Problem 5.15

$$\Omega = \{\text{cards}\};\ |\Omega| = 52$$

a.

$$A = \{\text{hearts}\};\ |A| = 13$$

$$B = \{\text{red cards}\};\ |B| = 26$$

$$P(A|B) = \frac{P(A \cap B)}{B} = \frac{1}{2}$$

b.

$$A = \{\text{ rank 11, 12, 13, 14 }\};$$

$$B = \{hearts\};$$

$$P(A|B) = \frac{4}{13}$$

c.

$$P(A|B) = \frac{2}{26} = \frac{1}{13}$$

Problem 5.16

$$\Omega = \{(i, j) \mid i, j \in \{1, 2, 3, 4, 5, 6\}\};\ |\Omega| = 36$$

$$A = \{\, (i, j) \in \Omega \mid i + j > 7 \,\};\ |A| = 5 + 4 + 3 + 2 + 1 = 15$$

a

$$B = \{(i, j) \in \Omega | i = 1\};\ |B| = 6$$

$$A \cap B = \emptyset$$

$$P(A|B) = 0$$

b

$$B = \{\, (i, j) \in \Omega \mid i < 5 \,\};\ |B| = 4 \cdot 6 = 24$$

$$|A \cap B| = 6$$

$$P(A|B) = \frac{1}{4}$$

Problem 5.17

$$\Omega = \{\,(i,j) \mid i,j \in \{\,0,1\,\}\,, 0 \text{ - tails, } 1 \text{ - heads}\,\}$$
$$A = \{\,(1,j) \mid j \in \{\,0,1\,\}\,\}$$
$$B = \{\,(i,1) \mid i \in \{\,0,1\,\}\,\}$$
$$C = \{\,(i,j) \in \Omega \mid i = j)\,\}$$
$$P(A) = 1/2$$
$$P(B) = 1/2$$
$$P(C) = 1/2$$
$$P(A \cap B) = \frac{1}{4} = P(A)P(B)$$
$$P(B \cap C) = \frac{1}{4} = P(B)P(C)$$
$$P(A \cap C) = \frac{1}{4} = P(A)P(C)$$
$$P(A \cap B \cap C) = P(\{\,(1,1)\,\}) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C)$$

Problem 5.18

$$\Omega = \{\,\{\,i,j,k\,\} \mid i,j,k \in \mathbb{N}, i,j,k \in [1,10], i \neq j \neq k\,\}$$
$$|\Omega| = C_{10}^{3} = \frac{10!}{3!7!} = 4 \cdot 3 \cdot 10$$
$$B = \{\,\{\,i,j,k\,\} \in \Omega \mid i \in [8,10] \text{ or } j \in [8,10] \text{ or } k \in [8,10]\,\}$$
$$A = \{\,\{\,i,j,k\,\} \in \Omega \mid \exists a,b \in \{\,i,j,k\,\} : a,b \in [1,7]\,\}$$

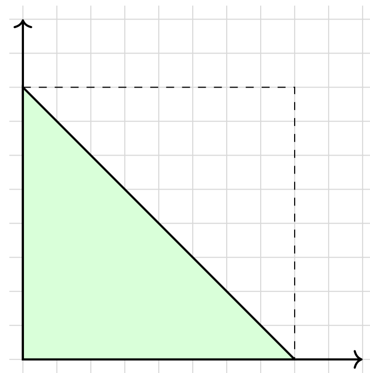
$$A \cap B = \{\,\{\,i,j,k\,\} \in \Omega \mid \text{two white, one black}\,\}$$
$$P(B) = 1 - P(\overline{B}) = 1 - \frac{|\overline{B}|}{|\Omega|} = 1 - \frac{C_7^3}{10 \cdot 4 \cdot 3} = 1 - \frac{7!}{3! \cdot 4! \cdot 10 \cdot 4 \cdot 3} =$$
$$= 1 - \frac{5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 10 \cdot 4 \cdot 3} = 1 - \frac{7}{24} = \frac{17}{24}$$

$$|A \cap B| = C_7^2 C_3^1 = \frac{7!}{2!5!}3 = \frac{7 \cdot 6}{2}3 = 7 \cdot 9$$
$$P(AB) = \frac{7 \cdot 9}{10 \cdot 4 \cdot 3} = \frac{21}{40}$$
$$P(A|B) = \frac{21 \cdot 24}{40 \cdot 17} = \frac{63}{85}$$

