Functional Analysis

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Contents

1 Lecture 1: Metric Spaces and Convergence

Definition 1. X is a set. Function $d : X \times X \to [0, \infty]$ is called a metric if three of the conditions are met:

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ — triangle inequality

(X, d) – is a metric space.

Example (1. Discrete space)**.** X — arbitrary.

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Example (2. Real numbers)**.** $X = \mathbb{R}$, $d(x, y) = |x - y|$

Example. $X = \mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}$ $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$

$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ – metric on $\mathbb{R}^n$

Proof. $d_1(x,z) = \sum_{i=1}^{n} |x_i - z_i| \le \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = d_1(x,y) + d_1(y,z)$ □

Example. $d_\infty(x,y) = \max_{1\le i\le n} |x_i - y_i|$ – metric on $\mathbb{R}^n$

Proof. $d_\infty(x,y) = 0 \Leftrightarrow \forall i x_i = y_i \Leftrightarrow x = y$

$d_\infty(x,z) = max_{1\le i\le n}|x_i - y_i| \le d_\infty(x,y) + d_\infty(y,z)$

$|x_i - z_i| \le |x_i - y_i| + |y_i - z_i| \le d_\infty(x,y) + d_\infty(y,z)$ □

Example. $1 \le p \le \infty$

$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right) \frac{1}{p}$ — metric on $\mathbb{R}^n$

$0 \le p \le 1 : d_p(x,y) = \sum_{i=1}^{n} |x_i - y_i|^p$ metric on $\mathbb{R}^n$

Example. $C[a,b]$ – a set of all continuous functions $f : [a,b] \to \mathbb{R}$

$d(f,g) = sup_{a\le t\le b}|f(t) - g(t)|$ — metric on $C[a,b]$

Example. $C_b(\mathbb{R})$ — a set of all continuous and bounded functions $f : \mathbb{R} \to \mathbb{R}$.

$d(f,g) = sup_{t\in R}|f(t) - g(t)|$

Example. (X,d) — metric space; $Y \subset X$

$d(y_1,y_2), \quad y_1,y_2 \in Y$

(Y,d) — subspace X

Definition 2. (X,d) – metric space, $(x_n : n \ge 1)$ – sequence of elements X. $(x_n, n \ge 1)$ converges to $x \in X$ if $\lim_{n\to\infty} d(x_n,x) = 0$.

$(\forall \varepsilon > 0 \ \exists N \ \forall n \ge N \ \ d(x_n,x) < \varepsilon)$

$$x = \lim_{n\to\infty} x_n$$

Theorem 1. In metric space sequence that converges has only ONE limit.

Proof. Let $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$

$$d(x,y) \le d(x,x_n) + d(x_n,y) \to 0$$

$$\Rightarrow d(x,y) = 0 \to x = y.$$

□

$(X,d_x),(Y,d_y)$ — metric spaces. $f : X \to Y$

Definition 3. f – continuous in point $x_0 \in X$, if

$$x_n \to x_0 \text{ in x } \Rightarrow f(x_n) \to f(x_0) \text{ in Y}$$

Definition 4. f continuous on X if f is continuous in every point $x_0 \in X$.

Exercise

f is continuous in point $x_0 \in X$ if and only if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : d_x(x, x_0) < \varepsilon \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

Definition 5. $f : X \to Y$ homogeneous (гомеоморфізм) if f is bijective, continuous and f^{-1} is continuous.

Definition 6. $f : X \to Y$ isometric if $d_y(f(x), f(x')) = d_x(x, x')$ (isometrie is always continuous)

$x \in X, \; r > 0$

Definition 7. Open ball $\mathbf{B}(x, r) = \{y \in X : d(y, x) < r\}$

Definition 8. Closed ball $\overline{B}(x, r) = \{y \in X : d(y, x) \leq r\}$

$$x_n \to x \Leftrightarrow \forall \varepsilon > 0 : \; \exists N \;\; \forall n \geq N : x_n \in \mathbf{B}(x, \varepsilon)$$

Definition: $A \subset X$. Point x tangent to the set A, if $\forall \varepsilon > 0$

$$\mathbf{B}(x, \varepsilon) \cap A \neq \varnothing$$

Example: $X = \mathbb{R}$. $A = (a, b)$ a and b tangent to A

![[Drawing 2023-09-05 20.44.54.excalidraw]]

2.

$$\overline{A} = \{x \in X : x \text{ дотична до } A\}$$

closed set A

Theorem 2 1. $A \subset \overline{A}$ 2. $\overline{\overline{A}} = \overline{A}$ 3. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ 4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof: 1. $x \in A \Rightarrow B(x, \varepsilon) \cap A \neq \varnothing$ as does not contain x 3. $xinn\overline{A} => B(x, \varepsilon) \cup A \neq \varnothing \Rightarrow B(x, \varepsilon) \cup B \neq \varnothing \Rightarrow x \in \overline{B}$ 2. $\overline{A} \subset \overline{\overline{A}}$ need to show that $\overline{\overline{A}} \subset \overline{A}$ $x \in \overline{\overline{A}}, \varepsilon > 0$ $B(x, \varepsilon) \cap \overline{A} \neq \varnothing$ exists such a point that $y \in B(x, \varepsilon) \cap \overline{A}$![[Drawing 2023-09-05 20.52.53.excalidraw]] show that $B(y, \varepsilon - d(x, y)) \subset B(x, \varepsilon)$ $z \in B(y, \varepsilon - d(x, y))$. $d(z, y) < \varepsilon) - d(x, y)$ $\varepsilon > d(z, y) + d(y, x) \geq d(z, x) \Rightarrow z \in B(x, \varepsilon)$ $B(y, \varepsilon - d(x, y)) \cap A \neq \varnothing \Rightarrow B(x, \varepsilon) \cap A \neq \varnothing$ $x \in \overline{A}$

4. $a \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$; $\overline{B} \subset \overline{A \cup B}$

$\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ Let $x \in \overline{A \cup B}$ $x \notin \overline{A}$, $x \notin \overline{B} \Rightarrow \varepsilon_1 > 0 : B(x, \varepsilon_1) \cap A = \varnothing \Rightarrow \varepsilon_2 > 0 : B(x, \varepsilon_2) \cap B = \varnothing$

$\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ $B(x, \varepsilon), \cap (A \cup B) = \varnothing$ $\overline{A \cup B} = \overline{A} \cup \overline{B}$ —

Theorem 3 $x \in \overline{A} \Leftrightarrow$ in set A there is a sequence $(x_n : n \geq 1)$ that converges to x

Proof: $(\Rightarrow)$ Let $x \in \overline{A}$ $\forall \varepsilon > 0$ $B(x, \varepsilon) \cap A \neq \varnothing$, $\quad \varepsilon_n \dfrac{1}{n}$ $\forall n \geq 1$ there is a point $x_n \in A \cap B(x, \dfrac{1}{n})$ $0 \leq d(x, x_n) < \dfrac{1}{n} \to 0$ $\lim\limits_{n \to \infty} x_n = x$

(<=) let $\lim_{n\to\infty} x_n = x, \;\; x_n \in A$

$$\forall\varepsilon > 0 \; \exists N \; \forall n \geq N \; d(x_n, x) < \varepsilon$$

$$x_n \in B(x, \varepsilon) \cap A$$

$x \in \overline{A}$

Definition 1. A is dense in a set B if $B \subset \overline{A}$ 2. A is dense everywhere if $\overline{A} = X$ 3. Metric space (X, d) separable if there is a countable everywhere dense set in it.

Examples: 1. $\mathbb{R}$ separable space. $\overline{\mathbb{Q}} = \mathbb{R}$ 2. $\mathbb{R}^n$ separable related to any metric $d_p, 0 < p \leq \infty$ 3. X, d – discrete. $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\} = x \; B(x, \varepsilon) \cap A \neq \varnothing \Leftrightarrow x \in A$ $\overline{A} = A$ The only everywhere dense set is X. 4. $C[a, b]$; $d(f, g) = sup_{t\in[a,b]}|f(t) - g(t)|$ by theorem of Weierstrasse $\forall f \in C[a, b] \; \forall\varepsilon > 0$ there is a polynomial $P(t) = a_0 + a_1 t + \cdots + a_d t^d$: $sup_{t\in[a,b]}|f(t) - P(t)| < \varepsilon$ *Countable everywhere dense set is a set of polynomials with rational coefficients.* 5. $C_b(\mathbb{R}), d(f, g) = sup_{t\in\mathbb{R}}|f(t) - g(t)|$ — not separable metric set. ![[Drawing 2023-09-05 21.43.21.excalidraw]] $A \subset \mathbb{Z}$

$$f_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \in \mathbb{Z}\backslash A \end{cases}$$

$A \neq A'$; $n \in A\backslash A'$ or $n \in A'\backslash A$ $d(f_A, f_{A'}) = 1$ $B\left(f_A, \frac{1}{2}\right) \cap B(f_{A'}, \frac{1}{2}) = \varnothing$

In space $C_b(\mathbb{R})$ exists continual family of open balls that do not intersect by pairs.

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(X, d) $A \subset X$ $\overline{A} = \{x \in X : \forall\varepsilon > 0 B(x, \varepsilon) \cap A \neq \varnothing\}$ Let x $in\overline{A}$, $y \neq x$. $\varepsilon < d(x, y) \Rightarrow B(X, \varepsilon)$ does not contain y. if for any $\varepsilon > 0$ $\; B(X, \varepsilon) \cap A$ finite then:

$$\exists\delta > 0 : B(X, \delta) \cap A = \{x\}$$

in this case point x is called isolated point of the set A

If $x \in \overline{A}$ and is not isolated, then x is called гранична

x is гранична to the set $A \Leftrightarrow \forall\varepsilon : B(X, \varepsilon) \cap A$ infinite

Example: 1. X is discrete. $B(X, 1) = \{x\}$ $\overline{A} = A$ is filled with only isolated points 2. $X = \mathbb{R}$. $A = (a, b)$. $\overline{A} = [a, b]$ is composed out of cluster points.

