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Research paper

The generalized Duffing oscillator

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ABSTRACT

A generalized Duffing oscillator is considered, which takes into account high-order derivatives and power nonlinearities. The Painlevé test is applied to study the integrability of the mathematical model. It is shown that the generalized Duffing oscillator passes the Painlevé test only in the case of the classic Duffing oscillator which is described by the second-order differential equation. However, in the general case there are the expansion of the general solution in the Laurent series with two arbitrary constants. This allows us to search for exact solutions of generalized Duffing oscillators with two arbitrary constants using the classical Duffing oscillator as the simplest equation. The algorithm of finding exact solutions is presented. Exact solutions for the generalized Duffing oscillator are found for equations of fourth, sixth, eighth and tenth order in the form of periodic oscillations and solitary pulse.

1. Introduction

The Duffing oscillator was first proposed in 1918 and is described by a second-order nonlinear differential equation [1–7]

$$\mu\, y_{tt} + A\, y + B\, y^3 = 0. \tag{1}$$

Eq. (1) is a mathematical model for describing many physical and biological processes in various areas of science. In particular, using this equation, we can describes the motion of a particle of mass μ under the action of a potential

$$U(y) = \frac{A\, y^2}{2} + \frac{B\, y^4}{4}. \tag{2}$$

One of remarkable features for processes described by Eq. (1) is that the chaotic behavior arises under the action of an external perturbation $F = F_0 \cos(\omega t)$.

This paper is devoted to consideration of the generalized Duffing oscillator in the form

$$\sum_{n=1}^{N} a_n\, y_{2n,t} + b_0\, y - \sum_{m=1}^{M} b_m\, y^{2m+1} = 0, \quad y_{2n,t} = \frac{d^{2n}y}{dt^{2n}}, \quad (M \le N), \tag{3}$$

where N and M are integers, a_n and b_m are parameters of Eq. (3). In the case $N = M = 1,\ a_1 = \mu,\ b_0 = A,\ b_1 = -B$ from Eq. (3) we have Eq. (1).

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At $N \neq 1$ and $M \neq 1$ Eq. (3) appears in the study of a number of mathematical models described by nonlinear partial differential equations. For example, if we use the traveling wave reduction $u(x,t) = y(t)$, $t = x + b_0\,\tau$ in the Kawahara equation [8–14]

$$u_\tau + a_2\, u_{xxxxx} + a_1\, u_{xxx} - 3\, b_1\, u^2\, u_x - 5\, b_2\, u^4\, u_x = 0, \tag{4}$$

we obtain after integration over t the nonlinear ordinary differential equation in the form

$$a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 - b_2\, y^5 = 0 \tag{5}$$

that is Eq. (3) at $N = M = 2$.

The Duffing oscillator is known to coincide with a second-order nonlinear differential equation for describing the real part of the nonlinear Schrödinger equation. Eq. (3) appear if we also use traveling wave reduction to search for soliton solutions to numerous generalizations of the nonlinear Schrodinger equation. In particular, if we look for soliton solutions of the generalized nonlinear Schrödinger equation [15–26]

$$q_\tau + a_1\, q_{xx} + i\,\beta_1\, q_{xxx} + a_2\, q_{xxxx} + i\,\beta_2\, q_{xxxxx} + a_3\, q_{xxxxxx} - b_1\, |q|^2\, q - b_2\, |q|^4\, q - b_3\, |q|^6\, q = 0 \tag{6}$$

and use the traveling wave variables $q(x,\tau) = y(t)\, e^{i(k\,x + b_0\,\tau)}$ again in Eq. (6) we have Eq. (3) at $N = M = 3$ in the form

$$a_3\, y_{6,t} + a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 - b_2\, y^5 - b_3\, y^7 = 0. \tag{7}$$

A special kind of Eq. (7) at $b_1 \neq 0$ and $b_2 = b_3 = 0$ can be used to interpret the phenomenon of seismic wave propagation [27,28]. We believe that the generalized Duffing oscillator can be also used for description of other physical processes.

Let us demonstrate that there is the first integral of Eq. (1). Multiplying Eq. (1) on y_t, we get

$$y_t^2 + \frac{A}{\mu}\, y^2 + \frac{B}{2\mu}\, y^4 = C_1. \tag{8}$$

Using in (8) new variable and parameters

$$y = \left(\frac{12\,\mu}{B}\right)^{1/2} R, \quad a = -\frac{A}{\mu}, \quad d = \frac{BC_1}{12\,\mu^2}, \tag{9}$$

we have the equation

$$R_t^2 - a\, R^2 + 6\, R^4 - d = 0. \tag{10}$$

The general solution of Eq. (10) and hence Eq. (1) is expressed in terms of the elliptic Weierstrass and Jacobi functions.

The purpose of this paper is to consider the integrability of hierarchy (3). Using the Painlevé test we demonstrate that Eq. (3) does not posses the Painlevé property in the general case but there are exact solutions of the generalized Duffing oscillator (3) which are expressed via the general solution of Eq. (10). We give a detailed study of Eq. (3) taking into account $N = 2$, $N = 3$, $N = 4$ and $N = 5$.

2. Application of the Painlevé test to Eq. (3).

The Panlevé test is one of the powerful approaches for studying the analytical properties of nonlinear differential equations. The current state of this approach is based on the brilliant works of the great nineteenth century scientists of H. Poincare, K. Weierstrass, S. Kovalevskaya, L. Fuchs, E. Picard, P. Painlevé, B. Gambier and a number of their followers [29–34]. The results of applying the Painlev e test led to impressive results in determining the integrability of differential equations and are now well known. It is known that the application of the Painlevé test allows us to obtain the necessary condition for the integrability of nonlinear ordinary differential equations [33–37].

