Functional Analysis

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Contents

1 Lecture 1: Metric Spaces and Convergence

Definition 1. X is a set. Function $d : X \times X \to [0, \infty]$ is called a metric if three of the conditions are met:

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ — triangle inequality

(X, d) – is a metric space.

Example (1. Discrete space)**.** X — arbitrary.

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Example (2. Real numbers)**.** $X = \mathbb{R}$, $d(x, y) = |x - y|$

Example. $X = \mathbb{R}^n = \{x = (x_1, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}$ $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ – metric on $\mathbb{R}^n$

Proof. $d_1(x, z) = \sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = d_1(x, y) + d_1(y, z)$ $\square$

Example. $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ – metric on $\mathbb{R}^n$

Proof. $d_\infty(x,y) = 0 \Leftrightarrow \forall i x_i = y_i \Leftrightarrow x = y$

$d_\infty(x,z) = max_{1\leq i\leq n}|x_i - y_i| \leq d_\infty(x,y) + d_\infty(y,z)$

$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq d_\infty(x,y) + d_\infty(y,z)$ □

Example. $1 \leq p \leq \infty$

$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right) \frac{1}{p}$ — metric on $\mathbb{R}^n$

$0 \leq p \leq 1 : d_p(x,y) = \sum_{i=1}^{n} |x_i - y_i|^p$ metric on $\mathbb{R}^n$

Example. $C[a,b]$ – a set of all continuous functions $f : [a,b] \to \mathbb{R}$

$d(f,g) = sup_{a\leq t\leq b}|f(t) - g(t)|$ — metric on $C[a,b]$

Example. $C_b(\mathbb{R})$ — a set of all continuous and bounded functions $f : \mathbb{R} \to \mathbb{R}$.

$d(f,g) = sup_{t\in R}|f(t) - g(t)|$

Example. (X,d) — metric space; $Y \subset X$

$d(y_1,y_2), \quad y_1,y_2 \in Y$

(Y,d) — subspace X

Definition 2. (X,d) – metric space, $(x_n : n \geq 1)$ – sequence of elements X. $(x_n, n \geq 1)$ converges to $x \in X$ if $\lim\limits_{n\to\infty} d(x_n,x) = 0$.

$(\forall\varepsilon > 0 \ \ \exists N \ \forall n \geq N \ \ d(x_n,x) < \varepsilon)$

$$x = \lim_{n\to\infty} x_n$$

Theorem 1. In metric space sequence that converges has only ONE limit.

Proof. Let $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} x_n = y$

$$d(x,y) \leq d(x,x_n) + d(x_n,y) \to 0$$

$$\Rightarrow d(x,y) = 0 \to x = y.$$

□

$(X,d_x),(Y,d_y)$ — metric spaces. $f : X \to Y$

Definition 3. f – continuous in point $x_0 \in X$, if

$$x_n \to x_0 \text{ in x } \Rightarrow f(x_n) \to f(x_0) \text{ in Y}$$

Definition 4. f continuous on X if f is continuous in every point $x_0 \in X$.

Exercise

f is continuous in point $x_0 \in X$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0 : d_x(x, x_0) < \varepsilon \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

Definition 5. $f : X \to Y$ homogeneous (гомеоморфізм) if f is bijective, continuous and f^{-1} is continuous.

Definition 6. $f : X \to Y$ isometric if $d_y(f(x), f(x')) = d_x(x, x')$ (isometrie is always continuous)

$x \in X,\ r > 0$

Definition 7. Open ball $\mathbf{B}(x, r) = \{y \in X : d(y, x) < r\}$

Definition 8. Closed ball $\overline{B}(x, r) = \{y \in X : d(y, x) \leq r\}$

$$x_n \to x \Leftrightarrow \forall \varepsilon > 0 :\ \exists N\ \ \forall n \geq N : x_n \in \mathbf{B}(x, \varepsilon)$$

Definition: $A \subset X$. Point x tangent to the set A, if $\forall \varepsilon > 0$

$$\mathbf{B}(x, \varepsilon) \cap A \neq \varnothing$$

Example: $X = \mathbb{R}$. $A = (a, b)$ a and b tangent to A

![[Drawing 2023-09-05 20.44.54.excalidraw]]

2.

$$\overline{A} = \{x \in X : x \text{ дотична до } A\}$$

closed set A

Theorem 2 1. $A \subset \overline{A}$ 2. $\overline{\overline{A}} = \overline{A}$ 3. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ 4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof: 1. $x \in A \Rightarrow B(x, \varepsilon) \cap A \neq \varnothing$ as does not contain x 3. $xinn\overline{A} => B(x, \varepsilon) \cup A \neq \varnothing \Rightarrow B(x, \varepsilon) \cup B \neq \varnothing \Rightarrow x \in \overline{B}$ 2. $\overline{A} \subset \overline{\overline{A}}$ need to show that $\overline{\overline{A}} \subset \overline{A}$ $x \in \overline{\overline{A}}, \varepsilon > 0$ $B(x, \varepsilon) \cap \overline{A} \neq \varnothing$ exists such a point that $y \in B(x, \varepsilon) \cap \overline{A}$![[Drawing 2023-09-05 20.52.53.excalidraw]] show that $B(y, \varepsilon - d(x, y)) \subset B(x, \varepsilon)$ $z \in B(y, \varepsilon - d(x, y))$. $d(z, y) < \varepsilon) - d(x, y)$ $\varepsilon > d(z, y) + d(y, x) \geq d(z, x) \Rightarrow z \in B(x, \varepsilon)$ $B(y, \varepsilon - d(x, y)) \cap A \neq \varnothing \Rightarrow B(x, \varepsilon) \cap A \neq \varnothing$ $x \in \overline{A}$