Definition A set A of metric space X is closed if $\overline{A} = A$.

Example: 1. X, $\varnothing$ are closed. 2. $\overline{B}(x, r)$ closed

$$\overline{\overline{B}(x, r)} \subset \overline{B}(x, r)$$

Let $y \notin \overline{B}(x, r)$ $d(x, y) > r$. $\varepsilon = d(x, y) - r$ If $z \in B(y, \varepsilon)$, then $d(y, z) < \varepsilon$ $d(z, x) \leq d(x, y) - d(z, y) > d(x, y) - \varepsilon = r$ $z \notin \overline{B}(x, r)$. $B(y, \varepsilon) \cap \overline{B}(x, r) = \varnothing$ and $y \notin \overline{\overline{B}(x, r)}$.

3. $\overline{A}$ closed ($\overline{\overline{A}} = \overline{A}$) 4. $\overline{A}$ – smallest closed set the contains A. (if B is closed and $A \subset B$ then $\overline{A} \subset B$)

Theorem 1. Intersection of any arbitrary closed sets is a closed set 2. Union of finite number of closed sets is a closed set

Proof: 1. Consider $(A_i)_{i\in I}$ — closed sets

$$A = \bigcap_{i\in I} A_i$$

$$\forall i \in I : \overline{A_i} = A_i$$

$A \subset A_i$ $\overline{A} \subset \overline{A_i} = A_i$ $\overline{A} \subset \bigcap\limits_{i \in I} A_i = A \subset \overline{A} \Rightarrow \overline{A} = A$ and A is closed. 2. If A and B are closed, then $\overline{A \cup B} = \overline{A} \cup \overline{B} = A \cup B$

Example: $X = \mathbb{R}$. $A_n = [0, 1 - \frac{1}{n}] \quad n \geq 1$

$$\bigcup_{n=1}^{\infty} A_n = [0, 1)$$

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Definition 1. Point $x \in X$ is inner for the set A if

$$\exists \varepsilon > 0 : B(x, \varepsilon) \subset A$$

2. $A^o = \{x \in X : x \text{ inner for } A\}$ — ***interior*** 3. A is open if $A = A^o$

Example: 1. $B(x, r)$ is an open set. $y \in B(x, r)$, $d(x, y) < r$. $\varepsilon = r - d(x, y)$. if $z \in B(y, \varepsilon)$ then $d(y, z) < \varepsilon$

$$d(z, x) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon = r$$

2. $X = \mathbb{R}$. $A = [a, b], \quad a < b$ $a < x < b \Rightarrow x \in A^o$ $A^o = (a, b)$ 3. $X, \varnothing$ are open.

Theorem For any arbitrary set $A \subset X$ it is true that

$$X \setminus A^o = \overline{X \backslash A}$$

Proof

$$x \in X \setminus A^o \Rightarrow x \notin A^o \Leftrightarrow \forall \varepsilon > 0 \quad B(x, \varepsilon) \not\subset A \Leftrightarrow \forall \varepsilon > 0 B(x, \varepsilon) \cap (X \setminus A) \neq \varnothing \Leftrightarrow X \in \overline{X \setminus A}\}$$

Consequences 1. $A^o \subset A$, $(X \setminus A^o = \overline{X \backslash A} \subset X \backslash A)$ 2. $A \subset B \Rightarrow A^o \subset B^o$ 3. $(A^o)^o = A^o$ 4. $(A \cap B)^o = A^o \cap B^o$ 5. A is open $\Leftrightarrow X \backslash A$ is closed $(A^o = A \Leftrightarrow X \backslash A^o = X \backslash A = \overline{X \backslash A})$ 6. Union of arbitrary family of open sets is an open set. 7. Intersection of finite number of open sets is an open set

Example 1. X — discrete space. All the sets are open. 2. $X = \mathbb{R}$. Set is open $\Leftrightarrow$ a set is a union of intervals sequence (open intervals)

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$X : d$ – metric on X. A set of all open sets is called a topology of the space X.

$$\lim_{n \to \infty} x_n = x \Leftrightarrow \forall \varepsilon > 0 \varepsilon \ \exists N \forall n \geq N x_n \in B(x, \varepsilon)$$
$$\Leftrightarrow \forall \text{ open set } U \text{ that contains } x, \exists N \ \forall n \geq N \ x_n \in U$$

Theorem $d_1 : d_2$ — metric on X. d_1 and d_2 define the same topology on X if and only if the convergence on these metrics is the same (in other words $d_1(x_n, x) \to 0 \Leftrightarrow d_2(x_n, x) \to 0$)

Proof: 1. Let the open sets relatively d_1 and d_2 coincide. Let $d_1(x_n, x) \to 0$

$$\forall \varepsilon > 0 : B_{d_2}(x, \varepsilon) \text{ open relatively } d_2 \Rightarrow B_{d_2}(x, \varepsilon) \text{ open relatively } d_1$$

$$\Rightarrow \exists \delta > 0 : B_{d_1}(x, \delta) \subset B_{d_2}(x, \varepsilon)$$
$$\exists N : \forall n \geq N : d_1(x_n, x) < \delta \Rightarrow d_2(x_n, x) < \varepsilon$$

2. Let the convergence in d_1 and d_2 be equivalent. Consider the the set $A \subset X$ exists that is open relatively to d_1 and not open relatively to d_2. $\exists x \in A$: x not inner for A relatively d_2.

$$\forall n \geq 1 : B_{d_2}\left(x, \frac{1}{n}\right) \not\subset A. \quad \forall n \geq 1 \exists x_n \notin A$$

$d_2(x_n, x) < \dfrac{1}{n}$ $d_2(x_n, x) \to\Rightarrow d_1(x_n, x) \to 0$ Relatively d_1 A is open, $x \in A \Rightarrow \exists N \quad \forall n \geq N$ $x_n \in A$. *Contradiction.*

Definition Two metrics: d_1 and d_2 on a set X are equivalent if they define the same topology (define the same convergent sequences).

Exercise: On $\mathbb{R}^n$ all the metrics $d_p, 0 < p \leq \infty$ are equivalent.

Topology of subspace

(X, d) — metric space, $Y \subset X$. $(Y, d|_{Y\times Y})$ — subspace

$y \in Y, r > 0$. $B_Y(y, r) = \{y' \in Y : d(y, y') < r\} = Y \cap B_X(y, r)$

$A \subset Y$ $\overline{A}_Y = \{y \in Y : \forall\varepsilon > 0 \;\; B(y, \varepsilon) \cap A \neq \varnothing\} = Y \cap \overline{A}_X$

$A \subset Y$ is closed relatively to Y if and only if $A = Y \cap F$ where F is closed in X.

$A \subset Y$ is open relatively to Y if and only if $A = Y \cap G$ where G is open in X.

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Definition A sequence $(x_{n\geq 1}^{\infty})$ in metric space (X, d) is fundamental (Cauchy sequence) if

$$d(x_n, x_m) \to 0 \;\; n, m \to \infty$$

$$\forall\varepsilon > 0 \;\; \exists N \;\; \forall n, m \geq N \;\; d(x_n, x_m) < \varepsilon$$

Corollary Convergent sequence in fundamental.

Proof: Let $x_n \to x, \;\; n \to \infty \Rightarrow d(x_n, x) \to 0, \;\; n \to \infty$ $\forall\varepsilon > 0 \;\; \exists N \;\; \forall n \geq N \;\; d(x_n, x) < \dfrac{\varepsilon}{2}$

If $n, m \geq N$ then $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon$

Definition: (X, d) — full, if in X any fundamental sequence is convergent.

Example: 1. $X = \mathbb{R}$, $d(x, y) = |x - y|$ — full metric space 2. $X = \mathbb{R}^n$, $d_2(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ Let $(x^{(k)})_{k\geq 1}$ fundamental sequence in $(\mathbb{R}^n, d_2)$ $x^{(k)} = \left(x_1^{(k)}, \ldots, x_n^{(k)}\right)$

$$0 \leftarrow d_2(x^{(k)}, x^{(m)}) = \sqrt{\sum_{i=1}^{n}(x_i^{(k)} - x_i^{(m)})^2} \geq \left|x_i^{(k)} - x_i^{(m)}\right|, \quad k, m \to \infty$$

$(x_i^{(k)})_{k\geq 1}$ — fundamental in $\mathbb{R}$.

$\exists \lim\limits_{k\to\infty} x_i^{(k)} = x_i, \quad 1 \leq i \leq n \quad x = (x_1, \ldots, x_n)$ $d_2(x^{(k)}, x) = \sqrt{\sum_{i=1}^{n} \underbrace{(x_i^{(k)} - x_i)}_{0}^{2}} \to_{k\to\infty} 0$

Exercise: $\forall p \in (0, \infty] \quad (\mathbb{R}^n, d_p)$ — full space

3. $X = C[a, b]$

$$d(f, g) = \sup_{a\leq t\leq b} |f(t) - g(t)|$$

$(C[a,b],d)$ — full metric space Let $(f_n)_{n\geq 1}$ fundamental sequence in that full metric space $0 \leftarrow d(f_n,f_m) = \sup\limits_{a\leq t\leq b} |f_n(t)-f_m(t)| \geq \geq |f_n(t)-f_m(t)|$ fixed t $(f_n(t))_{n\geq 1}$ — fundamental sequence in $\mathbb{R}$ $\exists \lim\limits_{n\to\infty} f_n(t) =: f(t)$ $f:[a,b]\to\mathbb{R}$

$\forall\varepsilon>0 \quad \exists N \quad \forall n,m\geq N \quad \forall t\in[a,b] \quad |f_n(t)-f_m(t)|\leq\varepsilon$ $m\to\infty$

$$\Rightarrow \forall\varepsilon>0 \ \exists N\forall n\geq N \ \ \underbrace{\forall t\,|f_n(t)-f(t)|\leq\varepsilon}_{d(f_n,f)\leq\varepsilon}$$

Lets show that f is continuous by t $t_0\in[a,b]$. Need to prove that $\forall\varepsilon>0\exists\delta>0 : |t-t_0|<\delta \Rightarrow |f(t)-f(t_0)|<\varepsilon$

$$\exists N : \forall n\geq N : \sup_S |f_n(s)-f(s)| < \frac{\varepsilon}{3}$$

$$|f_N(t)-f_N(t_0)| < \frac{\varepsilon}{3} \text{ if } |t-t_0|<\delta$$

$$|f(t)-f(t_0)| \leq |f_N(t)-f(t)| + |f_N(t_0)-f(t_0)| + \left|f_{N(t)} - f_N(t_0)\right| < \varepsilon$$

Example. $\mathbb{R}$, $d_1(x,y)=|e^x-e^y|$, $d(x,y)=|x-y|$

metrics d_1 and d are equivalent.

