Applied Probability Models for CS EX1

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Question 1

1. Let X represent the score of a student on a 20-question assignment.
2. Let W represent the event that a student uses Wikipedia.
3. Let E the event that a student uses an encyclopedia.
4. Each question is worth 5 points if answered correctly, and zero otherwise.

Given data:

- $P(W) = 0.7$ and $P(E) = 0.3$
- Wikipedia accuracy: $P(\text{correct} \mid W) = 0.85$
- Encyclopedia accuracy: $P(\text{correct} \mid E) = 0.95$
- Each question is worth 5 points.

a: Expected grade of a student who uses Wikipedia

Let X_W represent the scores of students who use Wikipedia.
Since there are 20 questions, and each question has an accuracy of 85% when using Wikipedia, the expected number of correct answers for a Wikipedia user is:

$$\mathbb{E}(\text{correct answers} \mid W) = 20 \times 0.85 = 17$$

Each correct answer is worth 5 points, so the expected score for a Wikipedia user is:

$$\mathbb{E}(X_W) = 17 \times 5 = 85$$

Thus, the expected grade for a student who uses Wikipedia is:

$$\mathbb{E}(X_W) = 85$$

b: Probability that a randomly chosen student got 90

To get a score of 90, a student must answer 18 out of 20 questions correctly, since $18 \times 5 = 90$.
Let's calculate the probability that a student got exactly 18 correct answers using each source, then apply the Law of Total Probability.

1. **Probability of 18 correct answers if the student uses Wikipedia:**

   The number of correct answers follows a binomial distribution, so:

$$P(X = 90 \mid W) = P(18 \text{ correct answers} \mid W) = \binom{20}{18}(0.85)^{18}(0.15)^2 = 0.22934$$

2. **Probability of 18 correct answers if the student uses an encyclopedia:**

$$P(X = 90 \mid E) = P(18 \text{ correct answers} \mid E) = \binom{20}{18}(0.95)^{18}(0.05)^2 = 0.18868$$

3. **Total probability that a student scores 90:**

   Using the Law of Total Probability,

$$P(X = 90) = P(X = 90 \mid W) \cdot P(W) + P(X = 90 \mid E) \cdot P(E) =$$

$$0.22934 \cdot 0.7 + 0.18868 \cdot 0.3 = 0.217142$$

c: Probability that Bob used Wikipedia given he scored 90

This is a conditional probability, $P(W \mid X = 90)$, which can be calculated using Bayes' theorem:

$$P(W \mid X = 90) = \frac{P(X = 90 \mid W) \cdot P(W)}{P(X = 90)} = \frac{0.22934 \cdot 0.7}{0.217142} = 0.73932$$

Question 2

1. Let N represent the number of cupcakes
2. Let FC represent the number of frosted cupcakes
3. Let NC represent the number of normal cupcakes
4. Let ki represent the k turn I get to pick a cupcake

Given data:

- $N = 40$
- $FC = 15$
- $NC = 25$
- $ki = 20$

a. What is the probability that you will be the first to get a frosted cupcake?

Meaning:
P(The 20th student will pick the first frosted cupcake) =
P(The first 19 students picked only normal cupcakes)$\times P$(The 20th student picks a frosted cupcake)

Step 1: Calculate P(The first 19 students picked only normal cupcakes)
To calculate this, we use the hypergeometric distribution. The probability is given by:

$$P(\text{The first 19 students picked only normal cupcakes}) = \frac{\binom{\text{NC}}{k} \times \binom{N-\text{NC}}{n-k}}{\binom{N}{n}}$$

where:

- $k = 19$ (the number of normal cupcakes selected),
- $n = 19$ (the total number of students who picked so far).

Substituting the values:

$$P(\text{The first 19 students picked only normal cupcakes}) = \frac{\binom{25}{19} \times \binom{40-25}{19-19}}{\binom{40}{19}}$$

Simplify the binomial coefficients:
P(The first 19 students picked only normal cupcakes) =
$\frac{\binom{25}{19}\times\binom{15}{0}}{\binom{40}{19}} = \frac{177100\times 1}{1.312\times 10^{11}} \approx \frac{7}{5189028} \approx 1.349 \times 10^{-6}$

Step 2: Calculate $P(\text{The 20th student picks a frosted cupcake})$
After the first 19 students, there are 15 frosted cupcakes and 6 normal cupcakes remaining. The probability that the 20th student picks a frosted cupcake is:

$$P(\text{The 20th student picks a frosted cupcake}) = \frac{\text{Number of frosted cupcakes left}}{\text{Total cupcakes left}} = \frac{15}{21}$$

Step 3: Combine the probabilities
Substituting the values:

$$P(\text{The 20th student will pick the first frosted cupcake}) \approx 1.349 \times 10^{-6} \times \frac{15}{21} \approx 9.636 \times 10^{-7}$$

Final Solution:

$$P(\text{The 20th student will pick the first frosted cupcake}) \approx 9.636 \times 10^{-7}$$

b. What is the probability that all the frosted cupcakes were taken already?