4. $a \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$; $\overline{B} \subset \overline{A \cup B}$

$\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ Let $x \in \overline{A \cup B}$ $x \notin \overline{A},\ x \notin \overline{B} \Rightarrow \varepsilon_1 > 0 : B(x, \varepsilon_1) \cap A = \varnothing \Rightarrow \varepsilon_2 > 0 : B(x, \varepsilon_2) \cap B = \varnothing$

$\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ $B(x, \varepsilon), \cap (A \cup B) = \varnothing$ $\overline{A \cup B} = \overline{A} \cup \overline{B}$ —

Theorem 3 $x \in \overline{A} \Leftrightarrow$ in set A there is a sequence $(x_n : n \geq 1)$ that converges to x

Proof: $(\Rightarrow)$ Let $x \in \overline{A}$ $\forall \varepsilon > 0$ $B(x, \varepsilon) \cap A \neq \varnothing$, $\quad \varepsilon_n \dfrac{1}{n}$ $\forall n \geq 1$ there is a point $x_n \in A \cap B(x, \dfrac{1}{n})$

$0 \leq d(x, x_n) < \dfrac{1}{n} \to 0$ $\lim\limits_{n \to \infty} x_n = x$

$(<=)$ let $\lim\limits_{n \to \infty} x_n = x,\ \ x_n \in A$

$$\forall \varepsilon > 0\ \exists N\ \forall n \geq N\ d(x_n, x) < \varepsilon$$

$$x_n \in B(x, \varepsilon) \cap A$$

$x \in \overline{A}$

Definition 1. A is dense in a set B if $B \subset \overline{A}$ 2. A is dense everywhere if $\overline{A} = X$ 3. Metric space (X, d) separable if there is a countable everywhere dense set in it.

Examples: 1. $\mathbb{R}$ separable space. $\overline{\mathbb{Q}} = \mathbb{R}$ 2. $\mathbb{R}^n$ separable related to any metric $d_p, 0 < p \leq \infty$ 3. X, d – discrete. $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\} = x$ $B(x, \varepsilon) \cap A \neq \varnothing \Leftrightarrow x \in A$ $\overline{A} = A$ The only everywhere dense set is X. 4. $C[a, b]$; $d(f, g) = sup_{t \in [a,b]} |f(t) - g(t)|$ by theorem of Weierstrasse $\forall f \in C[a, b]$ $\forall \varepsilon > 0$ there is a polynomial $P(t) = a_0 + a_1 t + \cdots + a_d t^d$: $sup_{t \in [a,b]} |f(t) - P(t)| < \varepsilon$ *Countable everywhere dense set is a set of polynomials with rational coefficients.* 5. $C_b(\mathbb{R}), d(f, g) = sup_{t \in \mathbb{R}} |f(t) - g(t)|$ — not separable metric set. ![[Drawing 2023-09-05 21.43.21.excalidraw]] $A \subset \mathbb{Z}$

$$f_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \in \mathbb{Z} \backslash A \end{cases}$$

$A \neq A'$; $n \in A \backslash A'$ or $n \in A' \backslash A$ $d(f_A, f_{A'}) = 1$ $B\left(f_A, \dfrac{1}{2}\right) \cap B(f_{A'}, \dfrac{1}{2}) = \varnothing$

In space $C_b(\mathbb{R})$ exists continual family of open balls that do not intersect by pairs.

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(X, d) $A \subset X$ $\overline{A} = \{x \in X : \forall \varepsilon > 0 B(x, \varepsilon) \cap A \neq \varnothing\}$ Let x $in \overline{A}$, $y \neq x$. $\varepsilon < d(x, y) \Rightarrow B(X, \varepsilon)$ does not contain y. if for any $\varepsilon > 0$ $B(X, \varepsilon) \cap A$ finite then:

$$\exists \delta > 0 : B(X, \delta) \cap A = \{x\}$$

in this case point x is called isolated point of the set A

If $x \in \overline{A}$ and is not isolated, then x is called гранична

x is гранична to the set $A \Leftrightarrow \forall \varepsilon : B(X, \varepsilon) \cap A$ infinite

Example: 1. X is discrete. $B(X, 1) = \{x\}$ $\overline{A} = A$ is filled with only isolated points 2. $X = \mathbb{R}$. $A = (a, b)$. $\overline{A} = [a, b]$ is composed out of cluster points.

Definition A set A of metric space X is closed if $\overline{A} = A$.

Example: 1. X, $\varnothing$ are closed. 2. $\overline{B}(x, r)$ closed

$$\overline{\overline{B}(x, r)} \subset \overline{B}(x, r)$$

Let $y \notin \overline{B}(x, r)$ $d(x, y) > r$. $\varepsilon = d(x, y) - r$ If $z \in B(y, \varepsilon)$, then $d(y, z) < \varepsilon$ $d(z, x) \leq d(x, y) - d(z, y) > d(x, y) - \varepsilon = r$ $z \notin \overline{B}(x, r)$. $B(y, \varepsilon) \cap \overline{B}(x, r) = \varnothing$ and $y \notin \overline{\overline{B}(x, r)}$.

