1 The Differentials

1. Find dy and $\triangle y$ for the given values of x and $\triangle x$.

   1.1. $y = x^2$, $x = 2$, and $\triangle x = 0.5$

   1.2. $y = x^3$, $x = 2$, and $\triangle x = 0.5$

   1.3. $y = \sqrt[3]{x}$, $x = 8$, and $\triangle x = 1$

   1.4. $y = \sqrt{x}$, $x = 4$, and $\triangle x = 1$

2. Find (a) $\triangle y$, (b) dy, (c) $\triangle y - dy$.

   2.1. $y = x^2 - 3x$, $x = 2$, and $\triangle x = 0.03$

   2.2. $y = x^2 - 3x$, $x = -1$, and $\triangle x = 0.02$

   2.3. $y = \frac{1}{x}$, $x = -2$, and $\triangle x = -0.1$

   2.4. $y = \frac{1}{x}$, $x = 3$, and $\triangle x = -0.2$

   2.5. $y = x^3 + 1$, $x = 1$, and $\triangle x = -0.5$

   2.6. $y = x^3 + 1$, $x = -1$, and $\triangle x = 0.1$

3. Find dy.

   3.1. $y = (3x^2 - 2x + 1)^3$

   3.2. $y = \frac{3x}{x^2+2}$

   3.3. $y = x^2\sqrt{2x + 3}$

   3.4. $y = \sqrt{4 - x^2}$

   3.5. $y = \frac{2+\cos(x)}{2-\sin(x)}$

   3.6. $y = \tan^2(x)\sec^2(x)$

4. Solve the following problems.

   4.1. The measurement of an edge of a cube is found to be 15 cm with a possible error of 0.01 cm. Use differentials to find the approximate error in computing from this measurement: (a) the volume; (b) the area of one of the faces.

   4.2. An open cylindrical tank is to have an outside coating of thickness 2 cm. If the inner radius is 6 m and the altitude is 10 m, find by differentials the approximate amount of coating material to be used.

   4.3. A burn on a person's skin is in the shape of a circle. Use differentials to find the approximate decrease in the area of the burn when the radius decreases from 1 cm to 0.8 cm.

4.4. A tumor in a person's body is spherical in shape. Use differentials to find the approximate increase in the volume of the tumor when the radius increases from 1.5 cm to 1.6 cm.

Answer Key

1. Find dy and $\triangle y$ for the given values of x and $\triangle x$.

   1.1. $dy = 2$, $\triangle y = 2.25$

   1.2. $dy = 6$, $\triangle y = 7.625$

   1.3. $dy = \frac{1}{12} \approx 0.083$, $\triangle y = \sqrt[3]{9} - 2 \approx 0.080$

   1.4. $dy = 0.25$, $\triangle y = \sqrt{5} - \sqrt{4} \approx 0.236$

2. Find (a) $\triangle y$, (b) dy, (c) $\triangle y - dy$.

   2.1. (a) 0.0309, (b) 0.03, (c) 0.0009

   2.2. (a) -0.0996, (b) -0.1, (c) 0.0004

   2.3. (a) $\frac{1}{42} \approx 0.0238$, (b) $\frac{1}{40} = 0.025$, (c) $-\frac{1}{840} \approx -0.0012$

   2.4. (a) $\frac{1}{42} \approx 0.0238$, (b) $\frac{1}{45} = 0.022$, (c) $\frac{1}{630} \approx -0.0016$

   2.5. (a) -0.875, (b) -1.5, (c) 0.625

   2.6. (a) 0.271, (b) 0.3, (c) -0.029

3. Find dy.

   3.1. $dy = 3(3x^2 - 2x + 1)^2(6x - 2)\,dx$

   3.2. $dy = \frac{3(2-x^2)}{(x^2+2)^2}\,dx$

   3.3. $dy = \frac{x(5x+6)}{(2x+3)^{1/2}}\,dx$

   3.4. $dy = \frac{-x}{\sqrt{4-x^2}}\,dx$

   3.5. $dy = \frac{1-2\sin(x)+2\cos(x)}{(2-\sin(x))^2}\,dx$

   3.6. $dy = 2\tan(x)\sec^2(x)x(2\tan^2(x) + 1)\,dx$

4. Solve the following problems.

   4.1. (a) $6.75\,cm^3$, (b) $0.3\,cm^2$

   4.2. $\frac{12}{5}\pi\,m^3$

   4.3. $0.4\pi\,cm^2$

   4.4. $0.9\pi\,cm^3$

2 Antidifferentiation: Indefinite Integration

1. $\int (x^2 + x^{-2})\, dx$
2. $\int (\sqrt{x^3} + \sqrt[3]{x^2})\, dx$
3. $\int \left(x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2\right)\, dx$
4. $\int (y^3 + 1.8y^2 - 2.4y)\, dy$
5. $\int (1 - t)(2 + t^2)\, dt$
6. $\int v(v^2 + 2)^2\, dv$
7. $\int \frac{x^3 - 2\sqrt{x}}{x}\, dx$
8. $\int (\theta - \csc\theta \cot\theta)\, d\theta$
9. $\int \sec t(\sec t + \tan t)\, dt$
10. $\int (1 + \tan^2\alpha)\, d\alpha$
11. $\int \frac{\sin 2x}{\sin x}\, dx$