$(\mathbb{R},d_1)$ is not complete. $x_n=-n, n\geq 1$

$$d_1(x_n,x_m) = |e^{x_n}-e^{x_m}| = \left|e^{-n}-e^{-m}\right| \to 0, \quad n,m\to\infty$$
$$d_1(x_n,x) = |e^{x_n}-e^{x}| = \left|e^{-n}-e^{x}\right| \to e^x$$

e^e set mutually unambiguous correspondence between $\mathbb{R}$ and $(0,\infty)$

Example. $C[a,b]$, $d_1(f,g)=\displaystyle\int_a^b |f(t)-g(t)|\,dt$

$(C[a,b],d_1)$ is not complete metric space.

$$f_n(t) = \begin{cases} 1 & t\geq c \\ 0 & t\leq c-\dfrac{1}{n} \\ \text{linear on } [c-\dfrac{1}{n}, c] & \end{cases}$$

$$d_1(f_n,f_m) = \int_a^b |f_n(t)-f_m(t)|\,dt \leq \int_{c-\frac{1}{n}}^{c} 2dt = \frac{2}{n} \to_{n,m\to\infty} 0$$

If $d_1(f_n,f)\to 0$, $n\to\infty$, then $f(t)=\begin{cases}1 & t\leq c\\ 0 & t<c\end{cases}$ which cannot be true for continuous f.

Example.

$$l^2 = \{x=(x_1,\ldots \mid \sum_{i=1}^{\infty} x_i^2 < \infty\}$$

$$d(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$$

(l^2, d) — complete metric space

$(x^{(k)})_{k\geq 1}$ — fundamental sequence in l^2

$$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots$$

$$d(x^{(k)}, x^{(m)}) = \sqrt{\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i^{(m)}\right)^2} \to 0, \quad n,m \to \infty$$

Let's freeze the number of n.

$$\left|x_n^{(k)} - x_n^{(m)}\right| \leq \sqrt{\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i^{(m)}\right)^2} = d(x^{(k)}, x^{(m)}) \to 0 \;\; k,m \to \infty$$

$$\exists \lim_{k\to\infty} x_n^{(k)} := x_n$$

$$\varepsilon > 0 : \exists N : \forall k,m \geq \mathrm{N} : d(x^{(k)}, x^{(m)}) \leq \varepsilon$$

$$\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i^{(m)}\right)^2 \leq \varepsilon^2, \quad k,m \geq N$$

$$\sum_{i=1}^{M}\left(x_i^{(k)} - \underbrace{x_i^{(m)}}_{x_i,\text{within } m\to\infty}\right)^2 \leq \varepsilon^2, \quad k,m \geq N, M \geq 1$$

$$\sum_{i=1}^{M}\left(x_i^{(k)} - x_i\right)^2 \leq \varepsilon^2, \quad k \geq N, M \geq 1$$

$$\sum_{i=1}^{\infty}\left(x_i^{(k)} - x_i\right)^2 \leq \varepsilon^2$$

$$\Rightarrow \begin{cases} \sum_{i=1}^{\infty} x_i^2 < \infty, \quad x \in l^2 \\ d(x^{(k)}, x) \leq \varepsilon, \quad k \geq N \end{cases}$$

Corollary 1. 1. Closed subspace of a complete space is complete.

2. Complete subspace of a metric space is closed.

Proof. 1. (X,d) is complete. Y — closed subset of X.

$(x_n)_{n\geq 1}$ — fundamental in $Y \Rightarrow (x_n)_{n\geq 1}$ — fundamental in $X \Rightarrow (x_n)_{n\geq 1}$ converges to $x \in X \Rightarrow x \in Y$ and $(x_n)_{n\geq 1}$ is convergent in Y.

2. Let Y — a subspace of space X, Y is complete.

   $y \in \overline{Y} \Rightarrow$ exists sequence $(y_n)_n$ in Y that converges to $y \Rightarrow (y_n)$ fundamental $\Rightarrow (y_n)$ converges in $Y \Rightarrow y \in Y$.

$\square$

Theorem 2 (about nested balls)**.** (X, d) metric space. X is complete if and only if any arbitrary sequence of nested closed balls which have $R \to 0$ has non-empty intersection.

Proof. ($\Rightarrow$) Let X is a complete. $B_n = \overline{B}(X_n, r_n), B_1 \supset B_2 \supset B_3 \ldots, r_n \to 0$

$d(x_n, x_m) \leq^{n \leq m} r_n \to 0, \quad n \to \infty$

$\exists \lim\limits_{n\to\infty} x_n := x$

$n \geq N \Rightarrow x_n \in B_N, \quad n \geq N \Rightarrow x \in B_N$

$\bigcap\limits_{n=1}^{\infty} B_n \neq \varnothing$

($\Leftarrow$)

Let $(x_n)_{n\geq 1}$ — fundamental in X

$$\exists n_1 \quad \forall n, m \geq n_1 : d(x_n, x_m) \leq \frac{1}{2}$$
$$\exists n_2 \geq n_1 : \forall n_1, m \geq n_2 : d(x_n, x_m) \leq \frac{1}{4}$$
$$\ldots$$

$$1 \leq n_1 < n_2 < n_3 \ldots : \forall n, m \geq n_k : d(x_n, x_m) \leq \frac{1}{2^k}$$
$$d(x_{n_k}, x_{n_{k+1}}) \leq 2^{-k}$$

$$B_k = \overline{B}(x_{n_k}, 2^{-k+1}$$

Let's show that $B_{k+1} \subset B_k$

$y \in B_{k+1} : d(y, x_{n_{k+1}}) \leq 2_{-k}$

$$d(y, x_{n_k}) \leq d(y, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \leq 2^{-k+1}$$
$$\exists x \in \cap_{k\geq 1} B_k$$
$$d(x_{n_k}, x) \leq 2^{-k+1} \to 0$$

$$x_{n_k} \to x, \quad k \to \infty$$

$$\varepsilon > 0. \quad \exists N : \forall n, m \geq N : d(x_n, x_m) < \frac{\varepsilon}{2}$$
$$\exists n_k \geq N : d(x_{n_k}, x) \leq \frac{\varepsilon}{2}$$

if $n \geq N$ then $d(x_n, x) \leq \varepsilon$

$\square$

2 Completation of Metric Space

Definition 9. Complete metric space $(\hat{X}, \hat{d})$ is a complitation of metric space (X, d) if X is isometric to dense everywhere subset of X.

Theorem 3. For any arbitrary metric space X its completation exists and only one with the precision to isometrie.

Proof. (Oneness)

$(\hat{X}, \hat{d})$ and $(\tilde{X}, \tilde{d})$ — a completion (X, d).

$f : X \to \hat{X}$ isometrie between X and $f(x), \overline{f(X)} = \hat{X}$

$g : X \to \tilde{X}$ isometrie between X and $g(x), \overline{g(X)} = \tilde{X}$

$\hat{x} \in \hat{X}$. $\hat{x} = \lim\limits_{n\to\infty} f(x_n)$.

$(f(x_n))$ convergent $\Rightarrow$ fundamental $\Rightarrow$ (x_n) fundamental $\Rightarrow$ $(g(x_n))$ fundamental in $\tilde{X}$

$\varphi(\hat{x}) = \lim\limits_{n\to\infty} g(x_n)$

Further need to show that φ is isometric

(Existence)

$S(X)$ set of all fundamental sequences in X.

$s \in S(X) \Rightarrow s = (x_1, x_2, \ldots)$. $\quad d(x_n, x_m) \to n, m \to \infty$

$S \sim S' \Leftrightarrow \lim\limits_{n\to\infty} d(x_n, x_m) \to, n, m \to \infty$

$$|d(x_n m x'_n) - d(x_m, x'_m)| \leq d(x_n, x_m) + d(x'_n, x'_m) \to 0 \;\; n, m \to \infty$$

$d(x_n, m x'_n)$ — fundamental in $\mathbb{R}$.

$\exists \lim\limits_{n\to\infty} d(x_n, x'_n)$

$s \sim s, s \sim s' \to s' \sim s$

$s \sim s', s' \sim s'' \Rightarrow s \sim s''$

$S(X)/\cancel{A}\hat{X}$ a set of equivalence classes

$\forall s \in S(X) \qquad [S]$ — equivalence class

$d([S], [S']) = \lim\limits_{n\to\infty} d(x_n, x'_n)$

$s = (x_1, x_2, \ldots), t = (y_1, y_2, \ldots) \quad t \sim s$

$s' = (x'_1, x'_2, \ldots), t' = (y'_1, y'_2, \ldots) \quad t' \sim s'$

$|d(x_n, x'_n) - d(y_n, y'_n)| \leq \underbrace{d(x_n, y_n)}_{(t\sim s)} + \underbrace{d(x'_n, y'_n)}_{t'\sim s'} \to 0$

$\hat{d}([S], [S"]) = 0 \Rightarrow \lim\limits_{n\to\infty} d(x_n, x'_n) = 0 \to s \sim s' \Rightarrow [S] = [S']$

$f : X \to \hat{X}$

$x \in X \to s = (x_1, x_2, \ldots) \Rightarrow f(x) = [S]$

$x, y \in X$. $\hat{d}(f(x), f(y)) = \lim\limits_{n\to\infty}$

$\overline{f(x)} = \hat{X}$?

