Statistics with SpaRRows II

Many models, matrices, and magic

Julia Schroeder

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Bivariate models, and what they have to do with variance/covariance

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Bivariate analysis

From Wikipedia, the free encyclopedia

Bivariate analysis is one of the simplest forms of quantitative (statistical) analysis.[1] It involves the analysis of two variables (often denoted as *X*, *Y*), for the purpose of determining the empirical relationship between them.[1]

Bivariate models, and what they have to do with variance/covariance

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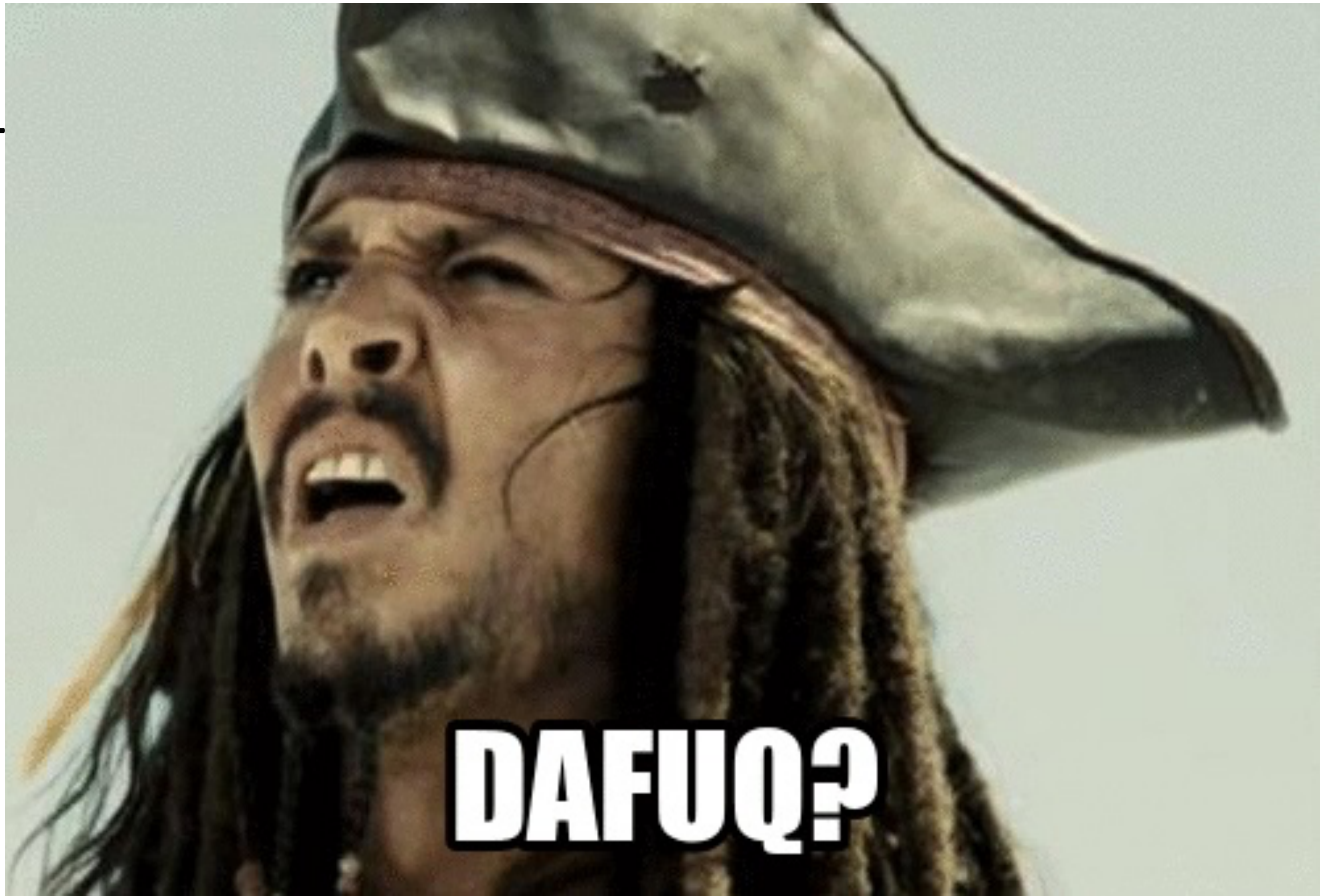

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Mean of tarsus

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Variance of tarsus

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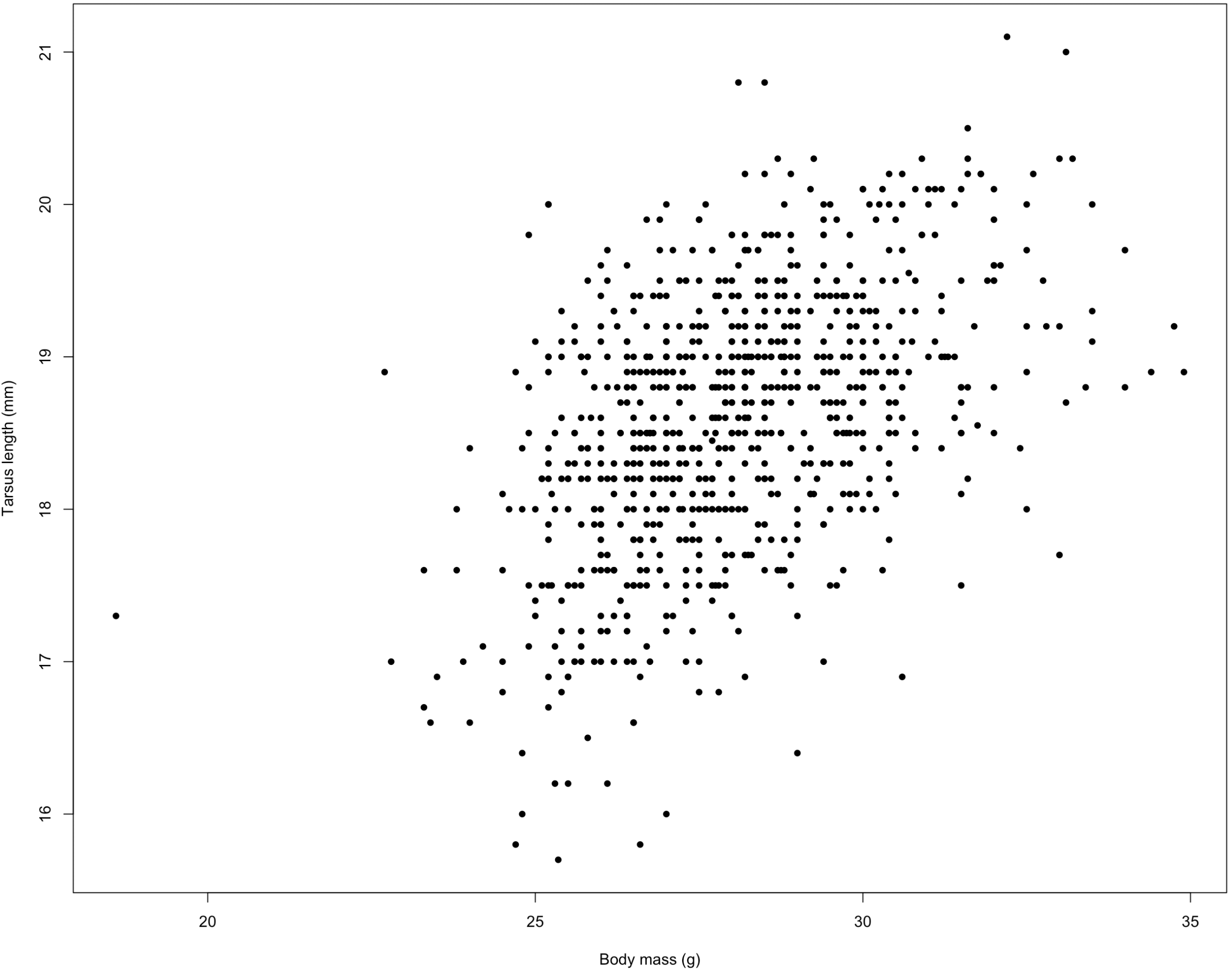

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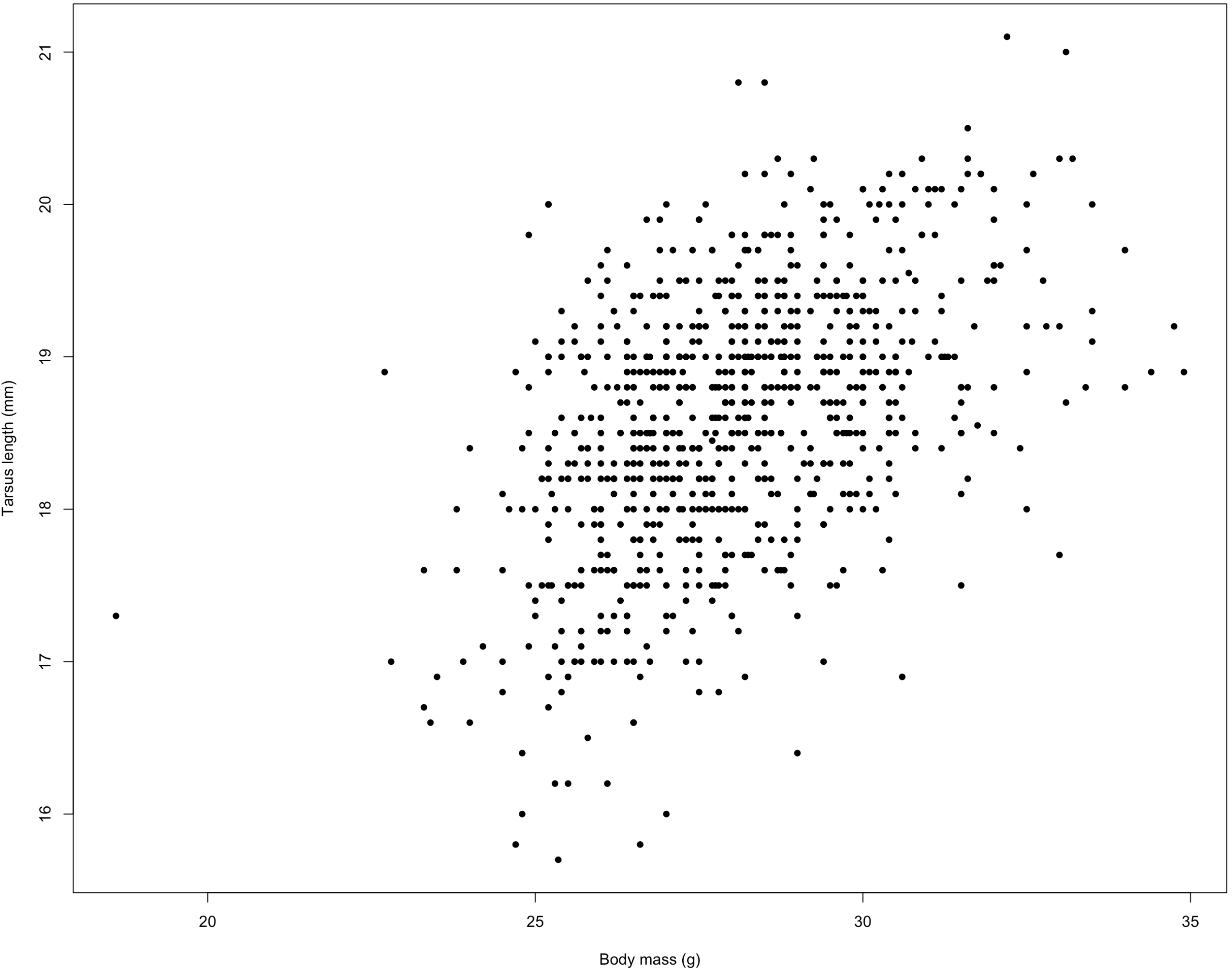

