SEMESTER PROJECT(6th)
Generation of spin currents by interaction of Cooper pairs and magnons

submitted by

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Chapter 1

Heisenberg ferromagnet and spin wave theory

1.1 Introduction

In the initial part of my reading project my first task was to understand ferromagnetism and spin wave theory.So i studied some sections from the book Solid State Physics by Ashcroft/Mermin.

Ferromagnets are the solid substances which have a non vanishing magnetic moment or "spontaneous magnetization" in some direction even in the absence of external magnetic field. It must be the interaction among the individual magnetic sites that is giving rise to this spontaneous magnetic ordering.If there was no such interaction,in the absence of external magnetic field,then the individual magnetic moments would be thermally disordered,would point in a random direction and could not sum to a net moment for the solid as whole.

1.2 Heisenberg ferromagnet

One way that i learned from the book to describe this interaction is to use the **Heisenberg Ferromagnet Hamiltonian** given as:

$$H = -\sum_{<ij>} J_{ij} \vec{S}_i . \vec{S}_j \tag{1.1}$$

Here i,j are the lattice sites ,S_i is the total spin at site i and the summation is over the bonds or interactions.$J_{ij} = J_{ji} \geq 0$ because a positive interaction factor J favours parallel spin alignment. Let us assume that the interactions are only present between nearest neighbous so that we can take the J outside the sum.This gives

$$H = -J \sum_{<ij>} \vec{S}_i.\vec{S}_j \tag{1.2}$$

Here the summation is over the nearest neighbour interactions.

We take the direction of spontaneous magnetization as positive z axis. In terms of x,y,z components H can be written as

$$H = -J \sum_{<ij>} [S_i^z.S_j^z + S_i^x.S_j^x + S_i^y.S_j^y]$$

We have raising and lowering spin operators defined as

$$\begin{aligned} S_i^+ &= S_i^x + iS_i^y \\ S_i^- &= S_i^x - iS_i^y \end{aligned} \tag{1.3}$$

$$\begin{aligned} S_i^x &= \frac{1}{2}(S_i^+ + iS_i^-) & \Rightarrow S_i^x.S_j^x &= \frac{1}{4}(S_i^+ + S_i^-)(S_j^+ + S_j^-) \\ S_i^y &= \frac{1}{2i}(S_i^+ - iS_i^-) & \Rightarrow S_i^y.S_j^y &= \frac{-1}{4}(S_i^+ - S_i^-)(S_j^+ - S_j^-) \end{aligned}$$

Here i that appears in the product is $\sqrt{-1}$. Now we plug these into equation 1.2 and get the following form

$$H = -J \sum_{<ij>} \left[S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+) \right] \tag{1.4}$$

1.2.1 Ground State:

With intuition from classical ground state i.e all the spin sites have maximum spin S along Z axis,We construct the quantum mechanical ground state $|0\rangle$ as:

$$\begin{aligned} |0\rangle &= \prod_i |S\rangle_i \\ S_i^z |S\rangle_i &= S\, |S\rangle_i \end{aligned} \tag{1.5}$$

We now prove that this is an eigenstate of H.

$$H|0\rangle = -J\sum_{<ij>}\left[S_i^zS_j^z(\prod_k |S\rangle_k) + \frac{1}{2}(S_i^+S_j^-(\prod_k |S\rangle_k) + S_i^-S_j^+(\prod_k |S\rangle_k))\right]$$

since all the $S_i^+ |S\rangle_i = 0$ only contribution will come from S^z

$$= -J\sum_{<ij>}\left[S_i^zS_j^z(\prod_k |S\rangle_k)\right] = -J\sum_{<ij>} S^2 \prod_k |S\rangle_k = -J\sum_{<ij>} S^2|0\rangle = -\frac{NqJS^2}{2}|0\rangle$$

.where q is the coordination number of the crystal stucture.Hence $|0\rangle$ is an eigen state of H with eigen value $E_0 = -\frac{NqJS^2}{2}$.

We now prove that this eigen state is also the ground state. Let X be a Hermitian operator with complete eigen basis $|x\rangle$ and eigen values E_x.let $|y\rangle$ be any other normalized state.y can be written as

$$|y\rangle = \sum_x C_x |x\rangle$$

The largest Matrix diagonal element that X can have in any state is

$$= max(\langle y| X |y\rangle) = max(\sum_x E_x|C_x|^2) = max(E_x)$$

since $0 \le |C_x|^2 \le 1$.So the largest diagonal matrix element Hermitian operator can have is equal to its largest eigen value.

$$\begin{aligned} S_i.S_j &= \frac{1}{2}[(S_i+S_j)^2 - S_i^2 - S_j^2] \\ \Rightarrow max(< S_i.S_j >) &= \frac{1}{2}[max < (S_i+S_j)^2 > - < S_i^2 > - < S_j^2 >] \\ &= \frac{1}{2}[(2S)(2S+1) - (S)(S+1) - (S)(S+1)] = S^2 \end{aligned}$$

Therefore we have the upper bound $< S_i.S_j > \le S^2$.

Now ,suppose $\left|0'\right\rangle$ be any other eigenstate of H with eigen value E_0'.

$$\Rightarrow E_0' = \left\langle 0'\right| H \left|0'\right\rangle$$

But from the last argument about the upper bound of $< S_i.S_j >$ we can have lower bound on $E_0^{'}$.Since J is a positive quantity we have

$$E_0^{'} \geq -J \sum_{<ij>} max < S_i.S_j >= -J \sum_{<ij>} S^2 = -\frac{NqJS^2}{2}$$

We have already found the energy state E_0 with the lowest possible energy.There can not be any state $E_0^{'}$ with energy lower than this. Thus $E_0 = -\frac{NqJS^2}{2}$ is the ground state energy.

1.2.2 Low Lying Excited states:spin waves(Magnons)

We now search for low lying excited states at low temperatures.Since the temperature is low we expect those states to be such that the spins are mainly directed along z axis with small fluctuations only.So we examine the following constructed state $|i\rangle$ differing from the ground state $|0\rangle$ only in that the spin at site i has its z-component reduced from S to S-1.

$$|i\rangle = \frac{1}{\sqrt{2S}} S_i^- |0\rangle \tag{1.6}$$

$S_i^- S_j^+$ terms in the hamiltonian will shift the site at which the spin is reduced from j to i.

$$S_i^- S_j^+ |j\rangle = 2S |i\rangle \tag{1.7}$$

$$\begin{aligned} H|i\rangle &= -J \sum_{jk} \left[S_j^z S_k^z |i\rangle + \frac{1}{2}(S_j^+ S_k^- |i\rangle + S_j^- S_k^+ |i\rangle) \right] \\ &= -J \left[(qS(S-1) + \frac{(N-2)qS^2}{2}) |i\rangle + S \sum_{\delta} |i+\delta\rangle \right] \\ &= -\frac{JqS(NS-2)}{2} |i\rangle - JS \sum_{\delta} |i+\delta\rangle \end{aligned}$$

Here sum over δ is the sum over nearest neighbour vectors.

So $|i\rangle$ is not an eigen state of H but $H|i\rangle$ is linear combination of $|i\rangle$ and other similar states with one spin lowered at some site.Therefore we now look for a states which is linear combination of $|i\rangle$s.