One of the well-known algorithms for testing differential equations on the Painlevé test is the Kovalevskaya approach, which can be used to determine whether the expansion of the general solution into a Laurent series has the required number of arbitrary constants or not [37–40]. The differential equation pass the Painleve test when the number of arbitrary constants in the expansion is equal to the order of the differential equation. In this case, one can hope that we can find the general solution for differential equation and we have the integrability of this nonlinear differential equation [33]. However, nonlinear differential equations are often encountered for solutions of which the expansion in the Laurent series exist, but the expansion has less the arbitrary constants of than the order of the equation. In this case, one can look for solutions to the non-linear differential equation with less number of arbitrary constants.

A widely known algorithm for the Painlevé test is currently a modification of the Kovalevskaya algorithm proposed in work [38–40]. This modification consists of three steps. At the first step, the pole of the general solution of the nonlinear differential equation and the number of branches of the general solution are determined. Algorithmically, this is determined by substituting $y(t) = s_0\,(t - t_0)^k$ (where t_0 is the value of pole for general solution) into equation with the leading terms.

At the second step of the Kovalevskaya algorithm the the Fuchs indices for the expansion of the general solution in the Laurent series are determined [38–40]. For this purpose the formula is used

$$y(t) = \frac{s_0}{(t - t_0)^p} + s_j\,(t - t_0)^{j-p}. \tag{11}$$

Here s_j is one of coefficients of the expansion for expansion the solution $y(t)$ in the Laurent series which cannot be found. The Fuchs indices can be obtained by means of a substitution of expression (11) into the equation with the leading members and equating the expression with the first power s_j to zero.

At the third stage we substitute the expansion of the general solution in the Laurent series with indeterminate coefficients [38–40]

$$y(t) = \frac{s_0}{(t-t_0)^p} + \frac{s_1}{(t-t_0)^{p-1}} + \ldots + s_p + s_{p+1}\,(t-t_0) + \ldots \tag{12}$$

into the original differential equation. At this step we check the existence of the arbitrary constants in the Laurent series of the general solution for the equation studied. In the case when there are necessary number of arbitrary constants in the Laurent series we have the necessary condition for the integrable nonlinear differential equation. Considering results of the expansion for the general solution in the Laurent series we can draw conclusions regarding closed-form solutions.

Let us apply the Painlevé test to Eq. (3) to understand the integrability of this equation.

Proposition 1. *Eq. (3) passes the first step of the Painlevé test if the result of dividing N by M is an integer.*

Proof. The equation with the leading members corresponding to Eq. (3) can be written as the following

$$a_N\, y_{2N,t} - b_M\, y^{2M+1} = 0. \tag{13}$$

Using the first step of the Painlevé test and substituting $y(t) = s_0\,(t-t_0)^k$ into Eq. (13), we have the order of the pole for the general solution of Eq. (3) in the form

$$p = -\frac{N}{M}. \tag{14}$$

From condition (14) follows that Eq. (3) passes the first step of the Painlevé test if p is negative integer. For a example if $N=3$ then value M can be equal to $M_1 = 1$ and $M_3 = 3$. In the case N = 6 we have the following value for M: $M_1 = 1$, $M_2 = 2$, $M_3 = 3$ and $M_4 = 6$. □

Proposition 2. *Eq. (3) passes the Painlevé test only at $N = M = 1$. At $N \neq 1$ or $M \neq 1$ Eq. (3) does not pass the Painlev e test.*

Proof. For $N = M = 1$, Eq. (3) is reduced to second-order Eq. (1). The general solution of this equation is expressed in terms of an elliptic functions. Under additional conditions on the parameters of the equation, elliptic functions can be degenerated and solutions can be expressed in terms of trigonometric, hyperbolic and rational functions [41–43]. Therefore, the proof of the proposition is reduced to the proof of fact that at $N \neq 1$ Eq. (3) does not pass the Painlevé test. Let us show that there are two integers of the Fuchs indexes at the the expansion of the general solution for Eq. (3) in the Laurent series. □

First of all let us consider the case $M = N$. In this case the formula for finding the Fuchs indices for one of branches for the general solution takes the form

$$y(t) = \left(\frac{a_{2N}\,(2N)!}{b_N}\right)^{1/N} \frac{1}{t} + s_j\, t^{j-1}. \tag{15}$$

Substituting (15) into Eq. (13) at $M = N$ and equating expressions at s_j to zero, we obtain the following equation for the finding of the Fuchs indices

$$E_1(j) = (j-1)(j-2)\ldots(j-2N) - (2N+1)! = 0. \tag{16}$$

Eq. (16) is found taking into account the following considerations. The first term is obtained at first power s_j by calculating the $2N$ derivative of $y(t)$, and the second term is determined at first power s_j by calculating $y(t)^{2N+1}$ power. The form of the formula is confirmed by the induction.

From Eq. (16) we see that there are two only real solutions of Eq. (16) that are integers

$$j_1 = -1, \qquad j_2 = 2N+2, \tag{17}$$

but the remaining $N-2$ roots of Eq. (17) are complex numbers.

