem: Mahony-Scandarolli navigation algorithm

The navigation problem lays in the determination of an algorithm that possesses the ability to describe the spatial condition of a rigid body using a set of signals that convey the motion. In this frame, the solution arises in the navigation algorithm described in Fig.1 which entails the combination of two nonlinear observers: an attitude observer and a position observer. The measurement of the angular velocity Ω_s and the vectorial reconstruction of the rotation matrix R_y are inputs into the attitude observer who determines:

- $\hat{\Theta}$: Estimation of the Euler angles.
- $\hat{\Omega}$: Estimation of the angular velocity in $\{\mathcal{B}\}$, where $\{\mathcal{B}\}$ denotes the body fixed framework, with x pointing in front, y to the right and z downward.
- $\tilde{R}$: Rotation matrix error defined as the transformation from $\{\mathcal{E}\} \hookrightarrow \{\mathcal{B}\}$ where $\{\mathcal{E}\}$ denotes the estimation referential framework $\{\mathcal{B}\}$.
- $\hat{R}$: The estimation of the rotation matrix that also defines a frame transformation $\{\mathcal{E}\} \hookrightarrow \{\mathcal{A}\}$ where $\{\mathcal{A}\}$ denotes the inertial frame, with x pointing to the north, y to east and z downward.

Finally, the position complementary filter uses the rest of the variables (the measurement of the acceleration a_s and the position p_y) and the transformation matrices computed by the previous observer; it determines :

- $\hat{p}$: Position estimation in $\{\mathcal{A}\}$.
- $\hat{v}$: Linear velocity estimation in $\{\mathcal{A}\}$.

The group of estimation results are concatenated in what we denote *the Estimated Navigation Information* expressed as $X = [\hat{p}\ \hat{v}\ \hat{\Theta}\ \hat{\Omega}]$.

Under this scheme, the vectorial reconstruction feeds in the orientation observer with the rotation matrix's reconstruction that depends directly on the weighted sum of the vectorial measurements in the body framework $\{\mathcal{B}\}$ (V_i) as well as $\hat{R}$. The main disadvantage in this formulation is the sensibility to R_y, inasmuch it is involved in the mapping of Ω_s to $\{\mathcal{A}\}$, and for that reason, it plays a significant role. Further, they mention that the optimal solution can be determined in the minimization argument of:

$$R^* = \arg \min_R \left\{ \sum_i |V_{0,i} - RV_{m,i}|^2 \right\} \tag{1}$$

Where the optimal solution R^* depends on a loss function defined in the sum of Euclidean norms of the errors between the vectorial measurements (R$V_{m,i}$) rotated to $\{\mathcal{A}\}$ with respect to their theoretical values $v_{0,i}$ (Sub-indexes 0 stay by theoretical and m by measured.). However, Mahony-Scandaroli's algorithm solves R_y through a sub-optimal method depending on $\hat{R}$, so it relies on the Lyapunov's stability trajectory. We assume that the optimal solution of R_y could enhance the general performance of the algorithm.

III. Optimal reconstruction of Rotation Matrix

In this section, we present the general solution for optimal determination of the Rotation Matrix by estimating the inclination of the gravity vector field in $\{\mathcal{B}\}$ with an optimal observer in quaternions in the sense that the rotation matrix can be expressed in terms of a unitary quaternion. Hence the optimal observer reduces the deviation of the measurement of acceleration a_s with respect to its rotated theoretical value g_0 in $\{\mathcal{A}\}$.

Therefore, we define the loss function (2) where R^* depends on the unitary quaternion $\breve{q} \in \mathbb{Q} \subset \mathbb{H}$, where $\mathbb{H}$ denotes the Hamiltonian space of quaternion complex numbers $\mathbb{Q}$ is the sub-space of all quaternions with unitary norm:

$$R^*(\breve{q}) = R\left(q^* = \arg\min_{\breve{q}} \left\{ a_s - R^T(\breve{q}) g_0 \right\} \right) \tag{2}$$

Thereby, the solution is given in terms of the optimal unitary quaternion that rotates g_0 in the direction of a_s, in other words, which the estimated acceleration $\hat{a}_s$ in $\{\mathcal{B}\}$. This estimation can be expressed as the rotation in quaternions, where $\breve{g}_0$ denotes gravity in $\{\mathcal{A}\}$ as a pure quaternion [17]:

$$\hat{a}_s = \bar{\breve{q}} \otimes \breve{g}_0 \otimes \breve{q} \tag{3}$$

The approach to computing this quaternion derives from the reformulation (2) in the discrete time k considering two norms:

$$q_k^* = \arg \min_{q_k} \left\{ norm_0(\tilde{q}_0, \hat{v}_0) + \sum_{i=0}^{k} norm_1(w_i, v_i) \right\} \tag{4}$$