Answer Key

1. $\int (x^2 + x^{-2})\,dx$

$$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + C$$
$$= \frac{1}{3}x^3 - \frac{1}{x} + C$$

2. $\int (\sqrt{x^3} + \sqrt[3]{x^2})\,dx$

$$= \int (x^{3/2} + x^{2/3})\,dx$$
$$= \frac{x^{5/2}}{5/2} + \frac{x^{5/3}}{5/3} + C$$
$$= \frac{2x^{5/2}}{5} + \frac{3x^{5/3}}{5} + C$$

3. $\int \left(x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2\right)\,dx$

$$= \frac{x^5}{5} - \frac{1}{2}\frac{x^4}{4} + \frac{1}{4}\frac{x^2}{2} - 2x + C$$
$$= \frac{x^5}{5} - \frac{x^4}{8} + \frac{x^2}{8} - 2x + C$$

4. $\int (y^3 + 1.8y^2 - 2.4y)\,dy$

$$= \frac{y^4}{4} + 1.8\frac{y^3}{3} - 2.4\frac{y^2}{2} + C$$
$$= \frac{y^4}{4} + 0.6y^3 - 1.2y^2 + C$$

5. $\int (1-t)(2+t^2)\,dt$

$$= (2 - 2t + t^2 - t^3)\,dt$$
$$= 2t - \frac{2t^2}{2} + \frac{t^3}{3} - t^4 4 + C$$
$$= 2t - t^2 + \frac{t^3}{3} - \frac{t^4}{4} + C$$

6. $\int v(v^2+2)^2\,dv$

$$= \int v(v^4 + 4v^2 + 4)\,dv$$
$$= \int (v^5 + 4v^3 + 4v)\,dv$$
$$= \frac{v^6}{6} + \frac{4v^4}{4} + \frac{4v^2}{2} + C$$
$$= \frac{v^6}{6} + v^4 + 2v^2 + C$$

7. $\int \frac{x^3-2\sqrt{x}}{x}\,dx$

$$\begin{aligned}
&= \int \left(\frac{x^3}{x} - \frac{2x^{1/2}}{x}\right) dx \\
&= \int (x^2 - 2x^{-1/2})\,dx \\
&= \frac{x^3}{3} - \frac{2x^{1/2}}{1/2} + C \\
&= \frac{x^3}{3} - 4\sqrt{x} + C
\end{aligned}$$

8. $\int (\theta - \csc\theta\cot\theta)\,d\theta$

$$= \frac{\theta^2}{2} + \csc\theta + C$$

9. $\int \sec t(\sec t + \tan t)\,dt$

$$\begin{aligned}
&= \int (\sec^2 t + \sec t\tan t)\,dt \\
&= \tan t + \sec t + C
\end{aligned}$$

10. $\int (1 + \tan^2\alpha)\,d\alpha$

$$\begin{aligned}
&= \int \sec^2\alpha\,d\alpha \\
&= \tan\alpha + C
\end{aligned}$$

11. $\int \frac{\sin 2x}{\sin x}\,dx$

$$\begin{aligned}
&= \int \frac{2\sin x\cos x}{\sin x}\,dx \\
&= \int 2\cos x\,dx \\
&= 2\sin x + C
\end{aligned}$$

3 Integration by Substitution

1. $\int x \sin\left(x^2\right) dx$
2. $\int x^2(x^3+5)^9 \, dx$
3. $\int (3x-2)^{20} \, dx$
4. $\int (3t+2)^{2.4} \, dt$
5. $\int (x+1)\sqrt{2x+x^2} \, dx$
6. $\int \frac{x}{(x^2+1)^2} \, dx$
7. $\int \sin \pi t \, dt$
8. $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}} \, dx$
9. $\int \sec 2\theta \tan 2\theta \, d\theta$
10. $\int \frac{\cos\sqrt{t}}{\sqrt{t}} \, dt$
11. $\int \sqrt{x} \sin\left(1+x^{\frac{3}{2}}\right) dx$
12. $\int \cos\theta \sin^6\theta \, d\theta$
13. $\int (1+\tan\theta)^5 \sec^2\theta \, d\theta$
14. $\int \frac{z^2}{\sqrt[3]{1+z^3}} \, dz$
15. $\int \frac{\cos x}{\sin^2 x} \, dx$
16. $\int \sqrt{\cot x} \csc^2 x \, dx$
17. $\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} \, dx$
18. $\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$
19. $\int \sec^3 x \tan x \, dx$
20. $\int \sin t \sec^2(\cos t) \, dt$
21. $\int \frac{x^2}{\sqrt{1-x}} \, dx$
22. $\int \frac{x}{\sqrt[4]{x+2}} \, dx$