For example, at $N = 3$ we have the following Fuchs indices

$$j_1 = -1, \quad j_2 = 8, \quad j_{3,4,5,6} = \frac{7}{2} \mp \frac{1}{2}\sqrt{-23 \mp 24\, i\sqrt{6}}. \tag{18}$$

In the case $a_6 = 1$, $b_3 = 720$ and $b_1 = b_2 = 0$ the Laurent series for the general solution of Eq. (7) at $N = 3$ and $M = 3$ can be written as the following

$$\begin{aligned} y(t) = &\pm \frac{1}{t-t_0} \pm \frac{a_4}{210}\,(t-t_0) \pm \left(\frac{a_2}{2520} - \frac{{a_4}^2}{14700}\right)(t-t_0)^3 \\ & b \pm \left(\frac{b_0}{5040} - \frac{a_4 a_2}{88200} \pm \frac{13\,{a_4}^3}{9261000}\right)(t-t_0)^5 + s_8\,(t-t_0)^7 + \ldots. \end{aligned} \tag{19}$$

So, it can be seen that the expansion of the general solution of Eq. (3) into the Laurent series exists, and there are two arbitrary constants t_0 and s_8 in the expansion. It means that the generalized Duffing equation is non-integrable, but it can have exact solutions with two arbitrary constants corresponding to the general solution of the second-order ordinary differential equation.

In the case $M=1$ in Eq. (3), the order of pole for general solution is equal to N and we have the following formula for finding the Fuchs indices

$$y(t)=\left(\frac{a_{2N}\,(3\,N-1)\,!}{b_1\,(N-1)\,!}\right)^{1/2}\frac{1}{t^N}+c_j\,t^{j-N}. \tag{20}$$

The equation for finding the Fuchs indices at $M=1$ takes the form

$$E_2(j)=(j-1)(j-2)\dots(j-3\,N)\;-\;\frac{(3\,N-1)\,!}{(N-1)\,!}\;=0. \tag{21}$$

There are two real roots of Eq. (21)

$$j_1=-1,\qquad j_2=4\,N. \tag{22}$$

The remaining roots of Eq. (21) are complex numbers too. For example, at $N=3$ we have roots in the form

$$j_1=-1,\qquad j_2=12,\quad j_{3,4,5,6}=\frac{11}{2}\mp\frac{1}{2}\sqrt{-67\mp 4\,i\sqrt{1151}}. \tag{23}$$

We have found that there are two values of the Fuchs indices for the expansion of the general solution in the Laurent series that are integers. We checked that there are two arbitrary constants in the Laurent series for the general solution of low-order equations corresponding to the integer Fuchs indices.

The found Fuchs indices for solution of Eq. (3) provide important information about this equation. The Fuchs index $j_1=-1$ corresponds to an arbitrary value of pole t_0 for solution $y(t)$. The Fuchs index $j_2=2\,N+2$ at $M=N$ and $j_2=4\,N$ at $M=1$ indicates the value of the pole for the leading members of the first integral for Eq. (3). The first integral of Eq. (3) can be found by multiplying all monomials of Eq. (3) by y_t and integrating the expression by t. As a result we get

$$\sum_{n=1}^{N}a_n\left[\sum_{k=1}^{2n-1}(-1)^{k-1}\,y_{2n-k,t}\,y_{k,t}\right]+b_0\,y^2-\sum_{m=1}^{M}\frac{2\,b_m}{2m+2}\,y^{2m+2}=C_1. \tag{24}$$

For example from Eq. (24) at $N=M=1$ we obtain the first integral in the form

$$a_1\,y_t^2+b_0\,y^2-\frac{b_1}{2}\,y^4=C_1. \tag{25}$$

In the case $N=2,\ M=1$ the first integral takes the form

$$2\,a_2\,y_{ttt}\,y_t-a_2\,y_{tt}^2-a_1\,y_t^2+b_0\,y^2-\frac{b_1}{2}\,y^4=C_1. \tag{26}$$

Assuming $N=M=3$ in Eq. (24) we have the first integral of Eq. (7) in the form

$$\begin{aligned}&2\,a_3\,y_{5,t}-2a_3\,y_{4,t}\,y_{tt}+a_3\,y_{ttt}^2+2\,a_2\,y_{ttt}\,y_t-a_2\,y_{tt}^2\\&\quad+a_1\,y_t^2+b_0\,y^2-\frac{b_1}{2}\,y^4-\frac{b_2}{3}\,y^6-\frac{b_3}{4}\,y^8=C_1.\end{aligned} \tag{27}$$

The first integrals Eqs. (24), (26), and (27) can be used for finding and checking analytical solutions of Eq. (3).

3. Method for finding solutions of Eq. (3)

We have found that Eq. (3) is not integrable in general case because does have the expansion in the Laurent series with necessary numbers of arbitrary constants. However we are going to demonstrate that Eq. (3) can have some exact solutions with two arbitrary constants. In order to find exact solutions, we will use the idea of the simplest equation method [44–48], taking into account the modification presented in recent papers [49–51].

We look for exact solutions of Eq. (3) in the form

$$y(t)=\sum_{k=0}^{p}A_k\,R(t)^k, \tag{28}$$

where p is the order of pole for the general solution of Eq. (3), function $R(t)$ is the general solution of the first-order differential equation

$$R_t^2=-6\,R^4+a\,R^2+d, \tag{29}$$

We can see that Eq. (29) is the first integral (10) of the classical Duffing oscillator, where a and d are constants.

