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Pg. 584 #5-15 odd, 17-18, 21, 23

Evaluate.

5. $\int_0^5 \int_{-2}^{-1} (3x + y)\,dx\,dy$

$$\begin{aligned}
&= \int_0^5 \left[\frac{3x^2}{2} + xy\right]_{-2}^{-1} dy && (1)\\
&= \int_0^5 \left[\left(\frac{3\left(-1\right)^2}{2} + \left(-1\right)y\right) - \left(\frac{3\left(-2\right)^2}{2} + \left(-2\right)y\right)\right] dy && (2)\\
&= \int_0^5 \left[\frac{3}{2} - y - \frac{12}{2} + 2y\right] dy && (3)\\
&= \int_0^5 \left[y - \frac{9}{2}\right] dy && (4)\\
&= \left[\frac{y^2}{2} - \frac{9y}{2}\right]_0^5 && (5)\\
&= \left(\frac{\left(5\right)^2}{2} - \frac{9\left(5\right)}{2}\right) - \left(\frac{\left(0\right)^2}{2} - \frac{9\left(0\right)}{2}\right) && (6)\\
&= \frac{25}{2} - \frac{45}{2} = -\frac{20}{2} = -10 && (7)
\end{aligned}$$

7. $\int_{-1}^1 \int_x^1 xy\,dy\,dx$

$$\begin{aligned}
&= \int_{-1}^1 \left[\frac{xy^2}{2}\right]_x^1 dx && (8)\\
&= \int_{-1}^1 \left[\frac{y^2}{2} - \frac{xy^2}{2}\right] dx && (9)\\
&= y^2 \int_{-1}^1 \left[\frac{1}{2} - \frac{x}{2}\right] dx && (10)\\
&= y^2 \left[\frac{x}{2} - \frac{x^2}{4}\right]_{-1}^1 && (11)\\
&= \frac{y^2}{2}\left[x - \frac{x^2}{2}\right]_{-1}^1 && (12)\\
&= \frac{y^2}{2}\left[1 - \frac{1}{2} - 1 + \frac{1}{2}\right] && (13)\\
&= \frac{y^2}{2}\left(0\right) && (14)\\
&= 0 && (15)
\end{aligned}$$

9. $\int_0^1 \int_{x^2}^{x} (x+y)\,dy\,dx$

$$\begin{aligned}
&= \int_0^1 \left[xy + \frac{y^2}{2}\right]_{x^2}^{x} dx && (16)\\
&= \int_0^1 \left[x\,(x) + \frac{(x)^2}{2} - x\left(x^2\right) - \frac{\left(x^2\right)^2}{2}\right] dx && (17)\\
&= \int_0^1 \left[x^2 + \frac{x^2}{2} - x^3 - \frac{x^4}{2}\right] dx && (18)\\
&= \left[\frac{x^3}{3} + \frac{x^3}{6} - \frac{x^4}{4} - \frac{x^5}{10}\right]_0^1 && (19)\\
&= \frac{1}{3} + \frac{1}{6} - \frac{1}{4} - \frac{1}{10} && (20)\\
&= \frac{20}{60} + \frac{10}{60} - \frac{15}{60} - \frac{6}{60} && (21)\\
&= \frac{9}{60} && (22)\\
&= \frac{3}{20} && (23)
\end{aligned}$$

11. $\int_0^1 \int_1^{e^x} \frac{1}{y}\,dy\,dx$

$$\begin{aligned}
&= \int_0^1 \left[\ln|y|\right]_1^{e^x} dx && (24)\\
&= \int_0^1 \left[\ln|e^x| - \ln|1|\right] dx && (25)\\
&= \int_0^1 x\,dx && (26)\\
&= \left.\frac{x^2}{2}\right|_0^1 && (27)\\
&= \frac{1}{2} && (28)
\end{aligned}$$