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{i\vec{k}.\vec{i}} |i\rangle \tag{1.8}$$

$$\begin{aligned}
H|k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{i\vec{k}.\vec{i}} H|i\rangle = \frac{1}{\sqrt{N}} \sum_i e^{i\vec{k}.\vec{i}} \left[-\frac{JqS(NS-2)}{2} |i\rangle - JS \sum_\delta |i+\delta\rangle \right] \\
= -\frac{JqS(NS-2)}{2\sqrt{N}} \sum_i e^{i\vec{k}.\vec{i}} |i\rangle - \frac{JS}{\sqrt{N}} \sum_{i.\delta} e^{i\vec{k}.\vec{i}} |i+\delta\rangle \\
= -\frac{JqS(NS-2)}{2} |k\rangle - \frac{JS}{\sqrt{N}} \sum_{i.\delta} e^{i\vec{k}.\vec{i}} |i+\delta\rangle \\
= -\frac{JqS(NS-2)}{2} |k\rangle - \frac{JS}{\sqrt{N}} \sum_{i,\delta} e^{i\vec{k}.(\vec{i}+\vec{\delta})} e^{-i\vec{k}.\vec{\delta}} \\
= -\frac{JqS(NS-2)}{2} |k\rangle - JS \sum_\delta e^{-i\vec{k}.\vec{\delta}} \left(\sum_i \frac{1}{\sqrt{N}} e^{i\vec{k}.(\vec{i}+\vec{\delta})} |i+\delta\rangle \right) \\
= \left(-\frac{JqS(NS-2)}{2} |k\rangle - JS \sum_\delta e^{-i\vec{k}.\vec{\delta}} \right) |k\rangle \\
= \left(-\frac{JqNS^2}{2} + JqS \left[1 - \frac{1}{q} \sum_\delta e^{-i\vec{k}.\vec{\delta}} \right] \right) |k\rangle \\
= \left(E_0 + JqS \left[1 - \frac{1}{2q} \sum_\delta (e^{-i\vec{k}.\vec{\delta}} + e^{+i\vec{k}.\vec{\delta}}) \right] \right) |k\rangle = \left(E_0 + JqS \left[1 - \frac{1}{q} \sum_\delta cos(\vec{k}.\vec{\delta}) \right] \right) |k\rangle
\end{aligned}$$

$$H|k\rangle = \left(E_0 + JqS \left[1 - \frac{1}{q} \sum_\delta cos(\vec{k}.\vec{\delta}) \right] \right) |k\rangle \tag{1.9}$$

Hence the states give in equation 1.8 are the exact eigenstates of heisenberg hamiltonian. We now give a constant shift to the hamiltonian by the ground state energy E_0 so that the eigenvalues of the $|k\rangle$ gives the amount of excitation energy that they differ from the ground state.

$$H \longrightarrow H - E_0 I \tag{1.10}$$

$$\begin{gathered} H\left|k\right\rangle = \epsilon_k \left|k\right\rangle \\ \epsilon_k = JqS\left[1 - \frac{1}{q}\sum_{\delta} cos(\vec{k}.\vec{\delta})\right] \end{gathered} \tag{1.11}$$

We now calculate transverse spin correlation function in state $|k\rangle$(given in equation 1.8) defined by the expectation value of following

$$S_i^{\perp}.S_j^{\perp} = S_i^x S_j^x + S_i^y S_j^y \tag{1.12}$$

for $i \neq j$

$$\begin{aligned} \langle k| S_i^{\perp}.S_j^{\perp} |k\rangle &= \langle k| S_i^x S_j^x + S_i^y S_j^y |k\rangle = \langle k| \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+) |k\rangle \\ &= \frac{1}{2}\langle k| S_i^+ S_j^- |k\rangle + \frac{1}{2}\langle k| S_i^- S_j^+ |k\rangle \\ = \frac{1}{2N}\left(\sum_m e^{-ik.m}\langle m|\right) S_i^+ S_j^- &\left(\sum_l e^{ik.l}|l\rangle\right) + \frac{1}{2N}\left(\sum_m e^{-ik.m}\langle m|\right) S_i^- S_j^+ \left(\sum_l e^{ik.l}|l\rangle\right) \\ &= \frac{1}{2N}\sum_{l,m} e^{ik.(l-m)}\langle m| S_i^+ S_j^- |l\rangle + \frac{1}{2N}\sum_{l,m} e^{ik.(l-m)}\langle m| S_i^- S_j^+ |l\rangle \\ &= \frac{S}{N}e^{ik.(i-j)} + \frac{S}{N}e^{ik.(j-i)} = \frac{2S}{N}\cos\left(k.(i-j)\right) \end{aligned}$$

So we have for $i \neq j$,

$$\langle k| S_i^{\perp}.S_j^{\perp} |k\rangle = \frac{2S}{N}\cos\left(k.(i-j)\right) \tag{1.13}$$

Thus on average each spin has a small transverse component perpendicular to the direction of magnetization of size $\sqrt{\frac{2S}{N}}$ and the orientation of two spins at two differnet lattice sites i and j differ by an angle $\vec{k}\cdot(\vec{i}-\vec{j})$.This suggest the following picture figure 1.1 for microscopic magnetisation in each state $|k\rangle$ given by equation

1.8.Therefore these states $|k\rangle$ are called a spin wave or magnon of wave vector k and energy ϵ_k.

Ground state:

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

One possible excitation:

$\uparrow \quad \uparrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

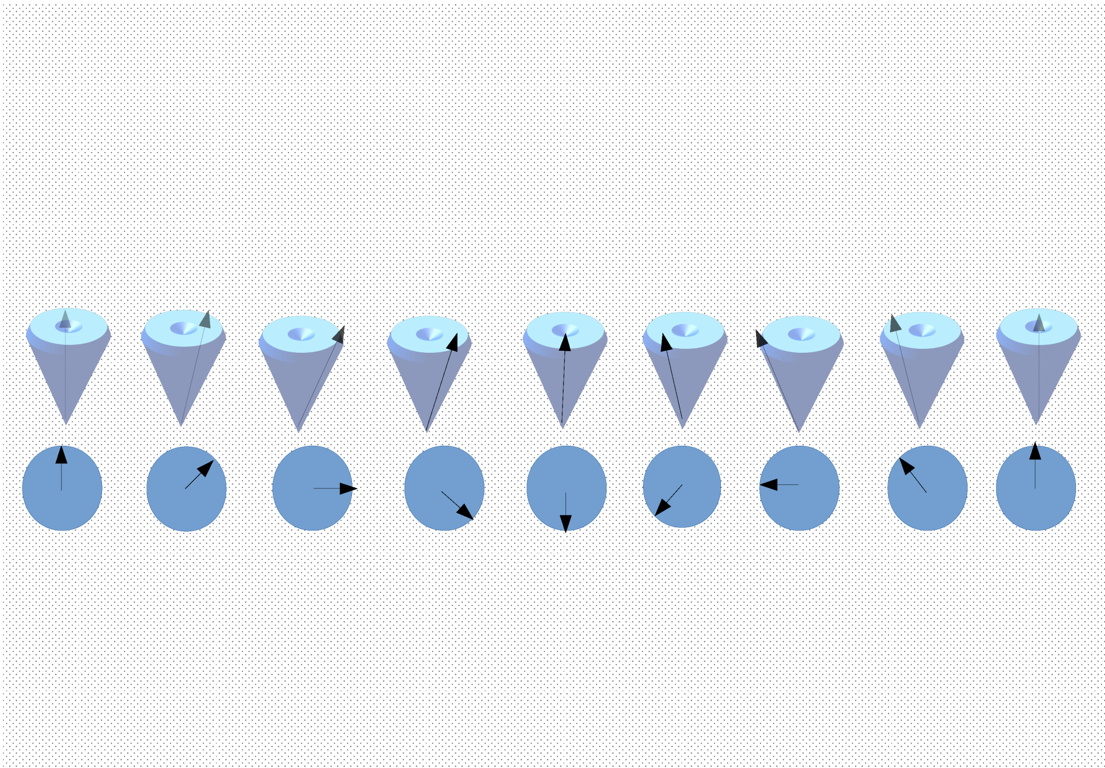


Figure 1.1: **Spin wave states**

Chapter 2

Heisenberg ferromagnet and Bosonic operators

2.1 Holstein-Primakoff(HP) transformation:

This Transformation maps spin operators of spin S moments on a lattice to bosonic creation and annihilation operators as:

$$S_j^z = S - n_j \tag{2.1}$$

$$S_j^+ = \sqrt{2S - n_j} a_j \tag{2.2}$$

$$S_j^- = a_j^\dagger \sqrt{2S - n_j}. \tag{2.3}$$

$n_j = a_j^\dagger . a_j$ is the number operator. Here a_j(annhilation) and a_j^t(creation) operators at site j satisfies the bosonic commutation relations

$$[a_i, a_i^\dagger] = \delta_{ij} \tag{2.4}$$

$$[a_i, a_j] = 0 \tag{2.5}$$

$$\left[a_i^\dagger, a_j^\dagger\right] = 0 \tag{2.6}$$

In this mapping,the ground state has a spin of +S in the z direction and each Holstein-primakoff boson represnts a spin 1 moment in the -z direction.This represnts the deviation from the classical ferromagnetic ground state.The factor $\sqrt{2S - n_j}$ in the raising and lowering spin operator are there to limit the number of bosons we can have on a give site to 2S since the z-component of the spin moment at given site must be between -S and +S.

