*April 9th, 2018*

**Exercise**

1. The network notations are as follows:  
: the flow on link ,;

: the flow on path  connecting origin  and destination ,,,;

: indicator variable;

：the link-path incidence matrix with elements .

Assume that the first path from origin node 1 to destination node 4 uses link 4 and the second one uses link 1 and 3; similarly, assume that the first path from origin node 2 to destination node 4 uses link 2 and 3 and the second one uses link 5.

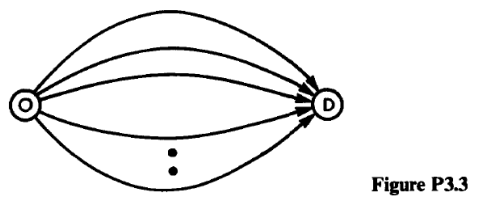
The link-path incidence matrix for this network is as follows:



The flow on link 3 can be written as:



2. The Figure 3.3 has one O-D pair and many paths connecting this O-D pair, and each path has only one link.



Let  denotes the number of paths and the number of links;  denotes the flow on path(or link) .

The equivalent minimization program in user-equilibrium station can be written as:  


Subject to



The Lagrangian of this program can be formulated as:



Subject to 

The first-order condition of Lagrangian can be written as:

for any 



That is



At the stationary point, the two possible combinations of path flow and travel time are:



At any event, the value of multiplier , which indicates that the value of multiplier is less than or equal to the travel time of any path of this O-D pair, that is, the value of multiplier equals to the minimum travel time of this O-D pair.

3. The network has one O-D pair, four paths and four links.

Let  denotes the number of links; denotes the number of paths;  denotes the flow on link ; denotes the flow on path ;denotes the travel time on link ;denotes the travel time on path .

Assume that the first path uses link 1 and 3; the second path uses link 1 and 4; the third path uses link 2 and 3; the four path uses link 2 and 4.

The equivalent minimization program can be written as:



That is:



The program above can be solved by GAMS. The result is as follows:



It can be found that: . Thus, the flow pattern satisfies the user equilibrium principle.

不能把 trip rate 当成4个流来看待，因为在大型交通网络中，路径中多增加一个流对该路径出行时间的影响微不足道。而在此问题中，如果把trip rate 当成4个流看待，增加一个流对路径出行时间的影响非常大。因此应该把流当成是连续变量来看待，也就是x和f可以取小数值。

4. The UE program can be written as:



The solution is as follows:



The first-order conditions are as follows:



The solution of the first order condition is as follows:



5.(a) the Hessian can be formulated as



The leading minor in the upper right corner of Hessian matrix are as follows:







Since leading minor in the upper right corner of Hessian matrix are not all positive, therefore, the Hessian is not a positive-definite marix.

(b)The first derivative of the UE objective function with respect to the path flow is as follows:



The path-flow Hessian of the UE objective function is as follows:



It can be guaranteed that



Because , all the links of the path for the O-D pair must belongs to the arc set. Now, the value of the elements in Hessian is discussed. That is, the value of has to be discussed.

Let denotes the set of all the links of the path for the O-D pair .

(1) The value of the elements in Hessian is positive, that is



It means that there are at least one link satisfying



The elements in Hessian



(2) The value of the elements in Hessian equals zero, that is



It means that all the links appearing in the path for the O-D pair  satisfies



The elements in Hessian



Hessian is positive definite if and only if . Specially, if all the off-diagonal elements of the Hessian, ,are zero and all diagonal elements are positive, then the Hessian must be positive definite.

For the diagonal elements, it can be guaranteed that



For the off-diagonal elements, if all the links appearing in the path for the O-D pair  satisfies will not appear in the path for the O-D pair , that is



Then it can be guaranteed that



Then Hessian must be positive definite.

Conclusion：usually, it is difficult to guarantee that all the off-diagonal elements of the Hessian zeros, thus the path-flow Hessian of the UE objective function is usually not a positive-definite matrix. To guarantee the Hessian be positive definite, any two paths for every two O-D pairs and and any two paths for each O-D pair must not has intersection, that is ,any two paths can't have common link.

6.Let denotes the increasing function, that is, the first derivative of the function is positive, which can be written mathematically as:



The integral of this function is . To demonstrate the integral is a convex function, then it must be satisfied that the Hessian of integral function is positive definite.

The first derivative of the integral function is as follows:



The second derivative of the integral function is as follows:



Thus, the Hessian is positive definite, and the integral of an increasing function is always a convex function.

