1 The Differentials

DEFINITION. These provide us with a way of estimating the amount a function changes as a result of a small change in input values. The equation

$$\triangle y \approx dy$$

can and will only be considered if $\triangle x$ is "close enough". The approximation of the equation becomes better as $\triangle x$ becomes smaller.

NOTE. We equate $\triangle x = dx$.

1.1 The Differential of the Independent Variable

DEFINITION. If the function f is definited by the equation $y = f(x)$, then the differential of y, denoted by dy, is given by

$$dy = f'(x)\,dx \longrightarrow f'(x) = \frac{dy}{dx}$$

where x is any number in the domain of f' and $\triangle x$ is an arbitrary increment of x.

EXAMPLE 1.1.1. Find dy for $y = (x^3 + 5x - 1)^{2023}$.

$$f'(x) = 2023(x^3 + 5x - 1)^{2022}(3x^2 + 5)$$

$$\therefore dy = 2023(x^3 + 5x - 1)(3x^2 + 5x)\,dx$$

EXAMPLE 1.1.2. Find the differential dy of the function $y = 4x^2 + x + 3$.

$$(8x + 1)\,dx$$

EXAMPLE 1.1.3. Find the differential dy of the function $y = \cos(x)$.

$$dy = -\sin(x)\,dx$$

NOTE. The following figures represent the corresponding derivatives of trigonometric identities or functions.

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$

EXAMPLE 1.1.4. Compare the values of $\triangle y$ and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

$$\begin{aligned} x &= 2 \\ x + \triangle x &= 2.05 \\ \triangle x &= 0.05 \\ dx &= 0.05 \end{aligned} \qquad \begin{aligned} \triangle y &= f(x + \triangle x) - f(x) \\ &= f(2.05) - f(2) \\ &= 0.7176 \end{aligned} \qquad \begin{aligned} dy &= f'(x)\,dx \\ &= (3x^2 + 2x - 2)\,dx \\ &= (3(2)^2 + 2(2) - 2)(0.05) \\ &= 0.7 \end{aligned}$$

NOTE. The final equation utilized our solution at $x = 2$, $\triangle x = dx = 0.05$. Observe the approximation of $\triangle y \approx dy$ becomes better as $\triangle x$ becomes smaller.

Practice Exercises

1. Find dy and $\triangle y$ for the given values of x and $\triangle x$.

    1.1. $y = x^2$, $x = 2$, and $\triangle x = 0.5$

    1.2. $y = x^3$, $x = 2$, and $\triangle x = 0.5$

    1.3. $y = \sqrt[3]{x}$, $x = 8$, and $\triangle x = 1$

    1.4. $y = \sqrt{x}$, $x = 4$, and $\triangle x = 1$

2. Find (a) $\triangle y$, (b) dy, (c) $\triangle y - dy$.

    2.1. $y = x^2 - 3x$, $x = 2$, and $\triangle x = 0.03$

    2.2. $y = x^2 - 3x$, $x = -1$, and $\triangle x = 0.02$

    2.3. $y = \frac{1}{x}$, $x = -2$, and $\triangle x = -0.1$

    2.4. $y = \frac{1}{x}$, $x = 3$, and $\triangle x = -0.2$

    2.5. $y = x^3 + 1$, $x = 1$, and $\triangle x = -0.5$

    2.6. $y = x^3 + 1$, $x = -1$, and $\triangle x = 0.1$

3. Find dy.

    3.1. $y = \left(3x^2 - 2x + 1\right)^3$

    3.2. $y = \frac{3x}{x^2+2}$

    3.3. $y = x^2\sqrt{2x+3}$

    3.4. $y = \sqrt{4 - x^2}$

    3.5. $y = \frac{2+\cos(x)}{2-\sin(x)}$

    3.6. $y = \tan^2(x)\sec^2(x)$

4. Solve the following problems.

    4.1. The measurement of an edge of a cube is found to be 15 cm with a possible error of 0.01 cm. Use differentials to find the approximate error in computing from this measurement: (a) the volume; (b) the area of one of the faces.

    4.2. An open cylindrical tank is to have an outside coating of thickness 2 cm. If the inner radius is 6 m and the altitude is 10 m, find by differentials the approximate amount of coating material to be used.

    4.3. A burn on a person's skin is in the shape of a circle. Use differentials to find the approximate decrease in the area of the burn when the radius decreases from 1 cm to 0.8 cm.