At low temperatures the number of perturbations from the classical ground state is very small i.e $\langle n_j \rangle \ll S$

2.1.1 Binomial approximation of transformation at low temperatures:

If temperatures are sufficiently low such that $\langle n_j \rangle \ll S$,we can use the binomial approximation for ($|x| < 1$)

$$(1+x)^a \approx 1 + ax$$

therfore

$$\sqrt{2S - n_j} = \sqrt{2S}\sqrt{1 - \frac{n_j}{2S}} \approx \sqrt{2S}\left(1 - \frac{n_j}{4S}\right) \tag{2.7}$$

So,equation 2.1 to 2.3 in approximated form can be written as

$$S_j^z = S - n_j \tag{2.8}$$

$$S_j^+ \approx \sqrt{2S}\left(1 - \frac{n_j}{4S}\right) a_j \tag{2.9}$$

$$S_j^- \approx a_j^\dagger \sqrt{2S}\left(1 - \frac{n_j}{4S}\right) \tag{2.10}$$

2.2 Heisenberg ferromagnet using HP transformation:

$$H = -J \sum_{<ij>} \vec{S}_i.\vec{S}_j = -J \sum_{<ij>} \left[S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+) \right]$$

where $J > 0$.

Now equation 2.8 to 2.10 will be used to rewrite the Heisenberg ferromagnet hamiltonian.

$$
\begin{aligned}
H = -J\sum_{<ij>}\left[(S-n_i)(S-n_j)+\frac{1}{2}(\sqrt{2S-n_i}a_ia_j^\dagger\sqrt{2S-n_j}+a_i^\dagger\sqrt{2S-n_i}\sqrt{2S-n_j}a_j)\right] \\
\approx -J\sum_{<ij>}\left[(S-n_i)(S-n_j)+S\left[(1-\frac{n_i}{4S})a_ia_j^\dagger(1-\frac{n_j}{4S})+a_i^\dagger(1-\frac{n_i}{4S})(1-\frac{n_j}{4S})a_j\right]\right] \\
= -J\sum_{<ij>}[S^2-Sn_i-Sn_j+n_in_j+Sa_ia_j^\dagger-\frac{1}{4}a_ia_j^\dagger n_j-\frac{1}{4}n_ia_ia_j^\dagger+\frac{1}{16S}n_ia_ia_j^\dagger n_j+Sa_i^\dagger a_j \\
-\frac{1}{4}a_i^\dagger n_ia_j-\frac{1}{4}a_i^\dagger n_ja_j+\frac{1}{16S}a_i^\dagger n_in_ja_j]
\end{aligned}
$$

Terms upto order of S^0 is kept,

$$
\begin{aligned}
\approx -J\sum_{<ij>}\left[S^2+S(a_i^\dagger a_j+a_j^\dagger a_i-n_i-n_j)+\frac{1}{4}(4n_in_j-a_i^\dagger(n_i+n_j)a_j-a_j^\dagger(n_i+n_j)a_i)\right] \\
= -\frac{NqJS^2}{2}+J\sum_{<ij>}\left[S(a_i^\dagger a_j+a_j^\dagger a_i-n_i-n_j)+\frac{1}{4}[4n_in_j-a_i^\dagger(n_i+n_j)a_j-a_j^\dagger(n_i+n_j)a_i]\right] \\
= -\frac{NqJS^2}{2}+J\sum_{<ij>}\left[S(a_i^\dagger-a_j^\dagger)(a_i-a_j)+\frac{1}{4}[a_i^\dagger a_j^\dagger(a_i-a_j)^2+(a_i^\dagger-a_j^\dagger)^2a_ia_j]\right]
\end{aligned}
$$

Here N is the number of lattice sites and q is the coordination number.At low temperatures since $< n_i > \ll S$ Higher order terms is ignored and we will keep terms quadratic in bosonic operators.The Hamiltonian then takes the form,

$$H = -\frac{NqJS^2}{2}+H_1 \tag{2.11}$$

$$
\begin{aligned}
H_1 &= JS\sum_{<ij>}(a_i^\dagger-a_j^\dagger)(a_i-a_j) \\
&= -JS\sum_{<ij>}(a_i^\dagger a_j+a_j^\dagger a_i)+qJS\sum_i n_i
\end{aligned} \tag{2.12}
$$

H_1 gives the magnon dispersion relation. Now,we will represent it in momentum basis by using the following relations.

$$a_j^\dagger = \frac{1}{\sqrt{N}}\sum_k e^{-ik.r_j}a_k^\dagger$$

$$a_j = \frac{1}{\sqrt{N}} \sum_k e^{ik.r_j} a_k \tag{2.13}$$

$$\begin{aligned} a_k^\dagger &= \frac{1}{\sqrt{N}} \sum_j e^{-ik.r_j} a_j^\dagger \\ a_k &= \frac{1}{\sqrt{N}} \sum_j e^{ik.r_j} a_j \end{aligned} \tag{2.14}$$

$$\sum_j e^{i(k-k^{'}).r_j} = N\delta_{kk'} \tag{2.15}$$

$$\sum_k e^{i(r_i - r_j).k} = N\delta_{ij} \tag{2.16}$$

Where the k vectors belong to first Brillouin zone.

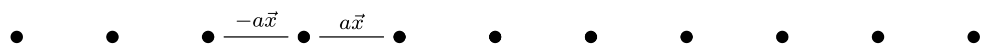
Then the magnon part of hamiltonian H_1 takes the form,

$$\begin{aligned} H_1 &= qJS \sum_k \left[1 - \frac{1}{q} \sum_\delta cos(k.\delta) \right] a_k^\dagger a_k \\ &= \sum_k \epsilon_k a_k^\dagger a_k \end{aligned} \tag{2.17}$$

$$\epsilon_k = qJS(1 - \frac{1}{q} \sum_\delta cos(k.\delta)) \tag{2.18}$$

sum over delta is the sum over nearest neighbour vectors.This is exactly the same that we got in equation 1.11 in the first chapter.

2.3 Magnon Dispersion relation in 1-D ferromagnetic lattice chain



nearest neighbour lattice vectors are $a\vec{x}$ and $-a\vec{x}.\vec{k} = k\vec{x}$.

$$\begin{aligned} \epsilon_k &= 2JS[1 - \frac{1}{2}(cos(ka) + cos(-ka))] \\ &= 2JS(1 - cos(ka)) \end{aligned} \tag{2.19}$$

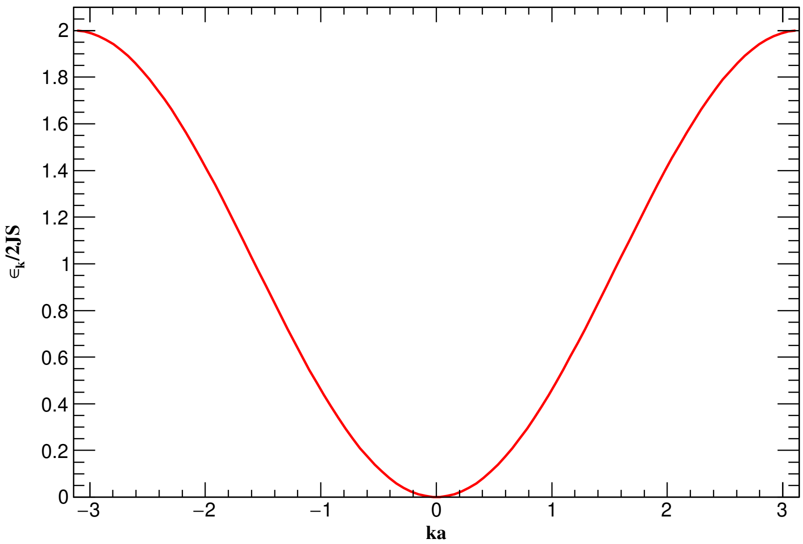
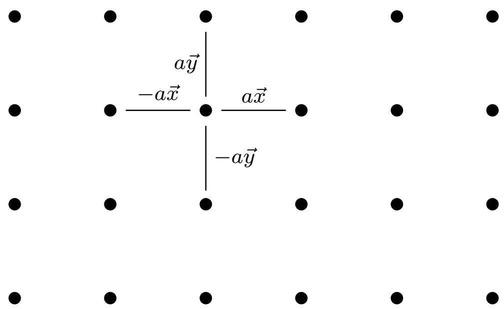


