1 Derivatives of Elementary Transcendental Functions

DEFINITION. In general, the term <mark>transcendental</mark> means non-algebraic. A transcendental function is a function that is not expressible as a finite combination of the algebraic opperations of addition, subtraction, multiplication, division, raising to a power, and extracting a root. An example includes the function $\log x$, $\sin x$, $\cos x$, e^x, and any functions containing them.

2 The Natural Logarithmic Function

DEFINITION. The function defined by

$$f(x) = \log_e x = \ln x$$

$(x > 0, e \approx 2.718281\dots)$ is called the <mark>natural logarithmic function</mark>.

NOTE. The equation $y = \ln x$ is equivalent to $e^y = x$.

2.1 Logarithmic Properties

DEFINITION. If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln 1 = 0$; if $x > 1$, then $y = \ln x > 0$, else vice versa.
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

2.2 Definition of the Natural Logarithmic Function

DEFINITION. The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt, x > 0$$

The domain of the natural logarithmic function is the set of all positive real numbers.

2.3 Derivative of the Natural Logarithmic Function

DEFINITION. Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, u > 0$

EXAMPLE 2.3.1. Solve for the equation $\frac{d}{dx}[\ln 2x]$.

$$= \frac{1}{2x}$$

EXAMPLE 2.3.2. Solve for the equation $\frac{d}{dx}[\ln(x^2 + 1)]$.

$$= \frac{1}{u}\frac{du}{dx} \qquad\qquad u = x^2 + 1$$
$$= \frac{1}{x^2 + 1}\frac{2xdx}{dx} \qquad\qquad du = 2x\,dx$$
$$= \frac{2x}{x^2 + 1}$$

EXAMPLE 2.3.3. Solve for the equation $\frac{d}{dx}[x \ln x]$.

$$= x \cdot \frac{1}{x}$$
$$= 1$$

EXAMPLE 2.3.4. Solve for the equation $\frac{d}{dx}[(\ln x)^3]$.

$$= 3(\ln x)^2 \cdot \frac{1}{x}$$
$$= \frac{3(\ln x)^2}{x}$$

EXAMPLE 2.3.5. Solve for the equation $y = \ln[(4x^2 + 3)(2x - 1)]$.

$$y = \ln(4x^2 + 3) + \ln(2x - 1)$$
$$\frac{dy}{dx} = \frac{8x}{4x^2 + 3} + \frac{2}{2x - 1}$$

EXAMPLE 2.3.6. Solve for the equation $y = \ln\left(\frac{x}{x+1}\right)$.

$$y = \ln(x)(x + 1)^{-1}$$
$$y = \ln(x) - \ln(x + 1)$$
$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x + 1}$$

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EXAMPLE 2.3.7. By implicit differentiation, find the dy/dx of $\ln\left(\frac{x}{y}\right) + xy = 1$.

$$\ln(x) - \ln(y) + xy = 1 \qquad\qquad f(y) = f(y(x))$$

$$\frac{d}{dx}\left(\ln(x) - \ln(y) + xy\right) = 1 \qquad\qquad = \ln y$$

$$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \qquad\qquad = \ln[y(x)]$$

$$\frac{dy}{dx}\left(x - \frac{1}{y}\right) = -y - \frac{1}{x} \qquad\qquad = \frac{1}{y}\cdot y' = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \left(\frac{-xy-1}{x}\right)\left(\frac{y}{xy-1}\right)$$

$$\frac{dy}{dx} = \frac{-xy^2 - y}{x^2y - x}$$

EXAMPLE 2.3.8. Solve for the equation $\ln(x+y) - \ln(x-y) = 4$.

3 Logarithmic Differentiation

DEFINITION. It is sometimes convenient to use logarithms as aids in differentiating non-logarithmic functions. This procedure is called <mark>logarithmic differentiation</mark>. This process uses the properties of natural logarithm to simplify the work involved in differentiating complicated expressions containing products, quotients, and powers.

EXAMPLE 3.0.1. Find the $\frac{dy}{dx}$ of the equation $y = \frac{(\sin x)^2(x^3+1)^4}{(x+3)^8}$.

$$\ln(y) = \ln\left[\frac{(\sin x)^2(x^3+1)^4}{(x+3)^8}\right]$$

$$\ln(y) = \ln(\sin x)^2 + \ln(x^3+1)^4 - \ln(x+3)^8$$

$$\ln(y) = 2\ln(\sin x) + 4\ln(x^3+1) - 8\ln(x+3)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\cos x}{\sin x} + \frac{4(3x^2)}{x^3+1} - \frac{8}{x+3}$$

$$\frac{dy}{dx} = y\left[\frac{2\cos x}{\sin x} + \frac{12x^2}{x^3+1} - \frac{8}{x+3}\right]$$

$$\frac{dy}{dx} = \frac{(\sin x)^2(x^3+1)^4}{(x+3)^8}\left[2\cot x + \frac{12x^2}{x^3+1} - \frac{8}{x+3}\right]$$

NOTE. The equation $\ln a = \ln b$ is equivalent to $a = b$.

EXAMPLE 3.0.2. Solve for the equation $y = x^3\sqrt{5-9x}$.

EXAMPLE 3.0.3. Solve for the equation $y = \frac{x^2(6+3x)^4}{\sqrt[3]{9-x^2}}$.

EXAMPLE 3.0.4. Solve for the equation $y = x^{\sin x}$.

$$\ln(y) = \ln(x^{\sin x})$$
$$\ln(y) = \sin x \ln(x)$$
$$\frac{1}{y}\frac{dy}{dx} = (\sin x)(\frac{1}{x}) + (\cos x)(\ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \cos x \ln(x)$$
$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \ln(x)\right]$$
$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \ln(x)\right]$$

EXAMPLE 3.0.5. Solve for the equation $y = \frac{\sqrt[3]{x+1}}{(x+2)\sqrt{x+3}}$.

EXAMPLE 3.0.6. Solve for the equation $y = \frac{(x-2)^2}{\sqrt{x^2+1}}$.