    4.4. A tumor in a person's body is spherical in shape. Use differentials to find the approximate increase in the volume of the tumor when the radius increases from 1.5 cm to 1.6 cm.

Answer Key

1. Find dy and $\triangle y$ for the given values of x and $\triangle x$.

    1.1. $dy = 2$, $\triangle y = 2.25$

    1.2. $dy = 6$, $\triangle y = 7.625$

    1.3. $dy = \frac{1}{12} \approx 0.083$, $\triangle y = \sqrt[3]{9} - 2 \approx 0.080$

    1.4. $dy = 0.25$, $\triangle y = \sqrt{5} - \sqrt{4} \approx 0.236$

2. Find (a) $\triangle y$, (b) dy, (c) $\triangle y - dy$.

    2.1. (a) 0.0309, (b) 0.03, (c) 0.0009

    2.2. (a) -0.0996, (b) -0.1, (c) 0.0004

    2.3. (a) $\frac{1}{42} \approx 0.0238$, (b) $\frac{1}{40} = 0.025$, (c) $-\frac{1}{840} \approx -0.0012$

    2.4. (a) $\frac{1}{42} \approx 0.0238$, (b) $\frac{1}{45} = 0.022$, (c) $\frac{1}{630} \approx -0.0016$

    2.5. (a) -0.875, (b) -1.5, (c) 0.625

    2.6. (a) 0.271, (b) 0.3, (c) -0.029

3. Find dy.

    3.1. $dy = 3(3x^2 - 2x + 1)^2(6x - 2)\,dx$

    3.2. $dy = \frac{3(2-x^2)}{(x^2+2)^2}\,dx$

    3.3. $dy = \frac{x(5x+6)}{(2x+3)^{1/2}}\,dx$

    3.4. $dy = \frac{-x}{\sqrt{4-x^2}}\,dx$

    3.5. $dy = \frac{1-2\sin(x)+2\cos(x)}{(2-\sin(x))^2}\,dx$

    3.6. $dy = 2\tan(x)\sec^2(x)x(2\tan^2(x) + 1)\,dx$

4. Solve the following problems.

    4.1. (a) $6.75\,cm^3$, (b) $0.3\,cm^2$

    4.2. $\frac{12}{5}\pi\,m^3$

    4.3. $0.4\pi\,cm^2$

    4.4. $0.9\pi\,cm^3$

1.2 Error Propagation

DEFINITION. In practice, differentials can be used in the estimation of errors propagated by physical measuring devices. If the measure value of x is used to compute another value $f(x)$, then the difference between $f(x + \triangle x)$ and f *(x)* is the propagated error.

$$f(x + \triangle x) - f(x) = \triangle y \approx dy$$

where:

$$\begin{aligned} x + \triangle x &= \text{Exact Value} \\ \triangle x &= \text{Measurement Error} \\ f(x) &= \text{Measured Value} \\ \triangle y &= \text{Propagated Error} \end{aligned}$$

NOTE. How do you know if the propagated error is large or small? The answer is best given in *relative* terms by comparing dA and A. The ratio is called the relative error and further be expressed as percentage error.

EXAMPLE 1.2.1 A radius of a sphere is to be 3 cm with a possible error of 0.02 cm. (1) Use differentials to approximate the error in calculating the volume. (2) What is the relative error and percentage error?

$$V = \frac{4}{3}\pi r^3$$

$$\begin{aligned} dV &= V'\,dr \\ &= 4\pi r^2\,dr \\ &= 4\pi(3)^2(\pm 0.02) \\ (1) &= \pm 2.2619\,cm^3 \end{aligned}$$

$$\begin{aligned} \frac{dV}{V} &= \frac{\pm 2.2619}{36\pi} \\ (2.1) &= \pm 0.02 \\ (2.2) &= \pm 2\% \end{aligned}$$

SUMMARY. The essence of differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

REMARK. Although the application of differentials to approximate function values is not very important in the age of technology, differentials are important as a convinient notational device for the computation of antiderivatives.

2 Antidifferentiation: Indefinite Integration

EXAMPLE 2.0.0 In order to find a function F whose derivative is $F'(x) = 3x^2$, we use our knowledge of derivatives to conclude the following:

$$F(x) = x^3 \text{ since } \frac{d}{dx}[x^3] = 3x^2$$

The function F is considered an antiderivative of f.