Figure 2.1: Magnon dispersion curve in 1D ferromagnetic chain in first Brillouin zone

2.4 Magnon Dispersion relation in square ferromagnetic lattice:



$$\vec{K} = k_x\vec{x} + k_y\vec{y}$$

$$\epsilon_k = 4JS\left[1 - \frac{1}{4}[cos(ak_x) + cos(-ak_x) + cos(ak_y) + cos(-ak_y)]\right]$$

$$\epsilon_k = 4JS\left[1 - \frac{1}{2}[cos(ak_x) + cos(ak_y)]\right] \tag{2.20}$$

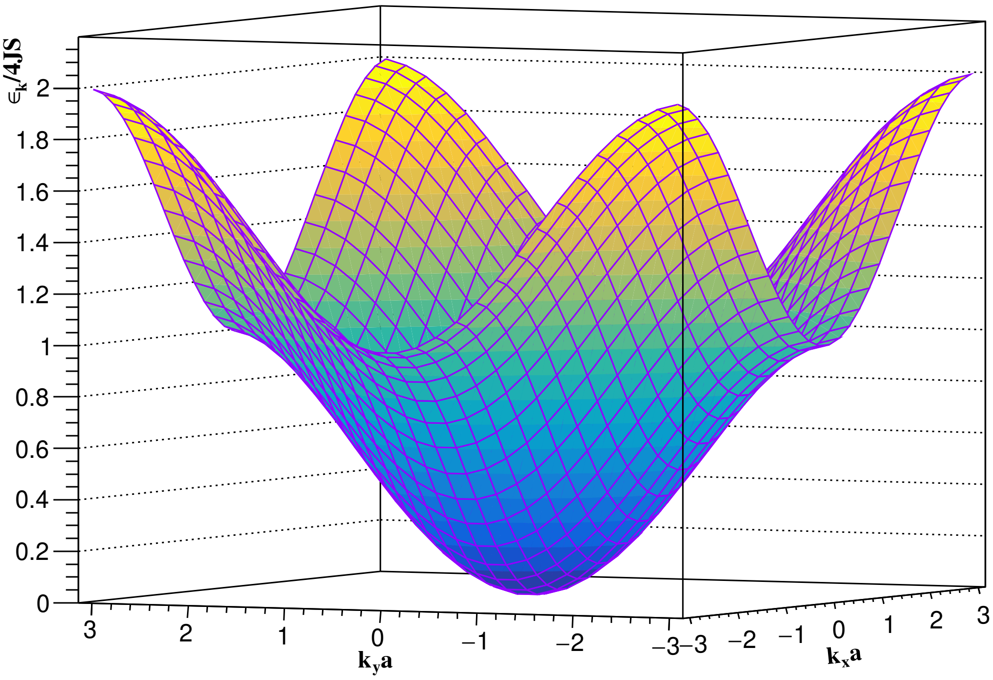


Figure 2.2: Magnon dispersion in square ferromagnetic lattice in first Brillouin zone

Chapter 3

Antiferromagnets

Antiferromagnets is a type of magnetically ordered substance in which individual magnetic ions have non vanishing average magnetic vector moment,but they do not add up to a net magnetization density for the solid as a whole.The following figure describes a example.

$$\uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \downarrow$$

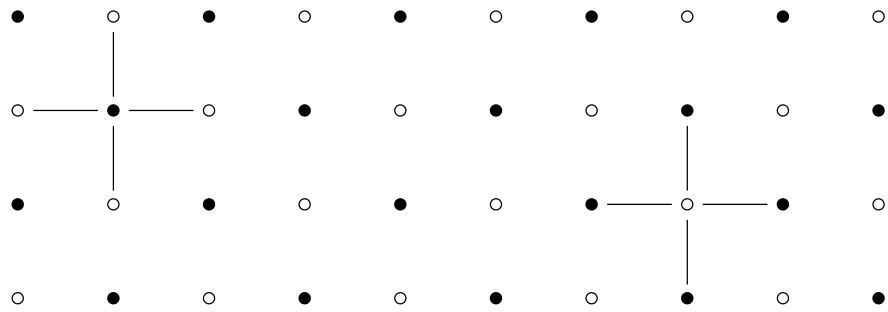
There is microscopic magnetic order but the array as a whole has zero magnetic moment.

3.1 Heisenberg Antiferromagnet:

Heisenberg antiferromagnetic Hamiltonian is similar to that of ferromagnet ,but there is no negative sign in the front so that the positive intercation term J favours antiparallel orientation of spins.

$$\begin{aligned} H &= J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \\ &= J \sum_{<ij>} \left[S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+) \right] \end{aligned} \tag{3.1}$$

We now consider a bipartite lattice in which spins reside on two types of sub lattices A and B.The Following figure is an example.Black bullets interact only with nearest white bullets and vice versa.The whole lattice can be partitioned into two sets of lattice.Therefore it is called bipartite lattice.



3.1.1 Holstein-Primakoff Transformation:

We use i(j) to index the spins on the sub lattice A(B).We map the spin operators on both sub lattice as:

Sub Lattice A:

$$\begin{aligned} S_i^z &= S - a_i^\dagger a_i \\ S_i^+ &= \sqrt{2S - n_i}\, a_i \\ S_i^- &= a_i^\dagger \sqrt{2S - n_i} \end{aligned} \tag{3.2}$$

Sub Lattice B:

$$\begin{aligned} S_j^z &= b_j^\dagger b_j - S \\ S_j^+ &= b_j^\dagger \sqrt{2S - n_j} \\ S_j^- &= \sqrt{2S - n_j}\, b_j \end{aligned} \tag{3.3}$$

$n_i(n_j)$ are number operators $a_i^\dagger a_i(b_j^\dagger b_j)$.

We now rewrite the hamiltonian using above set of equation and use the approximation $a_i^\dagger a_i(b_j^\dagger b_j) \ll S$ at low temperatures and keeping up to the term in $O(S^1)$.

$$\begin{aligned} H &\approx -\frac{NqJS^2}{2} + H_1 \\ H_1 = JS \sum_{<ij>} (a_i b_j + a_i^\dagger b_j^\dagger) &+ qJS \sum_{i \in A} a_i^\dagger a_i + qJS \sum_{j \in B} b_j^\dagger b_j \end{aligned} \tag{3.4}$$

The first term is the classical ground state of heisenberg antiferromagnet.H_1 contains the antiferromagnet magnon dispersion.Just like we did ferromagnets ,H_1 can be written in momentum space by fourier transformation to give

$$\begin{gathered} H_1 = qJS\sum_k \left[(\gamma_k^* a_k b_{-k} + \gamma_k a_k^\dagger b_{-k}^\dagger) + a_k^\dagger a_k + b_{-k}^\dagger b_{-k}\right] \\ \gamma_k = \frac{1}{q}\sum_\delta e^{ik\cdot\delta} \end{gathered} \tag{3.5}$$

To diagonalize this hamiltonian we perform Bogoliubov transformation. we transform to new operators α_k and β_k related to old bosonic operators a_k and b_k given by:

$$\begin{pmatrix}\alpha_k \\ \beta_k^\dagger\end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix}\begin{pmatrix} a_k \\ b_{-k}^\dagger \end{pmatrix} \tag{3.6}$$

In order to preserve the commutation relations

$$\begin{gathered} [\alpha_k, \alpha_{k'}^\dagger] = [\beta_k, \beta_{k'}^\dagger] = \delta_{kk'} \\ [\alpha_r, \alpha_s] = [\beta_r, \beta_s] = [\alpha_r, \beta_s] = [\alpha_r, \beta_s^\dagger] = 0 \end{gathered} \tag{3.7}$$

u_k^2 and v_k^2 must satisfy

$$u_k^2 - v_k^2 = 1 \tag{3.8}$$

Inverse transformation is given by

$$\begin{pmatrix} a_k \\ b_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix}\begin{pmatrix}\alpha_k \\ \beta_k^\dagger\end{pmatrix} \tag{3.9}$$

$$
\begin{aligned}
H_1 = qJS\sum_k \Bigg[\gamma_k \Big[(u_k\alpha_k + v_k\beta_k^\dagger)(v_k\alpha_k^\dagger + u_k\beta_k) + (u_k\alpha_k^\dagger + v_k\beta_k)(v_k\alpha_k + u_k\beta_k^\dagger)\Big] \\
+ (u_k\alpha_k^\dagger + v_k\beta_k)(u_k\alpha_k + v_k\beta_k^\dagger) + (v_k\alpha_k + u_k\beta_k^\dagger)(u_k\alpha_k^\dagger + u_k\beta_k)\Bigg] \\
= qJS\sum_k \Bigg[(u_k^2 + v_k^2 + 2\gamma_k u_k v_k)(\alpha_k^\dagger\alpha_k + \beta_k^\dagger\beta_k) + [\gamma_k(u_k^2 + v_k^2) + 2u_k v_k](\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger) \\
+ 2v_k^2 + 2\gamma_k u_k v_k\Bigg]
\end{aligned}
\tag{3.10}
$$

