1 Area of a Plane Region

DEFINITION. It is using integrals to find areas of regions that lie between the graphs of two functions.

RECALL. The definite integral generalizes the concept of the area under a curve. If f is continuous and non-negative on $[a, b]$, then the area under the graph of f from $x = a$ to $x = b$ is given by the integral of f from $x = a$ to $x = b$.

$$\text{Area of S} = \int_a^b f(x)\,dx$$

EXAMPLE 1.0.1. Find the area of the region bounded by the parabola $y = 10 - x^2$, x-axis, y-axis, and $x = 2$.

1.1 Area of Plane Region Between 2 Curves

DEFINITION. If f and g continuous functions on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ is given by the definite integral

$$A = \int_a^b [f(x) - g(x)]\,dx$$

NOTE. The formula provided on the left equates to an approximate, while the formula on the right equates to the exact area of an area.

$$\sum_{i=1}^{n} f(x_i^*)\triangle x \qquad = \lim_{n\to\infty} f(x_i^*)\triangle x$$

$$= \int_a^b f(x)\,dx$$

EXAMPLE 1.1.1 Find the area bounded above by $y = 2x + 5$ and bounded below by $y = x^3$ on [0, 2].

$$
\begin{aligned}
A &= \int_a^b [f(x) - g(x)]\, dx \\
&= \int_a^b 2x + 5 - x^3\, dx \\
&= \left[x^2 + 5x - \frac{x^4}{4} \right]_0^2 \\
&= 4 + 10 - 4 - 0 \\
&= 10
\end{aligned}
$$

EXAMPLE 1.1.2. Find the area of the region bounded above by the parabola $y = 9 - x^2$ and the line $y = 2x + 1$.

$$
\begin{aligned}
&= \int_{-4}^{2} (9 - x^2 - 2x - 1)\, dx \\
&= \left[9x - \frac{x^3}{3} - x^2 - x \right]_{-4}^{2} \\
&= 36
\end{aligned}
$$

EXAMPLE 1.1.3. Find the area of the region bounded by the parabolas $y = x^2$ and $y = -x^2 + 4x$.

$$
\begin{aligned}
x^2 &= -x^2 + 4x \\
2x^2 &= 4x \\
x &= 2, 0
\end{aligned}
$$

$$
\begin{aligned}
1 &= -1 + 4 \\
1 &= 3
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^2 (-x^2 + 4x - x^2)\, dx \\
&= \int_0^2 (-2x^2 + 4x)\, dx \\
&= \left[-\frac{2x^3}{3} + 2x^2 \right]_0^2 \\
&= -\frac{16}{3} + 8 \\
&= \frac{8}{3}
\end{aligned}
$$

NOTE. When selecting for a value of substituting, the values of x must be between only those calculated.

EXAMPLE 1.1.3. Find the area bounded by $y = x^3$ and $y = x$.

$$\begin{aligned} x^3 &= x \\ x^2 &= 1 \\ x &= \pm 1, 0 \end{aligned}$$

EXAMPLE 1.1.4. Find the area bounded by $x = y^2$ and $y = x - 2$.

$$\begin{array}{lll} x = y^2 & y = x - 2 & = \displaystyle\int_c^d [f(y) - g(y)]\ dy \\ g(y) = y^2 & y + 2 = x & = \displaystyle\int_{-1}^2 \left(y + 2 - y^2\right)\ dy \\ & y + 2 = f(y) & = \left[\dfrac{y^2}{2} + 2y - \dfrac{y^3}{3}\right]_{-1}^2 \\ & & = \dfrac{9}{2} \end{array}$$

NOTE. There are some regions wherein they are best treated by regarding x as a function of y.

EXAMPLE 1.1.5. Find the area bounded by $x = 2y^2$ and $x = 4 + y^2$. You must decide whether to integrate with respect to x or y.

$$\begin{array}{lll} 2y^2 = 4 + y^2 & 0 = 4 + 0 & = \displaystyle\int_c^d [f(y) - g(y)]\ dy \\ y^2 = 4 & 0 = 4 & = \displaystyle\int_{-2}^2 \left[4 + y^2 - 2y^2\right]\ dy \\ y = \pm 2 & & = \displaystyle\int_{-2}^2 \left(4 - y^2\right)\ dy \\ & & = \dfrac{32}{3} \end{array}$$

EXAMPLE 1.1.6. Find the area bounded by $x = y^3$ and $x = -3y^2 + 4$.

$$\begin{aligned} y^3 &= -3y^2 + 4 & &= \int_{-2}^1 \left[-3y^2 + 4 - y^3\right]\ dy \\ y^3 + 3y^2 - 4 &= 0 \\ (y-1)(y+2)(y+2) &= 0 \\ y &= 1, -2 \end{aligned}$$

EXAMPLE 1.1.7. Find the area bound by $y = 5x - x^2$ and $y = x$.

$$\begin{aligned} 5x - x^2 &= x & &= \int_0^4 (5x - x^2 - x)\,dx \\ x^2 + x - 5x &= 0 & &= \int_0^4 (4x - x^2)\,dx \\ x^2 - 4x &= 0 & &= \left[2x^2 - \frac{x^3}{3}\right]_0^4 \\ x(x-4) &= 0 & &= 2(4)^2 - \frac{(4)^3}{3} \\ x &= 0, 4 & &= 32 - \frac{64}{3} \\ & & &= \frac{32}{3} \end{aligned}$$

