Problem 1: The differential equation for the growth of the mass of a body by accretion is given by Equation 1 and the mass of the body is given by Equation 2

Equation 1) $\frac{dM}{dt} = 4\pi\rho V R(t)^2$ Equation 2) $M(t) = \frac{4}{3}\pi D R(t)^3$

where R is the radius of the body at time t, V is the speed of the infalling material, ρ is the density of the infalling material, and D is the density of the body.

Solve Equation 2 for $R(t)$, substitute this into Equation 1 and simplify.

Solution:

$$R(t) = \left(\frac{3M}{4\pi D}\right)^{1/3} \tag{1}$$

$$\frac{dM}{dt} = 4\pi\rho V \left(\frac{3M}{4\pi D}\right)^{2/3} \tag{2}$$

Problem 2: Integrate your answer to Problem 1 to derive the formula for $M(t)$.

Solution:

$$\frac{dM}{M^{\frac{2}{3}}} = 4\pi\rho V \left(\frac{3}{4\pi D}\right)^{2/3} dt \tag{3}$$

$$\int M^{-\frac{2}{3}} dM = \int 4\pi\rho V \left(\frac{3}{4\pi D}\right)^{2/3} dt \tag{4}$$

$$3M^{\frac{1}{3}} = 4\pi\rho V \left(\frac{3}{4\pi D}\right)^{2/3} t \tag{5}$$

$$M(t) = \left(\frac{4}{3}\pi\rho V\right)^{3} \left(\frac{3}{4\pi D}\right)^{2} t^3 \tag{6}$$

$$\tag{7}$$

Raindrop Condensation: A typical raindrop might form so that its final mass is about 100 milligrams and $D = 1000$ kg/m^3, under atmospheric conditions where $\rho = 1$ kg/m^3 and $V = 1$ m/sec. How long would it take such a raindrop to condense?

Solution:

$$100 \text{ mg} = \left(\frac{4}{3}\pi \left(1 \text{ kg/m}^3\right) (1 \text{ m/sec})\right)^3 \left(\frac{3}{4\pi \left(1000 \text{ kg/m}^3\right)}\right)^2 t^3 \tag{8}$$

$$t^3 = \frac{0.0001 \text{ kg}}{\left(\frac{4}{3}\pi \left(1 \text{ kg/m}^3\right) (1 \text{ m/sec})\right)^3 \left(\frac{3}{4\pi\left(1000 \text{ kg/m}^3\right)}\right)^2} \tag{9}$$

$$t \approx 2.88 \text{ sec} \tag{10}$$

Planet Accretion: A typical rocky planet might form so that its final mass is about that of Earth of 5.9×10^{24} kg, and $D = 3000$ kg/m^3, under conditions where $\rho = 0.0000001$ kg/m^3 and $V = 1$ km/sec. How long would it take such a planet to accret using this approximate mathematical model?

Solution:

$$5.9 \times 10^{24} = \left(\frac{4}{3}\pi \left(0.000001 \text{ kg/m}^3\right) (1 \text{ km/sec})\right)^3 \left(\frac{3}{4\pi \left(3000 \text{ kg/m}^3\right)}\right)^2 t^3 \tag{11}$$

$$t^3 = \frac{0.0001 \text{ kg}}{\left(\frac{4}{3}\pi \left(0.000001 \text{ kg/m}^3\right) (1000 \text{ m/sec})\right)^3 \left(\frac{3}{4\pi\left(3000 \text{ kg/m}^3\right)}\right)^2} \tag{12}$$

$$t \approx 2.33 \times 10^{13} \text{ sec} \quad \text{or} \quad 738,864.28 \text{ years} \tag{13}$$