Taking into account the transformation in the form

$$R(t) = \frac{3\sqrt{d}}{\sqrt{9V(t) - 3a}}, \tag{30}$$

we have at d $\neq 0$ the equation for the Weierstrass elliptic function

$$V_t^2 = 4V^3 - \left(24d + \frac{4}{3}a^2\right)V + \left(\frac{8}{27}a^3 + 8ad\right). \tag{31}$$

The general solution of Eq. (31) takes the form

$$V(t) = \wp(t - t_0, g_2, g_3), \tag{32}$$

where $\wp(t - t_0, g_2, g_3)$ is the Weierstrass elliptic function and the invariants g_2 and g_3 are determined by formulas

$$g_2 = \frac{4}{3}a^2 + 24d, \qquad g_3 = -8ad - \frac{8}{27}a^3. \tag{33}$$

Therefore the general solution of Eq. (29) can be written in the form

$$R(t) = \pm\frac{3\sqrt{d}}{\sqrt{9\wp(t - t_0, g_2, g_3) - 3a}}. \tag{34}$$

Let us note that the general solution of Eq. (29) can be written taking into account the Jacobi elliptic functions $\mathrm{sn}(x;\, k)$ or $\mathrm{cn}(x;\, k)$ and other elliptic functions. For example, the general solution of Eq. (29) is expressed by formula

$$R(t) = \pm\frac{2d}{\sqrt{\chi d}}\,\mathrm{sn}\left\{\frac{\chi}{2}(t - t_0);\; K\right\}, \qquad \chi = 2\sqrt{a^2 + 24d} - 2a,$$
$$K = \frac{2\sqrt{6d(\chi a - 24d)}}{\chi a - 24d}. \tag{35}$$

In the case $d = 0$ we have the solution $R(t)$ in the form of the solitary pulse. In fact, assuming $d = 0$ in (29) we have the equation

$$R_t^2 = -6R^2 + aR^2. \tag{36}$$

The general solution of Eq. (36) is given by formula for the solitary perturbation

$$R(t) = \pm\frac{4a\,e^{\sqrt{a}(t-t_0)}}{1 + 24a\,e^{2\sqrt{a}(t-t_0))}}, \tag{37}$$

where t_0 is an arbitrary constant.

Solution (37) is the solitary wave solution on the real axis [24–26] but if we take into account the change of complex variables $a = -b$ and $t = iT$ we obtain the singular solutions. Here $i^2 = -1$ and T is new variable.

Proposition 3. *Solutions (34) and (35) satisfy the following nonlinear ordinary differential equations*

$$R_{tt} = aR - 12R^3, \tag{38}$$

$$R_{4,t} = 864R^5 - 120aR^3 + \left(a^2 - 72d\right)R, \tag{39}$$

$$R_{6,t} = -155520R^7 + 30240aR^5 + \left(18144d - 1092a^2\right)R^3 + \left(a^3 - 792ad\right)R, \tag{40}$$

$$\begin{aligned} R_{8,t} = {} & 52254720R^9 - 13063680aR^7 + \left(834624a^2 - 7838208d\right)R^5 \\ & + \left(777600ad - 9840a^3\right)R^3 + \left(a^4 - 7344a^2d + 108864d^2\right)R, \end{aligned} \tag{41}$$

$$\begin{aligned} R_{10,t} = {} & -28217548800R^{11} + 8622028800aR^9 + 5173217280dR^7 \\ & - 790352640a^2R^7 + \left(21574080a^3 - 800616960ad\right)R^5 \\ & + \left(23779008a^2d - 158070528d^2 - 88572a^4\right)R^3 \\ & + \left(4774464ad^2 - 66384a^3d + a^5\right)R. \end{aligned} \tag{42}$$

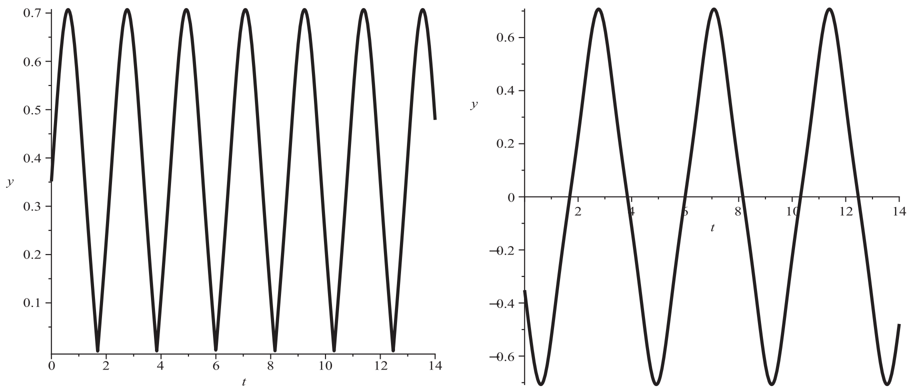

Fig. 1. Periodic oscillations (34) (on the left) and (35) (on the right) of Eq. (43) at $A_1 = 1.0$, $a = 2.0$, $d = 0.5$ and $z_0 = 6.0$ and $d = 0.5$.

Proof. The Proposition 3 is proved by direct calculations substituting solution (34) into (38)-(42).

We apply the method for finding exact solutions of Eq. (3) using the nonlinear differential Eqs. (38), (39), (40), (41) and (42).

Calculating derivatives $y(t)$ from formula (28) and substituting into them derivatives (38), (39), (40), (41) and (42), we obtain dependence function $y(t)$ and its derivatives on $R(t)$ in the polynomial form. In the next step we substitute $y(t)$ and its derivatives into Eq. (3). As a result, we obtain the algebraic equation of R(t). We find the solution of this equation for function $R(t)$ equating all coefficients of this expression to zero. This step allows us to obtain the conditions for parameters $a_n, a_{n-1}, \ldots a_1$ and b_0 of Eq. (3) when $R(t)$ is the solution of Eq. (29). □

4. Periodic solutions of Eq. (3) at $N = 2$ and $M \leq 2$

First of all let us consider Eq. (3) at $N = 2$ and $M = 2$. In this case Eq. (3) is reduced to the following fourth-order differential equation

$$a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 - b_2\, y^5 = 0. \tag{43}$$

Case $b_2 \neq 0$.

The order of pole for the general solution of Eq. (43) is $p = 1$ and without less of generality at $b_2 \neq 0$ we take $A_0 = 0$ and look for the solution of Eq. (43) in the form

$$y(t) = A_1\, R(t), \tag{44}$$

where $R(t)$ is the solution (34) or (35) of Eq. (29).

