# Question 1: Backpropagation by Hand

**This question involves a neural network with two input neurons, three hidden neurons, and one output neuron. For neuron $j$, its input current is $z_j$, its bias is $b_j$, and its activity is denoted $x_j$. All neurons use the same activation function, $\sigma(\cdot)$, except for the output neuron, which uses $\rho(\cdot)$. This connection weight from neuron $i$ to neuron $j$ is denoted $w_{ij}$. For example, the activity of neuron 5 can be calculated using, $x_5 = \sigma(w_{15}x_1 + w_{25}x_2 + w_{35}x_3 + b_5)$.**

a) **For the network above, suppose your cost function is the "Itakura-Saito distance", defined as**

$$L(x,t) = \frac{t}{x} - \ln\frac{t}{n} - 1$$

**and the activation function for the output node is $\rho(z) = z^2$. Derive an expression for $\frac{\partial L}{\partial z_6}$, the gradient of the cost with respect to the input current to the output neuron. Show your work, and simplify your answer as much as possible.**

As the activation function for the output node is $\rho(z)$, we have: $x_6 = \rho(z_6) = z_6^2$
Thus, we can define,

$$L(x_6,t) = \frac{t}{x_6} - \ln\left(\frac{t}{x_6}\right) - 1 = \frac{t}{z_6^2} - \ln\left(\frac{t}{z_6^2}\right) - 1 = \frac{t}{z_6^2} - \ln(t) + 2\ln(z_6) - 1$$

With this, we can take the derivative to get

$$\frac{\partial L}{\partial z_6} = \frac{-t(2)(z_6)}{z_6^4} + 2\left(\frac{1}{z_6}\right) = \frac{2}{z_6} - \frac{2t}{z_6^3} = 2\left(\frac{z_6^2 - t}{z_6^3}\right)$$

b) **Write down an expression for the update you would apply to the connection weight $w_{46}$. You may use your answer to question (a). Be as explicit as possible.**

Let $k$ be the step multiplier. I would apply the following update,

$$w_{46} = w_{46} - k\left(\frac{\partial L}{\partial w_{46}}\right)$$

Using the chain rule, we can define: $\frac{\partial L}{\partial w_{46}} = \frac{\partial L}{\partial z_6} \cdot \frac{\partial z_6}{\partial w_{46}}$
From (a), we have $\frac{\partial L}{\partial z_6} = 2\left(\frac{z_6^2 - t}{z_6^3}\right)$. As $z_6 = w_{36}x_3 + w_{46}x_4 + w_{56}x_5 + b_6$, we get $\frac{\partial z_6}{\partial w_{46}} = x_4$.
Then we have,

$$\frac{\partial L}{\partial w_{46}} = \frac{\partial L}{\partial z_6} \cdot \frac{\partial z_6}{\partial w_{46}} = 2x_4\left(\frac{z_6^2 - t}{z_6^3}\right)$$

Therefore, the update is:

$$w_{46} = w_{46} - 2kx_4\left(\frac{z_6^2 - t}{z_6^3}\right)$$

**c) Write down an expression for the update you would apply to the bias $b_6$. You may use your answer to question (a). Be as explicit as possible.**

Let $k$ be the step multiplier. I would apply the following update,

$$b_6 = b_6 - k\left(\frac{\partial L}{\partial b_6}\right)$$

Using the chain rule, we can define: $\frac{\partial L}{\partial b_6} = \frac{\partial L}{\partial z_6} \cdot \frac{\partial z_6}{\partial b_6}$

From (a), we have $\frac{\partial L}{\partial z_6} = 2\left(\frac{z_6^2 - t}{z_6^3}\right)$. As $z_6 = w_{36}x_3 + w_{46}x_4 + w_{56}x_5 + b_6$, we get $\frac{\partial z_6}{\partial b_6} = 1$.

Then we have,

$$\frac{\partial L}{\partial b_6} = \frac{\partial L}{\partial z_6} \cdot \frac{\partial z_6}{\partial b_6} = 2\left(\frac{z_6^2 - t}{z_6^3}\right)$$

Therefore, the update is:

$$b_6 = b_6 - 2k\left(\frac{z_6^2 - t}{z_6^3}\right)$$

**d) Write an expression for $\frac{\partial L}{\partial z_3}$. Again, you may use your answer to question (a). Show your work and simplify the formula as much as possible.**

Using the chain rule, we can define

$$\frac{\partial L}{\partial z_3} = \frac{\partial L}{\partial z_6} \cdot \frac{\partial z_6}{\partial z_3}$$

From (a), we have $\frac{\partial L}{\partial z_6} = 2\left(\frac{z_6^2 - t}{z_6^3}\right)$. For $\frac{\partial z_6}{\partial z_3}$, we have $z_6 = w_{36}x_3 + w_{46}x_4 + w_{56}x_5 + b_6$. Thus,

$$\frac{\partial z_6}{\partial z_3} = w_{36}\frac{\partial x_3}{\partial z_3} + w_{46}\frac{\partial x_4}{\partial z_3} + w_{56}\frac{\partial x_5}{\partial z_3}$$

To find $\frac{\partial x_3}{\partial z_3}$: $x_3 = \sigma(z_3) \implies \frac{\partial x_3}{\partial z_3} = \sigma'(z_3)$

To find $\frac{\partial x_4}{\partial z_3}$: $x_4 = \sigma(z_4) = \sigma(w_{14}x_1 + w_{24}x_2 + w_{34}x_3 + b_4)$

$\frac{\partial x_4}{\partial z_3} = \sigma'(z_4)\left(w_{34} \cdot \frac{\partial x_3}{\partial z_3}\right) = w_{34}\sigma'(z_3)\sigma'(z_4)$

To find $\frac{\partial x_5}{\partial z_3}$: $x_5 = \sigma(z_5) = \sigma(w_{15}x_1 + w_{25}x_2 + w_{35}x_3 + b_5)$

$\frac{\partial x_5}{\partial z_3} = \sigma'(z_5)\left(w_{35} \cdot \frac{\partial x_3}{\partial z_3}\right) = w_{35}\sigma'(z_3)\sigma'(z_5)$

Plugging everything back in, we get

$$\begin{aligned}\frac{\partial z_6}{\partial z_3} &= w_{36}\sigma'(z_3) + w_{34}w_{46}\sigma'(z_3)\sigma'(z_4) + w_{35}w_{56}\sigma'(z_3)\sigma'(z_5) \\ &= \sigma'(z_3)\left(w_{36} + w_{34}w_{46}\sigma'(z_4) + w_{35}w_{56}\sigma'(z_5)\right)\end{aligned}$$

Therefore,

$$\frac{\partial L}{\partial z_3} = \frac{\partial L}{\partial z_6} \cdot \frac{\partial z_6}{\partial z_3} = 2\left(\frac{z_6^2 - t}{z_6^3}\right)\sigma'(z_3)\left(w_{36} + w_{34}w_{46}\sigma'(z_4) + w_{35}w_{56}\sigma'(z_5)\right)$$