

## 1 The Differentials

1. Find  $dy$  and  $\Delta y$  for the given values of  $x$  and  $\Delta x$ .
  - 1.1.  $y = x^2$ ,  $x = 2$ , and  $\Delta x = 0.5$
  - 1.2.  $y = x^3$ ,  $x = 2$ , and  $\Delta x = 0.5$
  - 1.3.  $y = \sqrt[3]{x}$ ,  $x = 8$ , and  $\Delta x = 1$
  - 1.4.  $y = \sqrt{x}$ ,  $x = 4$ , and  $\Delta x = 1$
2. Find (a)  $\Delta y$ , (b)  $dy$ , (c)  $\Delta y - dy$ .
  - 2.1.  $y = x^2 - 3x$ ,  $x = 2$ , and  $\Delta x = 0.03$
  - 2.2.  $y = x^2 - 3x$ ,  $x = -1$ , and  $\Delta x = 0.02$
  - 2.3.  $y = \frac{1}{x}$ ,  $x = -2$ , and  $\Delta x = -0.1$
  - 2.4.  $y = \frac{1}{x}$ ,  $x = 3$ , and  $\Delta x = -0.2$
  - 2.5.  $y = x^3 + 1$ ,  $x = 1$ , and  $\Delta x = -0.5$
  - 2.6.  $y = x^3 + 1$ ,  $x = -1$ , and  $\Delta x = 0.1$
3. Find  $dy$ .
  - 3.1.  $y = (3x^2 - 2x + 1)^3$
  - 3.2.  $y = \frac{3x}{x^2 + 2}$
  - 3.3.  $y = x^2 \sqrt{2x + 3}$
  - 3.4.  $y = \sqrt{4 - x^2}$
  - 3.5.  $y = \frac{2 + \cos(x)}{2 - \sin(x)}$
  - 3.6.  $y = \tan^2(x) \sec^2(x)$
4. Solve the following problems.
  - 4.1. The measurement of an edge of a cube is found to be 15 cm with a possible error of 0.01 cm. Use differentials to find the approximate error in computing from this measurement: (a) the volume; (b) the area of one of the faces.
  - 4.2. An open cylindrical tank is to have an outside coating of thickness 2 cm. If the inner radius is 6 m and the altitude is 10 m, find by differentials the approximate amount of coating material to be used.
  - 4.3. A burn on a person's skin is in the shape of a circle. Use differentials to find the approximate decrease in the area of the burn when the radius decreases from 1 cm to 0.8 cm.

- 4.4. A tumor in a person's body is spherical in shape. Use differentials to find the approximate increase in the volume of the tumor when the radius increases from 1.5 cm to 1.6 cm.

## Answer Key

1. Find  $dy$  and  $\Delta y$  for the given values of  $x$  and  $\Delta x$ .
  - 1.1.  $dy = 2$ ,  $\Delta y = 2.25$
  - 1.2.  $dy = 6$ ,  $\Delta y = 7.625$
  - 1.3.  $dy = \frac{1}{12} \approx 0.083$ ,  $\Delta y = \sqrt[3]{9} - 2 \approx 0.080$
  - 1.4.  $dy = 0.25$ ,  $\Delta y = \sqrt{5} - \sqrt{4} \approx 0.236$
2. Find (a)  $\Delta y$ , (b)  $dy$ , (c)  $\Delta y - dy$ .
  - 2.1. (a) 0.0309, (b) 0.03, (c) 0.0009
  - 2.2. (a) -0.0996, (b) -0.1, (c) 0.0004
  - 2.3. (a)  $\frac{1}{42} \approx 0.0238$ , (b)  $\frac{1}{40} = 0.025$ , (c)  $-\frac{1}{840} \approx -0.0012$
  - 2.4. (a)  $\frac{1}{42} \approx 0.0238$ , (b)  $\frac{1}{45} = 0.022$ , (c)  $\frac{1}{630} \approx -0.0016$
  - 2.5. (a) -0.875, (b) -1.5, (c) 0.625
  - 2.6. (a) 0.271, (b) 0.3, (c) -0.029
3. Find  $dy$ .
  - 3.1.  $dy = 3(3x^2 - 2x + 1)^2(6x - 2) dx$
  - 3.2.  $dy = \frac{3(2-x^2)}{(x^2+2)^2} dx$
  - 3.3.  $dy = \frac{x(5x+6)}{(2x+3)^{1/2}} dx$
  - 3.4.  $dy = \frac{-x}{\sqrt{4-x^2}} dx$
  - 3.5.  $dy = \frac{1-2\sin(x)+2\cos(x)}{(2-\sin(x))^2} dx$
  - 3.6.  $dy = 2\tan(x)\sec^2(x)x(2\tan^2(x)+1) dx$
4. Solve the following problems.
  - 4.1. (a)  $6.75 \text{ cm}^3$ , (b)  $0.3 \text{ cm}^2$
  - 4.2.  $\frac{12}{5}\pi \text{ m}^3$
  - 4.3.  $0.4\pi \text{ cm}^2$
  - 4.4.  $0.9\pi \text{ cm}^3$