Substituting (44) and its derivatives into Eq. (43), we have the algebraic equation in the form

$$\begin{aligned}&\left(864\, a_2\, A_1 - b_2\, {A_1}^5\right) R^5 - \left(b_1\, {A_1}^3 + 12\, a_1\, A_1 + 120\, a_2\, A_1\, a\right) R^3\\ &\quad + \left(A_1 a^2\, a_2 - 72\, A_1\, d\, a_2 + b_0 A_1 + a_1 A_1\, a\right) R = 0.\end{aligned} \tag{45}$$

From Eq. (45) we obtain the conditions for parameters a_4, a_2 and b_0 for existence of solution (44)

$$a_2 = \frac{1}{864}\, b_2 {A_1}^4, \tag{46}$$

$$a_1 = -\frac{1}{12}\, {A_1}^2 b_1 - \frac{5}{432}\, {A_1}^4 b_2 a, \tag{47}$$

$$b_0 = \frac{1}{96}\, {A_1}^4\, b_2\, a^2 + \frac{1}{12}\, {A_1}^4\, b_2\, d + \frac{1}{12}\, b_1\, a\, {A_1}^2. \tag{48}$$

Solution (44) is periodic wave described by formulas (34) and (35) but in the case $d = 0$ this solution is the solitary pulse (37).

In Fig. 1 we illustrate periodic oscillations of Eq. (43) expressed by formula $y(t) = A_1\, R(t)$, where in the left hand side were used the function (34) and in the right hand side the function (35). The same values $A_1 = 1.0$, $a = 2.0$, $z_0 = 6.0$ and $d = 0.5$ for parameters of solutions were used for graphical representation. The solutions on the left and on the right are expressed by different functions and at first glance we can think that there are different solutions. However, this is not the case, the solutions are the same. However, when constructing a solution using the Maple symbolic computation program for the Weierstrass function, the program automatically selected the positive branch of the solution, and the program with the Jacobi elliptic function selected both positive and negative values of the function. Because of the symmetry of the solutions,

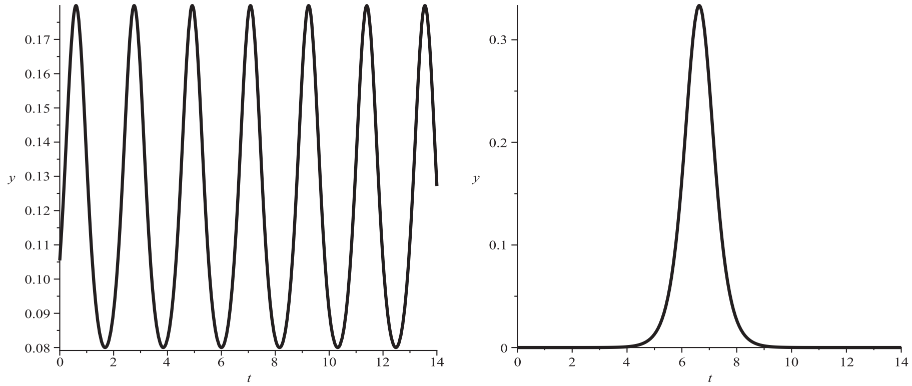

Fig. 2. Periodic oscillations of Eq. (49) at $A_0 = 0.005$, $A_2 = 1.0$, $a = 20.0$, $d = 0.2$ and $z_0 = 7.8$ (on the left) and solitary wave of Eq. (49) (on the right) at $d = 0$, $A_0 = 0.0$, $A_2 = 1.0$, $a = 2.0$ and $z_0 = 8.0$.

the solution equations are the same, but an inexperienced researcher may think that two different solutions have been obtained.

Case $b_2 = 0,\ b_1 \neq 0$**.**

In the case $b_2 = 0$ and $b_1 \neq 0$, Eq. (43) is written as the following

$$a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 = 0. \tag{49}$$

The order of pole for the general solution of Eq. (49) is $p = 2$ and we look for solution of Eq. (43) in the form

$$y(t) = A_0 + A_2\, R(t)^2. \tag{50}$$

Substituting (50) into Eq. (49) and taking into account derivatives of $y(t)$, we obtain the algebraic equations with respect to $R(t)$. We get the following values for parameters of Eq. (43)

$$a_2 = \frac{1}{4320}\, b_1 {A_2}^2, \tag{51}$$

$$a_1 = -\frac{1}{216}\, b_1\, {A_2}^2\, a - \frac{1}{12}\, b_1\, A_2\, A_0, \tag{52}$$

$$b_0 = 3\, b_1\, {A_0}^2 + \frac{1}{10}\, b_1\, {A_2}^2\, d + \frac{2}{135}\, b_1\, {A_2}^2\, a^2 + \frac{1}{3}\, b_1\, A_2\, a\, A_0, \tag{53}$$

$$d = \frac{A_0\left(45\, A_2\, a\, A_0 + 270\, {A_0}^2 + 2\, {A_2}^2\, a^2\right)}{{A_2}^2\, (A_2\, a + 9\, A_0)}. \tag{54}$$

The solution of Eq. (43) is expressed by formula

$$y_1^{(w)}(t) = A_0 + \frac{3\, A_2\, d}{3\, \wp(t - t_0,\, g_2,\, g_3) - a}, \quad g_2 = \frac{4}{3}\, a^2 + 24\, d,$$
$$g_3 = -8\, a\, d - \frac{8}{27}\, a^3 \tag{55}$$

or

$$y_1^{(s)}(t) = A_0 + \frac{4\, A_2\, d}{\chi}\, \mathrm{sn}^2\left\{\frac{\chi}{2}\,(t - t_0);\ K\right\}, \qquad \chi = 2\sqrt{a^2 + 24\, d} - 2\, a,$$
$$K = \frac{2\sqrt{6\, d\, (\chi\, a - 24\, d)}}{\chi\, a - 24\, d}, \tag{56}$$

where a, t_0, A_0 and A_2 are arbitrary constants.