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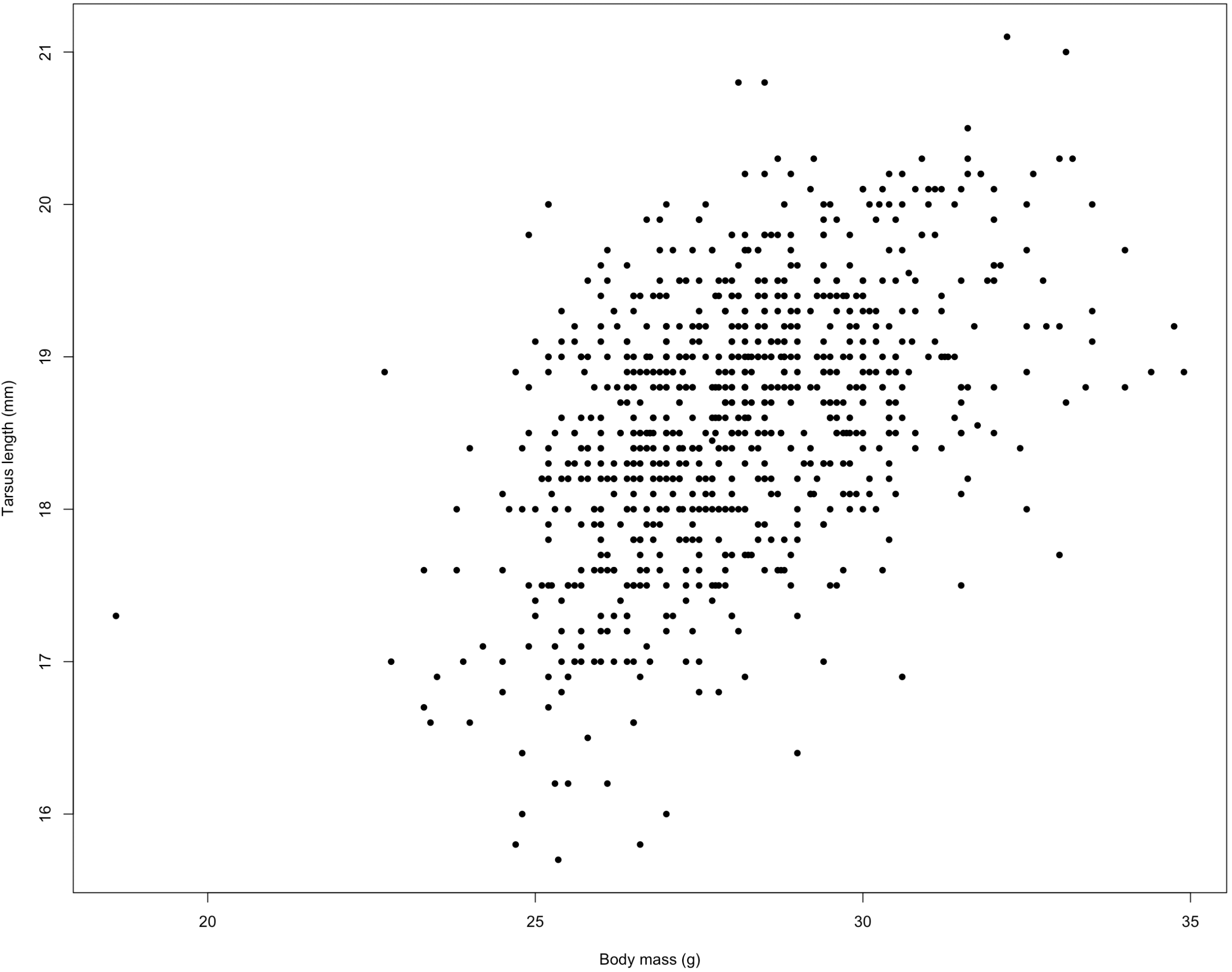


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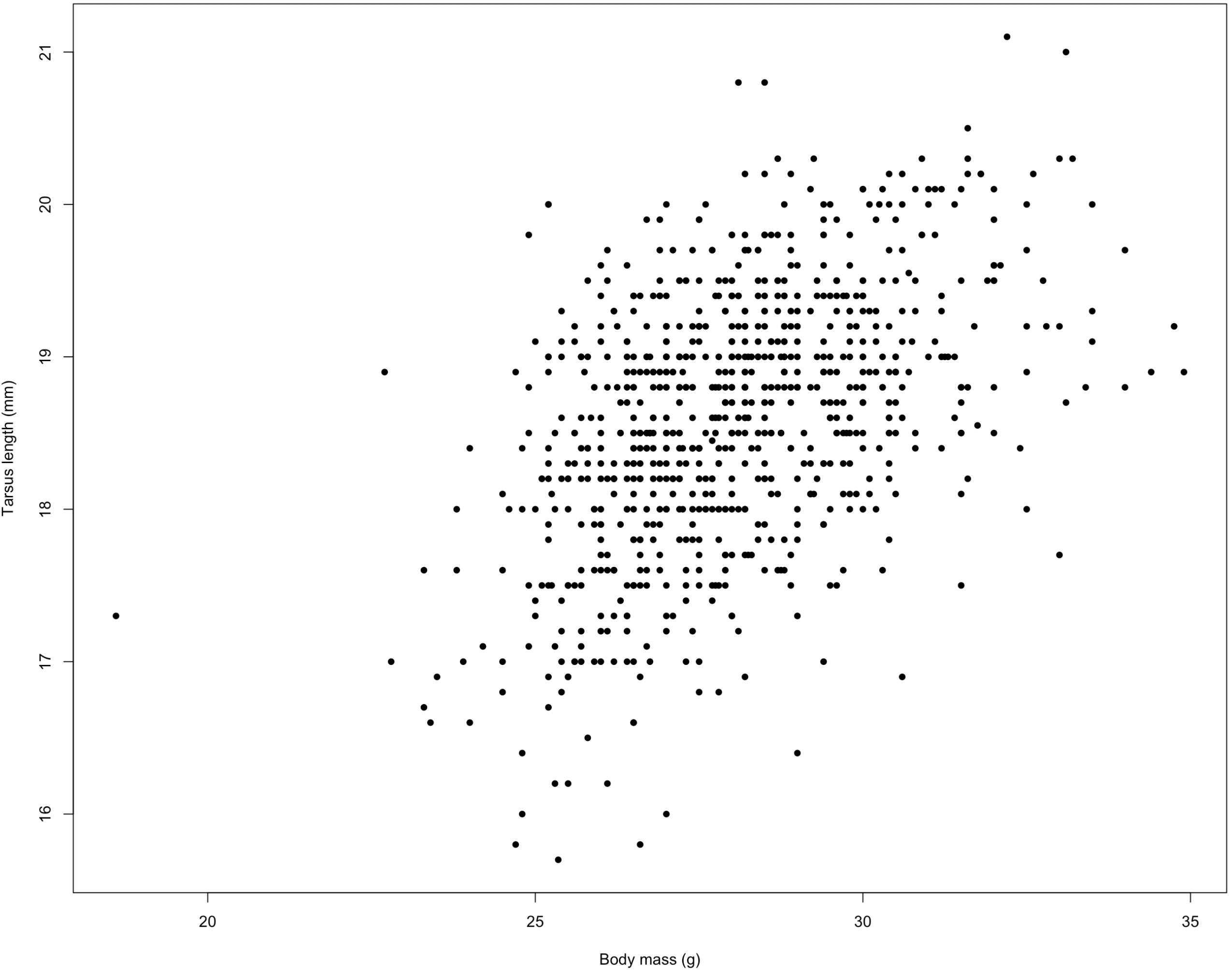

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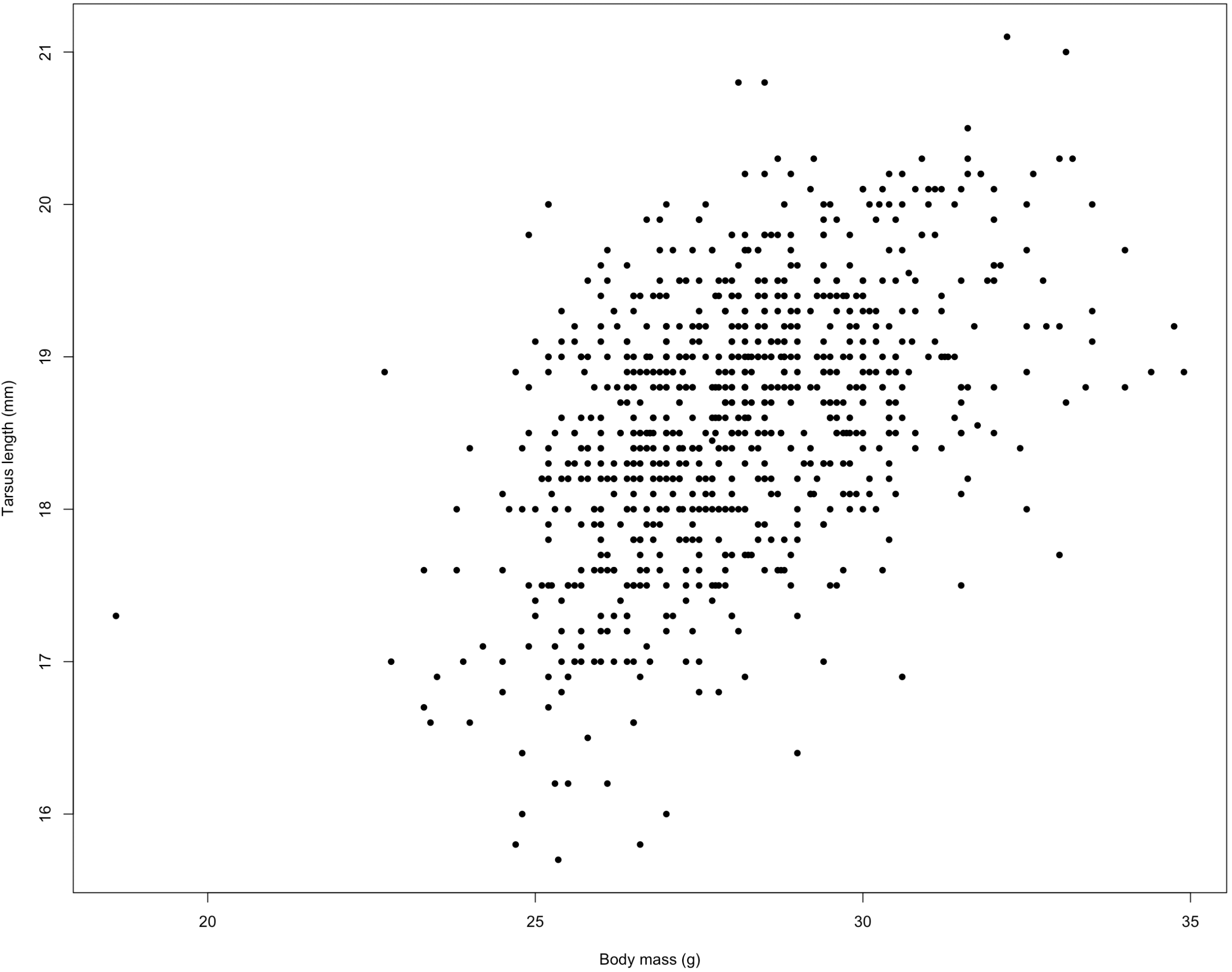