13. $\int_0^2 \int_0^x \left(x + y^2\right) dy\, dx$

$$= \int_0^2 \left[xy + \frac{y^3}{3}\right]_0^x dx \tag{29}$$

$$= \int_0^2 \left[x^2 + \frac{x^3}{3}\right] dx \tag{30}$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{12}\right]_0^2 \tag{31}$$

$$= \frac{2^3}{3} + \frac{2^4}{12} \tag{32}$$

$$= \frac{8}{3} + \frac{16}{12} \tag{33}$$

$$= \frac{32}{12} + \frac{16}{12} \tag{34}$$

$$= \frac{48}{12} \tag{35}$$

$$= 4 \tag{36}$$

15. Find the volume of the solid capped by the surface $z = 1 - y - x^2$ over the region bounded on the xy-plane by $y = 1 - x^2$, $y = 0$, $x = 0$, and $x = 1$, by evaluating the integral

$$\int_0^1 \int_0^{1-x^2} \left(1 - y - x^2\right) dy\, dx.$$

$$= \int_0^1 \left[y - \frac{y^2}{2} - x^2 y\right]_0^{1-x^2} dx \tag{37}$$

$$= \int_0^1 \left[\left(1 - x^2\right) - \frac{\left(1 - x^2\right)^2}{2} - x^2\left(1 - x^2\right)\right] dx \tag{38}$$

$$= \int_0^1 \left[1 - x^2 - \frac{1 - 2x^2 + x^4}{2} - x^2 + x^4\right] dx \tag{39}$$

$$= \int_0^1 \left[1 - x^2 - \frac{1}{2} + x^2 - \frac{x^4}{2} - x^2 + x^4\right] dx \tag{40}$$

$$= \int_0^1 \left[\frac{1}{2} - x^2 + \frac{x^4}{2}\right] dx \tag{41}$$

$$= \left[\frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1 \tag{42}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{10} \tag{43}$$

$$= \frac{15}{30} - \frac{10}{30} + \frac{3}{30} \tag{44}$$

$$= \frac{4}{15} \tag{45}$$

For Exercises 17 and 18, suppose that a continuous random variable has a joint probability density function given by

$$f(x,y) = x^2 + \frac{1}{3}xy, \quad 0 \le x \le 1, \quad 0 \le y \le 2.$$

17. $\int_0^2 \int_0^1 f\,(x,y)\,dx\,dy$

$$= \int_0^2 \left[\frac{x^3}{3} + \frac{1}{6}x^2 y\right]_0^1 dy \tag{46}$$

$$= \frac{1}{3}\int_0^2 \left[1 + \frac{1}{2}y\right] dy \tag{47}$$

$$= \frac{1}{3}\left[y + \frac{1}{4}y^2\right]_0^2 \tag{48}$$

$$= \frac{1}{3}\left[2 + \frac{1}{4}\,(2)^2\right] \tag{49}$$

$$= \frac{1}{3}\,[2+1] \tag{50}$$

$$= 1 \tag{51}$$

18. Find the probability that a point (x,y) is in the region bounded by $0 \le x \le \frac{1}{2}, 1 \le y \le 2$, by evaluating the integral

$$\int_1^2 \int_0^{1/2} f(x,y)\,dx\,dy.$$

$$= \frac{1}{3}\int_1^2 \left[x^3 + \frac{1}{2}\right]_0^{1/2} dy \tag{52}$$

$$= \frac{1}{3}\int_1^2 \left[\left(\frac{1}{2}\right)^3 + \frac{1}{2}\right] dy \tag{53}$$

$$= \frac{1}{3}\int_1^2 \left[\frac{1}{8} + \frac{4}{8}\right] dy \tag{54}$$

$$= \frac{1}{3}\int_1^2 \frac{5}{8}\,dy \tag{55}$$

$$= \frac{1}{3}\left[\frac{5y}{8}\right]_1^2 \tag{56}$$

$$= \frac{1}{3}\left[\frac{10}{8} - \frac{5}{8}\right] \tag{57}$$

$$= \frac{5}{24} \tag{58}$$

A triple iterated integral such as

$$\int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

is evaluated in much the same way as a double iterated integral. We first evaluate the inside x-integral, treating y and z as constants. Then we evaluate the middle y-integral, treating z as a constant. Finally, we evaluate the outside z-integral. Evaluate these triple integrals.