In the case $d = 0$ and $b_2 = 0$, we have $A_0 = 0$ and the solution of Eq. (43) takes the solitary perturbation

$$y_2(t) = \frac{16\, A_2\, a^2\, e^{2\sqrt{a}(t - t_0)}}{\left(1 + 24\, a\, e^{2\sqrt{a}\,(t - t_0)}\right)^2}. \tag{57}$$

We have found that there are periodic and solitary solutions of the generalized Duffing oscillator of Eq. (49), which are expressed by formulas (55), (56) and (57).

Periodic oscillations Eq. (49) are demonstrated in Fig. 2 at $A_0 = 0.08$, $A_2 = 1.0$, $a = 2.0$, $d = 0.5$ and $z_0 = 6.0$ (on the left) and solitary perturbation of Eq. (49) (on the right) at $d = 0$, $A_0 = 0.0$, $A_2 = 1.0$, $a = 2.0$ and $z_0 = 8.0$. Periodic solutions in

Fig. 2 are the same for formulas (55) and (56). In order to compare the solutions of Eqs. (43) and (49), we used the same parameters for mathematical model. It can be noted that even though the solutions of the equations are expressed in terms of the elliptic Weierstrass function, the periods and amplitudes of the waves are different.

5. Periodic solutions of Eq. (3) at $N = 3$ and $M \leq 3$.

In the case $M = N = 3$ Eq. (3) takes the form

$$a_3\, y_{6,t} + a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 - b_2\, y^5 - b_3\, y^7 = 0. \tag{58}$$

Case $b_3 \neq 0$.

At $b_3 \neq 0$ the general solution of Eq. (58) has the first order of pole and we look for the solution of Eq. (43) by formula (44) again. Substituting (44) and its derivatives into Eq. (58) at $b_3 \neq 0$, we have the algebraic equation in the form

$$\begin{aligned}&\left(-155520\, a_3\, A_1 - b_3\, {A_1}^7\right) R^7 + (30240\, a_3\, A_1\, a + 864\, a_2\, A_1\\ &\quad -b_2\, {A_1}^5\big)\, R^5 + \big(18144\, a_3\, A_1\, d - 12\, a_1\, A_1 - 1092\, a_3\, A_1\, a^2 - 120\, a_2\, A_1\, a\\ &\quad -b_1\, {A_1}^3\big) R^3 + \big(a_3\, A_1\, a^3 - 792\, a_3\, A_1\, a\, d + a_1\, A_1\, a + a_2\, A_1\, a^2\\ &\quad -72\, a_2\, A_1\, d + b_0\, A_1\big)\, R = 0.\end{aligned} \tag{59}$$

From Eq. (59) we obtain the conditions for parameters a_6 a_4, a_2 and b_0 for existence of solution (44) in the form

$$a_3 = -\frac{1}{155520}\, b_3 {A_1}^6, \tag{60}$$

$$a_2 = \frac{7}{31104}\, b_3 {A_1}^6 a + \frac{1}{864}\, {A_1}^4 b_2, \tag{61}$$

$$a_1 = -\frac{7}{720}\, {A_1}^6 b_3 d - \frac{259}{155520}\, {A_1}^6\, b_3\, a^2 - \frac{5}{432}\, {A_1}^4\, a\, b_2 - \frac{1}{12}\, {A_1}^2\, b_1, \tag{62}$$

$$\begin{aligned}b_0 = &\ \frac{1}{48}\, {A_1}^6\, b_3\, a\, d + \frac{5}{3456}\, b_3\, a^3\, {A_1}^6 + \frac{1}{12}\, {A_1}^4\, d\, b_2\\ &+ \frac{1}{96}\, a^2\, b_2\, {A_1}^4 + \frac{1}{12}\, a\, b_1\, {A_1}^2.\end{aligned} \tag{63}$$

Periodic solution at $b_3 \neq 0$ and $d \neq 0$ of Eq. (58) is expressed by formula

$$\begin{aligned}&y_3^{(w)}(t) = \pm \frac{3\, A_1\, \sqrt{d}}{\sqrt{9\wp(t - t_0, g_2, g_3) - 3\, a}}, \quad g_2 = \frac{4}{3}\, a^2 + 24\, d,\\ &\quad g_3 = -8\, a\, d - \frac{8}{27}\, a^3\end{aligned} \tag{64}$$

or

$$\begin{aligned}&y_3^{(s)}(t) = \pm \frac{2\, A_1\, d}{\sqrt{\chi\, d}}\, \mathrm{sn}\left\{\frac{\chi}{2}\, (t - t_0);\ K\right\}, \qquad \chi = 2\, \sqrt{a^2 + 24\, d} - 2\, a,\\ &\quad K = \frac{2\, \sqrt{6\, d\, (\chi\, a - 24\, d)}}{\chi\, a - 24\, d},\end{aligned} \tag{65}$$

where a, d and A_1 are arbitrary constants.

In Fig. 3 we illustrate periodic oscillations of Eq. (58) expressed by formulas (64) and (65), where the function (34) in the left hand side were used and the function (35) in the right hand side. The same values $A_1 = 1.0$, $a = 10.0$, $z_0 = 7.8$ and $d = 2.2$ for parameters of equation were used for graphical representation.