3. $\overline{A}$ closed ($\overline{\overline{A}} = \overline{A}$) 4. $\overline{A}$ – smallest closed set the contains A. (if B is closed and $A \subset B$ then $\overline{A} \subset B$)

Theorem 1. Intersection of any arbitrary closed sets is a closed set 2. Union of finite number of closed sets is a closed set

Proof: 1. Consider $(A_i)_{i \in I}$ — closed sets

$$A = \bigcap_{i \in I} A_i$$

$$\forall i \in I : \overline{A_i} = A_i$$

$A \subset A_i$ $\overline{A} \subset \overline{A_i} = A_i$ $\overline{A} \subset \bigcap_{i \in I} A_i = A \subset \overline{A} \Rightarrow \overline{A} = A$ and A is closed. 2. If A and B are closed, then $\overline{A \cup B} = \overline{A} \cup \overline{B} = A \cup B$

Example: $X = \mathbb{R}$. $A_n = [0, 1 - \frac{1}{n}]$ $\;n \geq 1$

$$\bigcup_{n=1}^{\infty} A_n = [0, 1)$$

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Definition 1. Point $x \in X$ is inner for the set A if

$$\exists \varepsilon > 0 : B(x, \varepsilon) \subset A$$

2. $A^o = \{x \in X : x \text{ inner for } A\}$ — ***interior*** 3. A is open if $A = A^o$

Example: 1. $B(x, r)$ is an open set. $y \in B(x, r)$, $d(x, y) < r$. $\varepsilon = r - d(x, y)$. if $z \in B(y, \varepsilon)$ then $d(y, z) < \varepsilon$

$$d(z, x) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon = r$$

2. $X = \mathbb{R}$. $A = [a, b]$, $\;a < b$ $a < x < b \Rightarrow x \in A^o$ $A^o = (a, b)$ 3. $X, \varnothing$ are open.

Theorem For any arbitrary set $A \subset X$ it is true that

$$X \setminus A^o = \overline{X \setminus A}$$

Proof

$$x \in X \setminus A^o \Rightarrow x \notin A^o \Leftrightarrow \forall \varepsilon > 0 \;\; B(x, \varepsilon) \not\subset A \Leftrightarrow \forall \varepsilon > 0 B(x, \varepsilon) \cap (X \setminus A) \neq \varnothing \Leftrightarrow X \in \overline{X \setminus A}\}$$

Consequences 1. $A^o \subset A$, $(X \setminus A^o = \overline{X \setminus A} \subset X \setminus A)$ 2. $A \subset B \Rightarrow A^o \subset B^o$ 3. $(A^o)^o = A^o$ 4. $(A \cap B)^o = A^o \cap B^o$ 5. A is open $\Leftrightarrow X \setminus A$ is closed $(A^o = A \Leftrightarrow X \setminus A^o = X \setminus A = \overline{X \setminus A})$ 6. Union of arbitrary family of open sets is an open set. 7. Intersection of finite number of open sets is an open set

Example 1. X — discrete space. All the sets are open. 2. $X = \mathbb{R}$. Set is open $\Leftrightarrow$ a set is a union of intervals sequence (open intervals)

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$X : d$ – metric on X. A set of all open sets is called a topology of the space X.

$$\lim_{n \to \infty} x_n = x \Leftrightarrow \forall \varepsilon > 0 \varepsilon \;\; \exists N \forall n \geq N x_n \in B(x, \varepsilon)$$
$$\Leftrightarrow \forall \text{ open set } U \text{ that contains } x, \exists N \; \forall n \geq N \; x_n \in U$$

Theorem $d_1 : d_2$ — metric on X. d_1 and d_2 define the same topology on X if and only if the convergence on these metrics is the same (in other words $d_1(x_n, x) \to 0 \Leftrightarrow d_2(x_n, x) \to 0$)

Proof: 1. Let the open sets relatively d_1 and d_2 coincide. Let $d_1(x_n, x) \to 0$

$$\forall \varepsilon > 0 : B_{d_2}(x, \varepsilon) \text{ open relatively } d_2 \Rightarrow B_{d_2}(x, \varepsilon) \text{ open relatively } d_1$$

$$\Rightarrow \exists \delta > 0 : B_{d_1}(x, \delta) \subset B_{d_2}(x, \varepsilon)$$
$$\exists N : \forall n \geq N : d_1(x_n, x) < \delta \Rightarrow d_2(x_n, x) < \varepsilon$$

2. Let the convergence in d_1 and d_2 be equivalent. Consider the the set $A \subset X$ exists that is open relatively to d_1 and not open relatively to d_2. $\exists x \in A$: x not inner for A relatively d_2.

$$\forall n \geq 1 : B_{d_2}\left(x, \frac{1}{n}\right) \not\subset A. \quad \forall n \geq 1 \exists x_n \notin A$$

$d_2(x_n, x) < \dfrac{1}{n}$ $d_2(x_n, x) \to\Rightarrow d_1(x_n, x) \to 0$ Relatively d_1 A is open, $x \in A \Rightarrow \exists N \quad \forall n \geq N$ $x_n \in A$. *Contradiction.*

Definition Two metrics: d_1 and d_2 on a set X are equivalent if they define the same topology (define the same convergent sequences).