If the hamiltonian is diagonalized then the coefficient of the off diagonal term $(\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger)$ must vanish.This gives

$$
\gamma_k(u_k^2 + v_k^2) + 2u_k v_k = 0 \tag{3.11}
$$

By solving equations 3.8 and 3.11 we get

$$
\begin{aligned}
u_k^2 &= \frac{1}{2}\left(\frac{1}{\sqrt{1-\gamma_k^2}} + 1\right) \\
v_k^2 &= \frac{1}{2}\left(\frac{1}{\sqrt{1-\gamma_k^2}} + 1\right) \\
u_k v_k &= -\frac{1}{2}\frac{\gamma_k}{\sqrt{1-\gamma_k^2}}
\end{aligned}
\tag{3.12}
$$

Hence we get diagonalized hamiltonian

$$
\begin{aligned}
H_1 &= qJS\sum_k \Big[(u_k^2 + v_k^2 + 2\gamma_k u_k v_k)(\alpha_k^\dagger\alpha_k + \beta_k^\dagger\beta_k) + 2v_k^2 + 2\gamma_k u_k v_k\Big] \\
&= qJS\sum_k \sqrt{1-\gamma_k^2}(\alpha_k^\dagger\alpha_k + \beta_k^\dagger\beta_k) - qJ\sum_k \left(1 - \sqrt{1-\gamma_k^2}\right)
\end{aligned}
\tag{3.13}
$$

The antiferromagnetic dispersion relation is therefore:

$$
\begin{aligned}
\epsilon_k &= qJS\sqrt{1-\gamma_k^2} \\
&= qJS\sqrt{1 - \left(\frac{1}{q}\sum_\delta e^{ik\cdot\delta}\right)^2}
\end{aligned}
\tag{3.14}
$$

3.2 Dispersion in 1D antiferromagnetic chain

In a 1 dimensional anti ferromagnetic chain

$$\begin{aligned} \epsilon_k &= 2JS\sqrt{1-(\frac{1}{2}(e^{ika}+e^{-ika}))} \\ &= 2JS|sin(ka)| \end{aligned} \tag{3.15}$$

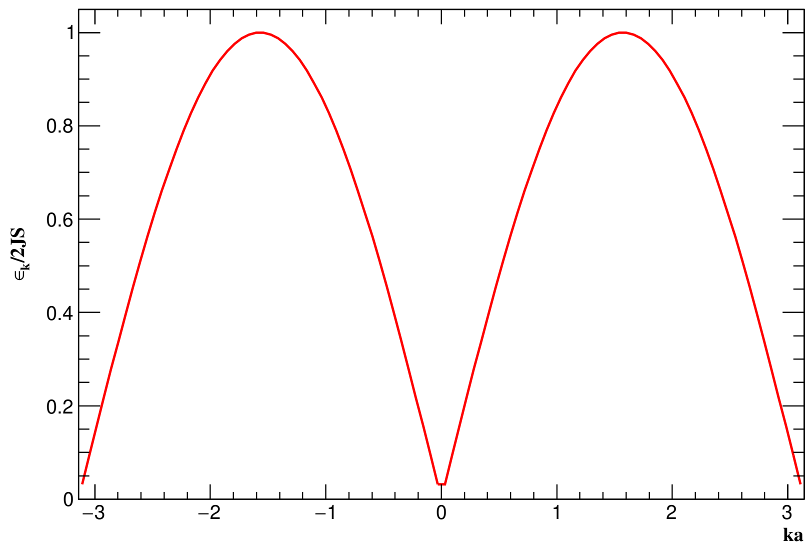


Figure 3.1: Magnon Dispersion curve in 1D Anti ferromagnetic chain in the first Brillouin zone

Chapter 4

Review of arXiv paper 2105.05861: Magnon spin current induced by triplet Cooper pair supercurrents

After learning about the formalism of spin waves and Holstein primakoff transformation i tried to understand from this paper how coupling between supercurrent in superconductor and ferromagnetic insulator lead to induced Magnon Spin current.

4.1 Introduction

Ferromagnetic insulators (FI) are of high relevance for spin transport applications due to their ability to carry pure spin currents over long distances.At the interface between a ferromagnetic insulator and a superconductor there is a coupling between the spins of two materials as shown in figure 4.1.

4.2 Model Hamiltonian

This system is described as two separate translationally invariant two-dimensional layers on top of each other,where coupling between layers exist for lattice sites corresponding to same location in plane.To describe the S/FI structure a Hamiltonian of form $H = H_s + H_{FI} + H_c$. where

- H_s describes superconductor layer.
- H_{FI} describes Ferromagnetic insulator.

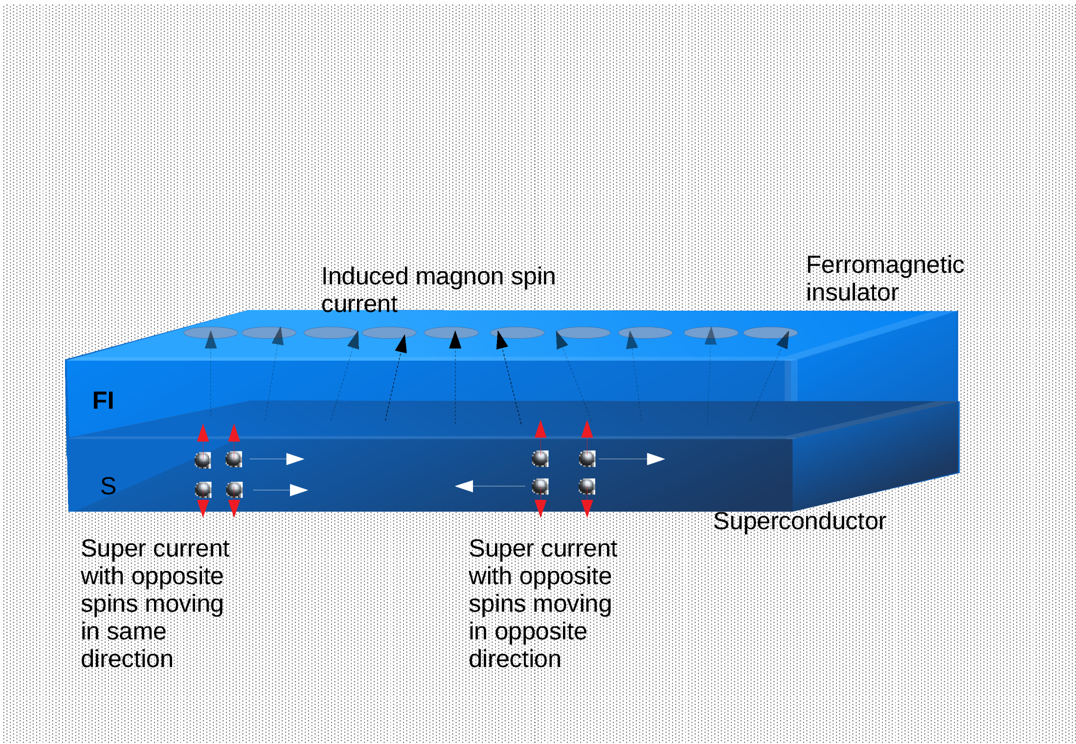


Figure 4.1: supercurrents induce magnon spin current in the FI

- H_c describes coupling between them.