Answer Key

1. $\int x \sin(x^2)\, dx$

$$
\begin{aligned}
&= \int \sin(x^2)\, x\, dx && u = x^2 \\
&= \frac{1}{2} \int \sin u\, du && du = 2x\, dx \\
&= \frac{1}{2}(-\cos u) + C \\
&= -\frac{1}{2}\cos(x^2) + C
\end{aligned}
$$

2. $\int x^2 (x^3+5)^9\, dx$

$$
\begin{aligned}
&= \int (x^3+5)^9 x^2\, dx && u = x^3 + 5 \\
&= \frac{1}{3} \int u^9\, du && du = 3x^2 \\
&= \frac{1}{3}\left(\frac{u^{10}}{10}\right) + C \\
&= \frac{u^{10}}{30} + C \\
&= \frac{(x^3+5)^{10}}{30} + C
\end{aligned}
$$

3. $\int (3x-2)^{20}\, dx$

$$
\begin{aligned}
&= \frac{1}{3} \int u^{20}\, du && u = 3x - 2 \\
&= \frac{1}{3}\left(\frac{u^{21}}{21}\right) + C && du = 3\, dx \\
&= \frac{u^{21}}{63} + C \\
&= \frac{(3x-2)^{21}}{63} + C
\end{aligned}
$$

4. $\int (3t+2)^{2.4}\,dt$

$$\begin{aligned} &= \frac{1}{3}\int u^{2.4}\,du && u = 3t+2 \\ &= \frac{1}{3}\left(\frac{u^{3.4}}{3.4}\right) + C && du = 3\,dt \\ &= \frac{u^{3.4}}{10.2} + C \end{aligned}$$

5. $\int (x+1)\sqrt{2x+x^2}\,dx$

$$\begin{aligned} &= \int (x+1)(2x+x^2)^{1/2}\,dx && u = 2x+x^2 \\ &= \frac{1}{2}\int u^{1/2}\,dx && du = 2+2x\,dx \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C \\ &= \frac{1}{3}u^{3/2} + C \\ &= \frac{(2x+x^2)^{3/2}}{3} + C \end{aligned}$$

6. $\int \frac{x}{(x^2+1)^2}\,dx$

$$\begin{aligned} &= \int x(x^2+1)^{-2}\,dx && u = x^2+1 \\ &= \int (x^2+1)^{-2}x\,dx && du = 2x\,dx \\ &= \frac{1}{2}\int u^{-2}\,du \\ &= \frac{1}{2}\left(-u^{-1}\right) + C \\ &= -\frac{1}{(2x^2+2)} + C \end{aligned}$$

7. $\int \sin \pi t\,dt$

8. $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$

$$
\begin{aligned}
&= \int (a+bx^2)(3ax+bx^3)^{-1/2}\,dx && u = 3ax+bx^3 \\
&= \frac{1}{3}\int u^{-1/2}\,du && du = 3a+3bx^2\,dx \\
&= \frac{1}{3}\left(2u^{1/2}\right)+C \\
&= \frac{2\sqrt{3ax+bx^3}}{3}+C
\end{aligned}
$$

9. $\int \sec 2\theta \tan 2\theta \, d\theta$
10. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$
11. $\int \sqrt{x}\sin\left(1+x^{\frac{3}{2}}\right)dx$
12. $\int \cos\theta \sin^6\theta \, d\theta$
13. $\int (1+\tan\theta)^5 \sec^2\theta \, d\theta$
14. $\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$

$$
\begin{aligned}
&= \int (1+z^3)^{-1/3} z^2\,dz && u = 1+z^3 \\
&= \frac{1}{3}\int u^{-1/3}\,du && du = 3z^2\,dz \\
&= \frac{1}{3}\left(\frac{3}{2}u^{2/3}\right)+C \\
&= \frac{u^{2/3}}{2}+C \\
&= \frac{\sqrt[3]{(1+z^3)^2}}{2}+C
\end{aligned}
$$

