P476 termpaper report
Stochastic Resetting

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Stochastic Resetting[1] is a stochastic process with resetting of variable under study.

1 Poissonian Resetting in 1-D Diffusion:

We study Single particle on the real line with initial position x_0 at t=0 and resetting with constant rate r to fixed position X_r.The Stochastic rule is:

$$x(t+dt) = \begin{cases} X_r & \text{, with probability rdt} \\ x(t) + \xi(t)(dt)^{1/2}, & \text{with probability (1-rdt)} \end{cases} \tag{1}$$

We take $\xi(t)$ as a r.v with some distriburion $g(\xi)$ s.t:

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)^2 \rangle = 2D$$

Then,

$$P(x, t+dt) = rdt\delta(x-X_r)+(1-rdt)\sum_{\xi(t)} g(\xi)P(x-\xi(t)(dt)^{1/2}, t)$$

$$= rdt\delta(x - X_r) + (1 - rdt)\int_{-\infty}^{\infty} d\xi P(x - \xi(t)(dt)^{1/2}, t)$$

We do taylor expansion of second term.Second term=:

$$(1 - rdt)\int_{-\infty}^{\infty} d\xi g(\xi)\left[P(x, t) - (dt)^{1/2}\frac{\partial P}{\partial x}\xi + \frac{dt}{2}\frac{\partial^2 P}{\partial x^2}\xi^2 + ...\right]$$

We neglect higher order terms ,rearrange the equation and take limit $dt \to 0$.We get the diffusion equation with resetting to X_r:

$$\frac{\partial P(x,t)}{\partial t} = r\delta(x - X_r) - rP(x, t) + D\frac{\partial^2 P(x,t)}{\partial x^2} \tag{2}$$

In comparison to simple Diffusion equation there are two extra terms. $r\delta(x - X_r)$ is the probability deposited at X_r due to resetting.$-rP(x, t)$ is the probability lost from x due to resetting.Now we need to solve it.but we know complete solution for simple diffusion equations ,can we use that solution for constructing the resetting solution.Consider the following Simple Diffusion equation in 1-D.

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}$$

Initial condition:$P(x, 0) = f(x)$ We construct the solution for simple Diffusion in terms of Greens function as:

$$P(x,t) = \int_{-\infty}^{\infty} G(x, x', t)f(x')dx'$$

Equation for Greens function:

$$\frac{\partial G(x, x', t)}{\partial t} = D\frac{\partial^2 G(x, x', t)}{\partial x^2}$$

We then do fourier transformation and solve it with given initial condition.

$$\frac{\partial \tilde{G}(k,t)}{\partial t} = -Dk^2\tilde{G}(k, t)$$

$$G(x, x', t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left\{\frac{-(x - x')^2}{4Dt}\right\}$$

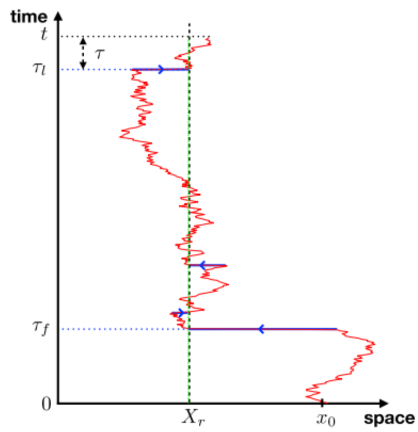
$P(x, t|x_0)$:Given particle was at x_0 initially,what is the probability of finding it at x at later time t

$$f(x) = \delta(x - x_0)$$

$$P(x, t|x_0) = \int_{-\infty}^{\infty} G(x, x', t)f(x')dx' = \frac{1}{\sqrt{4\pi Dt}} \exp\left\{\frac{-(x - x_0)^2}{4Dt}\right\}$$

$$\equiv G(x, t|x_0)$$

Getting Back to resetting problem.Probability density $P(x, t|x_0)$ in the presence of resetting can be written as sum of two contributions: No resetting has occurred till time t and summing over trajectories where last resetting(renewal) happend at $\tau_1 = t - \tau$.



Since we have Resetting at a constant rate r,we have the Poisson distribution.

$$\text{probability(k resetting or k success)} = \frac{t^k r^k e^{-rt}}{k!}$$

Probability of no resetting in time t=prob.(0 success)=e^{-rt}.So,No resetting contribution term=$e^{-rt} \times G(x,t|x_0)$. last resetting contribution=$r \times \int_0^t d\tau e^{-r\tau} \times G(x,\tau|X_r)$.We get the Last renewal equation:

$$\boxed{P(x,t|x_0) = e^{-rt}G(x,t|x_0) + \int_0^t d\tau r e^{-r\tau} G(x,\tau|X_r)}$$

Similarly,We can have First renewal equation;First resetting at time τ_f and then diffusion with resetting for remaining time $t-\tau_f$. First renewal equation:

$$\boxed{P(x,t|x_0) = e^{-rt}G(x,t|x_0) + \int_0^t d\tau r e^{-rt_f} P(x,t-\tau_f|X_r)}$$

By doing laplace transformation it can be shown that both the equation are equivalent.