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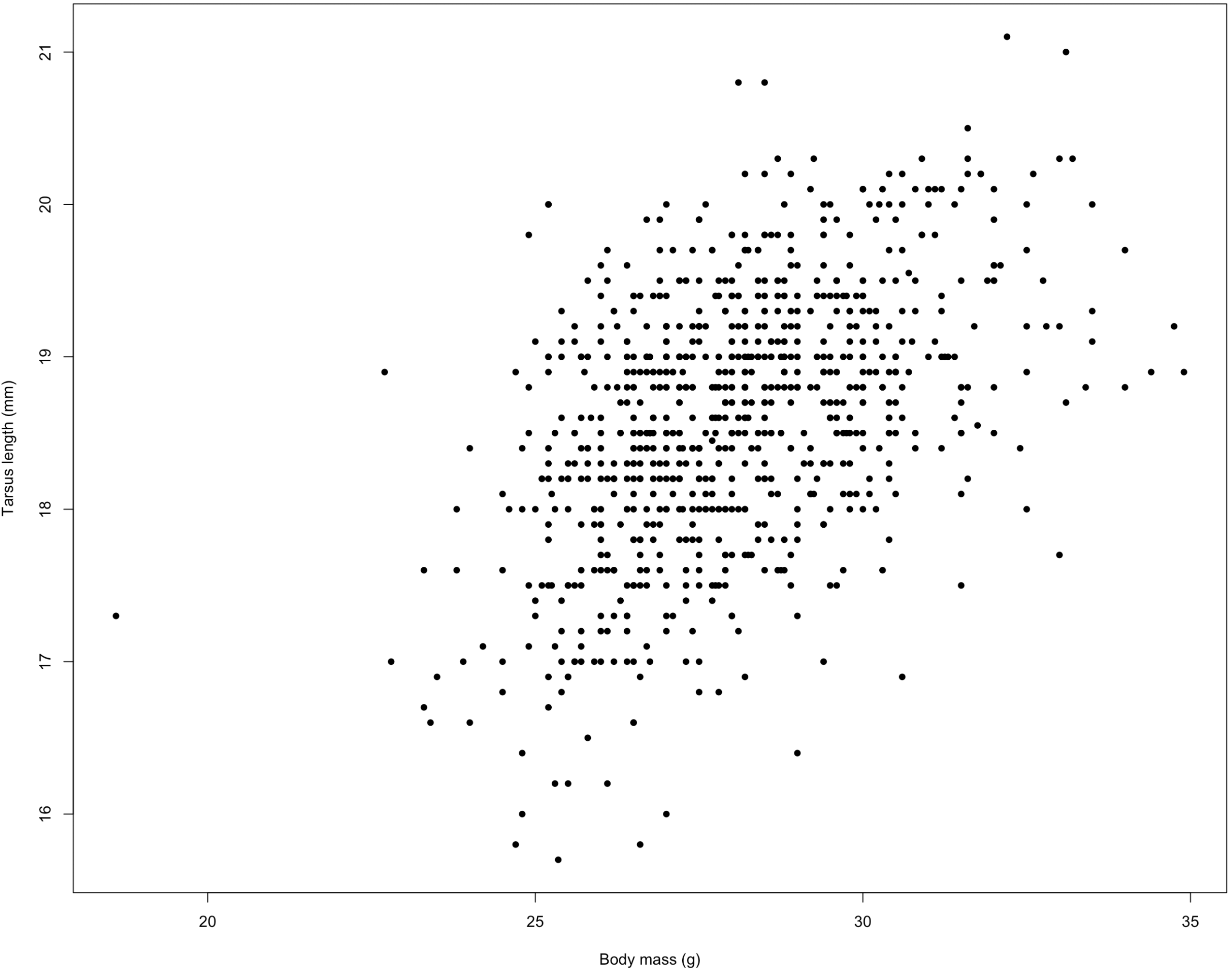


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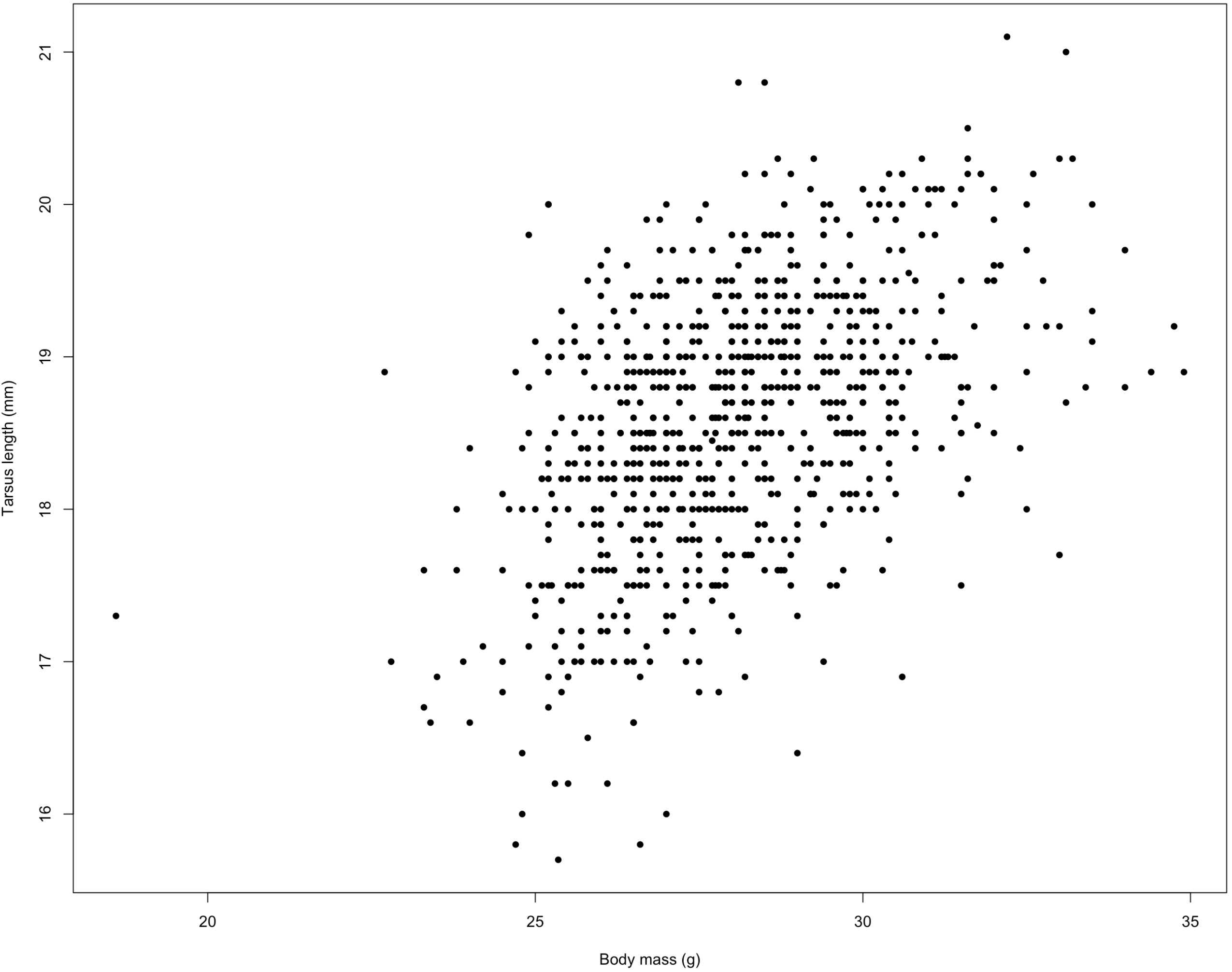

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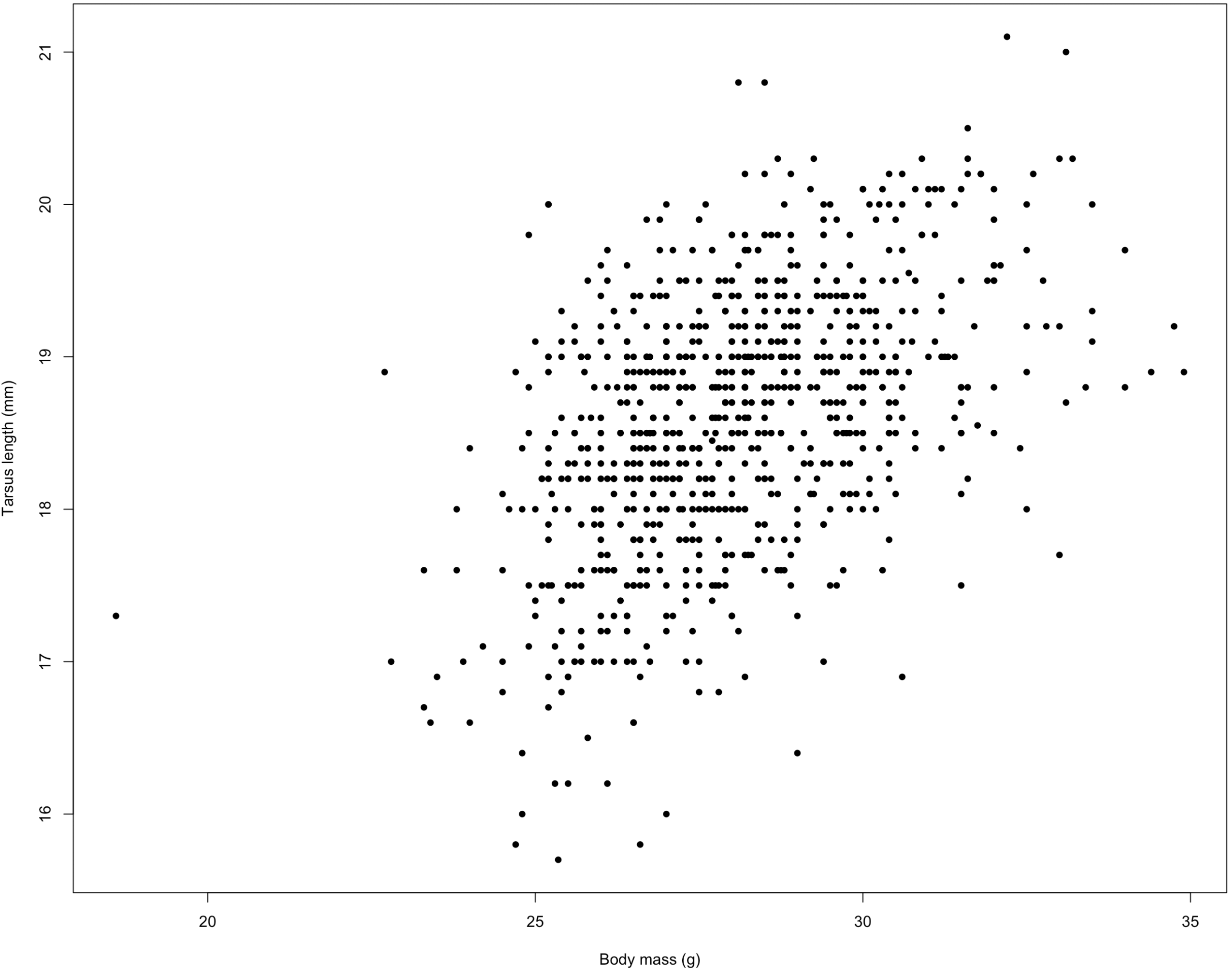