$s = (x_1, x_2, \ldots), \varepsilon > 0$

$\forall n, m \geq N \quad d(x_n, x_m) \leq \varepsilon$

$\hat{d}([s], f(x_n)) = \lim\limits_{m\to\infty} d(x_n, x_m) \leq \varepsilon$

Completeness $(\hat{X}, \hat{d})$. Let $([S^{(k)}])_{k\geq 1}$ fundamental sequence.

$\forall k \geq 1 : \exists x_k \in X : \hat{d}([S^{(k)}], d(x_k)) \leq \dfrac{1}{k}$

$s = (x_1, x_2, \ldots) \in S(X). \quad \lim\limits_{k\to\infty} f(x_k) = [S]$

$[S^{(k)}] \to [S]$

□

---

3 Baire Theorem

Definition 10. Set A is nowhere dense nowhere if A is not dense in any ball.

Equivalently:

$$int\overline{A} = \varnothing$$

Example. $X = \mathbb{R}, \quad A = \{a\}$ is dense nowhere

In a space of isolated points finite sets are nowhere dense.

Theorem 4 (Baire). (X, d) — complete metric space $(X \neq \varnothing$.

Then X cannot be represented as a countable union of nowhere dense sets.

Proof. Let $X = \bigcup\limits_{n=1}^{\infty} A_n$, every set A_n is nowhere dense set $(int\overline{A} = \varnothing)$.

$x_0 \in X$. x_0 — not an inner point of the set $\overline{A_1}$.

$B(x_0, 1)$ contains $x_1 \notin \overline{A_1}$

$$\exists r_1 < \frac{1}{2} : \overline{B}(x_1, r_1) \cap A_1 = \varnothing, \quad \overline{B}(x_1, r_1) \subset B(x_0, 1)$$

$B(x_1, r_1) \not\subset \overline{A_2}$

$B(x_1, r_1)$ contains $x_2 \notin \overline{A_2}$

$\exists r_2 < \dfrac{1}{4} : \overline{B}(x_2, r_2) \cap A_2 = \varnothing, \quad \overline{B}(x_2, r_2) \subset B(x_1, r_1)$

Exists such a sequence of closed balls $\overline{B}(x_n, r_n) : r_n < \dfrac{1}{2^n}, \quad \overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) :$ $\overline{B}(x_n, r_n) \cap A_n = \varnothing$

$$(X, d) \text{ complete } \Rightarrow \bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n) \ni x_*$$

$x_* \notin \bigcup_{n=1}^{\infty} A_n$. Contradiction. □

Corollary 2. (X, d) is a complete metric space without any isolated points. Then set X is not countable.

Corollary 3. $\mathbb{Q}$ — countable not complete space. There are no equivalent metric d_x that gives us (Q, d_x) as a complete space.

4 Continuous Mappings of Metric Spaces, Lipschitz Continuity

(X, d_x), (Y, d_y); $f : X \to Y$

Definition 11. f is continuous in a point x_0 if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

Alternatively:

$$\forall \varepsilon > 0 : \exists \delta > 0 : d_x(x, x_0) < \delta \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

Definition 12. f is continuous if it is continuous in every point $x \in X$.

Theorem 5 (Continuous Criteria)**.** The following conditions are equivalent:

1. $f : X \to Y$ continuous
2. $\forall$ open set $U \subset Y$ $\underbrace{\{x \in X : f(x) \in U\}}_{f^{-1}(U)}$ is open in X.
3. $\forall$ closed $F \subset Y : f^{-1}(F)$ — closed

Proof. (2) $\Leftrightarrow$ (3) F closed $\Leftrightarrow$ U open.

$X \backslash f^{-1}(F) = f^{-1}(U)$

(1) $\Rightarrow$ (2)

Let $f : X \to Y$ is continuous. Want to show that $\forall U \in Y$ is open.

$x_0 \in f^{-1}(U)$. Need to find such a radius $r > 0 : B(x_0, r) \subset f^{-1}(U)$.

$f(x_0) \in U . \exists \varepsilon > 0 \quad B(f(x_0), \varepsilon) \subset U$.

$$\exists \delta > 0 : d_x(x, x_0) < \delta \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

It means that

$$x \in B(x_0, \delta) \Rightarrow f(x) \in B(f(x_0), \varepsilon) \subset U \Rightarrow x \in f^{-1}(U)$$

$$B(x_0, \delta) \subset f^{-1}(U)$$

(2) ⇒ (1)

$f : X \to Y$; $x_0 \in X$

$\forall \varepsilon > 0; U = B(f(x_0), \varepsilon)$ — open set

$f^{-1}(U)$ — open set. $x_0 \in f^{-1}(U)$

$$\exists \delta > 0 \quad B(x_0, \delta) \subset f^{-1}(U)$$

$$d_x(x, x_0) < \delta \Rightarrow d_y(f(x), f(x_0)) < \varepsilon$$

□

Corollary 4. X, Y, Z — metric spaces. $f : X \to Y$, $g : Y \to Z$ — continuous. Then $g \circ f : X \to Z$ continuous.

Proof. $U \subset Z$ — open. $(g, f)^{-1}(U) = \underbrace{f^{-1}(g^{-1}(U))}_{\text{open in } X}$ □

Definition 13. $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0 : \exists \delta > 0 : d_x(x_1, x_2) < \delta \Rightarrow d_y(f(x_1), f(x_2)) < \varepsilon$$

Definition 14. $f : X \to Y$ satisfies Lipschitz condition with constant $c > 0$ if

$$d_y(f(x_1), f(x_2)) \leq c \cdot d_x(x_1, x_2)$$

Example. Let $A \subset X$, $A \neq \varnothing$.

$$d(x, A) := \inf_{y \in A} d(x, y)$$

$$d(\cdot, A) : X \to \mathbb{R} \text{ — Lipschitz function with } c = 1$$

$x_1, x_2 \in X$, $y \in A$.

$$d(x_1, A) \leq d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$$

$$d(x, A) - d(x_1, x_2) \leq d(x_2, y)$$

$$|d(x, A) - d(x_2, A)| \leq d(x_1, x_2)$$

Exercise

$\{x : d(x, A) = 0\} = \overline{A}$

5 Contraction mapping

Definition 15. $f : X \to Y$ is a contraction mapping if

$$\exists \alpha \in [0, 1) : d(f(x_1), f(x_2)) \le \alpha d(x_1, x_2)$$

For contraction mapping an equation $f(x) = x$ always has a solution.

$f(x) = x \Rightarrow x$ — fixed point of mapping f.

Theorem 6 (Banach)**.** (X, d) — complete metric space, $f : X \to Y$ — contraction mapping. Then f has only one fixed point.

Proof. **(Oneness)**

Let the two fixed point exist $x_1, x_2 \in X$.

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \le \alpha d(x_1, x_2) \Rightarrow x_1 = x_2$$

(Existence)

Arbitrary $x_0 \in X$.

$x_1 = f(x_0)$,
$x_2 = f(x_1)$
$\ldots$

$x_n = \underbrace{f(f(\ldots(f(x_0))\ldots))}_{n}$

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \le \alpha d(x_n, x_{n-1}) \le \alpha^2 d(x_{n-2}, x_n) \le \ldots \le \alpha^n d(x_0, x_n)$$

$$d(x_{n+p}, x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \le d(x_0, x_1)(\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+p-1} \le$$

$$\le d(x_0, x_1)\frac{\alpha^n}{1 - \alpha}$$

$$\lim_{n\to\infty} \sup_{p \ge 1} d(x_{n+p}, x_n) = 0$$

(x_n) Cauchy sequence.

$x_* = \lim\limits_{n\to\infty} x_n$
$x_n \to x_*$
$\underbrace{f(x_n)}_{x_{n+1}} \to f(x_*)$
$\Rightarrow f(x_*) = x_*$ ∎

Corollary 5. f — contraction mapping, $x_0 \in X$; $x_n = f(x_{n-1})$

$$d(x_*, x_n) \le d(x_0, x_1)\frac{\alpha^n}{1 - \alpha}$$

Applications

1. $f : [a, b] \to [a, b]$ continuous.
   $f : [0, 1] \to [0, 1]$; $\quad f(x) = 1 - x$ is Lipschitz mapping but not contraction mapping.

   If $|f'(x)| \le \alpha < 1$ then $|f(x_1) - f(x_2)| \le \alpha\, |x_1 - x_2|$

   $F : [a, b] \to \mathbb{R} :\ F(a) < 0, F(b) > 0, \quad F'(x) \in [k_1, k_2], 0 < l_1 \le k_2 < \infty$

   Then this function has only one 0. $F(x_*) = 0, \quad x_*$ — ?

   $f(x) = x - \lambda F(x)$
   $F(x_*) = 0 \Leftrightarrow x$ is fixed for f

   Need several things:

   (a) $f : [a, b] \to [a, b]$

   (b) $f'(x) = 1 - \lambda F'(x) \in [1 - \lambda k_2, 1 - \lambda k_1]$

2. Linear equations systems

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i$$

$$x = Ax + b =: f(x)$$

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

   The contraction mapping actually depends on the matrix A and picked metric function. So usually the metric function is picked the way that the mapping is contraction for a specific matrix A.

$$d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i|$$

$$d_\infty(f(x), f(y) = \max_{1\le i\le n} |\ldots| =$$

$$= \max_{1\le i\le n} \left| \sum_{j=1}^{n} a_{ij}(x_j - y_j) \right| \le \left(\max_i \sum_{j=1}^{n} |a_{ij}| \right) d_\infty(x, y)$$

   The mapping $f(x) = Ax + b$ is going to be contraction mapping relative to d_∞ if

$$\max_i \sum_{j=1}^{n} |a_{ij}| < 1$$

   .