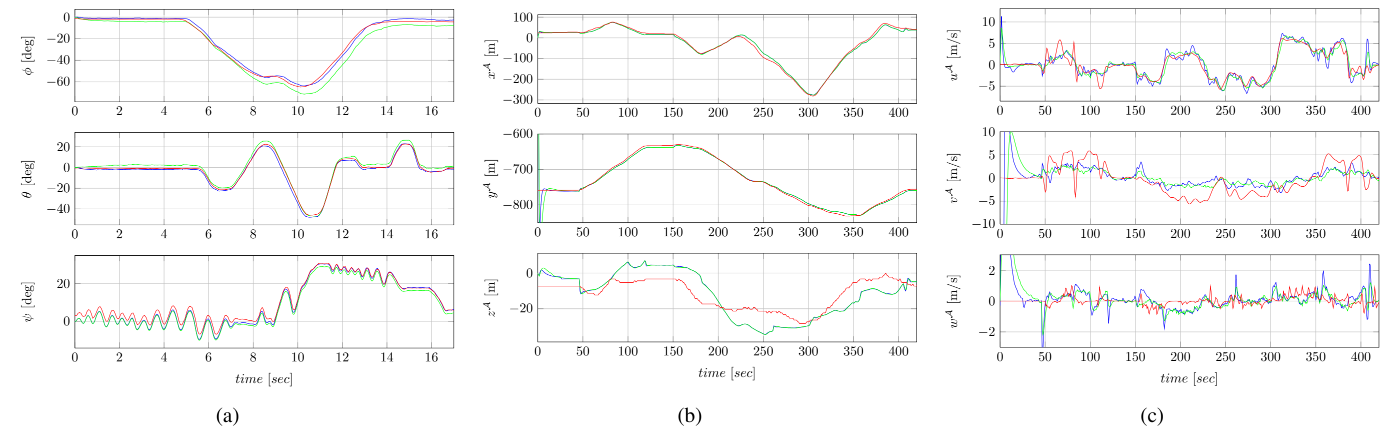

Fig. 2. Estimation results comparing the original and the modified algorithm. Typical results from repeated essays. The estimation results for the original algorithm in green, for the modified in blue and for the reference in red. The notation used for the Euler pairs ϕ for pitch, θ roll and ψ yaw. In the case of linear motion upper-indexes $\mathcal{A}$ ties variables to $\{\mathcal{A}\}$, $(x^{\mathcal{A}}, y^{\mathcal{A}}, z^{\mathcal{A}})$ for position and $(u^{\mathcal{A}}, v^{\mathcal{A}}, w^{\mathcal{A}})$ for velocity.

The first refers to the uncertainty in the initial conditions concreted in the following expression, where H refers linearized Jacobian matrix of the last term of (7):

$$norm_0(\tilde{q}_0, \hat{v}_0) = (\tilde{q}_0 + H\hat{v}_0)^T P(\tilde{q}_0 + H\hat{v}_0) \quad (5)$$

The second norm in (6) refers to the subsequent accumulation of errors, expressed by the model's uncertainty $\hat{w}_k$ and the measurement's uncertainty $\hat{v}_k$ in the equations (8) and (7).

$$norm_1(w_k, v_k) = \begin{bmatrix} w_k \\ v_k \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix} \quad (6)$$

The measurement uncertainty emerge as deviation of $\hat{a}_{s,k}$ to $a_{s,k}$ in the discrete time k, that also depends on $\mathring{\hat{q}}_k$.

$$\hat{v}_k = a_{s,k} - \underbrace{\hat{a}_{s,k}}_{\hat{a}_{s,k} = h_k(\mathring{\hat{q}}_k)} \quad (7)$$

The equation (8) refers to the kinematic model f_k that depends on the measurement of angular velocity expressed as pure quaternion Ω_k and the unitary quaternion $\mathring{\hat{q}}$, that is also linearized as a Jacobian matrix F.

$$\mathring{\hat{q}}_{k+1} = f_k(\mathring{\hat{q}}_k, \Omega_k) + \hat{w}_k \quad (8)$$

Hereby, we propose the solution of the optimization problem finding the next $\mathring{\hat{q}}_{k+1}$ in terms of successive corrections of $\hat{w}_k$ that can be obtained from a dynamical programming approach (v. [18]). Hence, we define the optimal solution, where Q, and R are positive definite matrices selected by design, and P_k in (9a) establishes the discrete-time Riccati equation; finally, F is the linearization of $f_k(\mathring{\hat{q}}_k, \Omega_k)$:

$$w_k = -\underbrace{P_k H^T (R + H P_k H^T)}_{K} \tilde{y}_k \quad (9a)$$

$$w_k^T F^{-T} P_k A^{-1} w_k = w_k^T (F^{-T} Q F^{-1} + P_k - K H P_k) w_k \quad (9b)$$