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 DIC: 5660.597

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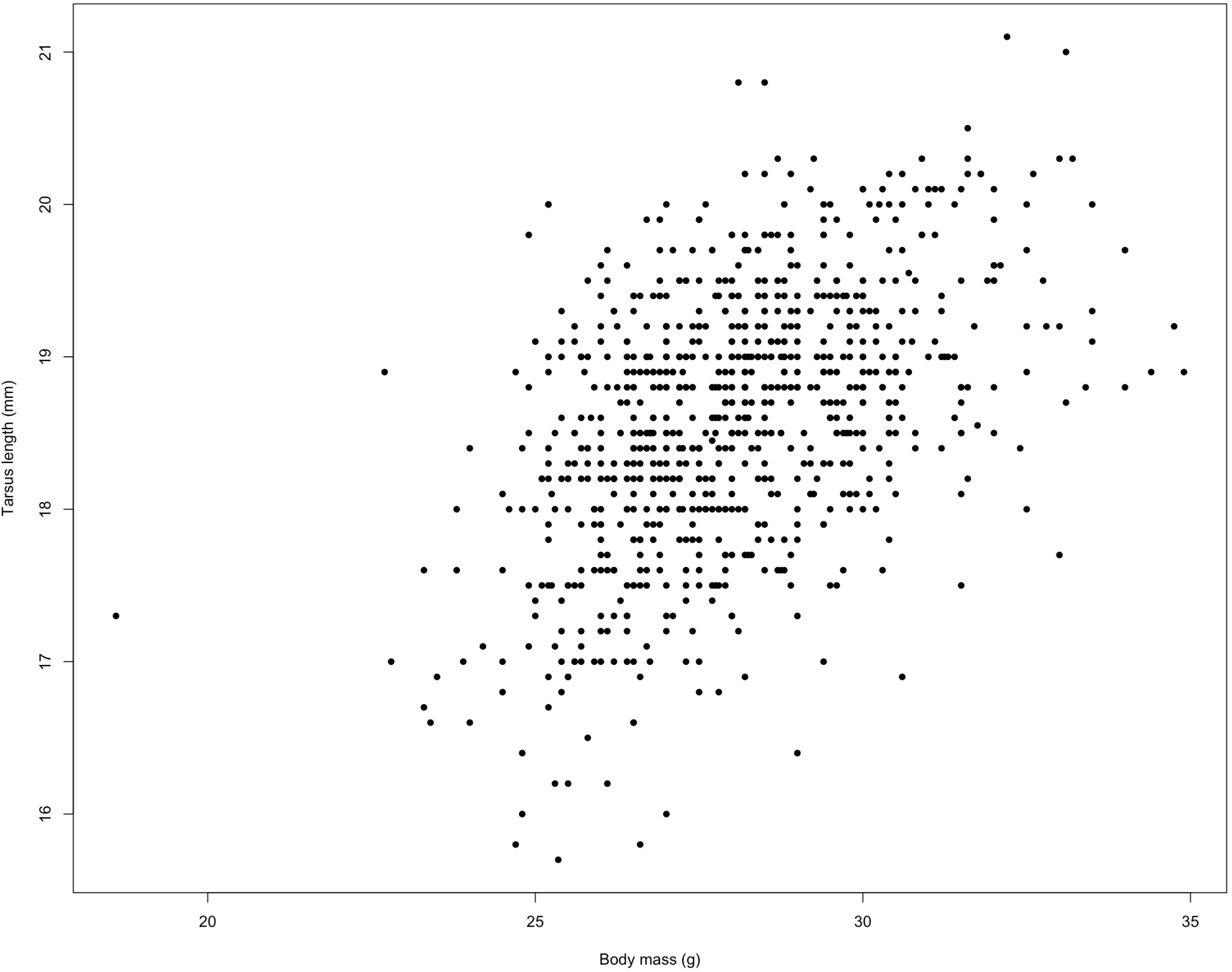


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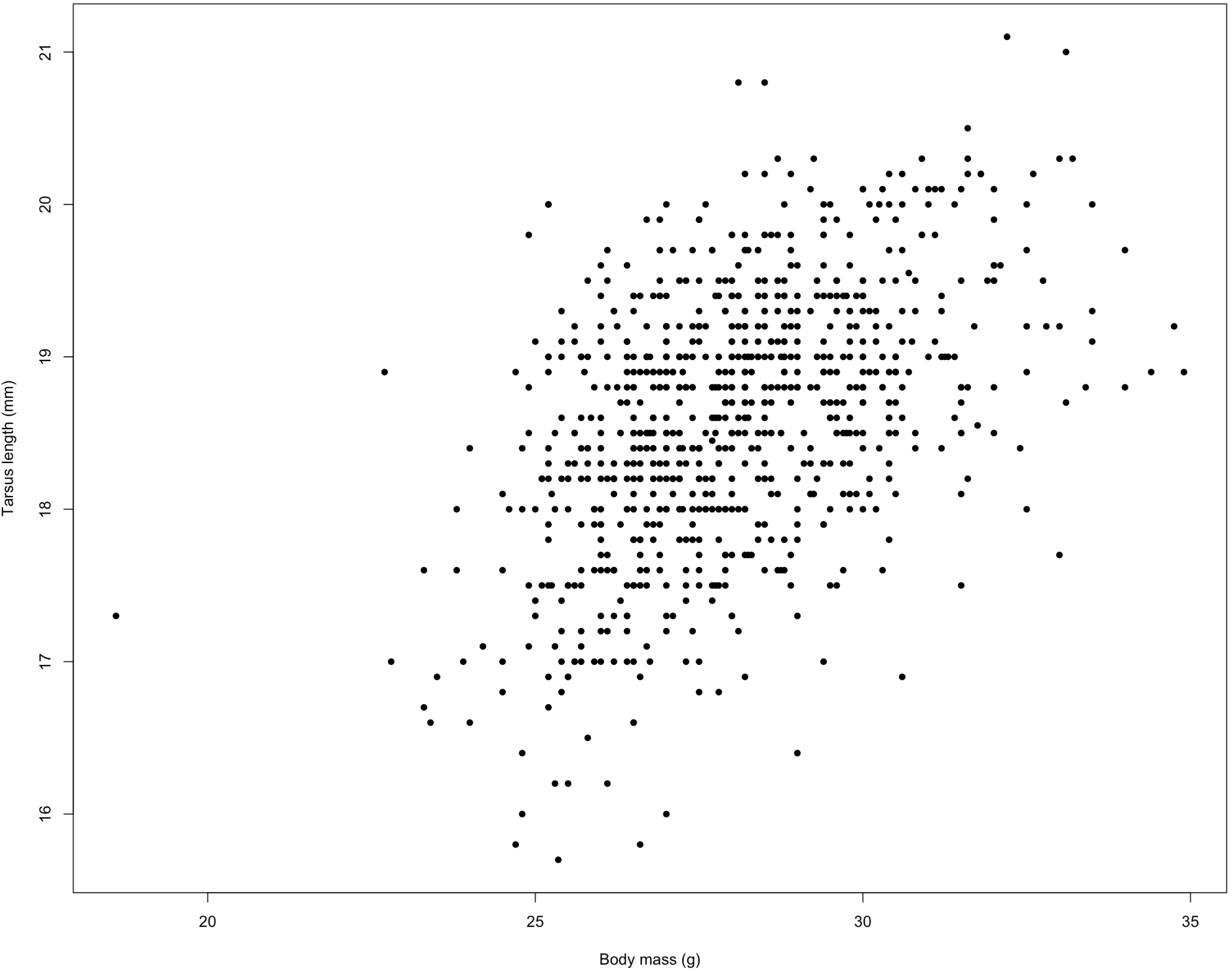


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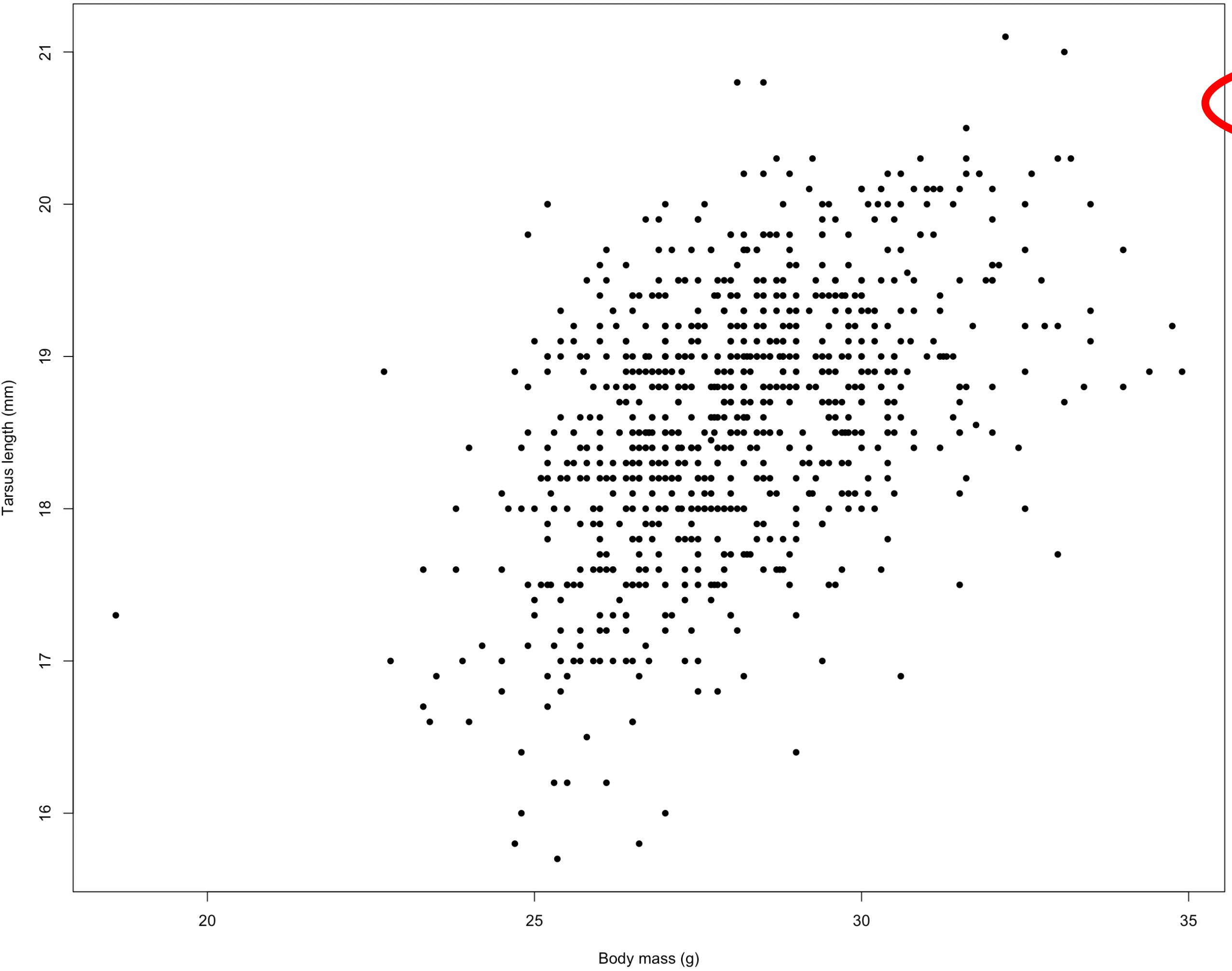