   $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$

$$d_1(f(x), f(y)) = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij}\,(x_j - y_j) \right| \le$$

$$\le \sum_{j=1}^{n} |x_j - y_j| \sum_{i=1}^{n} |a_{ij}| \le \left(\max_j \sum_{j=1}^{n} |a_{ij}| \right) d_1(x, y)$$

   If $\max_j \sum_{j=1}^{n} |a_{ij}| < 1$ then $f(x) = Ax + b$ is a contraction mapping relative to d_1.

3.
$$\begin{cases} \dfrac{\partial y}{\partial x} = f(x,y) \\ y(x_0) = y_0 \end{cases} \Leftrightarrow y(x) = \underbrace{y_0 + \int_{x_0}^{x} f(t, y(t))dt}_{F(y)}$$
$$|f(x_1, y_1) - f(x, y_2)| \le L\,|y_1 - y_2|$$

4. Fredholm equations
$$x(t) = \lambda \int_a^b K(t,s)x(s)ds + y(t), \quad a \le b$$
K is continuous on $[a,b]^2$, y is continuous on $[a,b]$.
$$f : C[a,b] \to C[a,b]$$
$C[a,b]$ is complete relative to $d(x_1, x_2) = \max_t |x_1(t) - x_2(t)|$
$$f(x)(t) = \lambda \int_a^b K(t,s)x(s)ds + y(t)$$
$$d(f(x_1), f(x_2)) = \max_t |f(x_1)(t) - f(x_2)(t)|$$
Let's fix point t... $M = \sup\limits_{(t,s)\in[a,b]^2} |K(t,s)|$
$|f(x_1)(t) - f(x_2)(t)| = \left|\lambda \int_a^b K(t,s)(x_1(s) - x_2(s))ds\right| \le |\lambda| \int_a^b Md(x_1,x_2)ds = |\lambda|\, M(b-a)d(x_1,x_2)$
$|\lambda| < \dfrac{1}{M(b-a)}$ then f is a contraction mapping.

6 Lecture 1: Cover and Compact Spaces

X — a set. $(A_i)_{i\in I}$ subsets of X.

Definition 16. $(A_i)_{i\in I}$ is a cover of the set X if $X = \bigcup\limits_{i\in I} A_i$.

Lemma 1 (Heihe-Borel)**.** From arbitrary open cover of an interval $[a,b]$ on $\mathbb{R}$ we may separate finite subcover.

Proof. **By contradiction**.

Exists such a sequence of open sets $(G_i))_{i\in I}$ in $\mathbb{R}$ such that

1. $[a,b] \subset \bigcup\limits_{i\in I} G_i$
2. $[a,b]$ is not covered by any finite number of G_i.

$[a_1, b_1] \subset [a,b], \quad b_1 - a_1 = \dfrac{b-a}{2}$
$[a_1, b_1]$ is not covered by a finite number of G_i.
$[a,b] \supset [a_1,b_1] \supset [a_2,b_2] \supset \dots$

$b_n - a_n = \dfrac{b-a}{2^n}$

$[a_n, b_n]$ is not covered by a finite number of G_i. $x \in \bigcap\limits_n [a_n, b_n]$

$\exists i_0 : x \in G_{i_0} \Rightarrow (x-\varepsilon, x+\varepsilon) \in G_{i_o}$

$[a_n, b_n] \subset G_{i_0} \Rightarrow$ contradiction. $\square$

Exercise

From arbitrary open cover of the closed rectangular in R^n we may separate a finite subcover.

Definition 17. Metric space (X, d) is called compact if its any open cover contains finite subcover.

Definition 18. Set A in a metric space (X, d) is compact if A is a compact subspace of (X, d).

Equivalently:

Arbitrary cover of A consisting of open sets of X contains finite subcover.

Definition 19. Set A is relatively compact to X if its closure is compact.

Definition 20. Collection of sets $\{A_i\}_{i \in I}$ has finite intersection property (centred system) if

$$\forall i_1, \ldots, i_n \in I : n \geq 1 : A_{i_1} \cap \ldots \cap A_{i_n} \neq \varnothing$$

Theorem 7. Metric space (X, d) is compact if and only if arbitrary centred collection of closed sets in X has non-empty intersection.

Proof. $(\Rightarrow)$

Let $\{F_i\}_{i \in I}$ — centred collection of closed sets in compact metric space (X, d).

$$U_i = X \backslash F_i \text{ are open}$$

$$\bigcup_{j=1}^{n} U_{i_j} = (X \backslash F_{i_1}) \cup \ldots \cup (X \backslash F_{i_n}) = X \backslash \underbrace{(F_{i_1} \cap \ldots \cap F_{i_n})}_{\neq \varnothing} \neq X$$

$$X \text{ is compact } \Rightarrow \bigcup_{i \in I} U_i \neq X \Leftrightarrow X \backslash \bigcap_{i \in I} F_i \neq X \Rightarrow \bigcap_{i \in I} F_i \neq \varnothing$$

$\square$

Corollary 6. If metric space (X, d) is compact and set $A \subset X$ is infinite then A has limit points in X (in other words, $\exists x \in X : \forall r > 0 : B(x, r) \cap A$ is infinite).

Proof. Let A does not have any limit points. Let's show that A is closed.

Let $\exists x \in \overline{A} \backslash A$. x is tangent to A, $x \notin A \Rightarrow \forall r > 0 \;\; B(x,r) \cap A$ is infinite, which cannot be true.

$\{x_1, x_2, \ldots\} \subset A$ where all x_n are distinct.
$F_n = \{x_n, x_{n+1}, \ldots\}$ is closed.
$F_1, \cap \ldots \cap F_n = F_n \neq \varnothing$
$\bigcap\limits_{n=1}^{\infty} F_n = \varnothing$.
Contradiction. □

Corollary 7. In a compact metric space any arbitrary sequence contains convergent subsequence.

Proof. $(x_1, x_2, \ldots)$ — sequence in a compact metric space (X, d).
$A = \bigcup\limits_{n=1}^{\infty} \{x_n\}$. If A is finite then there exists such a subsequence $x_{n_1} = x_{n_2} = x_{n_3} = \ldots$.
If A is infinite then A has a limit point x.
$B(x,1) \ni x_{n_1}$; $B(x, \frac{1}{2}) \ni x_{n_2} \;\; (n_2 > n_1)$; ...

$d(x_{n_k}, x) < \frac{1}{k} \to 0, \;\; k \to \infty$ □

Corollary 8. Compact metric space is complete (fundamental sequence with convergent subsequence is convergent by itself).

Corollary 9. Compact set in metric space is closed and bounded.

Proof. Let A be compact in X. Consider open balls $\{B(x,1) : x \in A\}$.

$\Rightarrow B(x_1, 1) \cap \ldots \cap B(x_n, 1) \supset A$. □

Theorem 8. X— compact metric space. Y — arbitrary metric space. $f : X \to Y$ is continuous. Then $f(X)$ — is compact in Y.

Proof. Let $\{V_i\}_{i \in I}$ — open cover $f(X)$.
$f(X) \subset \bigcup\limits_{i \in I} V_i$.
Criteria of continuity: $f^{-1}(V_i)$ is open in X.
$X = \bigcup\limits_{i \in I} f^{-1}(V_i) \Rightarrow X = f^{-1}(V_{i,1}) \cup \ldots \cup f^{-1}(V_{i,n}) \Rightarrow f(X) \subset V_{i,1} \cup \ldots \cup V_{i,n}$ □

Corollary 10. X is compact, $f : X \to \mathbb{R}$ is continuous. Then f is bounded and reaches it's minimal and maximal values.

Proof. $f(X)$ — compact in $\mathbb{R} \Rightarrow f(X)$ is bounded.
$\sup f(X) = f(x^*), \quad inf f(X) = f(x_*)$. □

Corollary 11. X compact. $f: X \to Y$ is continuous and bijective. Then f^{-1} is continuous too (f homogeneous).

Proof. $\square$

Definition 21 (ε grid)**.** $\{x_1, \ldots, x_n\}$ is ε-grid for (X, d) if $\cup B(x_i, \varepsilon) = X \quad (\forall x \in X : \exists i : d(x, x_i) < \varepsilon$.

Definition 22. (X, d) is totally bounded if $\forall \varepsilon > 0$ (X, d) has a finite ε-grid.

Example. in $\mathbb{R}^n$ bounded and totally bounded sets coincide.

Example. $l^1 = \{x = (x_1, x_2, \ldots) \mid \sum_{n=1}^{\infty} |x_n| < \infty\}$

$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|$

$\overline{B}(0, 1)$ is not totally bounded set as for $\varepsilon = \frac{1}{2}$ we cannot find a proper ε-grid.

Let such a finite $\frac{1}{2}$-grid exists.
$B(x_1, \frac{1}{2}), \ldots, B(x_n, \frac{1}{2})$.
$e_1 = (1, 0, 0 \ldots);\ e_2 = (0, 1, 0, 0 \ldots);\ \ldots$
$d(e_k, 0) = 1 \quad e_1, e_2, \ldots \in \overline{B}(0, 1)$
$e_i, e_j \in B(x_l, \frac{1}{2})\ \ (i \neq j)$
$2 = d(e_i, e_j) < 1$ — contradiction.

7 Lecture 3

Compact space is totally bounded.

Proof.

$$\varepsilon > 0 \quad \bigcup_{x \in X} B(x, \varepsilon) = X \Rightarrow B(x_1, \varepsilon) \cup \ldots \cup B(x_n, \varepsilon) = X$$

$\square$

Lemma 2. $(Y,)$ separable metric space. Then out of any open cover of Y we may select a countable cover.