The superconductor Hamiltonian is as follows

$$H_S = \sum_{k,\sigma} \epsilon_k c^\dagger_{k,\sigma} c_{k,\sigma} - \frac{1}{2} \sum_{k,\sigma} \left[\Delta^\delta_{k,\sigma} c^\dagger_{k+Q^\sigma,\sigma} c^\dagger_{-k+Q^\sigma,\sigma} + h.c \right] \tag{4.1}$$

$$\epsilon_k = -\mu - 2t \sum_\delta cos(k \cdot \delta) \tag{4.2}$$

$$\Delta^\delta_{k,\sigma} = f^\delta_\sigma sin(k \cdot \delta) \tag{4.3}$$

μ is chemical potential ,t is hopping integral.the strength of the superconducting order parameter $\Delta^\delta_{k,\sigma}$ is given by the real parameter f^δ_σ.The fermionic operators $C^{(\dagger)}_{k,\sigma}$ annhilates (creates) an electron with momentum k and spin σ.

We have already done the calculation for Hamiltonian describing ferromagnets in first and second chapter equation 2.17 and 2.18.

$$H_{FI} = \sum_q \omega_q a_q^\dagger a_q, \tag{4.4}$$

$$\omega_q = 4SK + J\sum_\delta [1 - cos(q\cdot\delta)] \tag{4.5}$$

The strength of coupling between spins on neighbouring lattice sites is $J > 0$.S is the magnitude of spin.This time we add a magnon gap whose magnitude is determined by $K > 0$ to account for the anisotropy favouring alignment along the z axis.The bosonic operators $a_q^{(\dagger)}$ annhilates (creates) a magnon of momenta q.

4.3 Calculation of coupling term H$_c$

The coupling term is introduced as following

$$H_c = \sum_i \Lambda_i s_i \cdot S_i, \tag{4.6}$$

$$s_i = \frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \boldsymbol{\sigma_{\sigma,\sigma'}} c_{i,\sigma'} \tag{4.7}$$

where s_i is the electron spin and $\boldsymbol{\sigma}$ is the vector of pauli matrices .S_i is the spin at ferromagnetic site.we assume that Λ_i is constant for i.e same for all i.

$$s_i \cdot S_i = \frac{1}{2}\left(\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \boldsymbol{\sigma_{\sigma,\sigma'}} c_{i,\sigma'}\right)\cdot S_i \tag{4.8}$$

$$= \frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^x S_i^x c_{i,\sigma'} + \frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^y S_i^y c_{i,\sigma'} + \frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^z S_i^z c_{i,\sigma'} \tag{4.9}$$

We now show that equation 4.7 preserve the commutation relation of spins of electron:

$$
\begin{aligned}
s_i^x &= \frac{1}{2}[c_{i\uparrow}^\dagger c_{i\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow}] \\
s_i^y &= \frac{1}{2}[-ic_{i\uparrow}^\dagger c_{i\downarrow} + ic_{i\downarrow}^\dagger c_{i\uparrow}] \\
s_i^z &= \frac{1}{2}[c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow}] \\
\Rightarrow s_i^+ &= c_{i\uparrow}^\dagger c_{i\downarrow}, s_i^- = c_{i\downarrow}^\dagger c_{i\uparrow}
\end{aligned}
$$

(I) $[s_i^z, s_i^+] = s_i^+$

We omit the i index to make it look notation wise less cumbersome.

$$
\begin{aligned}
[s^z, s^+] &= [\frac{1}{2}(c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c\downarrow), c_\uparrow^\dagger c_\downarrow] = \frac{1}{2}[c_\uparrow^\dagger c_\uparrow, c_\uparrow^\dagger c_\downarrow] - \frac{1}{2}[c_\downarrow^\dagger c_\downarrow, c_\uparrow^\dagger c_\downarrow] \\
\left[c_\uparrow^\dagger c_\uparrow, c_\uparrow^\dagger c_\downarrow\right] &= c_\uparrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow - c_\uparrow^\dagger c_\downarrow c_\uparrow^\dagger c_\uparrow = c_\uparrow^\dagger(1 - c_\uparrow^\dagger c_\uparrow)c_\downarrow - c_\uparrow^\dagger c_\downarrow c_\uparrow^\dagger c_\uparrow \\
&= c_\uparrow^\dagger c_\downarrow - c_\uparrow^\dagger c_\uparrow^\dagger c_\uparrow c_\downarrow + c_\uparrow^\dagger c_\uparrow^\dagger c_\downarrow c_\uparrow = c_\uparrow^\dagger c_\downarrow
\end{aligned}
$$

because for fermionic operator $\{c_\uparrow^\dagger, c_\uparrow^\dagger\}$=0 $\Rightarrow c_\uparrow^\dagger c_\uparrow^\dagger = 0$.similarly,

$$
\begin{aligned}
\left[c_\downarrow^\dagger c_\downarrow, c_\uparrow^\dagger c_\downarrow\right] &= c_\downarrow^\dagger c_\downarrow c_\uparrow^\dagger c_\downarrow - c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\downarrow = c_\downarrow^\dagger c_\downarrow c_\uparrow^\dagger c_\downarrow - c_\uparrow^\dagger(1 - c_\downarrow^\dagger c_\downarrow)c_\downarrow \\
&= -c_\downarrow^\dagger c_\uparrow^\dagger c_\downarrow c_\downarrow - c_\uparrow^\dagger c_\downarrow + c_\uparrow^\dagger c_\downarrow^\dagger c_\downarrow c_\downarrow = -c_\uparrow^\dagger c_\downarrow \\
\Rightarrow \left[s^z, s^+\right] &= \frac{1}{2}c_\uparrow^\dagger c_\downarrow - \frac{1}{2}(-c_\uparrow^\dagger c_\downarrow) = c_\uparrow^\dagger c_\downarrow = s^+
\end{aligned}
$$

(II) $\left[s_i^z, s_i^-\right] = -s_i^-$

$$
\begin{aligned}
\left[s^z, s^-\right] &= \left[\frac{1}{2}(c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow), c_\downarrow^\dagger c_\uparrow\right] = \frac{1}{2}[c_\uparrow^\dagger c_\uparrow, c_\downarrow^\dagger c_\uparrow] - \frac{1}{2}[c_\downarrow^\dagger c_\downarrow, c_\downarrow^\dagger c_\uparrow] \\
\left[c_\uparrow^\dagger c_\uparrow, c_\downarrow^\dagger c_\uparrow\right] &= c_\uparrow^\dagger c_\uparrow c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\uparrow = -c_\uparrow^\dagger c_\uparrow c_\uparrow c_\downarrow^\dagger - c_\downarrow^\dagger(1 - c_\uparrow^\dagger c_\uparrow)c_\uparrow \\
&= -c_\uparrow^\dagger c_\uparrow c_\uparrow c_\downarrow^\dagger - c_\downarrow^\dagger c_\uparrow + c_\uparrow^\dagger c_\downarrow^\dagger c_\uparrow c_\uparrow = -c_\downarrow^\dagger c_\uparrow \\
\left[c_\downarrow^\dagger c_\downarrow, c_\downarrow^\dagger c_\uparrow\right] &= c_\downarrow^\dagger c_\uparrow c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\uparrow c_\downarrow^\dagger c_\downarrow = c_\downarrow^\dagger(1 - c_\downarrow^\dagger c_\downarrow)c_\uparrow + c_\downarrow^\dagger c_\downarrow^\dagger c_\uparrow c_\downarrow \\
&= c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow^\dagger c_\downarrow c_\uparrow + c_\downarrow^\dagger c_\downarrow^\dagger c_\uparrow c_\downarrow = c_\downarrow^\dagger c_\uparrow \\
\Rightarrow \left[s^z, s^-\right] &= \frac{1}{2}(-c_\downarrow^\dagger c_\uparrow) - \frac{1}{2}(c_\downarrow^\dagger c_\uparrow) = -c_\downarrow^\dagger c_\uparrow = -s^-
\end{aligned}
$$

(III) $\left[s_i^+, s_i^-\right] = 2s_i^z$

$$\left[s^+, s^-\right] = \left[c_\uparrow^\dagger c_\downarrow, c_\downarrow^\dagger c_\uparrow\right] = c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow = c_\uparrow^\dagger c_\uparrow c_\downarrow c_\downarrow^\dagger - c_\downarrow^\dagger c_\downarrow c_\uparrow c_\uparrow^\dagger$$

$$= c_\uparrow^\dagger c_\uparrow(1 - c_\downarrow^\dagger c_\downarrow) - c_\downarrow^\dagger c_\downarrow(1 - c_\uparrow^\dagger c_\uparrow) = c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow - c_\uparrow^\dagger c_\uparrow c_\downarrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\downarrow c_\uparrow^\dagger c_\uparrow$$

$$= c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow = 2s^z$$

We will now calculate each of the three terms in equation 4.9.We use HP transformation to substitute ferromagnetic spin operators with bosonic operators.

