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Pg. 563 #3, 11, 13, 15, 19, 20, 23

Find the relative maximum and minimum values

1. $f(x, y) = x^2 + xy + y^2 - y$

Step 1

$$f_x = 2x + y \qquad f_y = x + 2y - 1 \qquad f_{xy} = 1$$
$$f_{xx} = 2 \qquad f_{yy} = 2$$

Step 2

$$\begin{aligned}
2x + y = 0 &\Rightarrow y = -2x && (1)\\
x + 2y = 1 &\Rightarrow x + 2(-2x) = 1 && (2)\\
&\Rightarrow x - 4x = 1 && (3)\\
&\Rightarrow x = -\frac{1}{3} && (4)\\
&\Rightarrow y = -2\left(-\frac{1}{3}\right) = \frac{2}{3} && (5)
\end{aligned}$$

Step 3

$$\begin{aligned}
D &= f_{xx}\left(-\frac{1}{3}, \frac{2}{3}\right) \cdot f_{yy}\left(-\frac{1}{3}, \frac{2}{3}\right) - \left[f_{xy}\left(-\frac{1}{3}, \frac{2}{3}\right)\right]^2 && (6)\\
&= 2 \cdot 2 - 1^2 = 3 && (7)
\end{aligned}$$

Step 4

$$f\left(-\frac{1}{3}, \frac{2}{3}\right) = \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right) = -\frac{1}{3} \quad (8)$$

f has a minimum $-\frac{1}{3}$ at $\left(-\frac{1}{3}, \frac{2}{3}\right)$, since $D > 0$ and $f_{xx}\left(-\frac{1}{3}, \frac{2}{3}\right) > 0$. (9)

(10)

3. $f(x, y) = 2xy - x^3 - y^2$

Step 1

$$f_x = 2y - 3x^2 \qquad f_y = 2x - 2y \qquad f_{xy} = 2$$
$$f_{xx} = -6x \qquad f_{yy} = -2$$

Step 2

$$2x - 2y = 0 \Rightarrow -2y = -2x \Rightarrow y = x \tag{11}$$

$$2y - 3x^2 = 0 \Rightarrow 2x - 3x^2 = 0 \Rightarrow x(2-3x) = 0 \Rightarrow x = 0 \text{ or } x = \frac{2}{3} \tag{12}$$

Step 3

$$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 \cdot 0 - [2]^2 = -4 \tag{13}$$

$$D = f_{xx}\left(\frac{2}{3}, \frac{2}{3}\right) \cdot f_{yy}\left(\frac{2}{3}, \frac{2}{3}\right) - \left[f_{xy}\left(\frac{2}{3}, \frac{2}{3}\right)\right]^2 = 0 \cdot 0 - [2]^2 = -4 \tag{14}$$

Step 4

$$f(0,0) = 0; \quad f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{4}{27}; \quad f\left(\frac{2}{3}, \frac{2}{3}\right) > f(0,0) \tag{15}$$

$$f \text{ has a maximum of } \frac{4}{27} \text{ at } \left(\frac{2}{3}, \frac{2}{3}\right), \text{ since } D > 0 \text{ and } f_{xx}\left(\frac{2}{3}, \frac{2}{3}\right) < 0. \tag{16}$$

11. $f(x,y) = 4x^2 - y^2$

Step 1

$$f_x = 8x \qquad f_y = -2y \qquad f_{xy} = 0$$
$$f_{xx} = 8 \qquad f_{yy} = -2$$

Step 2

$$8x = 0 \Rightarrow x = 0 \tag{17}$$

$$-2y = 0 \Rightarrow y = 0 \tag{18}$$

Step 3

$$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 \tag{19}$$

$$= 8 \cdot -2 = -16 \tag{20}$$

Step 4

$$f \text{ has a saddle point at } (0,0), \text{ since } D < 0. \tag{21}$$

13. $f(x,y) = e^{x^2+y^2+1}$

Step 1

$$f_x = 2xe^{x^2+y^2+1} \qquad f_y = 2ye^{x^2+y^2+1} \qquad f_{xy} = 4xye^{x^2+y^2+1}$$
$$f_{xx} = 2e^{x^2+y^2+1}\left(2x^2+1\right) \qquad f_{yy} = 2e^{x^2+y^2+1}\left(2y^2+1\right)$$

Step 2

$$2e^{x^2+y^2+1}\left(2x^2+1\right)=0 \Rightarrow x=0 \tag{22}$$
$$2e^{x^2+y^2+1}\left(2y^2+1\right)=0 \Rightarrow y=0 \tag{23}$$