Proof. $(U_i)_{i \in I},\ \cup_{i \in I} U_i = Y, U_i$ are open.
$\{y_1, y_2, \ldots\}$ are everywhere dense.
Let $\frac{1}{k} < \frac{\varepsilon}{2}$. $\exists j : d(x, y_j) < \frac{1}{k}$.
$x \in \underbrace{B(y_j, \frac{1}{k})}_{\Delta_{jk}} \subset B(x, \varepsilon) \subset U_i$

$U_i = \bigcup\limits_{(j,k)\in I(i)} \Delta_{jk}$

$\bigcup\limits_{i\in I} U_i = \bigcup\limits_{i\in I} \bigcup\limits_{(j,k)\in I(i)} \Delta_{jk} = \Delta_{j_1k_1} \cup \Delta_{j_2k_2} \cup \ldots \subset U_{i_1} \cup U_{i_2} \cup \ldots$ □

Theorem 9 (Hausdorf compact criteria)**.** Metric space (X,d) is compact $\Leftrightarrow$

1. (X,d) is complete
2. (X,d) is totally bounded

Proof. ($\Leftarrow$)
Let's prove that arbitrary sequence in X has convergent subsequence.
$(x_n : n \geq 1)$ — sequence in X.
$X = B(y_1, \varepsilon) \cup \ldots \cup B(y_m, \varepsilon)$

$$\exists (x_{n_k} : k \geq 1) : x_{n_k} \in B(y_i, \varepsilon)$$
$$d(x_{n_k}, x_{n_m}) < 2\varepsilon$$

$\varepsilon = 1$ — Exists subsequence $(x_n^{(1)} : n \geq 1) : d(x_n^{(1)}, x_k^{(1)}) < 1$ out of sequence x_n
$\varepsilon = \frac{1}{2}$ — Exists subsequence $(x_n^{(2)} : n \geq 1) : d(x_n^{(2)}, x_k^{(2)}) < \frac{1}{2}$ out of sequence $x_n^{(1)}$
$\varepsilon = \frac{1}{4}$ — Exists subsequence $(x_n^{(3)} : n \geq 1) : d(x_n^{(3)}, x_k^{(3)}) < \frac{1}{4}$ out of sequence $x_n^{(2)}$

$$\ldots$$
$$z_n = x_n^{(n)}$$

$(z_n, z_{n+1}, \ldots)$ — subsequence of x_n

$$\forall k_{ij} \geq n \quad d(z_k, z_j) = d(x_{r_k}^{(n)}, x_{r_j}^{(n)}) < \frac{1}{2^{n-1}}$$

$(z_k : k \geq 1)$ fundamental $\Rightarrow$ convergent subsequence.

Totally bounded space is separable. Indeed,

$$X = B(x_1, \frac{1}{n}) \cup \ldots \cup B(x_{k(n)}, \frac{1}{n}); \qquad D_n = \{x_1, x_{k(n)}\}$$

$$D = \bigcup_{n=1}^{\infty} D_n \text{ is a countable set}$$

$x \in X, \varepsilon > 0, \frac{1}{n} < \varepsilon$. We may find $y \in D_n : d(x,y) < \frac{1}{n} < \varepsilon$

Use Lemma 2.

Let $U_1, U_2, U_3, \ldots$ — countable open cover of X. Assume that finite sub cover does not exist. Then $X \bigvee (U_1 \cup U_2 \cup \ldots \cup U_n) \ni x_n \notin (U_1 \cup \ldots U_n)$.
Create a sequence of x_n where each x_i is not included in all of $U_j : j \leq i$.
$x_{n_k} \to x, \ k \to \infty$.
$x \in U_N, x_{n_m} \in U_N, m \geq k_0$
Contradiction.

□

Corollary 12. Metric space is compact $\Leftrightarrow$ arbitrary sequence has convergent subsequence.

Proof. ($\Leftarrow$) Completeness is fulfilled.

By contradiction. Assume that any sequence has a convergent subsequence. Now let X is not totally bounded.

$$\exists \varepsilon > 0 \ \varepsilon\text{-grid does not exist}$$

Pick some x_1.

$$x_1 \in X.$$

Open ball around x_1 does not cover all the space. Then pick x_2 that

$$\exists x_2 : d(x_2, x_1) \geq \varepsilon$$

Again, two balls out of x_1, x_2 do not cover all the space X.

$$\exists x_3 : d(x_3, x_2) \geq \varepsilon, d(x_3, x_1) \geq \varepsilon$$

$$\ldots$$

In result we got such a sequence of x_n that

$$d(x_n, x_k) \geq \varepsilon, n \neq k.$$

$(x_n : n \geq 1)$ does not have convergent subsequences. $\square$

Example. $A \subset \mathbb{R}^n$. A is compact $\Leftrightarrow$ A is closed and bounded.

Example. $l^1 = \{x = (x_1, x_2, \ldots) \mid \sum_{n=1}^{\infty} |x_n| < \infty\}$

$A \subset l^1$. A is compact $\Leftrightarrow$ A is closed, and A is bounded, and $\sup\limits_{x \in A} \sum\limits_{n=N}^{\infty} |x_n| \to_{N\to\infty} 0$

Proof. ($\Rightarrow$) $\varepsilon > 0 \qquad A \subset B(y^{(1)}, \varepsilon) \cup \ldots \cup B(y^{(m)}, \varepsilon)$
$y^{(i)} = \left(y_1^{(i)}, y_2^{(i)}, \ldots\right)$
$\sum\limits_{n=1}^{\infty} \left|y_n^{(i)}\right| < \infty$

$$\exists N : \sum_{n=N}^{\infty} \left|y_n^{(i)}\right| < \varepsilon, \ \ 1 \leq i \leq m$$

$$x \in A. d(x, y^{(i)}) < \varepsilon; \quad \sum_{n=1}^{\infty} \left|x_n - y_n^{(i)}\right| < \varepsilon$$

$$\sum_{n=N}^{\infty} |x_n| \leq \underbrace{\sum_{n=N}^{\infty} \left|x_n - x_n^{(i)}\right|}_{<\varepsilon} + \underbrace{\sum_{n=N}^{\infty} \left|y_n^{(i)}\right|}_{<\varepsilon} < 2\varepsilon$$

$$\sup_{x \in A} \sum_{n=N}^{\infty} |x_n| < 2\varepsilon$$

($\Leftarrow$) A is complete as it is closed in l^1.
$\varepsilon > 0 \qquad \exists N : \sup\limits_{x \in A} \sum\limits_{n=N}^{\infty} |x_n| < \varepsilon$

$\exists c > 0 : |x_n| < C, x \in A, n \geq 1$
exists $(y_1^{(i)}, \ldots, y_N^{(i)})$, $\quad 1 \leq i \leq M$
$u_l = -c + l\alpha, \quad 0 \leq l \leq \dfrac{2c}{\alpha}$
(u_{l_1}, u_{l_2})

$$\forall x \in A : \exists i :: \sum_{n=1}^{N} \left| y_n^{(i)} - x_n \right| \varepsilon$$

$$\left(y_1^{(i)}, \ldots, y_N^{(i)}, 0, 0, \ldots \right)$$

$$\forall x \in A : \exists i : d(X, y^{(i)}) < \varepsilon$$

$\square$

Example. $C[a,b]; \quad d(f,g) = \sup_{a \leq t \leq b} |f(t) - g(t)|$
$A \subset C[a,b]$ is compact $\Leftrightarrow$ A is closed, bounded and

$$\forall \varepsilon > 0 : \exists \delta > 0 : |t - s| \leq \varepsilon \Rightarrow \sup_{f \in A} |f(t) - f(s)| \leq \varepsilon$$

the last condition is called (одностайна рівномірна неперервність) Used Ascolli-Artsel theorem.

8 Linear Spaces

K — scalars field ($K = \mathbb{R}$ or $K = \mathbb{C}$).

Definition 23. Linear space upon field K is called a non-empty set X with operations:

1. $+ : X \times X \to X \quad ((x,y) \mapsto x + y)$
2. $\cdot : K \times X \to X \quad ((\alpha, x) \mapsto \alpha \cdot x = \alpha x)$

that satisfy the axioms:

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. $\exists 0 \in X : x + 0 = x$
4. $\forall x \in X : \exists (-x) : x + (-x) = 0$
5. $\alpha(\beta x) = (\alpha\beta) x$
6. $1 \cdot x = x$
7. $\alpha(x + y) = \alpha x + \alpha y$
8. $(\alpha + \beta) x = \alpha x + \beta x$

Example. $L^p(\Omega, \mathcal{F}, \mathbb{P})$ — collection of random variables ξ with $E\,|\xi|^p < \infty$

Definition 24. Let X be a linear space, $x_1, \ldots, x_n \in X$. Vectors $x_1, \ldots, x_n$ are linearly dependent, if there exist such scalars

$$\alpha_1, \ldots, \alpha_n \in K : \sum_{i=1}^{n} \alpha_i x_i = 0$$

and not every $\alpha_j = 0$.

Definition 25. Vectors $x_1, \ldots, x_n$ are linearly independent if they are not linearly dependent.

Definition 26. Basis of linear space is called maximum linear independent vector system.

Notice: In any arbitrary linear space there exists a basis.

Different basis' have similar cardinality, that is called a dimension of the space. If there is a finite basis (with n elements) then $dim X = n$ and X is finite-dimensional.

Definition 27. $X' \subset X$ — subspace, if $x, y \in X' \Rightarrow x + y \in X', \alpha x \in X'$

X' – subspace of linear space X.
$x \sim y \Leftrightarrow x - y \in X'$ — equivalence relation.
$[x] = \{y \in X : x \sim y\}$
$X//X'$ — a set of all equivalence classes. Let $\xi, \eta \in X//X' : \xi = [x], \eta = [y] \Rightarrow \xi + \eta = [x + y]$
$\alpha\xi = [\alpha x]$

$\dim X//X' \equiv$ codimension of X'.