2 Stationary State

$t \to \infty$:$P(x,t|x_0) \equiv P^*(x|x_0)$

$$P^*(x|x_0) = r.\int_0^\infty d\tau e^{-r\tau} G(x,\tau|X_r)$$

G is the simple diffusion equation propagator that we know. In words Stationary distribution under resetting rate r is equal to r times the laplace transform of propagator in absence of resetting with laplace transform variable being r. We just need to perform the following integral:

$$P^*(x|x_0) = r.\int_0^\infty d\tau e^{-r\tau} \frac{1}{\sqrt{4\pi D\tau}} \exp\left\{\frac{-(x-X_r)^2}{4D\tau}\right\}$$

Integral is of the form:

$$\int_0^\infty dt\ t^{v-1}\ e^{-\frac{\beta}{t}-\gamma t} = 2\left(\frac{\beta}{\gamma}\right)^{v/2} K_v(2\sqrt{\beta\gamma})$$

K_v is the modified Bessel function of second kind of order v v=1,$\gamma = r$,$\beta = \frac{(x-X_r)^2}{4D}$,$K_{1/2}(y) = (\pi/2y)^{1/2}e^{-y}$.We get Non equilibrium stationary state:

$$P^*(x|x_0) = \frac{\alpha_0}{2}e^{-\alpha_0|x-X_r|},\ \text{where}\ \alpha_0 = \sqrt{\frac{r}{D}} \tag{3}$$

.It Doesn't depend on initial position.It Exponentially decaying about X_r Note that Renewal equation is general.If we have diffusion with some potential all we need to do is workout the propagator G for the respective potential.

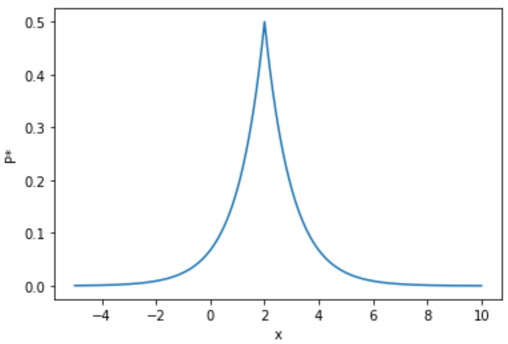


Figure 2 :Stationary state $\alpha_0 = 1$,resetting position X_r=2

Now How does the system relax to stationary state? We need to do integration in the renewal equation for large but finite t.we take $x_0 = X_r$ and use Saddle point Integration method.

$$p(x,t) \approx e^{-t\mathcal{I}(\frac{x-X_r}{t})}$$

where,

$$\mathcal{I}(y) = \begin{cases} \alpha_0|y| & \text{for } |y| < y^* \\ r + \frac{y^2}{4D} & \text{for } |y| > y^* \end{cases}$$

with $y^* = \sqrt{4Dr}$

If we have $|y| < y^* \equiv -\sqrt{4Dr}t < x - X_r < \sqrt{4Dr}t$,Then $p^*(x,t) \approx e^{-\alpha_0|x-X_r|}$.This is Independent of time.So it has already relaxed to NESS(Non Equilibrium stationary state).They have already undergone a large number of resetting.But for $|x-X_r| > \sqrt{4Dr}t$ its still time dependent.probability density is roughly of the form $\approx e^{-rt}G(x,t|X_r)$.This is term that we would get if the particle has diffused to x without any resetting.The boundary between these two cases (frontier $\xi(t)$) grows $\propto t$ at speed of $\sqrt{4Dr}$.In comparison to simple diffusion the important time scale was $\propto \sqrt{t}$ At the boundary the second derivative of $\mathcal{I}$ is discontinuous.It is a second order phase transition .

3 Summary:

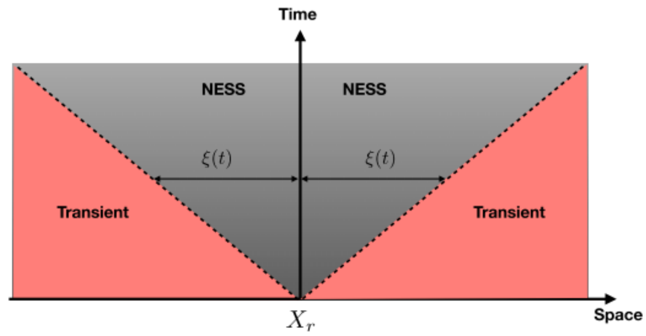


Figure 3

Stationary distribution under resetting rate r is equal to r times the laplace transform of propagator in absence of resetting with laplace transform variable being r. NESS gets established in a core region around the resetting centre $X_r r$ whose frontiers $\xi(t)$ grow with time as $\propto t$ at a speed of $\sqrt{4Dr}$. Outside the core region, the system is transient.

References

[1] Martin R Evans, Satya N Majumdar, and Grégory Schehr. "Stochastic resetting and applications". In: *Journal of Physics A: Mathematical and Theoretical* 53.19 (2020), p. 193001.