Step 3

$$\begin{aligned} D &= f_{xx}(0,0)\cdot f_{yy}(0,0)-\left[f_{xy}(0,0)\right]^2 && (24)\\ &= 2e\cdot 2e-[0]^2=4e && (25)\end{aligned}$$

Step 4

$$\begin{aligned} & f(0,0)=e^{0^2+0^2+1}=e && (26)\\ & f \text{ has a minimum } e \text{ at } (0,0), \text{ since } D>0 \text{ and } f_{xx}(0,0)>0. && (27)\\ & && (28)\end{aligned}$$

15. Maximizing profit. Safe Shades produces two kinds of sunglasses; one kind sells for \$17, and the other for \$21. The total revenue in thousands of dollars from the sale of x thousand sunglasses at \$17 each and y thousand at \$21 each is given by

$$R(x,y)=17x+21y.$$

The company determines that the total cost, in thousands of dollars, of producing x thousand of the \$17 sunglasses and y thousand of the \$21 sun glasses is given by

$$C(x,y)=4x^2-4xy+2y^2-11x+25y-3.$$

Find the number of each type of sunglasses that must be produced and sold in order to maximize profit.

$$\begin{aligned} P(x,y) &= R(x,y)-C(x,y)=17x+21y-\left(4x^2-4xy+2y^2-11x+25y-3\right) && (29)\\ &= 17x+21y-4x^2+4xy-2y^2+11x-25y+3 && (30)\\ &= -4x^2+4xy-2y^2+28x-4y+3 && (31)\end{aligned}$$

Step 1

$$P_x=-8x+4y+28 \qquad P_y=4x-4y-4 \qquad P_{xy}=4$$
$$P_{xx}=-8 \qquad P_{yy}=-4$$

Step 2

$$\begin{aligned} 4x-4y=4 &\Rightarrow 4x=4+4y \Rightarrow x=1+y && (32)\\ -8x+4y=-28 &\Rightarrow -8(1+y)+4y=-28 && (33)\\ &\Rightarrow -8-8y+4y=-28 && (34)\\ &\Rightarrow -4y=-20 && (35)\\ &\Rightarrow y=5 && (36)\\ 4x-20=4 &\Rightarrow x=6 && (37)\end{aligned}$$

Step 3

$$D = P_{xx}(5,6) \cdot P_{yy}(5,6) - [P_{xy}(5,6)]^2 \tag{38}$$
$$= -8 \cdot -4 - [4]^2 = 16 \tag{39}$$

Step 4

$$P(5,6) = -4(5)^2 + 4(5)(6) - 2(6)^2 + 28(5) - 4(6) + 3 = 67 \tag{40}$$
$$P \text{ has a maximum 67 at } (6,5), \text{ since } D > 0 \text{ and } P_{xx} < 0. \tag{41}$$

Safe Shades must produce 6 thousand $17 and 5 thousand $21 sunglasses to maximize profit.

19. Minimizing the cost of a container. A trash company is designing an open-top, rectangular container that will have a volume of 320 ft^3. The cost of making the bottom of the container is $5 per square foot, and the cost of the sides is $4 per square floor. Find the dimensions of the container that will minimize total cost. (*Hint:* Make a substitution using the formula for volume.)

$$320 = xyz \tag{42}$$
$$z = \frac{320}{xy} \tag{43}$$
$$C(x,y,z) = 5xy + 8yz + 8xz \tag{44}$$
$$= 5xy + 8y\left(\frac{320}{xy}\right) + 2x\left(\frac{320}{xy}\right) \tag{45}$$
$$= 5xy + \frac{2560}{x} + \frac{2560}{y} \tag{46}$$

Step 1

$$C_x = 5y - \frac{2560}{x^2} \qquad C_y = 5x - \frac{2560}{y^2} \qquad C_{xy} = 5$$
$$C_{xx} = \frac{5120}{x^3} \qquad C_{yy} = \frac{2560}{y^3}$$