DEFINITION. A function F is an antiderivative of f of an interval I when

$$F'(x) = f(x)$$

for all x in I. As for the previous example, $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$.

NOTE. An antiderivative of f is not unique because of the infinite possible values for the constant C.

2.1 General Antiderivative of a Function

DEFINITION. If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form

$$G(x) = F(x) + C$$

for all x in I, where C is a constant.

DEFINITION. The operation of finding the antiderivatves of a function is called antidifferentiation, or indefinite integration, and is denoted by an integral sign $\int$. Additionally, the equation below states that when we antidifferentiate the differential of a function, we obtain the function plus an arbitrary constant.

$$\int f(x)\, dx = F(x) + C$$

$\int$ = Integration Symbol
$f(x)$ = Integrand
dx = Differential of X
$F(x)$ = One Antiderivative
C = Constant of Integration

NOTE. The expression $\int f(x)\, dx$ is read as "the antiderivative of f with respect to x". The differential dx serves to identify x as the variable of integration. The term indefinite integral is a synonym for antiderivative.

2.2 Basic Integration Rules

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq\ -1$
$\frac{d}{dx}[\sin x] = \cos x$	$\int\ \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int\ \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int\ sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int\ \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int\ \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int\ \csc x \cot x\,dx = -\csc x + C$

2.3 Power Rule of Integrals

DEFINITION. As for the power rule of integrals, as long as $n \neq 1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

EXAMPLE 2.3.1. Find the antiderivative of $\int x^5\,dx$.

$$= \frac{x^6}{6} + C$$

EXAMPLE 2.3.2. Find the antiderivative of $\int \sqrt{x}\,dx$.

$$= \int x^{1/2}\,dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2x^{\frac{3}{2}}}{3} + C$$

EXAMPLE 2.3.3. Find the antiderivative of $\int 3x\,dx$.

$$
\begin{aligned}
&= 3\int x\,dx \\
&= 3\cdot\frac{x^2}{2} \\
&= \frac{3x^2}{2} + C
\end{aligned}
$$

EXAMPLE 2.3.4. Find the antiderivative of $\int \frac{1}{x^3}\,dx$.

$$
= -\frac{1}{2x^2} + C
$$

EXAMPLE 2.3.5. Find the antiderivative of $\int 2\sin x\,dx$.

$$
\begin{aligned}
&= 2\int \sin x\,dx \\
&= 2\cdot\ -\ \cos x \\
&= -2\cos x + C
\end{aligned}
$$

EXAMPLE 2.3.6. Find the antiderivative of $\int (3x^4 - 5x^2 + x)\,dx$.

$$
\begin{aligned}
&= 3\int x^4\,dx - 5\int x^2\,dx + \int x\,dx \\
&= \frac{3x^5}{5} - \frac{5x^3}{3} + \frac{2x^2}{2} + C
\end{aligned}
$$

EXAMPLE 2.3.7. Find the antiderivative of $\int \frac{x+1}{\sqrt{x}}\,dx$.

$$
\begin{aligned}
&= \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx \\
&= \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int \sqrt{x}\,dx \\
&= \int x^{\frac{1}{2}}\,dx \\
&= \frac{2\sqrt{x^3}}{3}
\end{aligned}
\qquad\qquad
\begin{aligned}
&= \int \frac{1}{\sqrt{x}}\,dx \\
&= \int x^{-\frac{1}{2}}\,dx \\
&= 2\sqrt{x}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\sqrt{x^3}}{3} + 2\sqrt{x} \\
&= \frac{2\sqrt{x}(x+3)}{3} + C
\end{aligned}
$$

NOTE. The following are reciprocal and pythagorean identities for trigonometric integrals.

$$
\begin{aligned}
\cot x &= \frac{1}{\tan x} = \frac{\cos x}{\sin x} & \sin^2 x + \cos^2 x &= 1 \\
\csc x &= \frac{1}{\sin x} & \sec^2 x &= 1 + \tan^2 x \\
\sec x &= \frac{1}{\cos x} & \csc^2 x &= 1 + \cot^2 x
\end{aligned}
$$

EXAMPLE 2.3.8. Find the antiderivative of $\int \frac{\sin x}{\cos^2 x}\, dx$.

$$
\begin{aligned}
&= \frac{1}{\cos x} + C \\
&= \sec x + C
\end{aligned}
$$