Question 3

$$
\begin{aligned}
\tau \frac{dv}{dt} &= v_{in} - v \\
\frac{\tau}{v_{in} - v} dv &= dt \\
\int \frac{\tau}{v_{in} - v} dv &= \int dt \\
-\tau \ln(v_{in} - v) &= t + c
\end{aligned}
$$

Since $v = 0$ at $t = 0$, $c = -\tau \ln v_{in}$. Now we solve for t_1, the time when $v = 1$.

$$
\begin{aligned}
-\tau \ln(v_{in} - 1) &= t_1 - \tau \ln v_{in} \\
\tau(\ln v_{in} - \ln(v_{in} - 1)) &= t_1
\end{aligned}
$$

Note that if $v_{in} \leq 1$, the $\ln(v_{in} - 1)$ term in the above equation is invalid, so v never reaches the threshold, hence the firing rate is 0. On the other hand, if $v_{in} > 1$,

$$
\begin{aligned}
t_{isi} &= \tau_{ref} + t_1 \\
&= \tau_{ref} + \tau(\ln v_{in} - \ln(v_{in} - 1)) \\
&= \tau_{ref} + \tau \ln \frac{v_{in}}{v_{in} - 1} \\
&= \tau_{ref} - \tau \ln \frac{v_{in} - 1}{v_{in}} \\
&= \tau_{ref} - \tau \ln(1 - \frac{1}{v_{in}})
\end{aligned}
$$

Since t_{isi} is the reciprocal of the firing rate,

$$
G(v_{in}) = \begin{cases} \frac{1}{\tau_{ref} - \tau \ln(1 - \frac{1}{v_{in}})} & \text{for } v_{in} > 1 \\ 0 & \text{otherwise} \end{cases}
$$