Logic
Review & Exercises

Outline

- Logical Entailment
- Soundness and Completeness (Theory only)
- PL exercises with truth tables
- FOL exercises with quantifiers

Logical Entailment

- $KB \vDash \alpha$ only if in every model where KB is *True*, α is also *True*
  - $KB \vDash \alpha \quad \text{iff} \quad M(KB) \subseteq M(\alpha)$
  - $x = 0 \vDash xy = 0$
  - *All cats are pink* $\vDash$ *My cat is pink*
  - $P \vDash (P \vee Q)$
  - $(P \wedge Q) \vDash (P \vee Q)$

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

- The truth of sentence(s) in KB guarantee the truth of the sentence α and the falsity of α guarantees the falsity of KB
- Exercise: Use truth tables to show that

$$((Q \Rightarrow P) \vee R) \nvDash (Q \Rightarrow P)$$

Entailment (Negation) Test

- A set of sentences KB entails α (KB $\models \alpha$) iff there is NO truth assignment that makes the sentences in KB True AND α False
  1. Assume $KB \models \alpha$
  2. Negate α, i.e. $\neg\alpha$
  3. Join KB and $\neg\alpha$, i.e $KB \wedge \neg\alpha$
  4. If the result is a contradiction, then KB $\models \alpha$ otherwise KB $\not\models \alpha$
- Examples:
  - "All cats are pink" $\models$ "My cat is pink" because the sentence "All cats are pink and my cat is not pink" doesn't make sense
  - "All cats are pink" $\not\models$ "My cat likes to nap" because the sentence "All cats are pink and my cat does not like to nap" is fine

Entailment Relations (Natural Language)

- Mutual entailment $(KB \vDash \alpha) \wedge (\alpha \vDash KB)$

KB	Mary loves John	All cats are pink	Mary is a woman
α	John is loved by Mary	There is no cat that is not pink	Mary is an adult female

- Contradiction $KB \nvDash \alpha$

KB	Mary loves John	All cats are pink	Mary is a woman
α	Mary does not love John	My cat is black	Mary is an adult male

- Inclusion $KB \vDash \alpha$

KB	Mary loves John	All cats are pink	Mary is a woman
α	Mary is fond of John	My cat is coloured	Mary is human

Soundness and Completeness of a Logical System (Theory)

- **Soundness**: Everything that is provable in the system is actually true
  - If $KB \vdash \alpha$ then $KB \models \alpha$
  - If something is derivable, then it is valid
  - The inference procedure/proofs don't include anything that's wrong
- **Completeness**: Everything that is true in the system is provable
  - If $KB \models \alpha$ then $KB \vdash \alpha$
  - If something is valid, then it is derivable
  - The inference procedure/proofs don't exclude anything that's correct
- Propositional Logic is sound and complete

Sentences in Propositional Logic

- Conjunction ($\wedge$) is normally translated to "and", also "but", "moreover", "however", "although", "while" and "even though"
- Sentences/Statements

  S The store is open today

  M Mary is going to the store today

  J John is going to the store today

- Define propositional sentences using the variables above and logical connectives ($\neg, \wedge, \vee,$) for the following:
- "Either John or Mary (or both) are going to the store today" $J \vee M$
- "John is going to the store today, **but** Mary isn't" $J \wedge \neg M$
- "The store is open today, **however**, **neither** John nor Mary is going"
  $S \wedge (\neg J \wedge \neg M)$ apply De Morgan's law $S \wedge \neg(J \vee M)$

Propositional Logic Truth Table Exercises

- Show if these formulae are valid, satisfiable (contingent) or unsatisfiable (invalid) using truth tables
  - $(P \vee \neg P)$
  - $(P \Rightarrow Q) \vee (Q \Rightarrow \mathrm{P})$
- Compute truth tables for the following formula
  - $(P \vee Q) \wedge (\neg Q \wedge \neg P)$
  - $(\neg P \Rightarrow Q) \vee ((P \wedge \neg R) \Leftrightarrow Q)$
- Check for logical equivalence using truth tables (equal truth assignments)
  - $P \vee (\neg P \Rightarrow Q) \equiv P \vee Q$?