Exercise: On $\mathbb{R}^n$ all the metrics $d_p, 0 < p \leq \infty$ are equivalent.

Topology of subspace

(X, d) — metric space, $Y \subset X$. $(Y, d|_{Y \times Y})$ — subspace

$y \in Y, r > 0$. $B_Y(y, r) = \{y' \in Y : d(y, y') < r\} = Y \cap B_X(y, r)$

$A \subset Y$ $\overline{A}_Y = \{y \in Y : \forall \varepsilon > 0 \;\; B(y, \varepsilon) \cap A \neq \varnothing\} = Y \cap \overline{A}_X$

$A \subset Y$ is closed relatively to Y if and only if $A = Y \cap F$ where F is closed in X.

$A \subset Y$ is open relatively to Y if and only if $A = Y \cap G$ where G is open in X.

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Definition A sequence $(x_{n\geq 1}^{\infty})$ in metric space (X, d) is fundamental (Cauchy sequence) if

$$d(x_n, x_m) \to 0 \;\; n, m \to \infty$$

$$\forall \varepsilon > 0 \;\; \exists N \;\; \forall n, m \geq N \;\; d(x_n, x_m) < \varepsilon$$

Corollary Convergent sequence in fundamental.

Proof: Let $x_n \to x, \;\; n \to \infty \Rightarrow d(x_n, x) \to 0, \;\; n \to \infty$ $\forall \varepsilon > 0 \;\; \exists N \;\; \forall n \geq N \;\; d(x_n, x) < \dfrac{\varepsilon}{2}$

If $n, m \geq N$ then $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon$

Definition: (X, d) — full, if in X any fundamental sequence is convergent.

Example: 1. $X = \mathbb{R}$, $d(x, y) = |x - y|$ — full metric space 2. $X = \mathbb{R}^n$, $d_2(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ Let $(x^{(k)})_{k\geq 1}$ fundamental sequence in $(\mathbb{R}^n, d_2)$ $x^{(k)} = \left(x_1^{(k)}, \ldots, x_n^{(k)}\right)$

$$0 \leftarrow d_2(x^{(k)}, x^{(m)}) = \sqrt{\sum_{i=1}^{n}(x_i^{(k)} - x_i^{(m)})^2} \geq \left|x_i^{(k)} - x_i^{(m)}\right|, \quad k, m \to \infty$$

$(x_i^{(k)})_{k\geq 1}$ — fundamental in $\mathbb{R}$.

$\exists \lim\limits_{k\to\infty} x_i^{(k)} = x_i, \quad 1 \leq i \leq n \quad x = (x_1, \ldots, x_n)$ $d_2(x^{(k)}, x) = \sqrt{\sum_{i=1}^{n} \underbrace{(x_i^{(k)} - x_i)}_{0}^{2}} \to_{k\to\infty} 0$

Exercise: $\forall p \in (0, \infty] \quad (\mathbb{R}^n, d_p)$ — full space

3. $X = C[a, b]$

$$d(f, g) = \sup_{a \leq t \leq b} |f(t) - g(t)|$$

$(C[a,b],d)$ — full metric space Let $(f_n)_{n\geq 1}$ fundamental sequence in that full metric space

$0 \leftarrow d(f_n, f_m) = \sup\limits_{a\leq t\leq b} |f_n(t) - f_m(t)| \geq \geq |f_n(t) - f_m(t)|$ fixed t $(f_n(t))_{n\geq 1}$ — fundamental sequence in $\mathbb{R}$ $\exists \lim\limits_{n\to\infty} f_n(t) =: f(t)$ $f : [a,b] \to \mathbb{R}$

$\forall \varepsilon > 0 \quad \exists N \quad \forall n, m \geq N \quad \forall t \in [a,b] \quad |f_n(t) - f_m(t)| \leq \varepsilon$ $m \to \infty$

$$\Rightarrow \forall \varepsilon > 0 \ \exists N \forall n \geq N \quad \underbrace{\forall t\, |f_n(t) - f(t)| \leq \varepsilon}_{d(f_n,f)\leq\varepsilon}$$

Lets show that f is continuous by t $t_0 \in [a,b]$. Need to prove that $\forall \varepsilon > 0 \exists \delta > 0 : |t - t_0| < \delta \Rightarrow |f(t) - f(t_0)| < \varepsilon$

$$\exists N : \forall n \geq N : \sup_S |f_n(s) - f(s)| < \frac{\varepsilon}{3}$$

$$|f_N(t) - f_N(t_0)| < \frac{\varepsilon}{3} \text{ if } |t - t_0| < \delta$$

$$|f(t) - f(t_0)| \leq |f_N(t) - f(t)| + |f_N(t_0) - f(t_0)| + \left|f_{N(t)} - f_N(t_0)\right| < \varepsilon$$

Example. $\mathbb{R}$, $d_1(x,y) = |e^x - e^y|$, $d(x,y) = |x-y|$

metrics d_1 and d are equivalent.