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Bayesian equivalent to AIC

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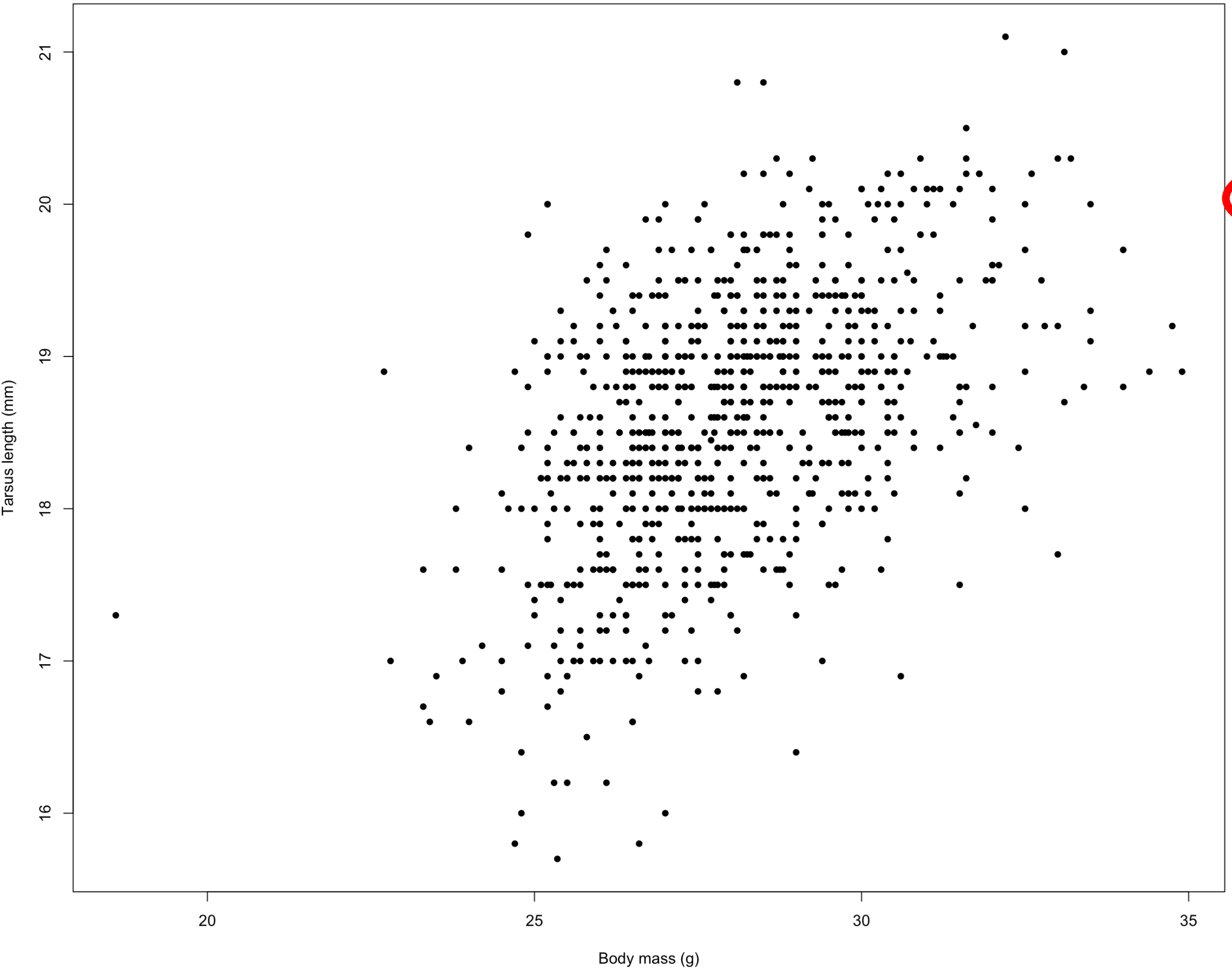


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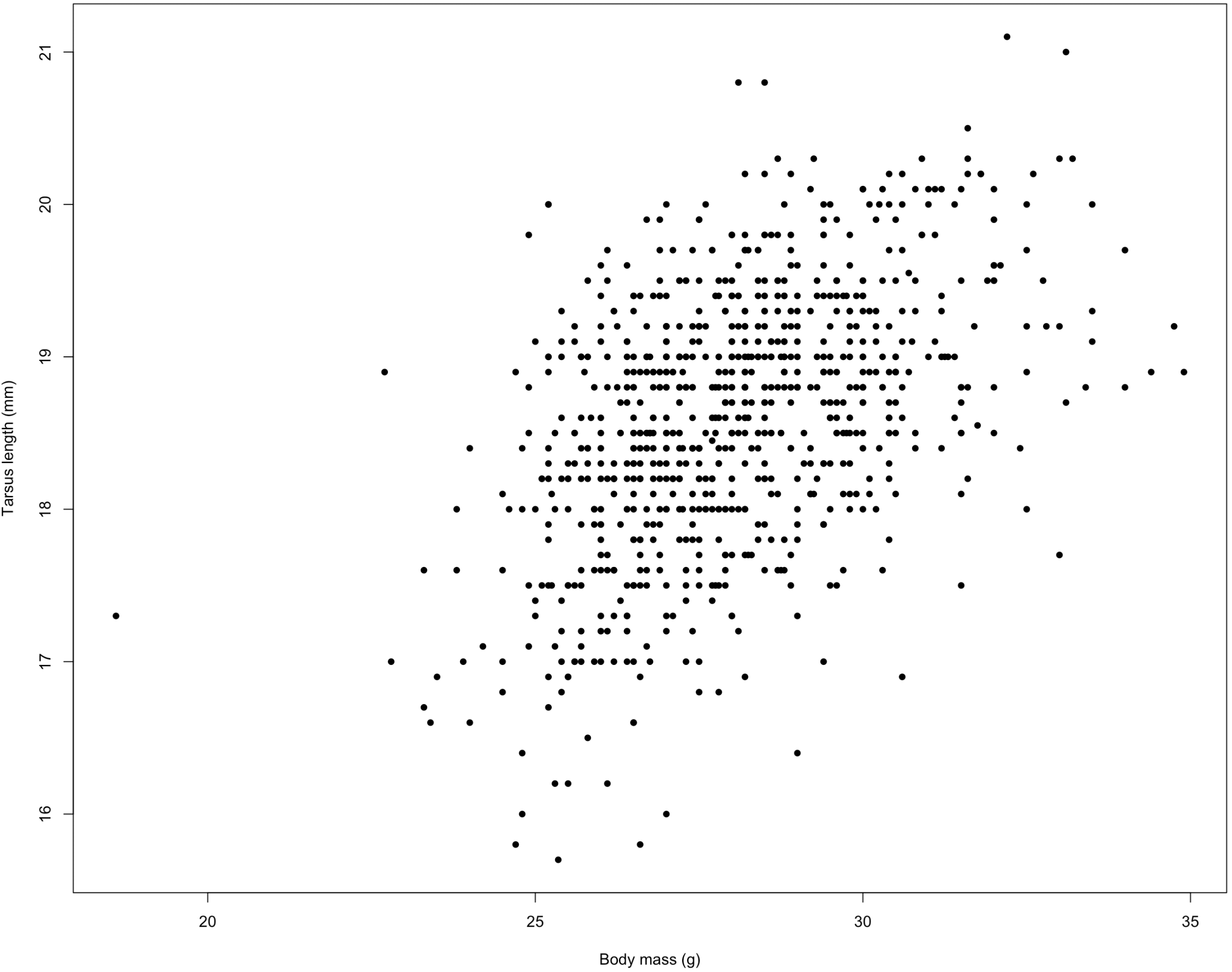