Problem 5.19

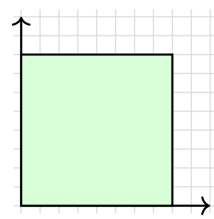
$$\Omega = \{\, (\zeta_1, \zeta_2) \in [0,1]^2 \,\}$$

$$B = \{\, (\zeta_1, \zeta_2) \in \Omega \mid \zeta_1 + \zeta_2 \leq 1 \,\}$$



a

$$A = \{\, (\zeta_1, \zeta_2) \in \Omega \mid |\, \zeta_1 - \zeta_2| < 1 \,\}$$



$$P(A|B) = \frac{P(B)}{P(B)} = 1$$

b

$$A = \{\, (\zeta_1, \zeta_2) \in \Omega \mid \zeta_1 \cdot \zeta_2 < \frac{1}{2} \,\}$$

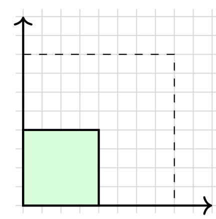
$$\zeta_2 < \frac{1}{2\zeta_1}$$

$$A \cap B = B$$

$$P(A|B) = 1$$

c

$$A = \{\, max(\zeta_1, \zeta_2) < 1/2 \,\}$$



$$A \cap B = A$$
$$P(B) = \frac{1}{2}$$
$$P(A) = \frac{1}{4}$$
$$P(A|B) = \frac{1}{2}$$

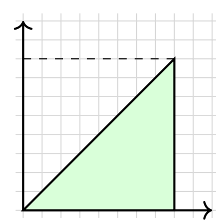
d

$$A = \{\, \zeta_1^2 + \zeta_2^2 < \frac{1}{4} \,\}$$

$$A \cap B = A$$
$$P(A) = \frac{\pi \frac{1}{4}}{4} = \frac{\pi}{16}$$
$$P(A|B) = \frac{\pi}{16} \cdot \frac{2}{1} = \frac{\pi}{8}$$

e

$$A = \{\, \zeta_1 > \zeta_2 \,\}$$



$$P(A|B) = \frac{1/4}{1/2} = \frac{1}{2}$$

Problem 5.20

$$\Omega = \{ (\varepsilon_1, \ldots, \varepsilon_n) \mid \varepsilon_i \in \{ 0, 1 \}, 1 \text{ - heads, } 0 \text{ - tails} \}; \ |\Omega| = 2^n$$

$$A = \{ (\varepsilon_1, \ldots, \varepsilon_n) \in \Omega \mid \varepsilon_1 = 1 \}$$

$$B_k = \{ (\varepsilon_1, \ldots, \varepsilon_n) \in \Omega \mid \sum \varepsilon_i = k \}$$

$$P(A) = p$$

$$P(B_k) = C_n^k p^k \cdot (1-p)^{n-k}$$

$$P(AB_k) = p(C_{n-1}^{k-1} p^{k-1} \cdot (1-p)^{n-k})$$

$$P(AB_k) = P(A)P(B_k)$$

$$p^k \frac{(n-1)!}{(k-1)!(n-k)!}(1-p)^{n-k} = p^{k+1} \frac{n!}{k!(n-k)!}(1-p)^{n-k}$$

$$\frac{k}{n} = p \Rightarrow k = pn$$

Problem 5.21

$$\Omega = \left\{ \begin{pmatrix} 1 & 2 & 3 & \ldots & 10 \\ k_1 & k_2 & k_3 & \ldots & k_{10} \end{pmatrix} \mid k_i \in \{ 1, \ldots, 10 \} \right\}$$

$$A = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & 10 \\ k_1 & k_2 & k_3 & 4 & k_5 & \ldots & k_{10} \end{pmatrix} \in \Omega \right\}$$

$$B = \left\{ \begin{pmatrix} 1 & 2 & 3 & \ldots & 10 \\ k_1 & k_2 & k_3 & \ldots & k_{10} \end{pmatrix} \in \Omega \mid k_{2n} = 1, 2, 3, \ k_{2(n+1)} = 2, 3, 1, \ k_{2(n+2)} = 3, 1, 2 \right\}$$

$$|B| = 3 \cdot 3! \cdot 7!$$

$$A \cap B = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 10 \\ k_1 & k_2 & k_3 & 4 & \ldots & k_10 \end{pmatrix} \in \Omega \mid k_6 = 1, 2, 3, \ k_8 = 2, 3, 1, \ k_10 = 3, 1, 2 \right\}$$

$$|A \cap B| = 3! \cdot 6!$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3! \cdot 6!}{3 \cdot 3! \cdot 7!} = \frac{1}{3 \cdot 7} = \frac{1}{21}$$