Proposition

Proposition 1. Let X' — subspace of codimension n. Then $\exists x_1, \ldots, x_n \in X$: arbitrary $x \in X$ is written in the only way:

$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n + y, \quad \alpha_1, \ldots, \alpha_n \in K, y \in X'$$

Proof. $\xi_1, \ldots, \xi_n$ — basis in $X//X'$.
$\xi_1 = [x_1]$

If $x \in X$, $\xi = [x]$. $\xi = \alpha_1\xi_1 + \ldots + \alpha_n\xi_n$

$[x] = [\alpha_1, x_1 + \ldots \alpha_n x_n] \Rightarrow x - \alpha_1 x_1 - \ldots - \alpha_n x_n = \xi$ □

Definition 28. X, Y — linear spaces on some field K. $A : X \to Y$. A is a linear operator if $A(x_1 + x_2) = Ax_1 + Ax_2,\ A(\alpha x) = \alpha Ax$

$$\mathrm{Ker}\, A - \{x \in X : Ax = 0\}$$
$$\mathrm{R}(A) = \{Ax : x \in X\}$$

Definition 29. If $Y = K$ then linear operator $f : X \to K$ is called a linear functional.

Proposition 2.
1. If $f : X \to Y$ is not null linear functional, then its kernel $\operatorname{Ker} f$ has codimension 1;
2. If $X' \subset X$ — subspace of codimension 1 then there exist a linear functional $f : X \to Y$ for which $X' = \operatorname{Ker} f$

Proof.
1. $f(x_0) \neq 0$. $x^* = \dfrac{x_0}{f(x_0)}$. $f(x^*) = \dfrac{f(x_0)}{f(x_0)} = 1$
   Consider $x \in X$. $y = x - f(x)x^*$. $f(y) = f(x) - f(x)f(x^*)$
   $x = \underbrace{f(x)x^*}_{\alpha} + \underbrace{y}_{\in \operatorname{Ker} f} \Rightarrow \dim X/\operatorname{Ker} f = 1$
2. $\dim X/X' = 1$. $\exists x_1 \in X : x = \alpha x_1 + y$ (the only way)
   $f(x) := \alpha$. $f : X \to K$

   Linearity: $x = \alpha x_1 + y$; $x' = \alpha' x_1 + y' \Rightarrow$
   $\beta x + \gamma x' = (\beta\alpha + \gamma\alpha')x_1 + \beta y + \gamma y'$
   $f(\beta x + \gamma x') = \beta\alpha + \gamma\alpha' = \beta f(x) + \gamma f(x')$
   $\operatorname{Ker} f = X'$

□

8.1 Normed vector spaces

Definition 30. Norm on a linear space is called a function $x \mapsto \|x\| (\|\cdot\| : X \to \mathbb{R}$ that satisfies the conditions:

1. $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0$
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

If $\|\cdot\|$ — norm on X, then we may define a metric

$$d(x, y) = \|x - y\|$$

Definition 31. Normed space $(X, \|\cdot\|)$ that is complete relatively to metric $d(x, y) = \|x - y\|$ is called Banach space.

Example. $\mathbb{R}^n$, $1 \leq p < \infty$. $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$

Minkovskiy inequiality

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}$$

Proposition 3. Let X' — a linear subspace of normed space X. Then $\overline{X'}$ is a subspace too.

Proof. Let $x, y \in \overline{X'}$. Then $\exists x_n, y_n \in X' : \|x_n - x\| \to 0, \|y_n - y\| \to 0$.
$x_n + y_n \in X'$.
$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$ Then $x + y \in \overline{X'}$.

$\alpha x_n \in X'$. $\quad \|\alpha x_n - \alpha x\| = |\alpha|\|x_n - x\| \to 0$
$\alpha x \in \overline{X'}$ $\square$

Definition 32. $\operatorname{span}\{x_i : i \in I\}$ — subspace of X.
If $\overline{\operatorname{span}\{x_i : i \in I\}} = X$, then the system $\{x_i : i \in I\}$ is called complete.

Example. $C[a,b]$. A sequence $1, t, t^2, t^3, \ldots$ is complete.

Lemma 3. Linear normed space $(X, \|\cdot\|)$ is Banach space $\Leftrightarrow \sum_{n=1}^{\infty} \|x_n\| < \infty \Rightarrow$ sequence $\sum_{n=1}^{\infty} x_n$ is convergent in X (in other words there exists $\lim_{N\to\infty} \sum_{n=1}^{\infty} x_n$).

Proof. $(\Rightarrow)$

$(X, \|\|)$ is Banach. Let $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

$S_N = \sum_{n=1}^{N} x_n$

$\|S_N - S_{N+p}\| = \| \sum_{n=N+1}^{N+p} x_n\| \le \sum_{n=N+1}^{N+p} \|x_n\| \le \sum_{n=N+1}^{\infty} \|x_n\| \to 0,\ N \to \infty$

$(\Leftarrow)$

Let $(x_n)_{n\ge1}$ — fundamental in X. In other words, $\|x_n - x_m\| \to 0,\ n, m \to \infty$
$\forall n, m \ge n_k \quad \|x_n - x_m\| \le 2^{-k} \quad (n_1 < n_2 < \ldots)$

$$\|x_{n_k} - x_{n_{k+1}}\| \le 2^{-k} \Rightarrow \sum_{k=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| \le \sum_{k=1}^{\infty} 2^{-k} = 1$$

$$\sum_{k=1}^{\infty} \left(x_{n_k} - x_{n_{k+1}}\right) \text{ convergent (to } y$$

$$y = \lim_{N\to\infty} \left(\sum_{k=1}^{N} \left(x_{n_k} - x_{n_{k+1}}\right) \right) = \lim_{N\to\infty} \left(x_{n_1} - x_{n_2} + x_{n_2} - x_{n_3} + \ldots + x_{n_N} - x_{n_{N+1}}\right) =$$
$$= x_{n_1} - \lim_{k\to\infty} x_{n_k}$$

$\square$

X — normed space. X' — closed subspace of X.
X/X' — collection of equivalence classes of the relation $x \sim y \Leftrightarrow x - y \in X'$.

$\xi \in X/X'$. $\quad \|\xi\| := \inf_{x\in\xi} \|x\|$

Theorem 10.
1. $\xi \mapsto \|\xi\|$ — is a norm on X/X';
2. if X is Banach space then X/X' is Banach too.

Proof. 1. $\|\xi\| \geq 0$. $\|0\| =?0$

In factor-space null is X'.

$\|0\| = \inf\limits_{x \in X'} \|x\| = 0$

Let $\|\xi\| = 0 = \inf\limits_{x \in \xi} \|x\|$. Exists $x_n \in \xi : \|x_n\| \to 0$.

$\underbrace{x_n - x_1}_{\in X'} \to x_1 \in X' \Rightarrow \xi = [x_1] = X' = 0$

2. $\alpha = 0 \Rightarrow \|\alpha\xi\| = |\alpha|\, dot \|\xi\|$

   Let $\alpha \neq 0$. $x \in \xi \Rightarrow \alpha\xi \in \alpha\xi$

   $\|\alpha\xi\| \leq \|\alpha x\| = |\alpha| \cdot \|x\| \Rightarrow \|\alpha\xi\| \leq |\alpha| \cdot \|\xi\|$

   $\|\xi\| = \|\frac{1}{\alpha}(\alpha\xi)\| \leq \frac{1}{|\alpha|}\|\alpha\xi\| \Rightarrow |\alpha|\,\|\xi\| \leq \|\alpha\xi\|$

3. $x \in \xi$, $y \in \eta \Rightarrow x + y \in \xi + \eta$

   $\|\xi + \eta\| \leq \|x + y\| \leq \|x\| + \|y\| \Rightarrow \|\xi + \eta\| \leq \|\xi\| + \|\eta\|$

   Let X — Banach space, X' — closed subspace X.

   Enough to show that $\sum\limits_{n=1}^{\infty} \|\xi_n\| < \infty \Rightarrow \sum\limits_{n=1}^{\infty} \xi_n$ is convergent.

   $\|\xi_n\| = \inf\limits_{x \in \xi_n} \|x\|$.

   $2\|\xi_n\| \geq \|x_n\|$ for some $x_n \in \xi_n$

   $\sum\limits_{n=1}^{\infty} \|x_n\| \leq 2 \sum\limits_{n=1}^{\infty} \|\xi_n\| < \infty \Rightarrow$ sequence $\sum\limits_{n=1}^{\infty} x_n$ is convergent.

   $x = \lim\limits_{N \to \infty} \sum\limits_{n=1}^{N} x_n$. $\xi = [x]$. $\|\xi - \sum\limits_{n=1}^{N} \xi_n\| \leq \|x - \sum\limits_{n=1}^{N} x_n\| \to 0$

   $\xi = \lim\limits_{N \to \infty} \sum\limits_{n=1}^{N} \xi_n$

□

$(X, \|\|), (Y, \|\|)$ normed spaces (with different norms) upon field K.
$A : X \to Y$ — linear operator.

Theorem 11. The following conditions are equivalent:

1. A is continuous
2. A is continuous in point 0
3. A is bounded in a ball with radius 1

$$\sup_{\|x\| \leq 1} \|Ax\| < \infty$$

4. $\exists c > 0$: and for any $x : \|Ax\| \leq c\|x\|$

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3)

A is continuous in $0 \Rightarrow \forall\varepsilon > 0 : \exists\delta > 0 : \|x\| \leq \delta \Rightarrow \|Ax\| \leq \varepsilon$
$\|x\| \leq 1 \Rightarrow \|\delta x\| \leq \delta \Rightarrow \|A\delta x\| \leq \varepsilon \Rightarrow \|Ax\| \leq \frac{\varepsilon}{\delta}$.

$\sup\limits_{\|x\|\le 1}\|Ax\| \le \dfrac{\varepsilon}{\delta}$

(3) ⇒ (4)

$\sup\limits_{\|x\|\le 1}\|Ax\| =: C < \infty$

Let $x \ne 0$. $\left\|\dfrac{x}{\|x\|}\right\| = 1 \Rightarrow \left\|A\dfrac{x}{\|x\|}\right\| \le x \Rightarrow \|Ax\| \le c\|x\|$ $(A0 = 0)$

(4) ⇒ (1)

$\|Ax - Ay\| = \|A(x-y)\| \le C\|x-y\|$
A is continuous. □

> **Definition 33.** Let $A : X \to Y$ — linear continuous operator. $\|A\| = \sup\limits_{\|x\|\le 1}\|Ax\|$ — norm of the operator.

