Functional Analysis

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Contents

1 Metric spaces

Definition 1 (Metric space)**.** X *set. Function* $d : x \times X \to [0, \infty)$ *is called a metric if the following condition are met:*

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ - *the inequality of the triangle*

Definition 2. (X, d) - *metric space.*

Example (Discrete space)**.** X - *arbitrary.*

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Example (Real line)**.** $X = \mathbb{R}, \quad d(x, y) = |x - y|$

Example (n-dimentional space)**.** $X = \mathbb{R}^n = \{x = (x_1, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}$

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Example. $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ *- metric on* $\mathbb{R}^n$

Proof.

$$d_1(x,z) = \sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} (|x_i -_i| + |y_i - z_i|) = d_1(x,y) + d_(y,z).$$

□

$d_\infty(x,y) = \max\limits_{1 \leq i \leq n} |x_i - y_i|$ *- metric on* $\mathbb{R}^n$

Proof.

$$d_\infty(x,y) = 0 \iff \forall i : x_i = y_i \iff x = y$$
$$d_\infty(x,z) = \max_{1 \leq i \leq n} |x_i - z_i| \leq d_\infty(x,y) + d_\infty(y,z)$$
$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$

□

$$1 \leq p < \infty : d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}.$$
$$0 < p < 1 : d_p(x,y) = \sum_{i=1}^{n} |x_i - y_i|^p.$$

Example. $C[a,b]$ *set of all continuous functions.* $f : [a,b] \to \mathbb{R}$

$$d(f,g) = \sup_{a \leq t \leq b} |f(t) - g(t)|.$$

$d(f,g)$ *is a metric on* $C[a,b]$*.*

Example. $C_b[\mathbb{R}]$ *- a set of all continuous and limited functions* $f : \mathbb{R} \to \mathbb{R}$*.*

$$d(f,g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)|.$$

Example. (X,d) *- metric space,* $Y \subset X$

$$d(y_1,y_2), y_1, y_2 \in Y.$$

(Y,d) *- subspace* X*.*

Definition 3. *(X, d) - metric space, $\{x_n : n \geq 1\}$ series of X elements. $\{x_n : n \geq 1\}$ converges to $x \in X$ if $\lim\limits_{n\to\infty} d(x_n, x) = 0$.*

$$(\forall \varepsilon > 0 \quad \exists N \quad \forall n \geq N \quad d(x_n, x) < \varepsilon).$$

$$x = \lim_{n\to\infty} x_n.$$

Theorem 1.1. *In metric space convergent sequence has only one boundary.*

Proof. Let $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} x_n = y$.

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0.$$

$$\Rightarrow d(x, y) = 0 \Rightarrow x = y.$$

□

$(X, d_x), (Y, d_y)$ - metric spaces. $f : X \to Y$.

Definition 4.

1. *f continuous in point $x_0 \in X$ if*

$$x_n \to x_0 \text{ in } X \ \Rightarrow f(x_n) \to f(x_0) \text{ in } Y .$$

2. *f continuous on X if f continuous in every point $x_0 \in X$.*

Remark. *f continuous in $x_0 \in X$ then and only then if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : d_x(x, x_0) < \delta \Rightarrow d_x(f(x), f(x_0)) < \varepsilon$$

Definition 5.

1. *$f : X \to Y$ is called homeomorphism if f is bijective, continuous and f^{-1} is continuous.*

2. *$f : X \to Y$ isometric if*

$$d_y(f(x), f(x')) = d_x(x, x').$$

Isometrie is always continuous.

