1 Area of a Plane Region

DEFINITION. It is using integrals to find areas of regions that lie between the graphs of two functions.

RECALL. The definite integral generalizes the concept of the area under a curve. If f is continuous and non-negative on $[a, b]$, then the area under the graph of f from $x = a$ to $x = b$ is given by the integral of f from $x = a$ to $x = b$.

$$\text{Area of S} = \int_a^b f(x)\,dx$$

EXAMPLE 1.0.1. Find the area of the region bounded by the parabola $y = 10 - x^2$, x-axis, y-axis, and $x = 2$.

1.1 Area of Plane Region Between 2 Curves

DEFINITION. If f and g continuous functions on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ is given by the definite integral

$$A = \int_a^b [f(x) - g(x)]\,dx$$

NOTE. The formula provided on the left equates to an approximate, while the formula on the right equates to the exact area of an area.

$$\sum_{i=1}^{n} f(x_i^*)\triangle x \qquad = \lim_{n\to\infty} f(x_i^*)\triangle x$$

$$= \int_a^b f(x)\,dx$$

EXAMPLE 1.1.1 Find the area bounded above by $y = 2x + 5$ and bounded below by $y = x^3$ on $[0, 2]$.

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\, dx \\ &= \int_a^b 2x + 5 - x^3\, dx \\ &= \left[x^2 + 5x - \frac{x^4}{4}\right]_0^2 \\ &= 4 + 10 - 4 - 0 \\ &= 10 \end{aligned}$$

EXAMPLE 1.1.2. Find the area of the region bounded above by the parabola $y = 9 - x^2$ and the line $y = 2x + 1$.

$$\begin{aligned} &= \int_{-4}^2 (9 - x^2 - 2x - 1)\, dx \\ &= \left[9x - \frac{x^3}{3} - x^2 - x\right]_{-4}^2 \\ &= 36 \end{aligned}$$

EXAMPLE 1.1.3. Find the area of the region bounded by the parabolas $y = x^2$ and $y = -x^2 + 4x$.

$$x^2 = -x^2 + 4x \qquad 1 = -1 + 4$$

$$2x^2 = 4x \qquad 1 = 3$$

$$x = 2, 0$$

$$\begin{aligned} &= \int_0^2 (-x^2 + 4x - x^2)\, dx \\ &= \int_0^2 (-2x^2 + 4x)\, dx \\ &= \left[-\frac{2x^3}{3} + 2x^2\right]_0^2 \\ &= -\frac{16}{3} + 8 \\ &= \frac{8}{3} \end{aligned}$$

NOTE. When selecting for a value of substituting, the values of x must be between only those calculated.

EXAMPLE 1.1.3. Find the area bounded by $y = x^3$ and $y = x$.

$$\begin{aligned} x^3 &= x \\ x^2 &= 1 \\ x &= \pm 1, 0 \end{aligned}$$