15. $\int \frac{\cos x}{\sin^2 x}\,dx$
16. $\int \sqrt{\cot x}\csc^2 x\,dx$
17. $\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$
18. $\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$
19. $\int \sec^3 x \tan x\,dx$
20. $\int \sin t \sec^2(\cos t)\,dt$
21. $\int \frac{x^2}{\sqrt{1-x}}\,dx$
22. $\int \frac{x}{\sqrt[4]{x+2}}\,dx$

4 Separable Differential Equations

1. Find the complete solution of the differential equation.

   1.1. $\frac{dy}{dx} = 4x - 5$

   1.2. $\frac{dy}{dx} = 6 - 3x^2$

   1.3. $\frac{dy}{dx} = 3x^2 + 2x - 7$

   1.4. $\frac{ds}{dt} = 5\sqrt{s}$

   1.5. $\frac{dy}{dx} = 3xy^2$

   1.6. $\frac{dy}{dx} = \frac{\sqrt{x}+x}{\sqrt{y}-y}$

   1.7. $\frac{du}{dv} = \frac{3v\sqrt{1+u^2}}{u}$

   1.8. $\frac{dy}{dx} = \frac{x^2\sqrt{x^3-3}}{y^2}$

   1.9. $\frac{dy}{dx} = \frac{\sec^2 x}{\tan^2 y}$

   1.10. $\frac{du}{dv} = \frac{\cos 2v}{\sin 3u}$

   1.11. $\frac{d^2y}{dx^2} = 5x^2 + 1$

   1.12. $\frac{d^2y}{dx^2} = \sqrt{2x - 3}$

   1.13. $\frac{d^2s}{dt^2} = \sin 3t + \cos 3t$

   1.14. $\frac{d^2u}{dv^2} = \tan v \sec^2 v$

2. Find the particular solution of the differential equation determined by the initial conditions.

   2.1. $\frac{dy}{dx} = x^2 - 2x - 4$; $y = -6$ when $x = 3$

   2.2. $\frac{dy}{dx} = (x + 1)(x + 2)$; $y = -\frac{3}{2}$ when $x = -3$

   2.3. $\frac{dy}{dx} = \frac{\cos 3x}{\sin 2y}$; $y = \frac{\pi}{3}$ when $x = \frac{\pi}{2}$

   2.4. $\frac{ds}{dt} = \cos\left(\frac{t}{2}\right)$; $s = 3$ when $t = \frac{\pi}{3}$

   2.5. $\frac{d^2u}{dv^2} = 4(1 + 3v)^2$; $u = -1$ and $\frac{du}{dv} = -2$ when $v = -1$

   2.6. $\frac{d^2y}{dx^2} = -\frac{3}{x^4}$; $y = \frac{1}{2}$ and $\frac{dy}{dx} = -1$ when $x = 1$

Answer Key

1. Find the complete solution of the differential equation.

   1.1. $y = 2x^2 - 5x + C$

   1.2. $y = 6x - x^3 + C$

   1.3. $y = x^3 + x^2 - 7x + C$

   1.4. $s = \left(\frac{5t+C}{2}\right)^2$

   1.5. $y = \frac{-2}{3x^2+C}$

   1.6. $\frac{2}{3}y^{3/2} - \frac{1}{2}y^2 = \frac{2}{3}x^{3/2} + \frac{1}{2}x^2 + C$

   1.7. $2\sqrt{1+u^2} = 3v^2 + C$

   1.8. $y = \left[\frac{2}{3}(x^3-3)^{3/2} + C\right]^{1/3}$

   1.9. $\tan y - y = \tan x + C$

   1.10. $-\frac{1}{3}\cos 3u = \frac{1}{2}\sin 2v + C$

   1.11. $y = \frac{5}{12}x^4 + \frac{1}{2}x^2 + C_1x + C_2$

   1.12. $y = \frac{1}{15}(2x-3)^{5/2} + C_1x + C_2$

   1.13. $s = -\frac{1}{9}\sin 3t - \frac{1}{9}\cos 3t + C_1t + C_2$

   1.14. $u = \frac{1}{2}\tan v + C_1v + C_2$

2. Find the particular solution of the differential equation determined by the initial conditions.

   2.1. $y = \frac{1}{3}x^3 - x^2 - 4x + 6$

   2.2. $y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x$

   2.3. $\cos 2y = -\frac{2}{3}\sin 3x - \frac{7}{6}$

   2.4. $s = 2\sin\left(\frac{t}{2}\right) + 2$

   2.5. $u = 3v^4 + 4v^3 + 2v^2 + 2v$

   2.6. $y = -\frac{1}{2}x^{-2} - 2x + 3$