$x \in X, \quad r > 0$

Definition 6. *Open ball*

$$\mathbb{B}(x, y) = \{y \in X : d(y, x) < r\}.$$

Definition 7. *Closed ball*

$$\overline{\mathbb{B}} = \{y \in X : d(y, x) \leq r\}.$$

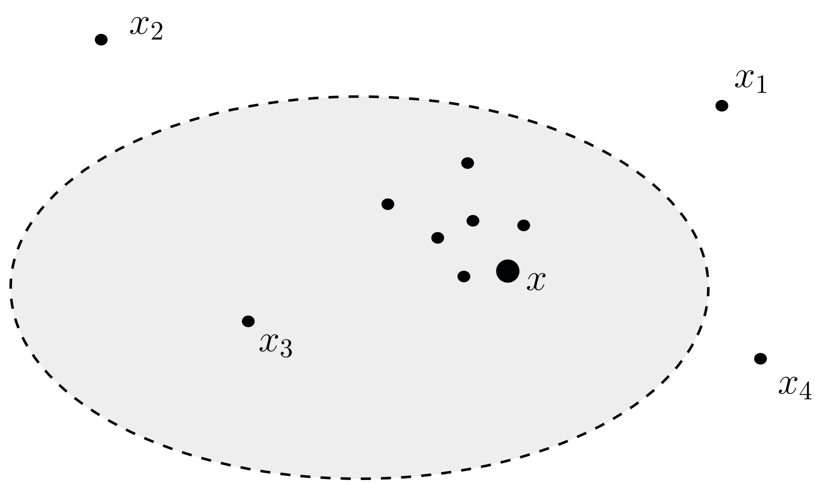

Figure 1: Convergence in terms of open and closed ball definitions

Convergence can be rewritten using the last two definitions:

$$x_n \to x \iff \forall \varepsilon > 0 \quad \exists N \quad \forall n \geq N \quad x_n \in \mathbb{B}(x, \varepsilon).$$

Definition 8. *$A \subset X$. Point x is tangent to set A if*

$$\forall \varepsilon > 0 \quad \mathbb{B}(x, \varepsilon) \cap A \neq \varnothing.$$

Example. *$X = \mathbb{R}$, $A = (a, b)$. a and b are tangent to A.*
All elements from set A are tangent to A.
If there's some $\exists c > b$ then we can pick some ball around c of radius r. In that ball there would be no elements from A.

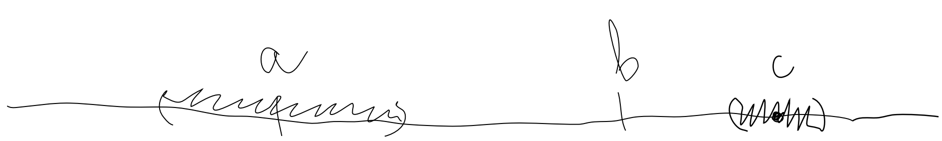

Figure 2: Tangent point example

Definition 9. *$\overline{A} = \{x \in X : x$ tangent to $A\}$ closure of set A.*

Theorem 1.2 (Properties of closure)**.** *Set A with closure $\overline{A}$ has the following properties:*

1. $A \subset \overline{A}$

2. $\overline{\overline{A}} = \overline{A}$ — *idempotence*

3. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. 1

$$x \in A \Rightarrow \mathbb{B}(x, \varepsilon) \cap A \neq \varnothing \text{ since it contains } x\ .$$

3

$$x \in \overline{A} \Rightarrow \mathbb{B}(x, \varepsilon) \cap A \neq \varnothing \Rightarrow$$
$$\Rightarrow \mathbb{B}(x, \varepsilon) \cap B \neq \varnothing \Rightarrow x \in \overline{B}.$$

2

$$\overline{A} \subset \overline{\overline{A}}$$
$$\text{need to show that } \overline{\overline{A}} \subset \overline{A}$$
$$x \in \overline{\overline{A}}, \varepsilon > 0. \quad \mathbb{B}(x, \varepsilon) \cap \overline{A} \neq \varnothing$$
$$\text{exists point } y \in \mathbb{B}(x, \varepsilon) \cap \overline{A}..$$

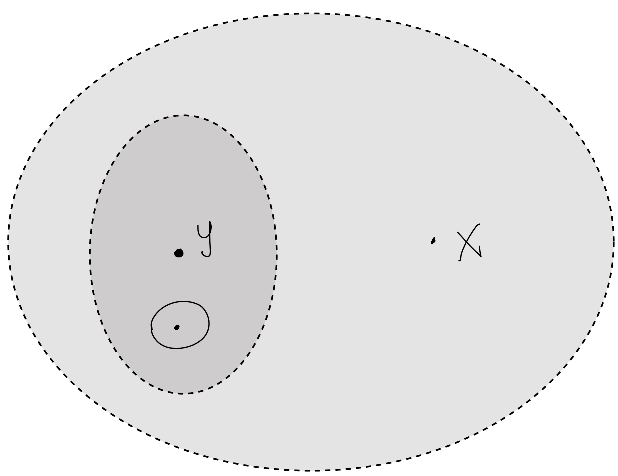

Figure 3: Eps-Ball Y in eps-Ball x with point from A

Lets show that $\mathbb{B}(y,\varepsilon-d(x,y))\subset\mathbb{B}(x,\varepsilon)$.