> **Theorem 12.**
> 1. $\mathcal{L}(X,Y) = \{A : X \to Y \mid A \text{ is linear and continuous}\}$, $\|A\| = \sup\limits_{\|x\|\le 1}\|Ax\|$
> $(\mathcal{L}(X,Y), \|\cdot\|)$ — normed space.
> 2. If Y is Banach space, then $\mathcal{L}(X,Y)$ is Banach too.

Proof. **2)** $\|A_n - A_m\| \to 0,\ n,m \to \infty$
$\|A_n x - A_m x\| \le \|A_n - A_m\| \cdot \|x\| \to 0$
$(A_n x : n \ge 1)$ — fundamental in Y
$\exists \lim\limits_{n\to\infty} A_n x =: Ax$ and $A : X \to Y$ is linear
$|\|A_n\| - \|A_m\|| \le \|A_n - A_m\| \to 0,\ n,m \to \infty$
$\exists \lim\limits_{n\to\infty} \|A_n\|$: $\|A_n\|$ are bounded . $\|A_n\| \le C$
$\|Ax\| = \lim\limits_{n\to\infty} \underbrace{\|A_n x\|}_{\le\|A_n\|\cdot\|x\|\le C\|X\|} \le C\|x\| \Rightarrow A$ is continuous

Let $\|x\| \le 1$.
$\exists N : \forall n,m \ge N : \|A_n - A_m\| \le \varepsilon$
$\|A_n x - Ax\| = \lim\limits_{m\to\infty} \underbrace{\|A_n x - A_m x\|}_{\le\|A_n - A_m\|\cdot\|x\|\le\varepsilon} \le \varepsilon$
$\|A_n - A\| \le \varepsilon, n \ge N$ □

$\mathcal{L}(X,K) =: X^*$ — collection of all linear continuous functionals.

X^* is Banach space (complement to X).

> **Theorem 13.** $f : X \to K$ — linear functional. Then f is continuous ⇔ $\operatorname{Ker} f$ is closed.

Proof. (⇒)

f — continuous. $\operatorname{Ker} f = \{x \in X : f(x) = 0\} = f^{-1}(\{0\})$ closed.

(⇐)

Let $\operatorname{Ker} f$ is closed. Assume that $f \ne 0$.
Exists $x_0 : f(x_0) = 1$

$x_0 \notin \operatorname{Ker} f \Rightarrow \exists \varepsilon > 0 : B(x_0, \varepsilon) \cap \operatorname{Ker} f = \varnothing$
$\|y\| \leq \varepsilon$. Let $|f(y)| > 1$. $x_0 - \dfrac{y}{f(y)} = x$
$\|x_0 - x\| = \|\dfrac{y}{f(y)}\| = \dfrac{\|y\|}{|f(y)|} < \varepsilon \Rightarrow x \in B(x_0, \varepsilon)$
$f(x) = f(x_0 - \dfrac{y}{f(y)}) = f(x_0) - \dfrac{f(y)}{f(y)} = 0$
x belongs to open ball, but x belongs to the kernel.
If $\|y\| \leq \varepsilon \Rightarrow |f(y)| \leq 1$
x is arbitrary. $\|\dfrac{\varepsilon x}{\|x\|}\| = \varepsilon \Rightarrow \left| f(\dfrac{\varepsilon x}{\|x\|} \right| < 1$
$\dfrac{\varepsilon}{\|x\|} |f(x)| \leq 1 \Leftrightarrow |f(x)| \leq \dfrac{1}{\varepsilon} \|x\|$ □

If $f : X \to K$ linear, then $\operatorname{Ker} f$ is either closed or everywhere dense.

$$\operatorname{Ker} f \underbrace{\subset}_{\text{or } =} \overline{\operatorname{Ker} f} \underbrace{\subset}_{\text{or } =} X$$

If we add another vector to kernel then we get whole the space X.

Theorem 14. Let X normed space, Y is a subspace of X. If Y is finite-dimensional then it's closed.

Proof. $dimY = n$. Induction by n.

1. $n = 0 \Rightarrow Y = \{0\}$ is closed.
2. Let the theorem be true for any subspace of dimension n. Let's check for $n + 1$. Y— subspace of dimension n. $\{e_1, e_2, \ldots, e_n\}$ — basis in Y.
   $Z = \operatorname{span}\{e_1, e_2, \ldots, e_{n-1}\}$. Z is a subspace of Y. $dimZ = n - 1$. And Z is closed.
   $y = z + te_n,\ z \in Z, t \in K$
   $f : Y \to K, \quad f(z + te_n) = t$.
   f is linear, $\operatorname{Ker} f = Z$ closed $\Rightarrow f$ is continuous.
   $\exists C > 0 : |f(y)| \leq C\|y\|$

   Prove that Y is closed: $y_k \in Y, y_k \to x \in Y$
   $y_k = z_k + t_k e_n = z_k + f(y_k)e_n$

   $$|t_k - t_m| = |f(y_k - y_m)| \leq C\|y_k - y_m\| \to_{k,m\to\infty} 0$$

   $(t_k)_{k\geq 1}$ fundamental $\Rightarrow$ is convergent.
   $t_k \to t$.

   $$z_k = y_k - t_k e_n \to x - te_n \in Z \text{ (as } Z \text{ is closed)}$$
   $$x = \underbrace{(x - te_n)}_{\in Z} + te_n \in Y$$

□

Corollary 13.

$$\dim X < \infty \Rightarrow \text{ all linear functionals are continuous}$$

Corollary 14. $\dim X < \infty$, Y is normed, $A : X \to Y$ linear. Then A is continuous.

Proof. $\{e_1, \ldots, e_n\}$ — basis in X. $x = t_1e_1 + \ldots + t_ne_n$.
$f_i(x) = t_i,\ \ 1 \le i \le n$. $|f_i(x)| \le C\|x\|$

$$\|Ax\| = \|A\left(f_1(x)e_1 + \ldots + f_n(x)e_n\right)\| = \|\sum_{i=1}^{n} f_i(x)Ae_i\| \le \sum_{i=1}^{n} |f_i(x)|\cdot\|Ae_i\| \le \sum_{i=1}^{n} C\|x\|\cdot\|Ae_i\|$$

$$\|Ax\| \le \left(C\sum_{i=1}^{n}\|Ae_i\|\cdot\|x\|\right)$$

□

Example. $X = C[0,1]$, $\|x\| = \displaystyle\int_0^1 |x(t)|\,dt$
$f(x) = x(0)$ is linear. If f is continuous, then $|f(x)| \le C\|x\|$. In other words, $|x(0)| \le C \cdot \displaystyle\int_0^1 |x(0)|\,dt$.
It's hard to determine such C constant. Then f is not continuous.

Theorem 15 (Hana-Banach, about continuity of linear continuous functional)**.** X — normed space. Y — its subspace. $f_0 : Y \to K$ linear continuous functional.
Whether we can continue f_0 on whole X while keeping linearity and continuity?

1. Geometric form of Hahn-Banach theorem. $K = \mathbb{R}$, X normed, real.
   $A \subset X$ is convex if $x, y \in A \Rightarrow tx + (1-t)y \in A$, $\forall t \in [0,1]$ *idea picture*

   X— read normed, A open convex set, M — subspace of X, $M \cap A = \varnothing$.
   Then exists closed hyperspace H (in other words with codimension 1):

   (a) $M \subset H$

   (b) $H \cap A = \varnothing$

Proof. $C = M + \bigcup\limits_{\lambda>0} \lambda A = \{y + \lambda a : y \in M, \lambda > 0, a \in A\}$
$-C = \{-y - \lambda a : y \in M, \lambda > 0, a \in A\} = M + \bigcup -\lambda a$
$C, -C, M$ are pairwise disjoint.
$x \in M \cap C. x = y + \lambda a \Rightarrow a = \dfrac{x-y}{\lambda} \in M$ — impossible
$M \cap (-C) = \varnothing$

$x \in C \cap (-C) \qquad y_1 + \lambda_1 a_1 = y_2 - \lambda_2 a_2$

$$\lambda_1 a_1 + \lambda_2 a_2 = y_2 - y_1$$

$$\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}a_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}a_2}_{\text{on segment from } a_1 \text{ to } a_2} = \frac{y_2 - y_1}{\lambda_1+\lambda_2} \in A \cap M \text{ — impossible}$$

Cases:

1. $M \cup C \cup (-C) \neq X$ choose $h \notin M \cup C \cup (-C)$
   $M_1 = \{y + th : y \in M, t \in \mathbb{R}\}$
   If $M_1 \cap A \neq \varnothing$ then $\exists a \in A : a = y + th.\ t \neq 0.h = -\dfrac{y}{t} + \left(\dfrac{1}{t}\right) a$

   $M_1 \cap A \neq \varnothing$

2. $M \cap C \cap (-C) = X$

   Let M codimension > 1
   $a \notin M, b \notin \operatorname{span}(M \cup \{a\})$
   $a \in C \cup (-C), b \in C \cup (-C)$
   $a \in C, b \in (-C) \qquad g(t) = ta + (1-t)b, 0 \leq t \leq 1$
   $g(0) = b \notin M, g(1) = a \notin M$
   If $0 < t < 1$ and $g(t) \in M$ then $b = \dfrac{g(t) - ta}{1 - t} \in \operatorname{span}(M \cup \{a\})$
   $\forall t : g(t) \in C \cup (-C)$
   $g^{-1}(C), g^{-1}(C)$ — are open and not empty.
   $t^* = \inf\{t > 0 : g(t) \in C\}$
   M has codimension 1.

proof is not finished □