Step 2

$$5y - \frac{2560}{x^2} = 0 \Rightarrow y = \frac{512}{x^2} \tag{47}$$

$$5x - \frac{2560}{y^2} = 0 \Rightarrow 5x - \frac{2560}{\left(\frac{512}{x^2}\right)^2} = 0 \tag{48}$$

$$\Rightarrow 5x = \frac{2560}{\left(\frac{512}{x^2}\right)^2} \tag{49}$$

$$\Rightarrow x\left(\frac{512}{x^2}\right)^2 = 512 \tag{50}$$

$$\Rightarrow 262144 = 512x^3 \tag{51}$$

$$\Rightarrow x = \sqrt[3]{512} = 8 \tag{52}$$

$$\Rightarrow y = \frac{512}{(8)^2} = 8 \tag{53}$$

Step 3

$$D = C_{xx}(8,8) \cdot C_{yy}(8,8) - \left[C_{xy}(8,8)\right]^2 = 75 \tag{54}$$

Step 4

$$320 = 8 \cdot 8 \cdot z \Rightarrow z = 5 \tag{55}$$

$$C(8,8,5) \approx \$720 \tag{56}$$

The dimensions 8x8 on the bottom and height of 5 ft will minimize the cost to \$720.

20. Two-variable revenue maximization. Boxowitz, Inc., a computer firm, markets two kinds of calculator that compete with one another. Their demand functions are expressed by the following relationships:

$$q_1 = 78 - 6p_1 - 3p_2, \qquad (1)$$

$$q_2 = 66 - 3p_1 - 6p_2, \qquad (2)$$

where p_1 and p_2 are the prices of the calculators, in multiples of \$10, and q_1 and q_2 are the quantities of the calculators demanded, in hundreds of units.

a) Find a formula for the total-revenue function, R, in terms of the variables p_1 and p_2. [*Hint:* $R = p_1q_1 + p_2q_2$; then substitute expressions from equations (1) and (2) to find $R(p_1, p_2)$.]

b) What prices p_1 and p_2 should be charged for each product in order to maximize total revenue?

c) How many units will be demanded?

d) What is the maximum total revenue?

a)

$$\begin{aligned} R(p_1, p_2) &= p_1(78 - 6p_1 - 3p_2) + p_2(66 - 3p_1 - 6p_2) && (57)\\ &= 78p_1 - 6{p_1}^2 - 3p_2p_1 + 66p_2 - 3p_1p_2 - 6{p_2}^2 && (58)\\ &= -6{p_1}^2 + 78p_1 - 6p_1p_2 + 66p_2 - 6{p_2}^2 && (59)\end{aligned}$$

b)

$$R_{p_1} = -12p_1 + 78 - 6p_2 \qquad R_{p_2} = -12p_2 + 66 - 6p_1 \qquad R_{p_1p_2} = -6$$

$$R_{p_1p_1} = -12 \qquad R_{p_2p_2} = -12$$

$$\begin{aligned} 12p_1 + 6p_2 = 78 &\Rightarrow p_2 = \frac{78 - 12p_1}{6} \Rightarrow 13 - 2p_1 && (60)\\ 6p_1 + 12p_2 = 66 &\Rightarrow 6p_1 + 12(13 - 2p_1) = 66 && (61)\\ &\Rightarrow 6p_1 + 156 - 24p_1 = 66 && (62)\\ &\Rightarrow -18p_1 = -90 && (63)\\ &\Rightarrow p_1 = 5 && (64)\\ 12(5) + 6p_2 = 78 &\Rightarrow p_2 = 3 && (65)\end{aligned}$$

c)

$$\begin{aligned} q_1 &= 78 - 6(5) - 3(3) = 39 && (66)\\ q_2 &= 66 - 3(5) - 6(3) = 33 && (67)\\ q_1 + q_2 &= 39 + 33 = 72 && (68)\end{aligned}$$

d)

$$R(5, 3) = 294 \qquad (69)$$

Find the relative maximum and minimum values and the saddle points.

23. $f(x,y) = e^x + e^y - e^{x+y}$

Step 1

$$f_x = e^x - e^{x+y} \qquad f_y = e^y - e^{x+y} \qquad f_{xy} = -e^{x+y}$$
$$f_{xx} = e^x - e^{x+y} \qquad f_{yy} = e^y - e^{x+y}$$

Step 2

$$e^x - e^{x+y} = 0 \Rightarrow x + y = \ln e^x \Rightarrow y = \ln\left(e^x\right) - x \tag{70}$$
$$e^y - e^{x+y} = 0 \Rightarrow e^{\ln(e^x)-x} - e^{x+\ln(e^x)-x} = 0 \Rightarrow x = 0 \tag{71}$$
$$y = \ln\left(e^0\right) - 0 = 0 \tag{72}$$

Step 3

$$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2 \tag{73}$$
$$= 0 \cdot 0 - [-1]^2 = -1 \tag{74}$$

Step 4

$$f \text{ has a saddle point at } (0,0), \text{ since } D < 0. \tag{75}$$