## 2 Antidifferentiation: Indefinite Integration

1.  $\int (x^2 + x^{-2}) dx$
2.  $\int (\sqrt{x^3} + \sqrt[3]{x^2}) dx$
3.  $\int (x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2) dx$
4.  $\int (y^3 + 1.8y^2 - 2.4y) dy$
5.  $\int (1 - t)(2 + t^2) dt$
6.  $\int v(v^2 + 2)^2 dv$
7.  $\int \frac{x^3 - 2\sqrt{x}}{x} dx$
8.  $\int (\theta - \csc \theta \cot \theta) d\theta$
9.  $\int \sec t(\sec t + \tan t) dt$
10.  $\int (1 + \tan^2 \alpha) d\alpha$
11.  $\int \frac{\sin 2x}{\sin x} dx$

## Answer Key

1.  $\int (x^2 + x^{-2}) dx$

$$\begin{aligned} &= \frac{x^3}{3} + \frac{x^{-1}}{-1} + C \\ &= \frac{1}{3}x^3 - \frac{1}{x} + C \end{aligned}$$

2.  $\int (\sqrt{x^3} + \sqrt[3]{x^2}) dx$

$$\begin{aligned} &= \int (x^{3/2} + x^{2/3}) dx \\ &= \frac{x^{5/2}}{5/2} + \frac{x^{5/3}}{5/3} + C \\ &= \frac{2x^{5/2}}{5} + \frac{3x^{5/3}}{5} + C \end{aligned}$$

3.  $\int (x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2) dx$

$$\begin{aligned} &= \frac{x^5}{5} - \frac{1}{2} \frac{x^4}{4} + \frac{1}{4} \frac{x^2}{2} - 2x + C \\ &= \frac{x^5}{5} - \frac{x^4}{8} + \frac{x^2}{8} - 2x + C \end{aligned}$$

4.  $\int (y^3 + 1.8y^2 - 2.4y) dy$

$$\begin{aligned} &= \frac{y^4}{4} + 1.8 \frac{y^3}{3} - 2.4 \frac{y^2}{2} + C \\ &= \frac{y^4}{4} + 0.6y^3 - 1.2y^2 + C \end{aligned}$$

5.  $\int (1 - t)(2 + t^2) dt$

$$\begin{aligned} &= (2 - 2t + t^2 - t^3) dt \\ &= 2t - \frac{2t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + C \\ &= 2t - t^2 + \frac{t^3}{3} - \frac{t^4}{4} + C \end{aligned}$$