In the case $\mathrm{d} = 0$ the solitary wave solution of Eq. (58) can be written in the form

$$y_4(t) = \pm \frac{4\, A_1\, a\, e^{\sqrt{a}(t - t_0)}}{1 + 24\, a\, e^{2\sqrt{a}(t - t_0)}}. \tag{66}$$

Case $\mathrm{b}_2 = \mathrm{b}_3 = 0,\ \mathrm{b}_1 \neq 0$.

In the case $\mathrm{b}_2 = \mathrm{b}_3 = 0$ Eq. (58) takes the form

$$a_3\, y_{6,t} + a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 = 0. \tag{67}$$

We have the third order of pole for the general solution of Eq. (67). In order to look for the periodic solution of Eq. (3), we can use the formula

$$y(t) = A_1\, R(t) + A_3\, R(t)^3. \tag{68}$$

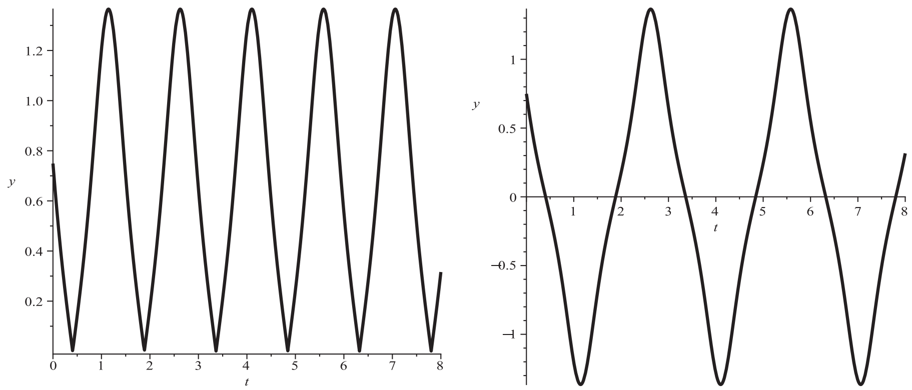

Fig. 3. Periodic oscillations of Eq. (58) with the Weierstrass function (34) (on the left) and with the Jacobi elliptic function (35) (on the right) at $A_0 = 0.0$, $A_1 = 1.0$, $a = 10.0$, $d = 2.2$ and $z_0 = 7.8$.

Substituting (68) into Eq. (67) and taking into account expressions (38), (39) and (40), we obtain the algebraic equation for $R(t)$ again. We have the following conditions for parameters of Eq. (67)

$$a_3 = -\frac{1}{4354560}\, b_1 {A_3}^2. \tag{69}$$

$$a_2 = \frac{83}{4354560}\, b_1 a {A_3}^2 + \frac{83}{362880}\, b_1 A_1 A_3, \tag{70}$$

$$\begin{aligned} a_1 = & -\frac{1387}{181440}\, b_1 A_3 a A_1 - \frac{1891}{4354560}\, b_1 {A_3}^2 a^2 - \frac{1177}{30240}\, b_1 {A_1}^2 \\ & -\frac{1}{480}\, b_1 {A_3}^2 d, \end{aligned} \tag{71}$$

$$\begin{aligned} b_0 = & \frac{211}{4480}\, b_1 A_3 a^2 A_1 + \frac{37}{1120}\, b_1 {A_3}^2 d\, a + \frac{35}{13824}\, b_1 {A_3}^2 a^3 \\ & + \frac{961}{3360}\, b_1 a {A_1}^2 + \frac{13}{40}\, b_1 A_3 d A_1 + \frac{1343}{2520}\, \frac{b_1 {A_1}^3}{} A_3, \end{aligned} \tag{72}$$

$$d_{1,2} = \left(\frac{18}{5}\, \frac{{A_1}^2}{{A_3}^2} - \frac{8\, a^2}{105} - \frac{4}{35}\, \frac{A_1 a}{A_3} \pm \frac{4\, S^{1/2}}{105\, {A_3}^2} \right), \tag{73}$$

where S takes the form

$$\begin{aligned} S = & \, 4\, {A_3}^4 a^4 + 152\, {A_3}^3 A_1 a^3 + 2256\, {A_3}^2 {A_1}^2 a^2 \\ & + 15778\, A_3 a {A_1}^3 + 44184\, {A_1}^4. \end{aligned} \tag{74}$$

Periodic solution of Eq. (58) at $b_1 \neq 0$ and $b_2 = b_3 = 0$ takes the form

$$\begin{aligned} & y_5^{(w)}(t) = \pm \frac{3\, A_1\, d^{1/2}}{(9\wp(t - t_0, g_2, g_3) - 3\, a)^{1/2}} \pm \frac{27\, A_3\, d\, \sqrt{d}}{(9\wp(t - t_0, g_2, g_3) - 3\, a)^{3/2}}, \\ & \quad g_2 = \frac{4}{3}\, a^2 + 24\, d, \qquad g_3 = -8\, a\, d - \frac{8}{27}\, a^3 \end{aligned} \tag{75}$$

and

$$\begin{aligned} & y_5^{(s)}(t) = \pm \frac{2\, A_1\, d}{\sqrt{\chi\, d}}\, \mathrm{sn}\left\{ \frac{\chi}{2}\, (t - t_0);\ K \right\} \pm \frac{8\, A_1^3\, d^3}{\sqrt{(\chi\, d)^3}}\, \mathrm{sn}^3 \left\{ \frac{\chi}{2}\, (t - t_0);\ K \right\}, \\ & \quad \chi = 2\sqrt{a^2 + 24\, d} - 2\, a, \qquad K = \frac{2\sqrt{6\, d\, (\chi\, a - 24\, d)}}{\chi\, a - 24\, d}, \end{aligned} \tag{76}$$

where a, A_1 and A_3 are arbitrary constants.