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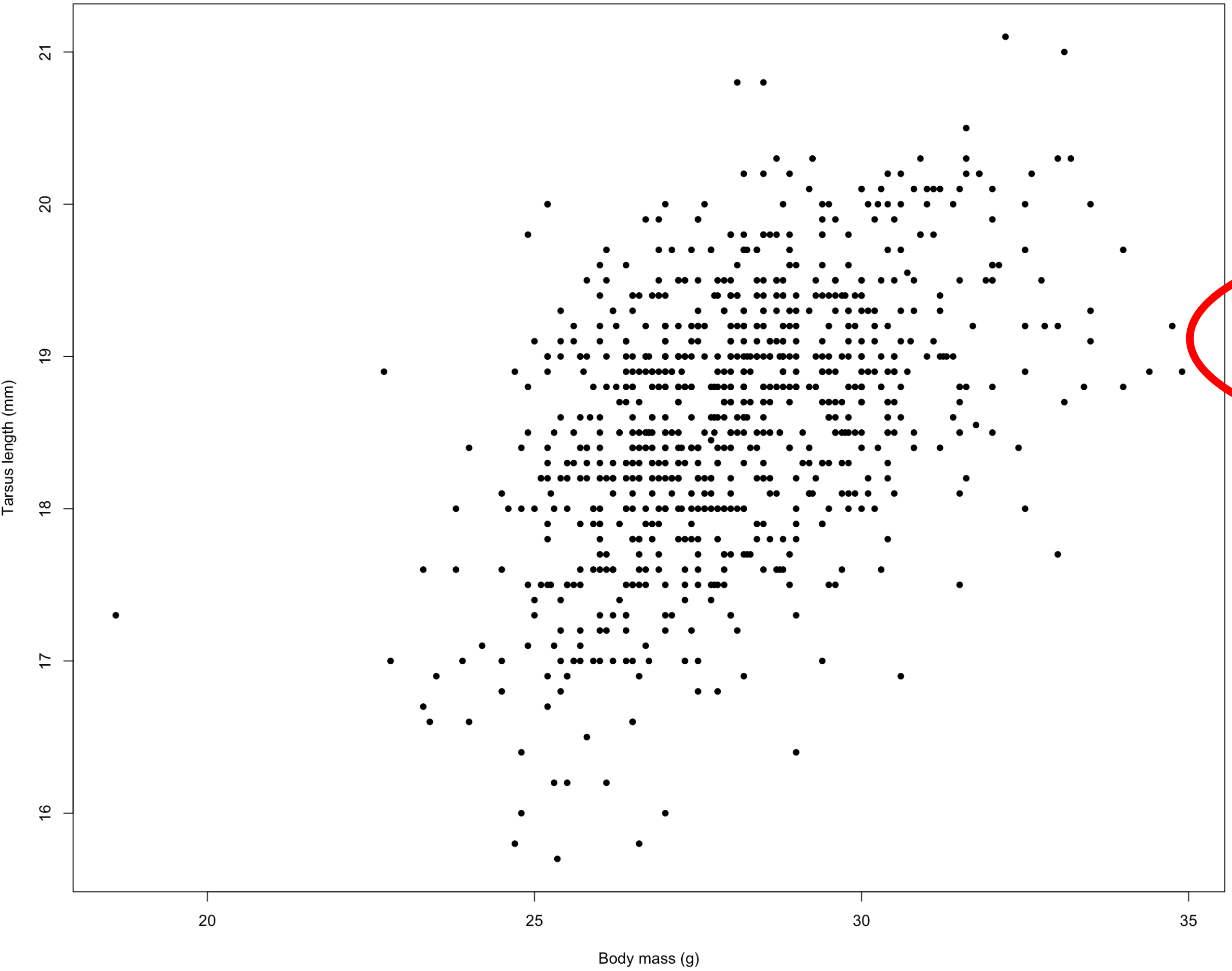


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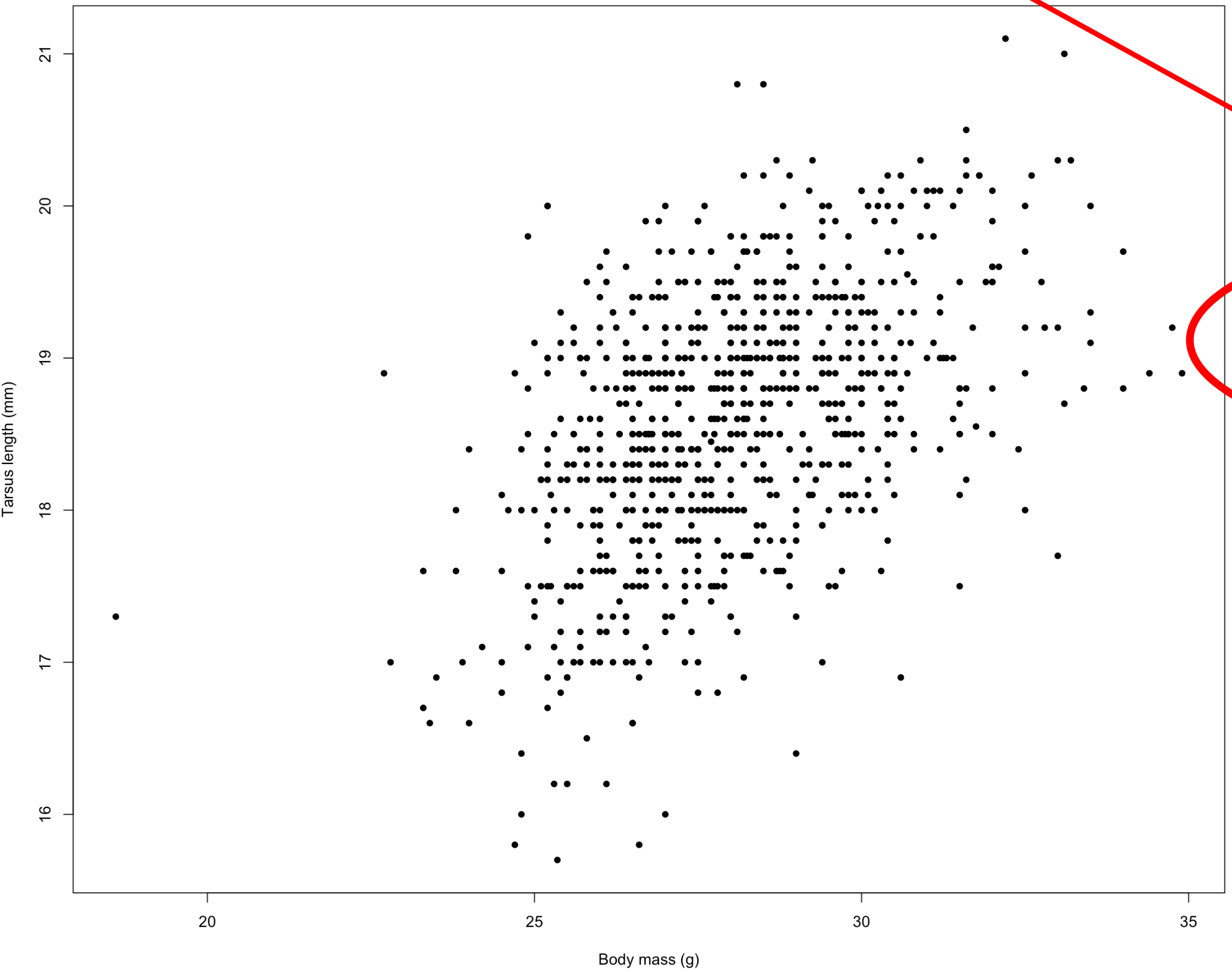


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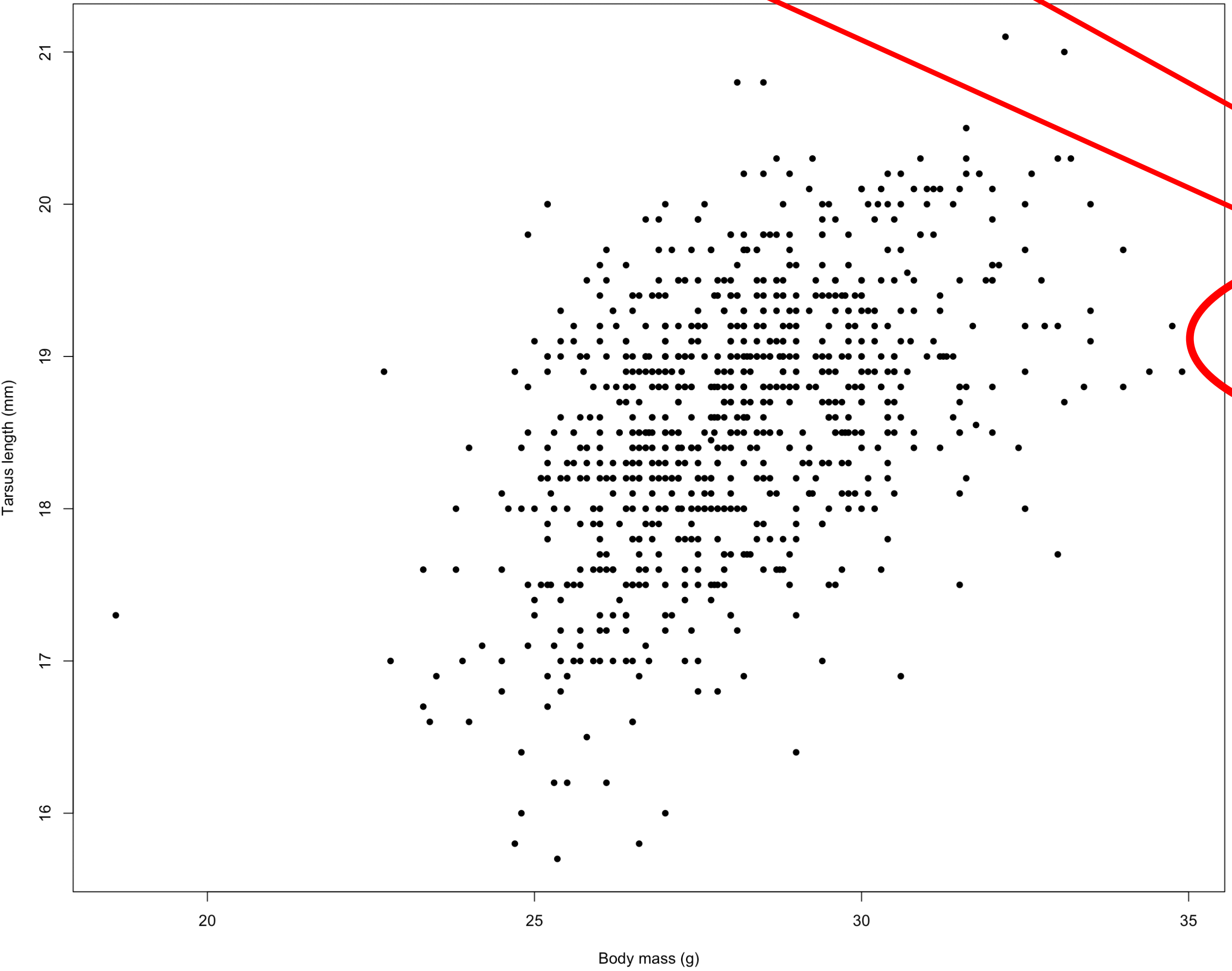
traitMass       28.06    27.92    28.19     1000 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```




```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~idh(trait):units, family=c("gaussian","gaussian"),data=dat)
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 5660.597

 R-structure:  ~idh(trait):units

                 post.mean l-95% CI u-95% CI eff.samp
traitTarsus.units    0.7275   0.6526   0.7913   1000.0
traitMass.units      4.0598   3.6602   4.4507    911.5

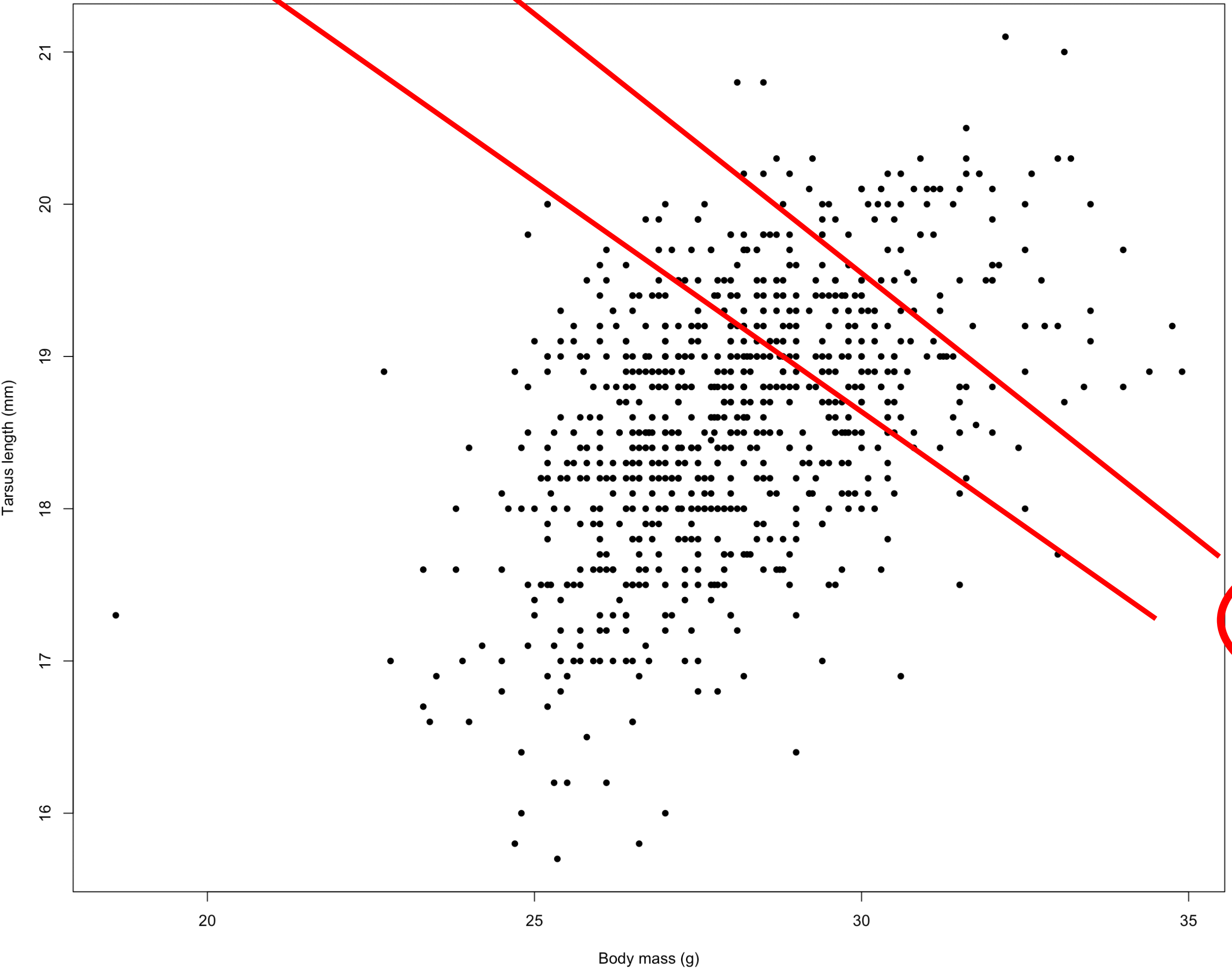
 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.60    18.55    18.66     1000 <0.001 ***
traitMass       28.06    27.92    28.19     1000 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```