7. Let  denotes the number of paths and the number of links;  denotes the flow on path(or link) ; denote the travel time on link .

The system-optimization program can be written as:



Subject to



The SO flow pattern minimizes the total travel time in the system. At the optimum, the marginal travel time on all used paths are equal and less than or equal to the marginal travel time on all unused paths on a given O-D pair.

However, the UE flow pattern has no intuitive interpretation. At the optimum, the travel time on all used paths are equal and less than or equal to the travel time on all unused paths on a given O-D pair.

In some special, the SO flow pattern is identical to the UE flow pattern, only if the objective functions are identical

8.The notation is the same as the problem 3.3.

The SO program can be formulated as



That is



The program above can be solved by GAMS. The result is as follows:



The total travel time in the system is 57.8433 resulting from the flow pattern of SO program. However, in the UE flow pattern, the total travel time in the system is 59.5160,which is higher than the total travel time in the SO flow pattern. The reason is that, in the UE equilibrium, the individual choice of route is carried out with no consideration of the effect of this action on other network users.

9. To demonstrate the SO program have a unique solution, the Hessian of the objective function in terms of flows must be positive definite.



If the link performance function is strictly increasing and convex, then the



And

.

Therefore, the diagonal elements in Hessian is positive.

As analysed in problem [3.5b], discuss the positive definite of Hessian from two conditions.

Firstly, if the off-diagonal elements are equals zero, that is



then



Otherwise,



Then, the off-diagonal elements



In this cases, it is very difficult to guarantee that the Hessian is positive definite.

10.(a) If the travel time



The objective function of the SO program is



Which is identical to the objective function of the UE program.

Therefore, the solution of the SO program with travel time is an UE flow pattern.

(b) If the travel time



The objective function of the UE program is



Which is identical to the objective function of the SO program.

Therefore, the solution of the UE program with travel time is an SO flow pattern.

11. As the flow over the network become smaller, the slope of the performance function is close to zero. And the travel time is a constant, it does not vary with the flow. Assuming that the travel time is .

For the UE program, the objective function is



For the SO program, the objective function is



Both the two objective functions are equal. Therefore, as the flows over the network becomes smaller, the SO flow pattern tends to grow in similarity to the UE flow pattern.

12.(a) The UE program can be formulated as:



Subject to



The solution is as follows:



The marginal travel time for the freeway is



The marginal travel time for the city street is



Since , it is not an SO solution.

(b) The SO program can be formulated as:



Subject to



The solution is as follows:



Since the travel time on freeway isn't equal to the travel time on city street, that is, , it is not an UE solution.

13.The system optimization program before the link addition can be written as:



Subject to



The result of this program calculated by GAMS is as follows:



The solution of SO program just is the UE flow pattern.

The total travel time of the SO flow pattern before the link addition is



The system optimization program after the link addition can be written as:



Subject to



The result of this program calculated by GAMS is as follows:



The total travel time of the SO flow pattern after the link addition is



*It can be concluded that* the total travel time on the SO flow patterns are identical before and after the link addition. After some links addition in the network, under the SO flow pattern, the total travel time would never be increased if the optimal station can't be improved.

14.(a)Let denotes the toll on paths . The product of multiplying has the same units of measurement as the travel time; denotes the Lagrangian multiplier of the UE program; denotes the Lagrangian multiplier of the SO program.

**At the UE flow pattern**, all the paths can be divided into two categories:

If the path carry flow, then



If the path does not carry flow, then



The KKT conditions of the UE program can be written as:



**At the SO flow pattern**, all the paths can be divided into two categories:

If the path carry flow, then



If the path does not carry flow, then



The KKT conditions of the SO program can be written as:



Only when the solutions of the two program are equal, the total travel time in the network can be minimized under the condition of setting a toll on each of the paths. Then



That is



In this case, the total travel time in the network is minimized.

(b) If the link performances are linear, that is the derivative of the performances function are constants, letting denoting the constants. That is



Then the toll expression is



(c) this approach applying in the real urban network is a good idea to minimize the total travel time over the network. Under this method using, every motorist minimize his or her travel cost(including travel time and toll). Under UE equilibrium, no traveler can improve his travel cost by unilaterally changing routes. By controlling the toll on each path, make the flow pattern equals to the SO flow pattern, then the total travel time over the network is minimized and the system is optimal. The crucial point is how to control the toll strictly.