6.  $\int v(v^2 + 2)^2 dv$

$$\begin{aligned} &= \int v(v^4 + 4v^2 + 4) dv \\ &= \int (v^5 + 4v^3 + 4v) dv \\ &= \frac{v^6}{6} + \frac{4v^4}{4} + \frac{4v^2}{2} + C \\ &= \frac{v^6}{6} + v^4 + 2v^2 + C \end{aligned}$$

$$7. \int \frac{x^3 - 2\sqrt{x}}{x} dx$$

$$\begin{aligned} &= \int \left( \frac{x^3}{x} - \frac{2x^{1/2}}{x} \right) dx \\ &= \int (x^2 - 2x^{-1/2}) dx \\ &= \frac{x^3}{3} - \frac{2x^{1/2}}{1/2} + C \\ &= \frac{x^3}{3} - 4\sqrt{x} + C \end{aligned}$$

$$8. \int (\theta - \csc \theta \cot \theta) d\theta$$

$$= \frac{\theta^2}{2} + \csc \theta + C$$

$$9. \int \sec t (\sec t + \tan t) dt$$

$$\begin{aligned} &= \int (\sec^2 t + \sec t \tan t) dt \\ &= \tan t + \sec t + C \end{aligned}$$

$$10. \int (1 + \tan^2 \alpha) d\alpha$$

$$\begin{aligned} &= \int \sec^2 \alpha d\alpha \\ &= \tan \alpha + C \end{aligned}$$

$$11. \int \frac{\sin 2x}{\sin x} dx$$

$$\begin{aligned} &= \int \frac{2 \sin x \cos x}{\sin x} dx \\ &= \int 2 \cos x dx \\ &= 2 \sin x + C \end{aligned}$$

### 3 Integration by Substitution

1.  $\int x \sin(x^2) dx$
2.  $\int x^2(x^3 + 5)^9 dx$
3.  $\int (3x - 2)^{20} dx$
4.  $\int (3t + 2)^{2.4} dt$
5.  $\int (x + 1)\sqrt{2x + x^2} dx$
6.  $\int \frac{x}{(x^2+1)^2} dx$
7.  $\int \sin \pi t dt$
8.  $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}} dx$
9.  $\int \sec 2\theta \tan 2\theta d\theta$
10.  $\int \frac{\cos \sqrt{t}}{\sqrt{t}} dt$
11.  $\int \sqrt{x} \sin(1 + x^{\frac{3}{2}}) dx$
12.  $\int \cos \theta \sin^6 \theta d\theta$
13.  $\int (1 + \tan \theta)^5 \sec^2 \theta d\theta$
14.  $\int \frac{z^2}{\sqrt[3]{1+z^3}} dz$
15.  $\int \frac{\cos x}{\sin^2 x} dx$
16.  $\int \sqrt{\cot x} \csc^2 x dx$
17.  $\int \frac{\cos(\frac{\pi}{x})}{x^2} dx$
18.  $\int \frac{dt}{\cos^2 t \sqrt{1+\tan t}}$
19.  $\int \sec^3 x \tan x dx$
20.  $\int \sin t \sec^2(\cos t) dt$
21.  $\int \frac{x^2}{\sqrt{1-x}} dx$
22.  $\int \frac{x}{\sqrt[4]{x+2}} dx$

## Answer Key

1.  $\int x \sin(x^2) dx$

$$\begin{aligned} &= \int \sin(x^2) x dx \\ &= \frac{1}{2} \int \sin u du \\ &= \frac{1}{2} (-\cos u) + C \\ &= -\frac{1}{2} \cos(x^2) + C \end{aligned}$$

$$u = x^2$$

$$du = 2x dx$$

2.  $\int x^2(x^3 + 5)^9 dx$

$$\begin{aligned} &= \int (x^3 + 5)^9 x^2 dx \\ &= \frac{1}{3} \int u^9 du \\ &= \frac{1}{3} \left( \frac{u^{10}}{10} \right) + C \\ &= \frac{u^{10}}{30} + C \\ &= \frac{(x^3 + 5)^{10}}{30} + C \end{aligned}$$