Problem 5.22

$$\Omega = \{(\varepsilon_1, \dots, \varepsilon_7) \mid \varepsilon_i \in \{\,0, 1\,\}\}$$
$$A_i = \{(\varepsilon_1, \dots, \varepsilon_7) \in \Omega \mid \varepsilon_i = 1\}$$
$$\overline{A_i} = \{(\varepsilon_1, \dots, \varepsilon_7) \in \Omega \mid \varepsilon_i = 0\}$$
$$P(A_i) = C_7^1 0.1 \Rightarrow P(\overline{A_i}) = C_7^1 0.9$$
$$P(\mathcal{A}_j) = [\text{probability of attack j times}] =$$
$$= P(A_a^{C_a} \cap A_b^{C_b} \cap A_c^{C_c} \cap A_d^{C_d} \cap A_e^{C_e} \cap A_f^{C_f} \cap A_g^{C_g}) = C_7^j 0.1^j \cdot 0.9^{7-j}$$
$$\sum C_k = i, \quad A_l^{C_l} = \begin{cases} A_l, \; C_k = 1 \\ \overline{A_l}, \; C_k = 0 \end{cases}$$
$$P(\mathcal{A}_2) = C_7^2 0.1^2 \cdot 0.9^5 \approx 0.124$$
$$P(\mathcal{A}_0) = 0.9^7 \approx 0.478$$
$$P(\mathcal{A}_0) + P(\mathcal{A}_1) + P(\mathcal{A}_2) = C_7^0 0.9^7 + C_7^1 0.1 \cdot 0.9^6 + C_7^2 0.1^2 \cdot 0.9^5 \approx 0.478 + 0.372 + 0.124 = 0.974$$

Problem 5.23

A_j - the coin is in j box. B_i - you haven't found for a coin in i box

$$P(B_i|A_j) = \begin{cases} (1 - a_i), & i = j \\ 1, \; i \neq j \end{cases}$$

$$P(B_i|A_j) = \frac{P(B_i A_j)}{P(A_j)} = \frac{P(B_i A_j)}{P(B_i)} = P(A_j|B_i)$$
$$P(B_i|A_j)P(A_j) = P(A_j|B_i)P(B_i)$$
$$P(A_j|B_i) = \frac{P(B_i|A_j)P(A_j)}{P(B_i)}$$

$$P(B_i) = P(B_i|A_j)P(A_j) + P(B_i|\overline{A_j})P(\overline{A_j}) = (1 - a_i)p_i + (1 - p_i) = 1 - a_i p_i$$

$$P(A_j|B_i) = \begin{cases} \frac{(1-a_i)p_i}{1-a_i p_i}, & i = j \\ \frac{p_j}{1-a_i p_i}, & i \neq j \end{cases}$$

Problem 5.24

$$P(A|B) = P(B|A)$$
$$P(A \cup B) = 1$$
$$P(A \cap B) > 0$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} = P(B|A) \;\Rightarrow\; P(A) = P(B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$1 = 2P(A) - P(A \cap B)$$
$$P(A) = \frac{1}{2} + \frac{1}{2}P(A \cap B) \;\Rightarrow\; P(A) > \frac{1}{2}$$

Problem 5.25

$$A \subset B$$
$$P(A \cap B) = [A \subset B] = P(A)$$
$$\Rightarrow P(A) = P(A)P(B)$$
$$P(B) = 1 : P(A) = P(A)P(B) = P(A)$$
$$P(A) = 0 : 0 = P(A) = P(A)P(B) = 0$$

Problem 5.26

$$P(A_1 \dots A_n|C) = \frac{P(A_1 \dots A_n C)}{P(C)}$$
$$P(A_1|C)P(A_2|A_1C) \dots P(A_n|A_1 \dots A_{n-1}C) = \frac{P(A_1C)}{P(C)} \cdot \frac{P(A_1A_2C)}{P(A_1C)} \dots \frac{P(A_1 \dots A_{n-1}A_nC)}{P(A_1 \dots A_{n-1}C)} =$$
$$= \frac{P(A_1 \dots A_{n-1}A_nC)}{P(C)}$$

Problem 5.27

$$P(A) = \begin{cases} 0 \\ 1 \end{cases}$$

1)

$$[P(A) = 0]$$

$$P(A)P(B) = 0$$
$$P(A \cap B) \leq P(A) \Rightarrow P(A \cap B) = 0$$

2)

$$[P(A) = 1]$$

$$P(A)P(B) = 1$$
$$P(\overline{A} \cap B) = 0$$
$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$
$$P(A \cap B) = P(B) = P(A)P(B) = 1$$