$(\mathbb{R}, d_1)$ is not complete. $x_n = -n, n \geq 1$

$$d_1(x_n, x_m) = |e^{x_n} - e^{x_m}| = \left|e^{-n} - e^{-m}\right| \to 0, \quad n, m \to \infty$$
$$d_1(x_n, x) = |e^{x_n} - e^x| = \left|e^{-n} - e^x\right| \to e^x$$

e^e set mutually unambiguous correspondence between $\mathbb{R}$ and $(0, \infty)$

Example. $C[a,b]$, $d_1(f,g) = \displaystyle\int_a^b |f(t) - g(t)|\, dt$

$(C[a,b], d_1)$ is not complete metric space.

$$f_n(t) = \begin{cases} 1 & t \geq c \\ 0 & t \leq c - \frac{1}{n} \\ \text{linear on } [c - \frac{1}{n}, c] & \end{cases}$$

$$d_1(f_n, f_m) = \int_a^b |f_n(t) - f_m(t)|\, dt \leq \int_{c-\frac{1}{n}}^c 2dt = \frac{2}{n} \to_{n,m\to\infty} 0$$

If $d_1(f_n, f) \to 0$, $n \to \infty$, then $f(t) = \begin{cases} 1 & t \leq c \\ 0 & t < c \end{cases}$ which cannot be true for continuous f.

Example.

$$l^2 = \{x = (x_1, \ldots \mid \sum_{i=1}^{\infty} x_i^2 < \infty\}$$

$$d(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$$

(l^2, d) — complete metric space

$(x^{(k)})_{k\geq 1}$ — fundamental sequence in l^2

$$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots$$

$$d(x^{(k)}, x^{(m)}) = \sqrt{\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i^{(m)}\right)^2} \to 0, \quad n, m \to \infty$$

Let's freeze the number of n.

$$\left|x_n^{(k)} - x_n^{(m)}\right| \leq \sqrt{\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i^{(m)}\right)^2} = d(x^{(k)}, x^{(m)}) \to 0 \;\; k, m \to \infty$$

$$\exists \lim_{k\to\infty} x_n^{(k)} := x_n$$

$$\varepsilon > 0 : \exists N : \forall k, m \geq \mathrm{N} : d(x^{(k)}, x^{(m)}) \leq \varepsilon$$

$$\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i^{(m)}\right)^2 \leq \varepsilon^2, \quad k, m \geq N$$

$$\sum_{i=1}^{M}\left(x_i^{(k)} - \underbrace{x_i^{(m)}}_{x_i, \text{within } m\to\infty}\right)^2 \leq \varepsilon^2, \quad k, m \geq N, M \geq 1$$

$$\sum_{i=1}^{M}\left(x_i^{(k)} - x_i\right)^2 \leq \varepsilon^2, \quad k \geq N, M \geq 1$$

$$\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i\right)^2 \leq \varepsilon^2$$

$$\Rightarrow \begin{cases} \sum_{i=1}^{\infty} x_i^2 < \infty, & x \in l^2 \\ d(x^{(k)}, x) \leq \varepsilon, & k \geq N \end{cases}$$

Corollary 1. 1. Closed subspace of a complete space is complete.

2. Complete subspace of a metric space is closed.

Proof. 1. (X, d) is complete. Y — closed subset of X.

$(x_n)_{n\geq 1}$ — fundamental in $Y \Rightarrow (x_n)_{n\geq 1}$ — fundamental in $X \Rightarrow (x_n)_{n\geq 1}$ converges to $x \in X \Rightarrow x \in Y$ and $(x_n)_{n\geq 1}$ is convergent in Y.

2. Let Y — a subspace of space X, Y is complete.

   $y \in \overline{Y} \Rightarrow$ exists sequence $(y_n)_n$ in Y that converges to $y \Rightarrow (y_n)$ fundamental $\Rightarrow (y_n)$ converges in $Y \Rightarrow y \in Y$.

$\square$

Theorem 2 (about nested balls)**.** (X, d) metric space. X is complete if and only if any arbitrary sequence of nested closed balls which have $R \to 0$ has non-empty intersection.

Proof. ($\Rightarrow$) Let X is a complete. $B_n = \overline{B}(X_n, r_n), B_1 \supset B_2 \supset B_3 \ldots, r_n \to 0$

$d(x_n, x_m) \leq^{n \leq m} r_n \to 0, \quad n \to \infty$

$\exists \lim\limits_{n \to \infty} x_n := x$

$n \geq N \Rightarrow x_n \in B_N, \quad n \geq N \Rightarrow x \in B_N$

$\bigcap\limits_{n=1}^{\infty} B_n \neq \varnothing$

($\Leftarrow$)

