1 Derivatives of Elementary Transcendental Functions

DEFINITION. In general, the term transcendental means non-algebraic. A transcendental function is a function that is not expressible as a finite combination of the algebraic opperations of addition, subtraction, multiplication, division, raising to a power, and extracting a root. An example includes the function $\log x$, $\sin x$, $\cos x$, e^x, and any functions containing them.

2 The Natural Logarithmic Function

DEFINITION. The function defined by

$$f(x) = \log_e x = \ln x$$

$(x > 0, e \approx 2.718281\ldots)$ is called the natural logarithmic function.

NOTE. The equation $y = \ln x$ is equivalent to $e^y = x$.

2.1 Logarithmic Properties

DEFINITION. If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln 1 = 0$; if $x > 1$, then $y = \ln x > 0$, else vice versa.
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

2.2 Definition of the Natural Logarithmic Function

DEFINITION. The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt, x > 0$$

The domain of the natural logarithmic function is the set of all positive real numbers.

2.3 Derivative of the Natural Logarithmic Function

DEFINITION. Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, u > 0$

EXAMPLE 2.3.1. Solve for the equation $\frac{d}{dx}[\ln 2x]$.

$$= \frac{1}{2x}$$

EXAMPLE 2.3.2. Solve for the equation $\frac{d}{dx}[\ln (x^2+1)]$.

$$\begin{aligned} &= \frac{1}{u} \cdot \frac{du}{dx} && u = x^2+1 \\ &= \frac{1}{x^2+1} \cdot \frac{2xdx}{dx} && du = 2x\,dx \\ &= \frac{2x}{x^2+1} \end{aligned}$$

EXAMPLE 2.3.3. Solve for the equation $\frac{d}{dx}[x \ln x]$.

$$\begin{aligned} &= x \cdot \frac{1}{x} \\ &= 1 \end{aligned}$$

EXAMPLE 2.3.4. Solve for the equation $\frac{d}{dx}[(\ln x)^3]$.

$$\begin{aligned} &= 3(\ln x)^2 \cdot \frac{1}{x} \\ &= \frac{3(\ln x)^2}{x} \end{aligned}$$

EXAMPLE 2.3.5. Solve for the equation $y = \ln[(4x^2+3)(2x-1)]$.

$$\begin{aligned} y &= \ln(4x^2+3) + \ln(2x-1) \\ \frac{dy}{dx} &= \frac{8x}{4x^2+3} + \frac{2}{2x-1} \end{aligned}$$

EXAMPLE 2.3.6. Solve for the equation $y = \ln\left(\frac{x}{x+1}\right)$.

$$\begin{aligned} y &= \ln(x)(x+1)^{-1} \\ y &= \ln(x) + \ln(x+1)^{-1} \\ \frac{dy}{dx} &= \frac{1}{x} - \frac{1}{x+1} \end{aligned}$$

EXAMPLE 2.3.7. By implicit differentiation, find the dy/dx of $\ln\left(\frac{x}{y}\right) + xy = 1$.

$$\begin{aligned}
\ln(x) - \ln(y) + xy &= 1 \\
\frac{d}{dx}(\ln(x) - \ln(y) + xy) &= 1 \\
\frac{1}{x} - \frac{1}{y} \cdot \frac{dy}{dx} + 1y + x \cdot \frac{dy}{dx} &= 0 \\
\frac{dy}{dx}\left(x - \frac{1}{y}\right) &= -y - \frac{1}{x}
\end{aligned}$$

$$\begin{aligned}
f(y) &= f(y(x)) \\
&= \ln y \\
&= \ln[y(x)] \\
&= \frac{1}{y} \cdot y'
\end{aligned}$$

EXAMPLE 2.3.8. Solve for the equation $\ln(x+y) - \ln(x-y) = 4$.

$$\begin{aligned}
\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) - \frac{1}{x-y}\left(1 - \frac{dy}{dx}\right) &= 0 \\
\frac{1}{x+y} + \frac{dy}{(x+y)\,dx} - \frac{1}{x-y} + \frac{dy}{(x-y)\,dx} &= 0 \\
\frac{dy}{dx} &= \frac{y}{x}
\end{aligned}$$

3 Logarithmic Differentiation

DEFINITION. It is sometimes convenient to use logarithms as aids in differentiating non-logarithmic functions. This procedure is called logarithmic differentiation. This process uses the properties of natural logarithm to simplify the work involved in differentiating complicated expressions containing products, quotients, and powers.

EXAMPLE 3.0.1. Find the $\frac{dy}{dx}$ of the equation $y = \frac{(\sin x)^2(x^3+1)^4}{(x+3)^8}$.

$$\begin{aligned}
\ln(y) &= \ln\left[\frac{(\sin x)^2(x^3+1)^4}{(x+3)^8}\right] \\
\ln(y) &= 2\ln(\sin x) + 4\ln(x^3+1) - 8\ln(x+3) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{2\cos x}{\sin x} + \frac{4(3x^2)}{x^3+1} - \frac{8}{x+3} \\
\frac{dy}{dx} &= y\left(2\cot x + \frac{12x^2}{x^3+1} - \frac{8}{x+3}\right) \\
\frac{dy}{dx} &= \frac{(\sin x)^2(x^3+1)^4}{(x+3)^8}\left[2\cot x + \frac{12x^2}{x^3+1} - \frac{8}{x-3}\right]
\end{aligned}$$

NOTE. The equation $\ln a = \ln b$ is equivalent to $a = b$.

EXAMPLE 3.0.2. Solve for the equation $y = x^3\sqrt{5-9x}$.

EXAMPLE 3.0.3. Solve for the equation $y = \frac{x^2(6+3x)^4}{\sqrt[3]{9-x^2}}$.

EXAMPLE 3.0.4. Solve for the equation $y = x^{\sin x}$.

$$\begin{aligned}
\ln(y) &= \ln\left[x^{\sin x}\right] \\
\ln(y) &= (\sin x)\ln(x) \\
\frac{1}{y}\frac{dy}{dx} &= (\sin x)(\frac{1}{x}) + (\cos x)(\ln x) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{\sin x}{x} + \cos x \ln x \\
\frac{dy}{dx} &= y\left[\frac{\sin x}{x} + \cos x \ln x\right] \\
\frac{dy}{dx} &= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \ln x\right]
\end{aligned}$$

EXAMPLE 3.0.5. Solve for the equation $y = \frac{\sqrt[3]{x+1}}{(x+2)\sqrt{x+3}}$.

EXAMPLE 3.0.6. Solve for the equation $y = \frac{(x-2)^2}{\sqrt{x^2+1}}$.