$$u = x^3 + 5$$

$$du = 3x^2$$

3.  $\int (3x - 2)^{20} dx$

$$\begin{aligned} &= \frac{1}{3} \int u^{20} du \\ &= \frac{1}{3} \left( \frac{u^{21}}{21} \right) + C \\ &= \frac{u^{21}}{63} + C \\ &= \frac{(3x - 2)^{21}}{63} + C \end{aligned}$$

$$u = 3x - 2$$

$$du = 3 dx$$



$$4. \int (3t + 2)^{2.4} dt$$

$$\begin{aligned} &= \frac{1}{3} \int u^{2.4} du \\ &= \frac{1}{3} \left( \frac{u^{3.4}}{3.4} \right) + C \\ &= \frac{u^{3.4}}{10.2} + C \end{aligned}$$

$$u = 3t + 2$$

$$du = 3 dt$$

$$5. \int (x + 1)\sqrt{2x + x^2} dx$$

$$\begin{aligned} &= \int (x + 1)(2x + x^2)^{1/2} dx \\ &= \frac{1}{2} \int u^{1/2} dx \\ &= \frac{1}{2} \left( \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{3} u^{3/2} + C \\ &= \frac{(2x + x^2)^{3/2}}{3} + C \end{aligned}$$

$$u = 2x + x^2$$

$$du = 2 + 2x dx$$

$$6. \int \frac{x}{(x^2+1)^2} dx$$

$$\begin{aligned} &= \int x(x^2 + 1)^{-2} dx \\ &= \int (x^2 + 1)^{-2} x dx \\ &= \frac{1}{2} \int u^{-2} du \\ &= \frac{1}{2} (-u^{-1}) + C \\ &= -\frac{1}{(2x^2 + 2)} + C \end{aligned}$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$7. \int \sin \pi t dt$$

$$8. \int \frac{a+bx^2}{\sqrt{3ax+bx^3}} dx$$

$$\begin{aligned}
&= \int (a + bx^2)(3ax + bx^3)^{-1/2} dx & u = 3ax + bx^3 \\
&= \frac{1}{3} \int u^{-1/2} du & du = 3a + 3bx^2 dx \\
&= \frac{1}{3} (2u^{1/2}) + C \\
&= \frac{2\sqrt{3ax + bx^3}}{3} + C
\end{aligned}$$

9.  $\int \sec 2\theta \tan 2\theta d\theta$

10.  $\int \frac{\cos \sqrt{t}}{\sqrt{t}} dt$

11.  $\int \sqrt{x} \sin(1 + x^{\frac{3}{2}}) dx$

12.  $\int \cos \theta \sin^6 \theta d\theta$

13.  $\int (1 + \tan \theta)^5 \sec^2 \theta d\theta$

14.  $\int \frac{z^2}{\sqrt[3]{1+z^3}} dz$

$$\begin{aligned}
&= \int (1 + z^3)^{-1/3} z^2 dz & u = 1 + z^3 \\
&= \frac{1}{3} \int u^{-1/3} du & du = 3z^2 dz \\
&= \frac{1}{3} \left( \frac{3}{2} u^{2/3} \right) + C \\
&= \frac{u^{2/3}}{2} + C \\
&= \frac{\sqrt[3]{(1 + z^3)^2}}{2} + C
\end{aligned}$$

15.  $\int \frac{\cos x}{\sin^2 x} dx$

16.  $\int \sqrt{\cot x} \csc^2 x dx$

17.  $\int \frac{\cos(\frac{\pi}{x})}{x^2} dx$

18.  $\int \frac{dt}{\cos^2 t \sqrt{1 + \tan t}}$

19.  $\int \sec^3 x \tan x dx$

20.  $\int \sin t \sec^2(\cos t) dt$

21.  $\int \frac{x^2}{\sqrt{1-x}} dx$

22.  $\int \frac{x}{\sqrt[4]{x+2}} dx$

## 4 Separable Differential Equations

1.