Let $(x_n)_{n \geq 1}$ — fundamental in X

$$\exists n_1 \ \ \forall n, m \geq n_1 : d(x_n, x_m) \leq \frac{1}{2}$$
$$\exists n_2 \geq n_1 : \forall n_1, m \geq n_2 : d(x_n, x_m) \leq \frac{1}{4}$$
$$\ldots$$

$$1 \leq n_1 < n_2 < n_3 \ldots : \forall n, m \geq n_k : d(x_n, x_m) \leq \frac{1}{2^k}$$
$$d(x_{nk}, x_{nk+1}) \leq 2^{-k}$$

$$B_k = \overline{B}(x_{nk}, 2^{-k+1}$$

Let's show that $B_{k+1} \subset B_k$

$y \in B_{k+1} : d(y, x_{n_{k+1}}) \leq 2_{-k}$

$$d(y, x_{nk}) \leq d(y, x_{nk+1}) + d(x_{nk+1}, x_{nk}) \leq 2^{-k+1}$$
$$\exists x \in \cap_{k \geq 1} B_k$$
$$d(x_{nk}, x) \leq 2^{-k+1} \to 0$$

$$x_{nk} \to x, \quad k \to \infty$$

$$\varepsilon > 0. \quad \exists N : \forall n, m \geq N : d(x_n, x_m) < \frac{\varepsilon}{2}$$
$$\exists n_k \geq N : d(x_{nK}, x) \leq \frac{\varepsilon}{2}$$

if $n \geq N$ then $d(x_n, x) \leq \varepsilon$

$\square$

2 Completation of Metric Space

Definition 9. Complete metric space $(\hat{X}, \hat{d})$ is a complitation of metric space (X, d) if X is isometric to dense everywhere subset of X.

Theorem 3. For any arbitrary metric space X its completation exists and only one with the precision to isometrie.

Proof. (Oneness)

$(\hat{X}, \hat{d})$ and $(\tilde{X}, \tilde{d})$ — a completion (X, d).

$f : X \to \hat{X}$ isometrie between X and $f(x), \overline{f(X)} = \hat{X}$

$g : X \to \tilde{X}$ isometrie between X and $g(x), \overline{g(X)} = \tilde{X}$

$\hat{x} \in \hat{X}$. $\hat{x} = \lim\limits_{n\to\infty} f(x_n)$.

$(f(x_n))$ convergent $\Rightarrow$ fundamental $\Rightarrow$ (x_n) fundamental $\Rightarrow$ $(g(x_n))$ fundamental in $\tilde{X}$

$\varphi(\hat{x}) = \lim\limits_{n\to\infty} g(x_n)$

Further need to show that φ is isometric

(Existence)

$S(X)$ set of all fundamental sequences in X.

$s \in S(X) \Rightarrow s = (x_1, x_2, \ldots)$. $\quad d(x_n, x_m) \to n, m \to \infty$

$S \sim S' \Leftrightarrow \lim\limits_{n\to\infty} d(x_n, x_m) \to, n, m \to \infty$

$$|d(x_n m x'_n) - d(x_m, x'_m)| \leq d(x_n, x_m) + d(x'_n, x'_m) \to 0 \;\; n, m \to \infty$$

$d(x_n, m x'_n)$ — fundamental in $\mathbb{R}$.

$\exists \lim\limits_{n\to\infty} d(x_n, x'_n)$

$s \sim s, s \sim s' \to s' \sim s$

$s \sim s', s' \sim s'' \Rightarrow s \sim s''$

$S(X)/\cancel{A}\hat{X}$ a set of equivalence classes

$\forall s \in S(X) \qquad [S]$ — equivalence class

$d([S], [S']) = \lim\limits_{n\to\infty} d(x_n, x'_n)$

$s = (x_1, x_2, \ldots), t = (y_1, y_2, \ldots) \quad t \sim s$

$s' = (x'_1, x'_2, \ldots), t' = (y'_1, y'_2, \ldots) \quad t' \sim s'$

$|d(x_n, x'_n) - d(y_n, y'_n)| \leq \underbrace{d(x_n, y_n)}_{(t\sim s)} + \underbrace{d(x'_n, y'_n)}_{t'\sim s'} \to 0$

$\hat{d}([S], [S"]) = 0 \Rightarrow \lim\limits_{n\to\infty} d(x_n, x'_n) = 0 \to s \sim s' \Rightarrow [S] = [S']$

$f : X \to \hat{X}$

$x \in X \to s = (x_1, x_2, \ldots) \Rightarrow f(x) = [S]$

$x, y \in X$. $\hat{d}(f(x), f(y)) = \lim\limits_{n\to\infty}$

$\overline{f(x)} = \hat{X}$?

$s = (x_1, x_2, \ldots), \varepsilon > 0$

$\forall n, m \geq N \;\; d(x_n, x_m) \leq \varepsilon$

$\hat{d}([s], f(x_n)) = \lim_{m \to \infty} d(x_n, x_m) \leq \varepsilon$

Completeness $(\hat{X}, \hat{d})$. Let $([S^{(k)}])_{k \geq 1}$ fundamental sequence.

$\forall k \geq 1 : \exists x_k \in X : \hat{d}([S^{(k)}], d(x_k)) \leq \frac{1}{k}$

$s = (x_1, x_2, \ldots) \in S(X). \quad \lim_{k \to \infty} f(x_k) = [S]$

$[S^{(k)}] \to [S]$

$\square$

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3 Baire Theorem

Definition 10. Set A is nowhere dense nowhere if A is not dense in any ball.

Equivalently:

$$int\overline{A} = \varnothing$$

Example. $X = \mathbb{R}, \quad A = \{a\}$ is dense nowhere

In a space of isolated points finite sets are nowhere dense.