21. $\displaystyle\int_0^1 \int_1^3 \int_{-1}^2 (2x+3y-z)\,dx\,dy\,dz$

$$= \int_0^1 \int_1^3 \left[x^2+3xy-xz\right]_{-1}^2 dy\,dz \tag{59}$$

$$= \int_0^1 \int_1^3 \left[(2)^2+3\,(2)\,y-(2)\,z-(-1)^2-3\,(-1)\,y+(-1)\,z\right] dy\,dz \tag{60}$$

$$= \int_0^1 \int_1^3 [4+6y-2z-1+3y-z]\,dy\,dz \tag{61}$$

$$= \int_0^1 \int_1^3 [9y-3z+3]\,dy\,dz \tag{62}$$

$$= \int_0^1 \left[\frac{9y^2}{2}-3yz+3y\right]_1^3 dz \tag{63}$$

$$= \int_0^1 \left[\frac{9\,(3)^2}{2}-3\,(3)\,z+3\,(3)-\frac{9\,(1)^2}{2}+3\,(1)\,z-3\,(1)\right] dz \tag{64}$$

$$= \int_0^1 \left[\frac{81}{2}-9z+9-\frac{9}{2}+3z-3\right] dz \tag{65}$$

$$= \int_0^1 [-6z+42]\,dz \tag{66}$$

$$= \left[-3z^2+42z\right]_0^1 \tag{67}$$

$$= -3\,(1)^2+42\,(1) \tag{68}$$

$$= -3+42 \tag{69}$$

$$= 39 \tag{70}$$

23. $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz\, dz\, dy\, dx$

$$= \int_0^1 \int_0^{1-x} \left[\frac{xyz^2}{2}\right]_0^{2-x} dy\, dx \tag{71}$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[xy\,(2-x)^2\right] dy\, dx \tag{72}$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[xy\,(2-x)\,(2-x)\right] dy\, dx \tag{73}$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[xy\left(4-4x+x^2\right)\right] dy\, dx \tag{74}$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[4xy-4x^2y+x^3y\right] dy\, dx \tag{75}$$

$$= \frac{1}{2} \int_0^1 \left[2xy^2-2x^2y^2+\frac{x^3y^2}{2}\right]_0^{1-x} dx \tag{76}$$

$$= \frac{1}{2} \int_0^1 \left[2x\,(1-x)^2-2x^2\,(1-x)^2+\frac{x^3\,(1-x)^2}{2}\right] dx \tag{77}$$

$$= \frac{1}{2} \int_0^1 \left[2x\,(1-x)\,(1-x)-2x^2\,(1-x)\,(1-x)+\frac{x^3\,(1-x)\,(1-x)}{2}\right] dx \tag{78}$$

$$= \frac{1}{2} \int_0^1 \left[2x\left(1-2x+x^2\right)-2x^2\left(1-2x+x^2\right)+\frac{x^3\,(1-2x+x^2)}{2}\right] dx \tag{79}$$

$$= \frac{1}{2} \int_0^1 \left[2x-4x^2+2x^3-2x^2+4x^3-2x^4+\frac{x^3-2x^4+x^5}{2}\right] dx \tag{80}$$

$$= \frac{1}{2} \left[x^2-\frac{4x^3}{3}+\frac{x^4}{2}-\frac{2x^3}{3}+x^4-\frac{2x^5}{5}+\frac{x^4}{8}-\frac{x^5}{5}+\frac{x^6}{12}\right]_0^1 \tag{81}$$

$$= \frac{1}{2} \left[1-\frac{4}{3}+\frac{1}{2}-\frac{2}{3}+1-\frac{2}{5}+\frac{1}{8}-\frac{1}{5}+\frac{1}{12}\right] \tag{82}$$

$$= \frac{1}{2} \left[\frac{240}{240}-\frac{320}{240}+\frac{120}{240}-\frac{160}{240}+\frac{240}{240}-\frac{96}{240}+\frac{30}{240}-\frac{48}{240}+\frac{20}{240}\right] \tag{83}$$

$$= \frac{1}{2} \left[\frac{26}{240}\right] \tag{84}$$

$$= \frac{13}{240} \tag{85}$$