Meaning:
We are calculating the probability that the 20th pick will be the 5th normal cupcake picked. This can be modeled using the negative hypergeometric distribution. The formula for the probability is:

$$P(X = m) = \frac{\binom{k-1}{m-1} \times \binom{N-k}{D-m}}{\binom{N}{D}}$$

where:

- $N = 40$ (total cupcakes),
- $k = 20$ (total draws made),
- $D = 25$ (normal cupcakes),
- $N - k = 15$ (frosted cupcakes),
- $m = 5$ (the 5th normal cupcake picked).

Substitute the Values

$$P(X = 5) = \frac{\binom{20-1}{5-1} \times \binom{40-20}{25-5}}{\binom{40}{25}} = \frac{\binom{19}{4} \times \binom{20}{20}}{\binom{40}{25}} = \frac{3876 \times 1}{40225345056} = \frac{1}{10378056} \approx 9.6357 \times 10^{-8}$$

Final Solution

$$P(\text{All the frosted cupcakes were taken by the 20th pick}) \approx 9.6357 \times 10^{-8}$$

(C):

i. What is your expected reward if you don't accept the deal?

We are selecting the 20th cake, and it is known that only 2 frosted cakes were chosen already.

$$\mathbb{E}(\text{no deal}) = P(\text{frosted cake at the 20th selection}) \times r_f +$$

$P(\text{normal cake at the 20th selection}) \times r_n = \dfrac{15-2}{40-19} \times r_f + \dfrac{25-17}{40-19} \times r_n = \dfrac{13}{21} \times r_f + \dfrac{8}{21} \times r_n$

ii. What is your expected reward if you accept the deal?

We have 3 cases, the first is if we get a normal cupcake, the second is if both we and our friend get a frosted cupcake, and the third one is if we get a frosted cupcake and our friend doesn't. The probability our friend doesn't get a normal cupcake equals the negative hyper geometric distribution for $N = 20, D = 12, k = 20, m = 12$:

- $A = \{\text{Our friend gets a frosted cupcake}\}$
- $B = \{\text{I get a frosted cupcake}\}$

$$P(A \mid B) = \frac{\binom{20-1}{12-1} \times \binom{20-20}{12-12}}{\binom{20}{12}} = 0.6$$

Meaning:

$$P(\overline{A} \mid B) = 1 - P(A \mid B) = 1 - 0.6 = 0.4$$

Together:

$\mathbb{E}(\text{deal}) = P(\text{frosted cake at the 20th selection}) \times (P(A \mid B) \times r_f + P(\overline{A} \mid B) \times (r_n + 2))$

$P(\text{normal cake at the 20th selection}) \times r_n = \dfrac{15-2}{40-19} \times (0.6 \times r_f + 0.4 \times (r_n + 2)) + \dfrac{25-17}{40-19} \times r_n =$

$= \dfrac{13}{21} \times (0.6 r_f + 0.4 r_n + 0.8) + \dfrac{8}{21} r_n = \dfrac{13}{35} r_f + \dfrac{26}{105} r_n + \dfrac{52}{105} + \dfrac{8}{21} r_n = \dfrac{13}{35} r_f + \dfrac{22}{35} r_n + \dfrac{52}{105}$

Final Solution:

$$\mathbb{E}(\text{deal}) = \frac{13}{35} r_f + \frac{22}{35} r_n + \frac{52}{105}$$

iii. For which value of r_f should you accept the deal? (write an expression that depends on r_n).

Meaning, we need to check when

$$\mathbb{E}(\text{no deal}) < \mathbb{E}(\text{deal})$$

Substitute the Values

$$\frac{13}{21}r_f + \frac{8}{21}r_n < \frac{13}{35}r_f + \frac{22}{35}r_n + \frac{52}{105}$$

$$\frac{26}{105}r_f < \frac{26}{105}r_n + \frac{52}{105}$$

Final Solution:

$$r_f < r_n + 2$$

Question 3

a. Prove that for any random variables X and Y $E[X+Y]=E[X]+E[Y]$

$$E[X+Y]=\sum_x\sum_y(x+y)P_{XY}(x,y)=\sum_x x(\underbrace{\sum_y P_{XY}(x,y)}_{P_X(x)})+\sum_y y(\underbrace{\sum_x P_{XY}(x,y)}_{P_Y(y)})$$

Because of Law of marginal probability:

$$=\sum_x xP_X(x)+\sum_y yP_Y(y)=E[X]+E[Y]$$

b. Compute Alice's expected reward

Given:

- $X = \{\text{Alice's grade}\}$
- $r_n = 1$
- $r_f = 2$
- Alice is the first student to pick a cupcake
- $Y = \{\text{Alice's reward from eating a cupcake}\}$