$$z\in\mathbb{B}(y,\varepsilon-d(x,y))$$
$$\text{for } z \text{ the following is met } d(z,y)<\varepsilon-d(x,y)$$
$$\varepsilon>d(z,y)+d(y,x)\geq d(z,x)\Rightarrow z\in\mathbb{B}(x,\varepsilon)$$
$$\mathbb{B}(y,\varepsilon-d(x,y))\cap A\neq\varnothing\Rightarrow\mathbb{B}(x,\varepsilon)\cap A\neq\varnothing$$
$$\Rightarrow x\in\overline{A}.$$

4

$$A\subset A\cup B\Rightarrow\overline{A}\subset\overline{A\cup B};\ \overline{B}\subset\overline{A\cup B}$$
$$\Rightarrow\overline{A}\cup\overline{B}\subset\overline{A\cup B}.$$

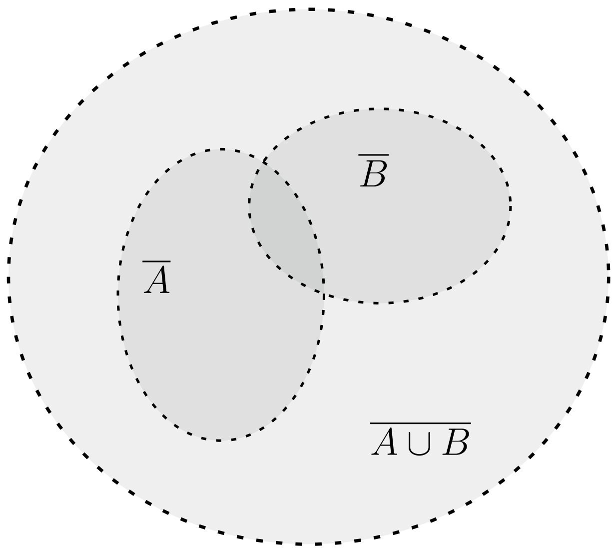

Figure 4: For stupid idiots

Need to prove $\overline{A\cup B}\subset\overline{A}\cup\overline{B}$

(by contradiction)
Let $x\in\overline{A\cup B}$ and $x\notin\overline{A}$, $x\notin\overline{B}$.

$$\exists\varepsilon_1>0:\mathbb{B}\left(x,\varepsilon_1\right)\cap A=\varnothing.$$

$$\exists\varepsilon_2>0:\mathbb{B}\left(x,\varepsilon_2\right)\cap B=\varnothing.$$

$$\varepsilon=\min(\varepsilon_1,\varepsilon_2).\quad\mathbb{B}(x,\varepsilon)\cap(A\cup B)=\varnothing.$$
$$\Rightarrow\overline{A\cup B}=\overline{A}\cup\overline{B}.$$

□

Theorem 1.3. *$x \in \overline{A} \iff$ in set A exists series $(x_n : n \geq 1)$ that converges to x.*

Proof. $(\Rightarrow)$ Let $x \in \overline{A}$

$$\forall \varepsilon > 0 \quad \mathbb{B}(x, \varepsilon) \cap A \neq \varnothing.$$

let $\varepsilon_n = \frac{1}{n}$
$\forall n \geq 1$ exists point $x_n \in A \cap \mathbb{B}(x, \frac{1}{n}$

$$0 \leq d(x, x_n) < \frac{1}{n} \to 0. \quad \lim_{n\to\infty} x_n = x.$$

$(\Leftarrow)$ Let $\lim_{n\to\infty} x_n = x$ and $x_n \in A$.

$$\forall \varepsilon > 0 \quad \exists N \quad \forall n \geq N \quad d(x_n, x) < \varepsilon.$$

$$x_n \in \mathbb{B}(x, \varepsilon) \cap A \neq \varnothing.$$

$$\Rightarrow x \in \overline{A}.$$

□

Definition 10. *A is dense in set B if $B \subset \overline{A}$ (any B element can be approached to elements of A)*