EXAMPLE 1.1.8. Find the area bound by $x = y^2 - 4y$ and $x = 2y - y^2$.

$$\begin{aligned} y^2 - 4y &= 2y - y^2 & &= \int_0^3 (2y - y^2 - y^2 + 4y)\,dy \\ 2y^2 - 6y &= 0 & &= \int_0^3 (6y - 2y^2)\,dy \\ y^2 - 3y &= 0 & &= \left[3y^2 - \frac{2y^3}{3}\right]_0^3 \\ y(y-3) &= 0 & &= 3(3)^2 - \frac{2(3)^3}{3} \\ y &= 0, 3 & &= 9 \end{aligned}$$

2 Volumes of Solids of Revolution

DEFINITION It is using integration to find out the volume of a solid of revolution. We have an intuitive idea of what volume means. In calculus, we make this idea precise to give an exact definition of volume.

2.1 Solid of Revolution

DEFINITION. It is a solid obtained by revolving a plane region about a fixed line called the axis of revolution.

2.2 Disk Method

DEFINITION. Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved around the x-axis, the volume of the resulting solid is

$$V = \int_a^b \pi f(x)^2 \, dx = \pi \int_a^b f(x)^2 \, dx$$

Additionally, the set-up goes:

- For a solid S that isn't a cylinder, we first "cut" S into pieces and approximate each piece by a thin cylinder or disk.
- We estimate the volume of S by adding the volumes of the disk.
- We arrive at the exact volume of S through a limiting process in which the number of pieces become large.

EXAMPLE 2.2.1. Given that the axis of revolution is x-axis, find the volume of the solid obtained by rotating about the x-axis of the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

$$\begin{aligned} V &= \int_a^b A(x) \, dx \\ &= \int_0^1 \pi(\sqrt{x})^2 \, dx \\ &= \pi \int_0^1 x \, dx \\ &= \pi \left[\frac{x^2}{2} \right]_0^1 \\ &= \frac{\pi}{2} \end{aligned}$$

EXAMPLE 2.2.2. Given that it is rotated around the x-axis, write the integral that would be used to find the volume of the region bounded by $x = -1$, $x = 2$, $y = 0$, and $y = \frac{1}{2}x^2 + 2$.

$$\begin{aligned} V &= \int_a^b \pi f(x)^2 \, dx \\ &= \pi \int_{-1}^2 \left(\frac{1}{2}x^2 + 2 \right)^2 dx \\ &= \pi \int_{-1}^2 \left(\frac{1}{2}x^2 + 2 \right)^2 dx \end{aligned}$$

EXAMPLE 2.2.3. Given that the axis of revolution is not a coordinate axis, find the volume of the solid when the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ is revolved about the line $y = 1$.

$$\begin{aligned} V &= \pi \int_{-1}^{1} (1 - x^2)^2 \, dx \\ &= \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, dx \\ &= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1} \\ &= \frac{16\pi}{15} \end{aligned}$$

EXAMPLE 2.2.4. Given that the axis of revolution is y-axis, find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

$$\begin{aligned} V &= \int_{y_1}^{y_2} \pi f(y)^2 \, dy \\ &= \pi \int_{0}^{8} (\sqrt[3]{y})^2 \, dy \\ &= \pi \int_{0}^{8} y^{2/3} \, dy \\ &= \pi \left[\frac{y^{5/3}}{5/3} \right]_{0}^{8} \\ &= \frac{3\pi}{5} \left[y^{5/3} \right]_{0}^{8} \\ &= \frac{96\pi}{5} \end{aligned}$$

EXAMPLE 2.2.5. Write the integral that would be used to find the volume of the solid obtained by revolving the region bounded by $x = y^2 - 4$, $x = 0$, and $y = 0$ about the y-axis.

$$\begin{aligned} r &= 0 - (y^2 - 4) \\ r^2 &= (y^2 - 4)^2 \end{aligned}$$

$$\begin{aligned} A(y) &= \pi (y^2 - 4)^2 \\ V &= \int_{y_1}^{y_2} A(y) \, dy \\ &= \int_{0}^{2} \pi (y^2 - 4)^2 \, dy \end{aligned}$$

EXAMPLE 2.2.6. Given that the axis of revolution is not the y-axis, find the volume of the solid when the region bounded by $x = y^2$ and $x = 1$ is revolved about the line $x = 1$.

$$\begin{aligned} V &= \int_{-1}^{1} \pi(1 - y^2)^2 \, dy \\ &= \pi \int_{-1}^{1} (1 - 2y^2 + y^4) \, dy \\ &= \pi \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_{-1}^{1} \\ &= \frac{16\pi}{15} \end{aligned}$$

DEFINITION. To find the volume of a solid of revolution with the disk method, use one of the formulas below.

Horizontal Axis of Revolution	Vertical Axis of Revolution
$V = \pi \int_a^b [r(x)]^2 \, dx$	$V = \pi \int_c^d [r(y)]^2 \, dy$

NOTE. You can determine the variable of integration by placing a representative rectangle in the plane region perpendicular to the axis of revolution. When the width of the rectangle is $\triangle x$ integrate with respect to x, and when the width of the rectangle is $\triangle y$, integrate with respect to y.

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