This defines the observer as follows:

$$\hat{q}_{k+1} = F\hat{q}_k + K(a_{s,k} - H\hat{q}_k) \quad (10)$$

IV. Experimental evaluation

In general, the experimental evaluation aims to investigate the reduction of estimation errors by including the optimal observer stage into the original navigation algorithm; the resulting algorithm has been named *the modified navigation algorithm*. We study the performance of these two algorithms by providing the same input and collecting the results in $\hat{X}_{mdf}$ for the modified algorithm and $\hat{X}_{mh}$ for the original algorithm; these results are compared with a reference measurement X_{real} obtained from an external source. We reduce the whole experiment into two different study cases:

1) Estimation of Euler angles for independent axes. In this case, the rotations were performed around one axis at the time; the reference was obtained with a camera that tracked the trajectory of two black marks placed on the box containing the navigation sensors (a low-cost IMU and GPS).
2) Estimation of position in an urban field. In this case, the experiments were conducted along closed paths in a wheeled vehicle were we mount the navigation sensors and a commercial GPS tracker, the former was used as input for the navigation algorithms and the latter as the reference.

A. Experimental results

We present a sample of many experiments in Fig.2. For the estimation of the Euler angles (2a), where we observe in the worst case an error of $10.58°$ for the modified approach in contrast to 18.97 for the original algorithm. It is also important to remark the bottom graph where the modified algorithm reduces the error progressively, compared to the original algorithm which seems to remain in $1.5[°]$ along the time window that encloses this particular trial. This latter is unusual since we experiment with abrupt changes of direction and the gravity vector parallel to rotating axis, what difficulties the vectorial reconstruction. The results in the case of the linear motion are presented in Fig.2b and Fig.2c. Before the

TABLE I
MODIFIED AND ORIGINAL ALGORITHMS COMPARISON.

Estimation	**MAPE modified**	**MAPE original**	**PRD**
X position (x)	5.8862[m]	5.9427[m]	0.9596%
Y position (y)	3.5445[m]	4.7739[m]	34.6828%
Z position (z)	6.1698[m]	6.3369[m]	2.7080%
X-velocity (v_x)	0.8214[m/s]	0.7871[m/s]	-12.1305%
Y-velocity (v_y)	3.1564[m/s]	3.3291[m/s]	5.4710%
Z-velocity (v_z)	0.4445[m/s]	0.5697[m/s]	28.1664%
Pitching (ϕ)	1.3579 [°]	3.7560 [°]	177.97%
Rolling (θ)	1.4635 [°]	2.801 [°]	91.394%
Yawing (ψ)	1.3268 [°]	2.1855 [°]	44.53%

transitory stage, we observed that the errors vary around $2[m]$ y $10[m]$ in the axis x^A and y^A in $\{\mathcal{A}\}$, besides the errors in the vertical axis remain to bellow $18[m]$. Regarding the linear velocity, we noticed some peaks for both algorithms but they seem to follow the reference, we didn't find big differences.

B. Comparative Analysis

To compare the results, we used two different metrics: the mean absolute percentage error (MAPE) to calculate each variable's performance and the percentage relative difference (PRD) to compare results between algorithms (modified relative to original). The results are ordered in the table I where we identify a better performance for the modified algorithm compared to the original version. The mean of all PRDs (MPRD) shows a 40.92% better performance for the modified algorithm, namely we obtained an average 42.92% better performance by including an optimal pre-computing of R^* compared to the original approach.

In addition, we perform a comparison of processing times, where we contrast the times to process samples sets of 2957 points for both algorithms. In ten trials, we obtained an average time of $1.0158[ms]$ for the modified algorithm in contrast to $0.7954[ms]$ for the original algorithm, i.e a MPRD of -21.6933%, which seems to indicate that the original algorithm uses 21.6933% less time than the modified.

V. CONCLUSION

In this work we proposed an observer architecture composed of non-linear complementary filters on the $SO(3)$ and an optimal observer of the rotation matrix, this considers the signals were coming from an inertial measurement unit (IMU) and a GPS receptor as inputs. Besides, we carried out an experimental exploration of the properties of both algorithms in the same conditions to obtain comparable information. Based on the experimental results, we can say that the navigation algorithm that includes an optimal estimation of the rotation matrix does better in estimation than the algorithm with suboptimal vectorial reconstruction, however, such modification involves a significant increment of complexity.

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