Definition 11. *A dense everywhere if $\overline{A} = X$.*

Definition 12. *Metric space (X, d) is separable if exists dense everywhere countable set.*

Example. *1. $\mathbb{R}$ - separable space. $\overline{\mathbb{Q}} = \mathbb{R}$*

2. $\mathbb{R}^n$ - separable space related to any metric $d_p,\ 0 < p \leq \infty$. $\overline{\mathbb{Q}^n} = \mathbb{R}^n$

3. X, d - discrete. $\mathbb{B}(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$. But if $0 < \varepsilon < 1$ then

$$\mathbb{B}(x, \varepsilon) \cap A \neq \varnothing \iff x \in A..$$

$$\Rightarrow \overline{A} = A.$$

The only dense everywhere set is X.

4. $C[a, b]$; $d(f, g) = \sup_{t\in[a,b]} |f(t) - g(t)|$
By Weierstrasse theorem $\forall f \in C[a, b] \quad \forall \varepsilon > 0$ exists polynomial

$$P(t) = a_0 + a_1 t + \ldots + a_d t^d : \sup_{t\in[a,b]} |f(t) - P(t)| < \varepsilon$$

Dense everywhere set is set of polymonials with rational coefficients.

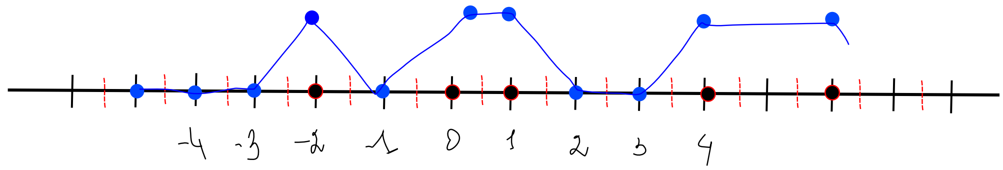

Figure 5: Line for example of not separable metric space

5. $C_b(\mathbb{R}), d(f,g) = \sup\limits_{t\in\mathbb{R}} |f(t) - g(t)|$ *- is not separable metric space.*

$$A \subset \mathbb{Z} \quad f_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \in \mathbb{Z}\backslash A \end{cases}.$$

$$A \neq A' \quad n \in A\backslash A' \textit{ or } n \in A'\backslash A.$$

$$d(f_A, f_{A'}) = 1.$$

$$\mathbb{B}(f_A, \frac{1}{2}) \cap \mathbb{B}(f_{A'}, \frac{1}{2}) = \varnothing.$$

In space $C_b(\mathbb{R})$ exists a continuum family of open balls that do not intersect.

If dense everywhere set exists than in every open ball must be an element of the one.

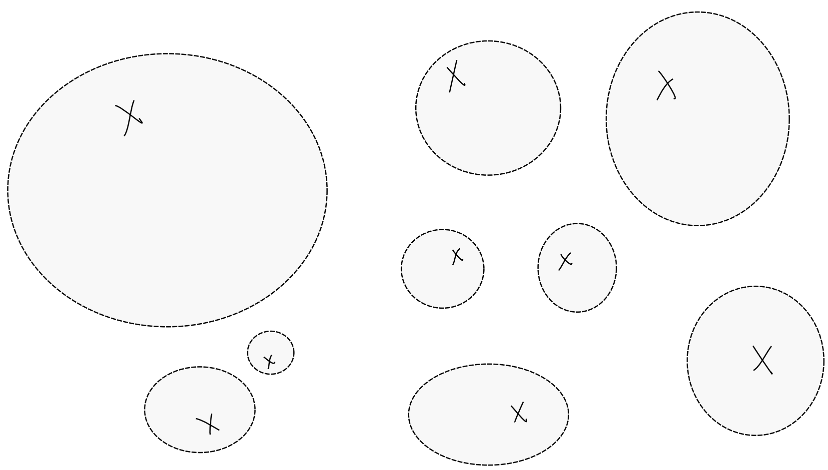

Figure 6: Family of open balls each containing an element