Theorem 4 (Baire)**.** (X, d) — complete metric space $(X \neq \varnothing$.

Then X cannot be represented as a countable union of nowhere dense sets.

Proof. Let $X = \bigcup_{n=1}^{\infty} A_n$, every set A_n is nowhere dense set $(int\overline{A} = \varnothing)$.

$x_0 \in X$. x_0 — not an inner point of the set $\overline{A_1}$.

$B(x_0, 1)$ contains $x_1 \notin \overline{A_1}$

$$\exists r_1 < \frac{1}{2} : B(x_1, r_1) \cap A_1 = \varnothing, \quad \overline{B}(x_1, r_1) \subset B(x_0, 1)$$

$B(x_1, r_1) \not\subset \overline{A_2}$

$B(x_1, r_1)$ contains $x_2 \notin \overline{A_2}$

$\exists r_2 < \frac{1}{4} : \overline{B}(x_2, r_2) \cap A_2 = \varnothing, \quad \overline{B}(x_2, r_2) \subset B(x_1, r_1)$

Exists such a sequence of closed balls $\overline{B}(x_n, r_n) : r_n < \frac{1}{2^n}, \quad \overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) :$ $\overline{B}(x_n, r_n) \cap A_n = \varnothing$

$$(X, d) \text{ complete } \Rightarrow \bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n) \ni x_*$$

$x_* \notin \bigcup_{n=1}^{\infty} A_n$. Contradiction. □

Corollary 2. (X, d) is a complete metric space without any isolated points. Then set X is not countable.

Corollary 3. $\mathbb{Q}$ — countable not complete space. There are no equivalent metric d_x that gives us (Q, d_x) as a complete space.

4 Continuous Mappings of Metric Spaces, Lipschitz Continuity

(X, d_x), (Y, d_y); $f : X \to Y$

Definition 11. f is continuous in a point x_0 if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

Alternatively:

$$\forall \varepsilon > 0 : \exists \delta > 0 : d_x(x, x_0) < \delta \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

Definition 12. f is continuous if it is continuous in every point $x \in X$.

Theorem 5 (Continuous Criteria)**.** The following conditions are equivalent:

1. $f : X \to Y$ continuous
2. $\forall$ open set $U \subset Y$ $\underbrace{\{x \in X : f(x) \in U\}}_{f^{-1}(U)}$ is open in X.
3. $\forall$ closed $F \subset Y : f^{-1}(F)$ — closed

Proof. (2) $\Leftrightarrow$ (3) F closed $\Leftrightarrow$ U open.

$X \backslash f^{-1}(F) = f^{-1}(U)$

(1) $\Rightarrow$ (2)

Let $f : X \to Y$ is continuous. Want to show that $\forall U \in Y$ is open.

$x_0 \in f^{-1}(U)$. Need to find such a radius $r > 0 : B(x_0, r) \subset f^{-1}(U)$.

$f(x_0) \in U. \exists \varepsilon > 0 \quad B(f(x_0), \varepsilon) \subset U$.

$$\exists \delta > 0 : d_x(x, x_0) < \delta \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

It means that

$$x \in B(x_0, \delta) \Rightarrow f(x) \in B(f(x_0), \varepsilon) \subset U \Rightarrow x \in f^{-1}(U)$$

$$B(x_0, \delta) \subset f^{-1}(U)$$

(2) ⇒ (1)

$f : X \to Y$; $x_0 \in X$

$\forall \varepsilon > 0; U = B(f(x_0), \varepsilon)$ — open set

$f^{-1}(U)$ — open set. $x_0 \in f^{-1}(U)$

$$\exists \delta > 0 \quad B(x_0, \delta) \subset f^{-1}(U)$$
$$d_x(x, x_0) < \delta \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

□

Corollary 4. X, Y, Z — metric spaces. $f : X \to Y$, $g : Y \to Z$ — continuous. Then $g \circ f : X \to Z$ continuous.

Proof. $U \subset Z$ — open. $(g, f)^{-1}(U) = \underbrace{f^{-1}(g^{-1}(U))}_{\text{open in } X}$ □

Definition 13. $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0 : \exists \delta > 0 : d_x(x_1, x_2) < \delta \Rightarrow d_y(f(x_1), f(x_2)) < \varepsilon$$

Definition 14. $f : X \to Y$ satisfies Lipschitz condition with constant $c > 0$ if

$$d_y(f(x_1), f(x_2)) \leq c \cdot d_x(x_1, x_2)$$

Example. Let $A \subset X$, $A \neq \varnothing$.

$$d(x, A) := \inf_{y \in A} d(x, y)$$
$$d(\cdot, A) : X \to \mathbb{R} \text{ — Lipschitz function with } c = 1$$

$x_1, x_2 \in X$, $y \in A$.

$$d(x_1, A) \leq d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$$
$$d(x, A) - d(x_1, x_2) \leq d(x_2, y)$$
$$|d(x, A) - d(x_2, A)| \leq d(x_1, x_2)$$

Exercise

$\{x : d(x, A) = 0\} = \overline{A}$

5 Contraction mapping

Definition 15. $f : X \to Y$ is a contraction mapping if

$$\exists \alpha \in [0, 1) : d(f(x_1), f(x_2)) \leq \alpha d(x_1, x_2)$$

For contraction mapping an equation $f(x) = x$ always has a solution.