```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~idh(trait):units, family=c("gaussian","gaussian"),data=dat)
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 5660.597

 R-structure:  ~idh(trait):units

                 post.mean l-95% CI u-95% CI eff.samp
traitTarsus.units    0.7275   0.6526   0.7913   1000.0
traitMass.units      4.0598   3.6602   4.4507    911.5

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.60    18.55    18.66     1000 <0.001 ***
traitMass       28.06    27.92    28.19     1000 <0.001 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example – Sparrow Size

- (Tarsus,Mass)~1

$$(y_i, z_i) = b_0 + \varepsilon_i$$

$$\begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \\ \dots & \dots \\ y_i & z_i \end{pmatrix}$$

$$\begin{pmatrix} \beta_{0y} \\ \beta_{0z} \end{pmatrix}$$

Mean of tarsus

Mean of mass

$$\begin{pmatrix} \sigma^2_{\varepsilon y} \\ \sigma^2_{\varepsilon z} \end{pmatrix}$$

Variance of tarsus

Variance of mass

Example – Sparrow Size

- (Tarsus,Mass)~1

$$(y_i, z_i) = b_0 + \varepsilon_i$$

$$\begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \\ \ldots & \ldots \\ y_i & z_i \end{pmatrix}$$

$$\begin{pmatrix} \beta_{0y} \\ \beta_{0z} \end{pmatrix}$$

Mean of tarsus

Mean of mass

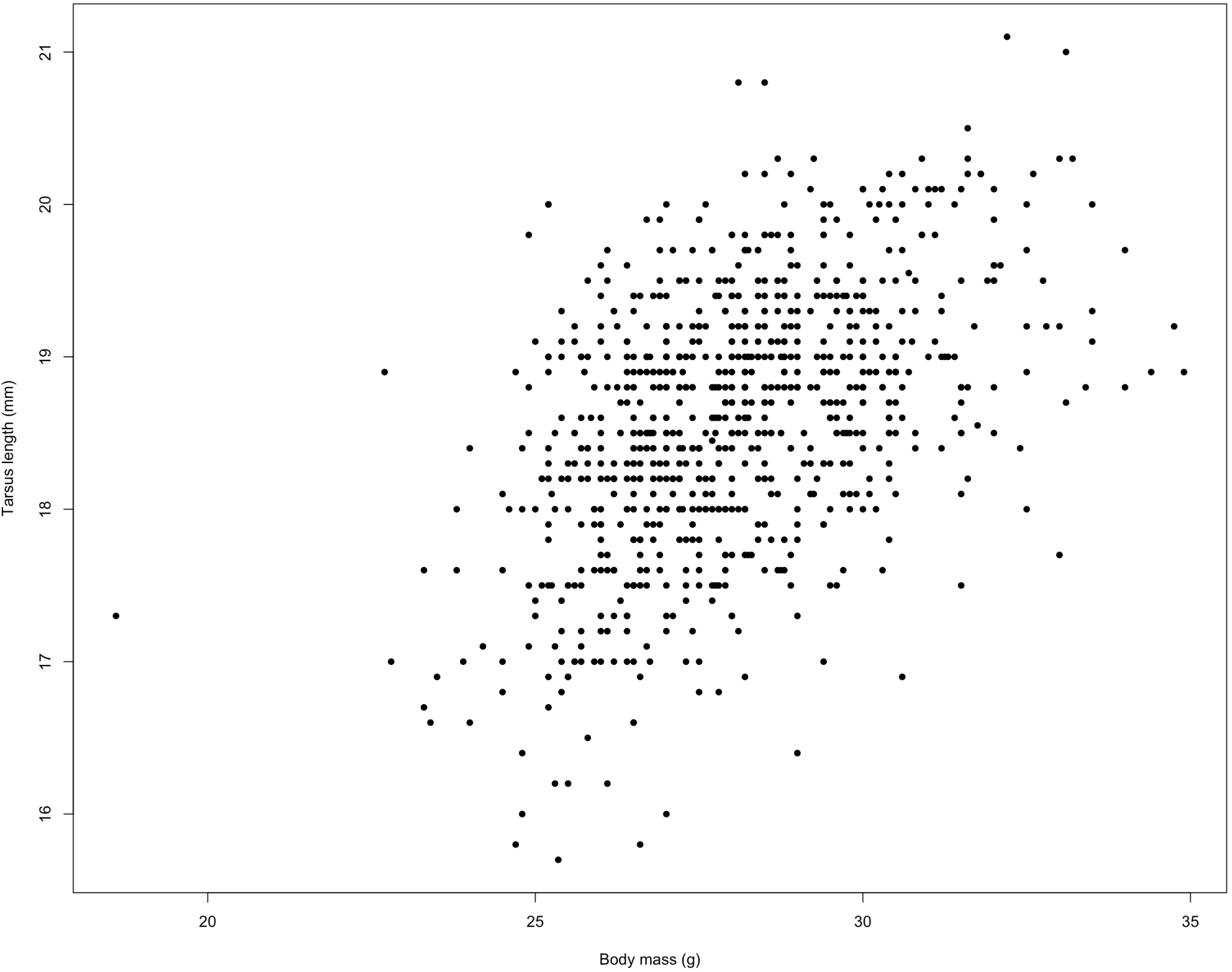
$$\begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```

```
> cov(dat$Tarsus,dat$Mass)
[1] 0.8068064
```



Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```

```
> cov(dat$Tarsus,dat$Mass)
[1] 0.8068064
```

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> summary(m)
```

Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```

```
> cov(dat$Tarsus,dat$Mass)
[1] 0.8068064
```

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1, rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> summary(m)
```

UNSTRUCTURED

Unre**S**tricted

Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```

```
> cov(dat$Tarsus,dat$Mass)
[1] 0.8068064
```

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000 

 DIC: 5453.385 

 R-structure:  ~us(trait):units

                             post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units    0.7303   0.6650   0.8052   1000.0
traitMass:traitTarsus.units      0.8128   0.6786   0.9403   1000.0
traitTarsus:traitMass.units      0.8128   0.6786   0.9403   1000.0
traitMass:traitMass.units        4.0794   3.6691   4.4330    899.3

 Location effects: c(Tarsus, Mass) ~ trait - 1 

            post.mean l-95% CI u-95% CI eff.samp  pMCMC    
traitTarsus     18.60    18.55    18.66     1000 <0.001 ***
traitMass       28.06    27.94    28.20     1170 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Example – Sparrow Size

```
> mean(dat$Tarsus)      > var(dat$Tarsus)
[1] 18.60054            [1] 0.7251402         > cov(dat$Tarsus,dat$Mass)
> mean(dat$Mass)        > var(dat$Mass)       [1] 0.8068064
[1] 28.06075            [1] 4.060524

> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 5453.385

 R-structure:  ~us(trait):units

                            post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units    0.7303   0.6650   0.8052   1000.0
traitMass:traitTarsus.units      0.8128   0.6786   0.9403   1000.0
traitTarsus:traitMass.units      0.8128   0.6786   0.9403   1000.0
traitMass:traitMass.units        4.0794   3.6691   4.4330    899.3

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.60    18.55    18.66     1000 <0.001 ***
traitMass       28.06    27.94    28.20     1170 <0.001 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```

```
> cov(dat$Tarsus,dat$Mass)
[1] 0.8068064
```

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 5453.385

 R-structure:  ~us(trait):units

                              post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units    0.7303   0.6650   0.8052   1000.0
traitMass:traitTarsus.units      0.8128   0.6786   0.9403   1000.0
traitTarsus:traitMass.units      0.8128   0.6786   0.9403   1000.0
traitMass:traitMass.units        4.0794   3.6691   4.4330    899.3

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.60    18.55    18.66     1000 <0.001 ***
traitMass       28.06    27.94    28.20     1170 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Variance-Covariance matrix

Example – Sparrow Size