Drawing Truth Tables

1. How many variables are there? Draw t.t. with 2^n rows, e.g. 2 variables will have 4 rows and 3 variables will have 8 rows
2. Write out all combinations of *T*s and *F*s for all variables, can use 1 for *T* and 0 for *F* as well in ascending order
3. Ensure that anything in brackets is calculated independently
4. Compute the result of the formula in the last column

Truth Table Example

$(\neg P \Rightarrow Q) \vee ((P \wedge \neg R) \Leftrightarrow Q)$

3 variables, so $2^3 = 8$ rows

P	Q	R	$\neg P$	$\neg P \Rightarrow Q$ (A)	$\neg R$	$P \wedge \neg R$ (B)	$B \Leftrightarrow Q$ (C)	$A \vee C$
F	F	F						
F	F	T						
F	T	F						
F	T	T						
T	F	F						
T	F	T						
T	T	F						
T	T	T						

FOL – Nested Quantifiers

Quantification	**When True**	When False
$\forall x \forall y\ P(x,y)$	***P(x,y)* is true for every pair *x,y***	There is a pair *x, y* for which *P(x,y)* is false
$\forall x \exists y\ P(x,y)$	**For every *x* there is a *y* for which *P(x,y)* is true**	There is an *x* such that *P(x,y)* is false for every *y*
$\exists x \forall y\ P(x,y)$	**There is an *x* for which *P(x,y)* is true for every *y***	For every *x* there is a *y* for which *P(x,y)* is false
$\exists x \exists y\ P(x,y)$	**There is a pair x, y for which *P(x,y)* is true**	*P(x,y)* is false for every pair *x, y*

FOL Nested Quantifiers Ex1

- Translate the following statement into English
- $\forall x \forall y (x + y = y + x)$
- Domain: real numbers
-

FOL Nested Quantifiers Ex2

- Translate the following statement into English
- $\forall x \exists y (x = -y)$
- Domain: real numbers
-

FOL Nested Quantifiers Ex3

- Translate the following statement into English
- $\forall x \forall y \, (\, (x > 0) \land (y < 0) \rightarrow xy < 0) \,)$
- Domain: real numbers
-

Quantifiers in Human Domain 1

- Let our universe of discourse be human beings, and let $Likes(x, y)$ mean x likes y.
  - $\forall x \exists y \text{ Likes}(x, y)$
  - $\exists y \forall x \text{ Likes}(x, y)$
  - $\forall y \exists x \text{ Likes}(x, y)$
  - $\exists x \forall y \text{ Likes}(x, y)$

Quantifiers in Human Domain 2

- "Men and women are welcome to attend."
  - $\forall x \; Man(x) \land Woman(x) \rightarrow CanAttend(x)$
  - $\forall x \; Man(x) \lor Woman(x) \rightarrow CanAttend(x)$
- Disjunction ($\lor$) means **at least one** of the terms is true
- If you don't want both to be true (exclusive OR), then the sentence should be $(p \lor q) \land \neg(p \land q)$

Key Points

- Entailment (using truth tables)
- Represent English sentences in Propositional Logic and vice versa
- Truth Tables for validity, satisfiability, unsatisfiability, logical equivalence
- Represent English sentences in First Order Logic and vice versa
- Nested quantifiers in FOL

Tips

- Order truth table values in ascending or descending order (use 0/1 if more comfortable than F/T)

P	Q	R		
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

P	Q	R		
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

P	Q	R		
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

P	Q	R		
1	1	1		
1	1	0		
1	0	1		
1	0	0		
0	1	1		
0	1	0		
0	0	1		
0	0	0		

FOL to English

- Give only ONE translation in English
- If you give more than one sentence with different meanings, then only your FIRST sentence will be accepted
- Avoid using variable names such as x or y in English (except for some cases involving real numbers)

English to FOL

- Should NOT use a predicate inside another predicate
  - `Big(Car(x))` Incorrect
  - `Big(x)` ∧ `Car(x)` Correct
- Use the same variable to refer to the same thing
  - `∀x Big(x) →` `∃y Car(y)`
  - This means `x` and `y` are most likely different things
  - `∀x Big(x) →` `Car(x)`