We already know that:

$$\mathbb{E}(\text{student that uses Wikipedia}) = 85$$

Meaning;

$$\mathbb{E}(X) = 85$$

Also:

$$\mathbb{E}[Y]=\frac{\text{Amount of frosted cupcakes}}{\text{Amount of cupcakes}}\times r_f+\frac{\text{Amount of normal cupcakes}}{\text{Amount of cupcakes}}\times r_n$$

Substitute the values:

$$=\frac{15}{40}\times 2+\frac{25}{40}\times 1=\frac{55}{40}=\frac{11}{8}=1.375$$

Together:

$$\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y]=85+1.375=86.375$$

Final Solution:

$$\mathbb{E}[X+Y]=86.375$$

Question 4

a. Prove $\mathbb{E}[X \times Y] = \mathbb{E}[X] \times \mathbb{E}[Y]$

Given:

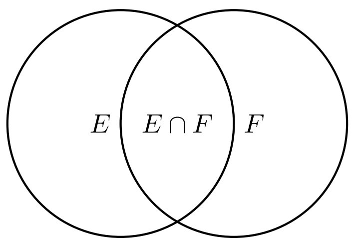
- X and Y are independent

$$\mathbb{E}[X \times Y] = \sum_x \sum_y X \times Y \times P_{xy}(X, Y) =$$

Because of Law of Joint distribution for independent variables:

$$= \sum_x \sum_y X \times Y \times P_x(X) \times P_y(Y) = \sum_x X \times P_x(X) \times \sum_y Y \times P_y(Y) = E[X] + E[Y]$$

b. Prove $P(E \cup F) = P(E) + P(F) - P(EF)$



From the venn diagram we can clearly see that by summing $P(E) + P(F)$ we are counting $P(E \cap F)$ twice, because $P(E) = P(E \setminus F) + P(E \cap F)$ and $P(F) = P(F \setminus E) + P(F \cap E)$ so we subtract $P(E \cap F)$ one time to avoid a double, to get $P(E \cup F)$, mathematically:

$$P(E) + P(F) = \big(P(E \setminus F) + P(E \cap F)\big) + \big(P(F \setminus E) + P(E \cap F)\big) =$$
$$P(E \setminus F) + P(F \setminus E) + P(E \cap F) + P(E \cap F) = P(E \cup F) + P(E \cap F)$$

c.

Given:

- $\Omega = \{1, 2, 3, 4\}$
- $E = \{1, 2\}$
- $F = \{1, 3\}$
- $G = \{1, 4\}$

i. Show that each pair of events is independent

To prove that a pair of events, A and B are independent, we need to prove:

$$P(A \cap B) = P(A) \times P(B)$$

First, we calculate each probability:

$P(E) = \frac{\|\{1,2\}\|}{\|\{1,2,3,4\}\|} = \frac{1}{2}$ $\quad P(F) = \frac{\|\{1,3\}\|}{\|\{1,2,3,4\}\|} = \frac{1}{2}$ $\quad P(G) = \frac{\|\{1,4\}\|}{\|\{1,2,3,4\}\|} = \frac{1}{2}$

$$P(E \cap F) = \frac{\|E \cap F\|}{\|\Omega\|} = \frac{\|\{1,2\} \cap \{1,3\}\|}{\|\{1,2,3,4\}\|} = \frac{\|\{1\}\|}{\|\{1,2,3,4\}\|} = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(E) \times P(F)$$

$$P(E \cap G) = \frac{\|E \cap G\|}{\|\Omega\|} = \frac{\|\{1,2\} \cap \{1,4\}\|}{\|\{1,2,3,4\}\|} = \frac{\|\{1\}\|}{\|\{1,2,3,4\}\|} = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(E) \times P(G)$$

$$P(G \cap F) = \frac{\|G \cap F\|}{\|\Omega\|} = \frac{\|\{1,4\} \cap \{1,3\}\|}{\|\{1,2,3,4\}\|} = \frac{\|\{1\}\|}{\|\{1,2,3,4\}\|} = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(G) \times P(F)$$

ii. Prove or dismiss that E, F, G are mutually independent

To prove that E, F, G are mutually independent, we need to prove:

$$P(E \cap F \cap G) = P(E) \times P(F) \times P(G)$$
$$P(E \cap F) = P(E) \times P(F)$$
$$P(E \cap G) = P(E) \times P(G)$$
$$P(F \cap G) = P(F) \times P(G)$$

We proved everything except:

$$P(E \cap F \cap G) = \frac{\|E \cap F \cap G\|}{\|\Omega\|} = \frac{\|\{1,2\} \cap \{1,3\} \cap \{1,4\}\|}{\|\{1,2,3,4\}\|} = \frac{\|\{1\}\|}{\|\{1,2,3,4\}\|} = \frac{1}{4}$$

While:

$$P(E) \times P(F) \times P(G) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Meaning:

$$P(E \cap F \cap G) \neq P(E) \times P(F) \times P(G)$$

E, F, G are not mutually independent.