1 The Differentials

DEFINITION. These provide us with a way of estimating the amount a function changes as a result of a small change in input values. The equation

$$\triangle y \approx dy$$

can and will only be considered if $\triangle x$ is "close enough". The approximation of the equation becomes better as $\triangle x$ becomes smaller.

NOTE. We equate $\triangle x = dx$.

1.1 The Differential of the Independent Variable

DEFINITION. If the function f is defined by the equation $y = f(x)$, then the differential of y, denoted by dy, is given by

$$dy = f'(x)dx \longrightarrow f'(x) = \frac{dy}{dx}$$

where x is any number in the domain of f' and $\triangle x$ is an arbitrary increment of x.

EXAMPLE 1.1. Find dy for $y = (x^3 + 5x - 1)^{2023}$.

$$f'(x) = 2023(x^3 + 5x - 1)^{2022}(3x^2 + 5)$$

$$\therefore dy = 2023(x^3 + 5x - 1)(3x^2 + 5)dx$$

EXAMPLE 1.2. Find the differential dy of the function $y = 4x^2 + x + 3$.

$$(8x + 1)dx$$

EXAMPLE 1.3. Find the differential dy of the function $y = \cos(x)$.

$$dy = -\sin(x)dx$$

EXAMPLE 1.4. Compare the values of $\triangle y$ and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

$$x = 2 \qquad x + \triangle x = 2.05$$

$$\Rightarrow \triangle x = 0.05 = dx$$

$$\triangle y = f(x + \triangle x) - f(x)$$

$$= f(2.05) - f(2)$$

$$= [(2.05)^3 + (2.05)^2 + 2(2.05) + 1] - [2^3 + 2^2 - 2.2 + 1]$$

$$\triangle y = 0.7176$$

$$dy = f'(x)dx$$

$$= (3x^2 + 2x - 2)dx$$

$$= (3.2^2 + 2.2 - 2)(0.05)$$

$$dy = 0.7$$

NOTE. The final equation utilized our solution at $x = 2, \triangle x = dx = 0.05$. Observe that the approximation of $\triangle y \approx dy$ becomes better as $\triangle x$ becomes smaller.

Practice Exercises

1. Find dy and $\triangle y$ for the given values of x and $\triangle x$.

   1.1. $y = x^2$, $x = 2$, and $\triangle x = 0.5$

   1.2. $y = x^3$, $x = 2$, and $\triangle x = 0.5$

   1.3. $y = \sqrt[3]{x}$, $x = 8$, and $\triangle x = 1$

   1.4. $y = \sqrt{x}$, $x = 4$, and $\triangle x = 1$

2. Find (a) $\triangle y$; (b) dy; (c) $\triangle y - dy$.

   2.1. $y = x^2 - 3x$, $x = 2$, and $\triangle x = 0.03$

   2.2. $y = x^2 - 3x$, $x = -1$, and $\triangle x = 0.02$

   2.3. $y = \frac{1}{x}$, $x = -2$, and $\triangle x = -0.1$

   2.4. $y = \frac{1}{x}$, $x = 3$, and $\triangle x = -0.2$

   2.5. $y = x^3 + 1$, $x = 1$, and $\triangle x = -0.5$

   2.6. $y = x^3 + 1$, $x = -1$, and $\triangle x = 0.1$

3. Find dy.

   3.1. $y = (3x^2 - 2x + 1)^3$

   3.2. $y = \frac{3x}{x^2+2}$

   3.3. $y = x^2\sqrt{2x + 3}$

   3.4. $y = \sqrt{4 - x^2}$

   3.5. $y = \frac{2+\cos x}{2-\sin x}$

   3.6. $y = \tan^2 x \sec^2 x$

4. Solve the following problems.

   4.1. The measurement of an edge of a cube is found to be 15 cm with a possible error of 0.01 cm. Use differentials to find the approximate error in computing from this measurement: (a) the volume; (b) the area of one of the faces.

   4.2. An open cylindrical tank is to have an outside coating of thickness 2 cm. If the inner radius is 6 m and the altitude is 10 m, find by differentials the approximate amount of coating material to be used.

   4.3. A burn on a person's skin is in the shape of a circle. Use differentials to find the approximate decrease in the area of the burn when the radius decreases from 1 cm to 0.8 cm.

   4.4. A tumor in a person's body is spherical in shape. Use differentials to find the approximate increase in the volume of the tumor when the radius increases from 1.5 cm to 1.6 cm.

Answer Key

1. Find dy and $\triangle y$ for the given values of x and $\triangle x$.

   1.1. $dy = 2$, $\triangle y = 2.25$

   1.2. $dy = 6$, $\triangle y = 7.625$

   1.3. $dy = \frac{1}{12} \approx 0.083$, $\triangle y = \sqrt[3]{9} - 2 \approx 0.080$

   1.4. $dy = 0.25$, $\triangle y = \sqrt{5} - \sqrt{4} \approx 0.236$

2. Find (a) $\triangle y$; (b) dy; (c) $\triangle y - dy$.

   2.1. (a) 0.0309, (b) 0.03, (c) 0.0009

   2.2. (a) -0.0996, (b) -0.1, (c) 0.0004

   2.3. (a) $\frac{1}{42} \approx 0.0238$, (b) $\frac{1}{40} = 0.025$, (c) $-\frac{1}{840} \approx -0.0012$

   2.4. (a) $\frac{1}{42} \approx 0.0238$, (b) $\frac{1}{45} = 0.022$, (c) $frac1630 \approx -0.0016$

   2.5. (a) -0.875, (b) -1.5, (c) 0.625

   2.6. (a) 0.271, (b) 0.3, (c) -0.029

3. Find dy.

   3.1. $dy = 3(3x^2 - 2x + 1)^2(6x - 2)dx$

   3.2. $dy = \frac{3(2-x^2)}{(x^2+2)^2}dx$

   3.3. $dy = \frac{x(5x+6)}{(2x+3)^{1/2}}dx$

   3.4. $dy = \frac{-x}{\sqrt{4-x^2}}dx$

   3.5. $dy = \frac{1-2\sin x+2\cos x}{(2-\sin x)^2}dx$

   3.6. $dy = 2\tan x \sec^2 x(2\tan^2 x + 1)dx$

4. Solve the following problems.

   4.1. (a) $6.75cm^3$, (b) $0.3cm^2$

   4.2. $\frac{12}{5}\pi m^3$

   4.3. $0.4\pi cm^2$

   4.4. $0.9\pi cm^3$