$f(x) = x \Rightarrow x$ — fixed point of mapping f.

Theorem 6 (Banach)**.** (X, d) — complete metric space, $f : X \to Y$ — contraction mapping. Then f has only one fixed point.

Proof. **(Oneness)**

Let the two fixed point exist $x_1, x_2 \in X$.

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \alpha d(x_1, x_2) \Rightarrow x_1 = x_2$$

(Existence)

Arbitrary $x_0 \in X$.

$x_1 = f(x_0)$,
$x_2 = f(x_1)$
...

$x_n = \underbrace{f(f(\ldots(f(x_0))\ldots))}_{n}$

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \alpha d(x_n, x_{n-1}) \leq \alpha^2 d(x_{n-2}, x_n) \leq \ldots \leq \alpha^n d(x_0, x_n)$$

$$d(x_{n+p}, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \leq d(x_0, x_1)(\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+p-1} \leq$$

$$\leq d(x_0, x_1) \frac{\alpha^n}{1 - \alpha}$$

$$\lim_{n\to\infty} \sup_{p\geq 1} d(x_{n+p}, x_n) = 0$$

(x_n) Cauchy sequence.

$x_* = \lim\limits_{n\to\infty} x_n$
$x_n \to x_*$
$\underbrace{f(x_n)}_{x_{n+1}} \to f(x_*)$
$\Rightarrow f(x_*) = x_*$ ∎

Corollary 5. f — contraction mapping, $x_0 \in X$; $x_n = f(x_{n-1})$

$$d(x_*, x_n) \leq d(x_0, x_1) \frac{\alpha^n}{1 - \alpha}$$

Applications

1. $f : [a, b] \to [a, b]$ continuous.
   $f : [0, 1] \to [0, 1]$; $\quad f(x) = 1 - x$ is Lipschitz mapping but not contraction mapping.

   If $|f'(x)| \le \alpha < 1$ then $|f(x_1) - f(x_2)| \le \alpha |x_1 - x_2|$

   $F : [a, b] \to \mathbb{R} : F(a) < 0, F(b) > 0, \quad F'(x) \in [k_1, k_2], 0 < l_1 \le k_2 < \infty$

   Then this function has only one 0. $F(x_*) = 0, \quad x_* - ?$

   $f(x) = x - \lambda F(x)$
   $F(x_*) = 0 \Leftrightarrow x$ is fixed for f

   Need several things:

   (a) $f : [a, b] \to [a, b]$

   (b) $f'(x) = 1 - \lambda F'(x) \in [1 - \lambda k_2, 1 - \lambda k_1]$

2. Linear equations systems

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i$$

$$x = Ax + b =: f(x)$$

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

The contraction mapping actually depends on the matrix A and picked metric function. So usually the metric function is picked the way that the mapping is contraction for a specific matrix A.

$$d_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|$$

$$d_\infty(f(x), f(y) = \max_{1 \le i \le n} |\ldots| =$$

$$= \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij}(x_j - y_j) \right| \le \left(\max_i \sum_{j=1}^{n} |a_{ij}| \right) d_\infty(x, y)$$

The mapping $f(x) = Ax + b$ is going to be contraction mapping relative to d_∞ if

$$\max_i \sum_{j=1}^{n} |a_{ij}| < 1$$

.

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_1(f(x), f(y)) = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} \left(x_j - y_j\right) \right| \le$$

$$\le \sum_{j=1}^{n} |x_j - y_j| \sum_{i=1}^{n} |a_{ij}| \le \left(\max_j \sum_{j=1}^{n} |a_{ij}| \right) d_1(x, y)$$

If $\max\limits_j \sum\limits_{j=1}^{n} |a_{ij}| < 1$ then $f(x) = Ax + b$ is a contraction mapping relative to d_1.

3.
$$\begin{cases} \dfrac{\partial y}{\partial x} = f(x,y) \\ y(x_0) = y_0 \end{cases} \Leftrightarrow y(x) = \underbrace{y_0 + \int_{x_0}^{x} f(t, y(t))dt}_{F(y)}$$

$$|f(x_1, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2|$$

4. Fredholm equations

$$x(t) = \lambda \int_a^b K(t,s)x(s)ds + y(t), \quad a \leq b$$

K is continuous on $[a,b]^2$, y is continuous on $[a,b]$.

$$f : C[a,b] \to C[a,b]$$

$C[a,b]$ is complete relative to $d(x_1, x_2) = \max_t |x_1(t) - x_2(t)|$

$$f(x)(t) = \lambda \int_a^b K(t,s)x(s)ds + y(t)$$

$$d(f(x_1), f(x_2)) = \max_t |f(x_1)(t) - f(x_2)(t)|$$

Let's fix point t... $M = \sup\limits_{(t,s)\in[a,b]^2} |K(t,s)|$

$|f(x_1)(t) - f(x_2)(t)| = \left|\lambda \int_a^b K(t,s)(x_1(s) - x_2(s))ds\right| \leq |\lambda| \int_a^b M d(x_1, x_2)ds = |\lambda|\, M(b-a)d(x_1, x_2)$

$|\lambda| < \dfrac{1}{M(b-a)}$ then f is a contraction mapping.