Numerical Methods for Computing Eigenvectors

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Abstract

In this document, I present some background on numerical methods for computing the eigenvectors and singular vectors of matrices. A Julia implementation of the power and QR methods is given, and the two algorithms are compared

1 Mathematical Background

It is assumed that the reader is familiar with linear algebra at the college introductory level, but a brief background of the relevant concepts is presented in this section.

1.1 Eigenvectors and Eigenvalues

Definition 1.1: eigenvectors and eigenvalues

Consider an arbitrary $n \times n$ matrix A. For some $\mathbf{x} \in \mathbb{R}^n$ (with $\mathbf{x} \neq \mathbf{0}$), we say that $\mathbf{x}$ is an *eigenvector* of A if, for some $\lambda \in \mathbb{R}$, $A\mathbf{x} = \lambda\mathbf{x}$. We denote λ as the corresponding *eigenvalue* of A.

From definition 1.1 follows immediately a way to compute the eigenvalues of a given matrix. Note that

$$A\mathbf{x} = \lambda\mathbf{x} \implies A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

and $\lambda\mathbf{x} = \lambda I\mathbf{x}$, so we have

$$A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

That is to say, $\lambda \in \mathbb{R}^n$ is an eigenvalue of A if and only if $A - \lambda I$ has a non-trivial null space. A matrix has a non-trivial null space if and only if it is singular, so we require that $\det(A - \lambda I) = 0$. This result is stated in theorem 1.1.

Theorem 1.1: computing eigenvalues as polynomial roots

Some $\lambda \in \mathbb{R}^n$ is an eigenvalue of the $n \times n$ matrix A if and only if $\det(A - \lambda I) = 0$. We call $\det(A - \lambda I)$ the *characteristic polynomial* for A. The problem then becomes identifying the roots of this polynomial.

Computing eigenvectors is useful for solving [blah blah]

1.2 Singular Vectors and Values

We have seen that eigenvectors and eigenvalues are useful for understanding the behavior of square matrices, but what about the more general case of $m \times n$ matrices? To handle this, we define the notion of singular vectors and values in definition 1.2.

Definition 1.2: singular vectors and values

Any $m \times n$ matrix A admits a decomposition $A = U\Sigma V^T$, where

- U is an $m \times m$ orthogonal matrix
- Σ is an $m \times n$ matrix that is "diagonal" in that $\Sigma_{ij} = 0$ if $i \neq j$
- V is an $n \times n$ orthogonal matrix.

We call the nonzero entries σ_i in

1.3 Numerical Methods

In general, we have seen that it is possible to compute eigenvectors of an $n \times n$ matrix as the roots of an n-degree polynomial. Unfortunately, it is a well-known result that polynomials of degree 5 and higher to not in general admit a solution by radicals [cite]...

The most obvious way to compute the eigenvectors of A

1.4 Power and QR Algorithms