```
> mean(dat$Tarsus)      > var(dat$Tarsus)
[1] 18.60054            [1] 0.7251402       > cov(dat$Tarsus,dat$Mass)
> mean(dat$Mass)        > var(dat$Mass)     [1] 0.8068064
[1] 28.06075            [1] 4.060524
```

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 5453.385

 R-structure:  ~us(trait):units

                             post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units    0.7303   0.6650   0.8052   1000.0
traitMass:traitTarsus.units      0.8128   0.6786   0.9403   1000.0
traitTarsus:traitMass.units      0.8128   0.6786   0.9403   1000.0
traitMass:traitMass.units        4.0794   3.6691   4.4330    899.3

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.60    18.55    18.66     1000 <0.001 ***
traitMass       28.06    27.94    28.20     1170 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Variance-Covariance matrix

$$\begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

Example – Sparrow Size

```
> mean(dat$Tarsus)
[1] 18.60054
> mean(dat$Mass)
[1] 28.06075
```

```
> var(dat$Tarsus)
[1] 0.7251402
> var(dat$Mass)
[1] 4.060524
```

```
> cov(dat$Tarsus,dat$Mass)
[1] 0.8068064
```

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1,  rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 5453.385

 R-structure:  ~us(trait):units

                            post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units    0.7303   0.6650   0.8052   1000.0
traitMass:traitTarsus.units      0.8128   0.6786   0.9403   1000.0
traitTarsus:traitMass.units      0.8128   0.6786   0.9403   1000.0
traitMass:traitMass.units        4.0794   3.6691   4.4330    899.3

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.60    18.55    18.66     1000 <0.001 ***
traitMass       28.06    27.94    28.20     1170 <0.001 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Variance-Covariance matrix

$$\begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

$$\begin{pmatrix} 0.73\ (0.67\text{–}0.81) & 0.81\ (0.68\text{–}0.94) \\ 0.81\ (0.68\text{–}0.94) & 4.08\ (3.67\text{–}4.43) \end{pmatrix}$$

Example – Sparrow Size - extensions

- (Tarsus,Mass)~1, random = BirdID

$$(y_{i,j}, z_{i,j}) = b_0 + \alpha_j + \varepsilon_i$$

$$\begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \\ \dots & \dots \\ y_i & z_i \end{pmatrix}$$

$$\begin{pmatrix} \beta_{0y} \\ \beta_{0z} \end{pmatrix}$$

Mean of tarsus

Mean of mass

$$\begin{pmatrix} \sigma^2_{\alpha y} \\ \sigma^2_{\alpha z} \end{pmatrix}$$

$$\begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

Example – Sparrow Size

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1, random=~idh(trait):BirdID, rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
```

```
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 3783.797

 G-structure:  ~idh(trait):BirdID

                  post.mean l-95% CI u-95% CI eff.samp
traitTarsus.BirdID    0.6477   0.5381   0.7674     1000
traitMass.BirdID      2.7315   2.1217   3.2869     1000

 R-structure:  ~us(trait):units

                             post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units   0.10717  0.09512  0.11923     1106
traitMass:traitTarsus.units     0.01168 -0.02326  0.04541     1000
traitTarsus:traitMass.units     0.01168 -0.02326  0.04541     1000
traitMass:traitMass.units       1.58148  1.40069  1.77900     1000

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.59    18.50    18.68   1034.3 <0.001 ***
traitMass       28.16    27.96    28.37    908.1 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

```
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000 

 DIC: 3783.797 

 G-structure:  ~idh(trait):BirdID

                 post.mean l-95% CI u-95% CI eff.samp
traitTarsus.BirdID    0.6477   0.5381   0.7674     1000
traitMass.BirdID      2.7315   2.1217   3.2869     1000

 R-structure:  ~us(trait):units

                            post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units   0.10717  0.09512  0.11923     1106
traitMass:traitTarsus.units     0.01168 -0.02326  0.04541     1000
traitTarsus:traitMass.units     0.01168 -0.02326  0.04541     1000
traitMass:traitMass.units       1.58148  1.40069  1.77900     1000

 Location effects: c(Tarsus, Mass) ~ trait - 1 

            post.mean l-95% CI u-95% CI eff.samp  pMCMC    
traitTarsus     18.59    18.50    18.68   1034.3 <0.001 ***
traitMass       28.16    27.96    28.37    908.1 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Example – Sparrow Size - extensions

- (Tarsus,Mass)~1, random = BirdID

$$(y_{i,j}, z_{i,j}) = b_0 + \alpha_j + \varepsilon_i$$

$$\begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \\ \dots & \dots \\ y_i & z_i \end{pmatrix}$$

$$\begin{pmatrix} \beta_{0y} \\ \beta_{0z} \end{pmatrix}$$

Mean of tarsus

Mean of mass

$$\begin{pmatrix} \sigma^2_{\alpha y} \\ \sigma^2_{\alpha z} \end{pmatrix}$$

$$\begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

Example – Sparrow Size - extensions

- (Tarsus,Mass)~1, random = BirdID

$$(y_{i,j}, z_{i,j}) = b_0 + \alpha_j + \varepsilon_i$$

$$\begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \\ \dots & \dots \\ y_i & z_i \end{pmatrix}$$

$$\begin{pmatrix} \beta_{0y} \\ \beta_{0z} \end{pmatrix}$$

Mean of tarsus

Mean of mass

$$\begin{pmatrix} \sigma^2_{\alpha y} & COV(\alpha_y, \alpha_z) \\ COV(\alpha_y, \alpha_z) & \sigma^2_{\alpha z} \end{pmatrix}$$

$$\begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

Example – Sparrow Size

- Hypothesis:
- Sparrows that get a lot of parental care grow to become larger adults

Example – Sparrow Size

```
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1, random=~us(trait):BirdID, rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
```

```
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 3740.01

 G-structure:  ~us(trait):BirdID

                             post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.BirdID    0.6545   0.5457   0.7688     1000
traitMass:traitTarsus.BirdID      0.9069   0.7146   1.1309     1000
traitTarsus:traitMass.BirdID      0.9069   0.7146   1.1309     1000
traitMass:traitMass.BirdID        2.7366   2.2478   3.3672     1000

 R-structure:  ~us(trait):units

                            post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units  0.106649  0.09488  0.11921   1116.0
traitMass:traitTarsus.units   -0.009273 -0.04226  0.03019    993.3
traitTarsus:traitMass.units   -0.009273 -0.04226  0.03019    993.3
traitMass:traitMass.units      1.584112  1.39463  1.78875   1000.0

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.59    18.50    18.69     1105 <0.001 ***
traitMass       28.15    27.94    28.33     1123 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

```
> summary(m)

 Iterations = 3001:12991
 Thinning interval  = 10
 Sample size  = 1000

 DIC: 3740.01

 G-structure:  ~us(trait):BirdID

                             post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.BirdID    0.6545   0.5457   0.7688     1000
traitMass:traitTarsus.BirdID      0.9069   0.7146   1.1309     1000
traitTarsus:traitMass.BirdID      0.9069   0.7146   1.1309     1000
traitMass:traitMass.BirdID        2.7366   2.2478   3.3672     1000

 R-structure:  ~us(trait):units

                            post.mean l-95% CI u-95% CI eff.samp
traitTarsus:traitTarsus.units  0.106649  0.09488  0.11921   1116.0
traitMass:traitTarsus.units   -0.009273 -0.04226  0.03019    993.3
traitTarsus:traitMass.units   -0.009273 -0.04226  0.03019    993.3
traitMass:traitMass.units      1.584112  1.39463  1.78875   1000.0

 Location effects: c(Tarsus, Mass) ~ trait - 1

            post.mean l-95% CI u-95% CI eff.samp  pMCMC
traitTarsus     18.59    18.50    18.69     1105 <0.001 ***
traitMass       28.15    27.94    28.33     1123 <0.001 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

```
> mReduced<-MCMCglmm(c(Tarsus, Mass)~trait-1, random=~us(trait):BirdID, rcov=~idh(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1, random=~us(trait):BirdID, rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE
```

$$\begin{pmatrix} \sigma^2_{\alpha y} & COV(\alpha_y,\alpha_z) \\ COV(\alpha_y,\alpha_z) & \sigma^2_{\alpha z} \end{pmatrix} \quad \begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y,\varepsilon_z) \\ COV(\varepsilon_z,\varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

$$\begin{pmatrix} \sigma^2_{\alpha y} & COV(\alpha_y,\alpha_z) \\ COV(\alpha_y,\alpha_z) & \sigma^2_{\alpha z} \end{pmatrix} \quad \begin{pmatrix} \sigma^2_{\varepsilon y} & 0 \\ 0 & \sigma^2_{\varepsilon z} \end{pmatrix}$$

```
> mReduced<-MCMCglmm(c(Tarsus, Mass)~trait-1, random=~us(trait):BirdID, rcov=~idh(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE)
> m<-MCMCglmm(c(Tarsus, Mass)~trait-1, random=~us(trait):BirdID, rcov=~us(trait):units, family=c("gaussian","gaussian"),data=dat, verbose=FALSE
```

```
> mReduced$DIC
[1] 3739.497
> m$DIC
[1] 3740.01
>
```

$$\begin{pmatrix} \sigma^2_{\alpha y} & COV(\alpha_y, \alpha_z) \\ COV(\alpha_y, \alpha_z) & \sigma^2_{\alpha z} \end{pmatrix} \quad \begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

$$\begin{pmatrix} \sigma^2_{\alpha y} & COV(\alpha_y, \alpha_z) \\ COV(\alpha_y, \alpha_z) & \sigma^2_{\alpha z} \end{pmatrix} \quad \begin{pmatrix} \sigma^2_{\varepsilon y} & 0 \\ 0 & \sigma^2_{\varepsilon z} \end{pmatrix}$$

Example – Sparrow Size - extensions

- (Tarsus,Mass)~Sex, random = BirdID

$$(y_{i,j}, z_{i,j}) = b_0 + b_0 X_1 + \alpha_j + \varepsilon_i$$

$$\begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \\ \dots & \dots \\ y_i & z_i \end{pmatrix}$$

Intercept of tarsus

Intercept of mass

$$\begin{pmatrix} \beta_{0y} \\ \beta_{0z} \end{pmatrix} \quad \begin{pmatrix} \beta_{1y} \\ \beta_{1z} \end{pmatrix}$$

Slope of tarsus

Slope of mass

$$\begin{pmatrix} \sigma^2_{\alpha y} & COV(\alpha_y, \alpha_z) \\ COV(\alpha_y, \alpha_z) & \sigma^2_{\alpha z} \end{pmatrix}$$

$$\begin{pmatrix} \sigma^2_{\varepsilon y} & COV(\varepsilon_y, \varepsilon_z) \\ COV(\varepsilon_z, \varepsilon_y) & \sigma^2_{\varepsilon z} \end{pmatrix}$$

More extensions

- GLMMs – mixed models with different error distributions
  - Poisson
  - Binomial
  - Mixed!

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More extensions

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  - Binomial
- Multivariate models (with more than 2 responses)
- GAM models with exotic link functions
- GLMMs with relatedness matrix as random effect (= animal models)
- GLMMS with phylogeny as random effect
- GLMMS for meta-analysis (phylogeny or other random effects)

EVEN! more extensions

- ZIP – Zero-inflated models
- ZAP – Zero altered models
- ZAT – Zero-truncated models

HO – bivariate models and others

- Use this time to keep us with other HO’s
- Only do bivariate HO once you feel comfortable with the other HO’s