$$\frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^x S_i^x c_{i,\sigma'} = \frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^x \frac{1}{2}\left(S_i^+ + S_i^-\right) c_{i,\sigma'} = \frac{\sqrt{2S}}{4}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^x (a_i + a_i^\dagger) c_{i,\sigma'}$$

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

σ^x is non zero when σ, σ' are different so,

$$= \frac{\sqrt{2S}}{4}\left[c_{i\uparrow}^+ a_i c_{i\downarrow} + c_{i\downarrow}^+ a_i c_{i\uparrow} + c_{i\uparrow}^+ a_i^\dagger c_{i\downarrow} + c_{i\downarrow}^+ a_i^\dagger c_{i\uparrow}\right]$$

We will now do similar thing with second and third term .

$$\frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^y S_i^y c_{i,\sigma'} = \frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^y \frac{1}{2i}\left(S_i^+ - S_i^-\right) c_{i,\sigma'} = \frac{\sqrt{2S}}{4i}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^y (a_i - a_i^\dagger) c_{i,\sigma'}$$

$$\sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

σ^y is non zero when $\sigma \neq \sigma'$.

$$= \frac{\sqrt{2S}}{4}\left[-c_{i\uparrow}^+ a_i c_{i\downarrow} + c_{i\downarrow}^+ a_i c_{i\uparrow} + c_{i\uparrow}^+ a_i^\dagger c_{i\downarrow} - c_{i\downarrow}^+ a_i^\dagger c_{i\uparrow}\right]$$

$$\frac{1}{2}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^z S_i^z c_{i,\sigma'} = \frac{\sqrt{2S}}{4i}\sum_{\sigma,\sigma'} c_{i,\sigma}^\dagger \sigma_{\sigma,\sigma'}^z (S - a_i^\dagger a_i) c_{i,\sigma'}$$

$$\sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

σ^z is non zero when $\sigma = \sigma'$.

$$= \frac{S}{2}\left[c^\dagger_{i\uparrow}c_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow}\right] - \frac{1}{2}\left[c^\dagger_{i\uparrow}a^\dagger_i a_i c_{i\uparrow} - c^\dagger_{i\downarrow}a^\dagger_i a_i c_{i\downarrow}\right]$$

Putting all these together

$$\begin{aligned} s_i \cdot S_i &= \frac{\sqrt{2S}}{2}\left[c^\dagger_{i\uparrow}a^\dagger_i c_{i\downarrow} + c^\dagger_{i\downarrow}a_i c_{i\uparrow}\right] + \frac{S}{2}\left[c^\dagger_{i\uparrow}c_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow}\right] - \frac{1}{2}\left[c^\dagger_{i\uparrow}a^\dagger_i a_i c_{i\uparrow} - c^\dagger_{i\downarrow}a^\dagger_i a_i c_{i\downarrow}\right] \\ &= \frac{\sqrt{2S}}{2}\left[c^\dagger_{i\uparrow}a^\dagger_i c_{i\downarrow} + c^\dagger_{i\downarrow}a_i c_{i\uparrow}\right] + \frac{S}{2}\sum_\sigma \sigma c^\dagger_{i,\sigma}c_{i,\sigma} - \frac{1}{2}\sum_\sigma \sigma c^\dagger_{i,\sigma}a^\dagger_i a_i c_{i,\sigma} \end{aligned}$$

$$\begin{aligned} H_c = \sum_i \Lambda s_i \cdot S_i = \sum_i \Lambda\left[\frac{\sqrt{2S}}{2}\left[c^\dagger_{i\uparrow}a^\dagger_i c_{i\downarrow} + c^\dagger_{i\downarrow}a_i c_{i\uparrow}\right] + \frac{S}{2}\sum_\sigma \sigma c^\dagger_{i,\sigma}c_{i,\sigma} - \frac{1}{2}\sum_\sigma \sigma c^\dagger_{i,\sigma}a^\dagger_i a_i c_{i,\sigma}\right] \\ = H_c^{2c} + H_c^{2c1a} + H_c^{2c2a} \end{aligned}$$

$$\begin{aligned} H_c^{2c} &= \frac{\Lambda S}{2}\sum_{i,\sigma}\sigma c^\dagger_{i,\sigma}c_{i,\sigma} \\ H_c^{2c1a} &= \sum_i \frac{\sqrt{2S}\Lambda}{2}\left[c^\dagger_{i\uparrow}a^\dagger_i c_{i\downarrow} + c^\dagger_{i\downarrow}a_i c_{i\uparrow}\right] \\ H_c^{2c2a} &= \frac{-\Lambda}{2}\sum_{i,\sigma}\sigma c^\dagger_{i,\sigma}a^\dagger_i a_i c_{i,\sigma} \end{aligned} \tag{4.10}$$

We will now rewrite these coupling terms in momentum space by doing Fourier transformation.

$$c_{i,\sigma} = \frac{1}{\sqrt{N}}\sum_k c_{k,\sigma}e^{ik\cdot i} \tag{4.11}$$

$$a_i = \frac{1}{\sqrt{N}}\sum_k a_k e^{ik\cdot i} \tag{4.12}$$

Where the k vectors belong to first brillouin zone.

$$H_c^{2c} = \frac{S\Lambda}{2N}\sum_{i,\sigma}\sigma\sum_{k,k'} c^\dagger_{k',\sigma}e^{-ik'\cdot i}c_{k,\sigma}e^{ik\cdot i} = \frac{\Lambda S}{2N}\sum_{\sigma,k,k'}\sigma c^\dagger_{k'\sigma}c_{k,\sigma}\left(\sum_i e^{i(k-k')\cdot i}\right)$$
$$= \frac{\Lambda S}{2N}\sum_{\sigma,k,k'}\sigma c^\dagger_{k'\sigma}c_{k,\sigma}N\delta^k_{k'} = \frac{\Lambda S}{2}\sum_{k,\sigma}\sigma c^\dagger_{k,\sigma}c_{k,\sigma} = \lambda\sqrt{\frac{NS}{2}}\sum_{k,\sigma}\sigma c^\dagger_{k,\sigma}c_{k,\sigma}$$

where

$$\lambda = \Lambda\sqrt{\frac{S}{2N}} \tag{4.13}$$

$$H_c^{2c1a} = \sum_i \frac{\sqrt{2S}\Lambda}{2}\left[c^\dagger_{i\uparrow}a^\dagger_i c_{i\downarrow} + c^\dagger_{i\downarrow}a_i c_{i\uparrow}\right]$$
$$= \frac{\sqrt{2S}\Lambda}{2}\sum_i\left[\sum_{k_1,k_2,k_3}\frac{1}{N^{3/2}}c^\dagger_{k_1,\uparrow}e^{-ik_1\cdot i}a^\dagger_{k_2}e^{-ik_2\cdot i}c_{k_3,\downarrow}e^{ik_3\cdot i} + \sum_{k_1,k_2,k_3}\frac{1}{N^{3/2}}c^\dagger_{k_1,\downarrow}e^{-ik_1\cdot i}a_{k_2}e^{ik_2\cdot i}c_{k_3,\uparrow}e^{ik_3\cdot i}\right]$$
$$= \frac{\sqrt{2S}\Lambda}{2N^{3/2}}\sum_i\left[\sum_{k_1,k_2,k_3}c^\dagger_{k_1,\uparrow}a^\dagger_{k_2}c_{k_3,\downarrow}e^{i(-k_1-k_2+k_3)\cdot i} + \sum_{k_1,k_2,k_3}c^\dagger_{k_1,\downarrow}a_{k_2}c_{k_3,\uparrow}e^{i(-k_1+k_2+k_3)\cdot i}\right]$$

we have

$$\sum_i e^{i(-k_1-K_2+K_3).i} = N\delta^{k_3}_{k_1+k_2}$$
$$\sum_i e^{i(-k_1+K_2+K_3).i} = N\delta^{k_3}_{k_1-k_2}$$

$$= \frac{\sqrt{2S}\Lambda}{2N^{3/2}}\left[\sum_{k_1,k_2,k_3}c^\dagger_{k_1,\uparrow}a^\dagger_{k_2}c_{k_3,\downarrow}N\delta^{k_3}_{k_1+k_2} + \sum_{k_1,k_2,k_3}c^\dagger_{k_1,\downarrow}a_{k_2}c_{k_3,\uparrow}N\delta^{k_3}_{k_1-k_2}\right]$$
$$= \frac{\sqrt{2S}\Lambda}{2N^{1/2}}\left[\sum_{k,q}\left(c^\dagger_{k,\uparrow}a^\dagger_q c_{k+q,\downarrow} + c^\dagger_{k,\downarrow}a_{-q}c_{k+q,\uparrow}\right)\right] = \lambda\sum_{k,q}\left(c^\dagger_{k,\uparrow}a^\dagger_q c_{k+q,\downarrow} + c^\dagger_{k,\downarrow}a_{-q}c_{k+q,\uparrow}\right)$$

$$H_c^{2c2a} = \frac{-\Lambda}{2}\sum_{i,\sigma}\sigma c^\dagger_{i,\sigma}a^\dagger_i a_i c_{i,\sigma} = -\frac{\Lambda}{2N^2}\sum_{i,\sigma}\sigma\left[\sum_{k_1,k_2,k_3,k_4}c^\dagger_{k_1,\sigma}e^{-ik_1\cdot i}c_{k_2,\sigma}e^{ik_2\cdot i}a^\dagger_{k_3}e^{-ik_3\cdot i}a_{k_4}e^{ik_4\cdot i}\right]$$
$$= -\frac{\Lambda}{2N^2}\sum_{i,\sigma}\sigma\left[\sum_{k_1,k_2,k_3,k_4}c^\dagger_{k_1,\sigma}c_{k_2,\sigma}e^{ik_2\cdot i}a^\dagger_{k_3}a_{k_4}e^{i(k_2+k_4-k_1-k_3)\cdot i}\right]$$

we have

$$\sum_i e^{i(k_2+k_4-k_1-k_3)\cdot i} = N\delta_{k_1+k_3}^{k_2+k_4}$$

Non zero contribution in the sum comes when $k_2 + k_4 = k_1 + k_3$.so Four variables ,one equation,therefore the sum can be represented in three independent variables. We take $k_2 = k - q^{'}, k_4 = q^{'}$ and $k_1 = k + q, k_3 = -q$.

$$H_c^{2c2a} = -\frac{\Lambda}{2N^2}\sum_\sigma \sigma\left[\sum_{k_1,k_2,k_3,k_4} c_{k_1,\sigma}^\dagger c_{k_2,\sigma} e^{ik_2\cdot i} a_{k_3}^\dagger a_{k_4} N\delta_{k_1+k_3}^{k_2+k_4}\right] = -\frac{\Lambda}{2N}\sum_{k,q,q',\sigma}\sigma\left[c_{k+q,\sigma}^\dagger c_{k-q',\sigma} a_{-q}^\dagger a_{q'}\right]$$
$$= \frac{\lambda}{\sqrt{2NS}}\sum_{k,q,q',\sigma}\sigma\left[c_{k+q,\sigma}^\dagger c_{k-q',\sigma} a_{-q}^\dagger a_{q'}\right]$$

So we find three terms of coupling H_c as

$$H_c^{2c} = \lambda\sqrt{\frac{NS}{2}}\sum_{k,\sigma}\sigma c_{k,\sigma}^\dagger c_{k,\sigma} \tag{4.14}$$

$$H_c^{2c1a} = \lambda\sum_{k,q}\left(c_{k,\uparrow}^\dagger a_q^\dagger c_{k+q,\downarrow} + c_{k,\downarrow}^\dagger a_{-q} c_{k+q,\uparrow}\right) \tag{4.15}$$

$$H_c^{2c2a} = \frac{\lambda}{\sqrt{2NS}}\sum_{k,q,q',\sigma}\sigma\left[c_{k+q,\sigma}^\dagger c_{k-q',\sigma} a_{-q}^\dagger a_{q'}\right] \tag{4.16}$$

- H_c^{2c} describe a spin splitting of the fermionic energy spectrum.This term is absorbed into H_S by letting $\epsilon_k \longrightarrow \epsilon_{k,\sigma} = \epsilon_k + \sigma\lambda\sqrt{\frac{NS}{2}}$

- H_c^{2c1a} transfers the spin between fermion and boson operators and turns out to be significant for inducing a magnon spin current.

- H_c^{2c2a} gives constant shift in the magnon energy spectrum to the first order in λ.

The modified H_S is now

$$H_S = \sum_{k,\sigma}\epsilon_{k,\sigma} c_{k,\sigma}^\dagger c_{k,\sigma} - \frac{1}{2}\sum_{k,\sigma}\left[\Delta_{k,\sigma}^\delta c_{k+Q^\sigma,\sigma}^\dagger c_{-k+Q^\sigma,\sigma}^\dagger + h.c\right] \tag{4.17}$$

4.4 Diagonalising H_S :

The Superconducting Part of hamiltonian H_S can be diagonalized by Bogoliubov Transformation.The old operators are related to new ones by

$$c_{k+Q^\sigma,\sigma} = u_{k,\sigma}\gamma_{k+Q^\sigma,\sigma} + sgn(\Delta_k^\sigma)v_{k,\sigma}\gamma^\dagger_{-k+Q^\sigma,\sigma} \tag{4.18}$$

We have $\{c^\dagger_{k+Q^\sigma.\sigma}, c_{k+Q^\sigma,\sigma}\} = 1$

Solving this gives

$$u_{k,\sigma}^2 + v_{k,\sigma}^2 = 1 \tag{4.19}$$

When we put equation 4.18 to H_S in equation 4.17 we need the coefficient of non diagonal terms to be zero.Together with this constraint and equation 4.19 we have

$$u_{k,\sigma}(v_{k.\sigma}) = \left[\frac{1}{2}\left(1 + (-)\frac{\frac{1}{2}(\epsilon_{k+Q^\sigma,\sigma} + \epsilon_{-k+Q^\sigma,\sigma})}{\sqrt{[\frac{1}{2}(\epsilon_{k+Q^\sigma,\sigma} + \epsilon_{-k+Q^\sigma,\sigma})]^2 + |\Delta_k|^2}}\right)\right]^{1/2} \tag{4.20}$$

The assumption is made that the superconducting order parameter $|\Delta_k|$ is spin independent. The Diagonalized H_S becomes

$$\begin{gathered} H_S = \sum_{k,\sigma} E_{k-Q^\sigma,\sigma}\gamma^\dagger_{k,\sigma}\gamma_{k,\sigma} \\ E_{k,\sigma} = \frac{1}{2}(\epsilon_{k+Q^\sigma,\sigma} - \epsilon_{-k+Q^\sigma,\sigma}) + \sqrt{[\frac{1}{2}(\epsilon_{k+Q^\sigma,\sigma} + \epsilon_{-k+Q^\sigma,\sigma})]^2 + |\Delta_k|^2} \end{gathered} \tag{4.21}$$

4.5 Future Calculations:

After this There will be a construction of effective hamiltonian H_{eff} from H_{FI} ,H_S and the coupling terms H_c^{2c1a},H_c^{2c2a}.Evaluating Expectation value with respect to fermions operators and diagonalizing with respect to Bosonic operators will give an effective Hamiltonian of the form $< H_{eff} >= \sum_q \Omega_q \alpha_q^\dagger \alpha_q$.Because of the other interaction terms the energy degeneracy between q and -q for magnonic frequencies is broken which eventually results in a nonzero spin current.The calculation of Lack of symmetry between Ω_q and Ω_{-q} will provide insight to the induced magnon spin current.

4.6 Summary

In this project I have learned about low excited spin wave states of ferromagnets.Then I learned the formalism for doing the spin wave calculation in terms of Holstein primakoff transformation and have studied the paper about the coupling of magnons and cooper pairs which can lead to generation of spin currents.

References

1. Solid state physics.Textbook by N. David Mermin and Neil Ashcroft.chapter 33.

2. Magnon spin current induced by triplet Cooper pair supercurrents.https://arxiv.org/pdf/2105.05861.pdf