If $A_1 = 0$ we have $d_1 = \frac{3\, a^2}{20}$ or $d_2 = 0$. Assuming $d = d_2 = 0$ we get the solution of Eq. (68) in the form of solitary pulse with $R(t)$ expressed by formula (37).

The periodic oscillations of Eq. (67) is demonstrated in Fig. 4 by means of the Jacobi elliptic function (76) on the left. The solitary wave solution (68) of Eq. (67) is illustrated on the right. The parameters are used $A_1 = 0.0$, $A_3 = 2.0$, $a = 2.0$ and $z_0 = 6.0$.

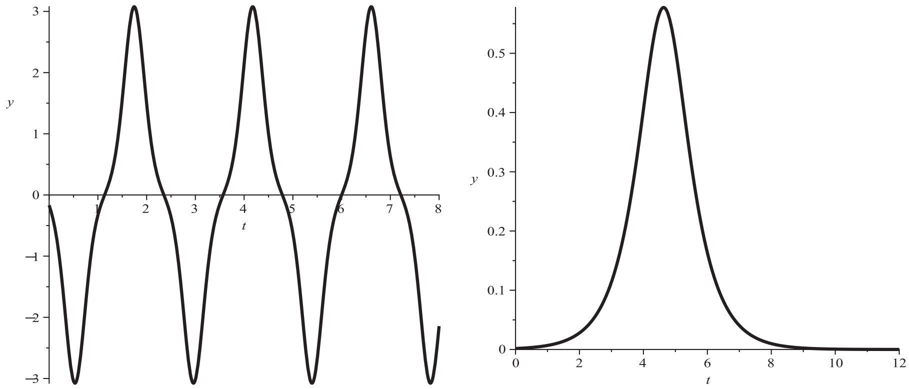

Fig. 4. Periodic oscillations of Eq. (67) with the Jacobi elliptic function (76) (on the left) at $A_1 = 0.0$, $A_3 = 2.0$, $a = 2.0$, $d = 0.5$ and $z_0 = 6.0$ and solitary wave solution (on the right) at $A_1 = 0.0$, $A_3 = 2.0$, $a = 2.0$ and $z_0 = 6.0$.

6. Exact solutions of Eq. (3) at $N = 4$ and $M \leq 4$.

Eq. (3) at $N = 4$ takes the form

$$\begin{aligned} & a_4\, y_{8,t} + a_3\, y_{6,t} + a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y \\ & \quad -b_1\, y^3 - b_2\, y^5 - b_3\, y^7 - b_4\, y^9 = 0. \end{aligned} \tag{77}$$

Case $b_4 \neq 0$.

The general solution of Eq. (77) has the first-order pole in the case $b_4 \neq 0$. We look for the exact solution by formula (44). Substituting $y(t) = A_1\, R(t)$ and its derivatives into Eq. (77), we have the algebraic equation with respect to function $R(t)$ in the form

$$\begin{aligned} & \left(52254720\, a_4\, A_1 - b_4\, {A_1}^9\right) R^9 - \left(b_3\, {A_1}^7 + 155520\, a_3\, A_1\right. \\ & \quad \left. +13063680\, a_4\, A_1\, a\right) R^7 + \left(864\, a_2 A_1 + 834624\, a_4\, A_1 a^2 + 30240\, a_3 A_1 a\right. \\ & \quad \left. -b_2\, {A_1}^5 - 7838208\, a_4 A_1 d\right) R^5 + \left(777600\, a_4\, A_1\, a\, d - 9840\, a_4\, A_1\, a^3\right. \\ & \quad \left. -12\, a_1\, A_1 - 120\, a_2\, A_1\, a + 18144\, a_3\, A_1\, d - 1092\, a_3\, A_1\, a^2 - b_1\, {A_1}^3\right) R^3 \\ & \quad +\left(a_2\, A_1\, a^2 - 72\, a_2\, A_1\, d - 7344\, a_4\, A_1\, a^2\, d - 792\, a_3\, A_1\, a\, d\right. \\ & \quad \left. +a_1\, A_1\, a + b_0\, A_1 + a_4\, A_1\, a^4 + 108864\, a_4\, A_1\, d^2 + a_3\, A_1\, a^3\right) R = 0. \end{aligned} \tag{78}$$

From Eq. (78) we get the following conditions for parameters a_6 a_4, a_2 and b_0 for existence of solution in the form (44)

$$a_4 = \frac{1}{52254720}\, b_4 {A_1}^8, \tag{79}$$

$$a_3 = -\frac{1}{155520}\, {A_1}^6\, b_3 - \frac{1}{622080}\, {A_1}^8\, b_4\, a, \tag{80}$$

$$a_2 = \frac{1}{5760}\, {A_1}^8\, b_4\, d + \frac{47}{1244160}\, {A_1}^8\, b_4\, a^2 + \frac{7}{31104}\, {A_1}^6 a\, b_3 + \frac{1}{864}\, {A_1}^4\, b_2, \tag{81}$$

$$\begin{aligned} a_1 = & -\frac{3229}{13063680}\, b_4\, a^3\, {A_1}^8 - \frac{59}{20160}\, {A_1}^8\, b_4\, a\, d - \frac{7}{720}\, {A_1}^6\, d\, b_3 \\ & -\frac{259}{155520}\, a^2\, b_3\, {A_1}^6 - \frac{5}{432}\, a\, b_2\, {A_1}^4 - \frac{1}{12}\, b_1\, {A_1}^2, \end{aligned} \tag{82}$$

$$\begin{aligned} b_0 = & \frac{1}{96}\, {A_1}^8\, b_4\, d^2 + \frac{5}{1152}\, {A_1}^8\, b_4\, a^2\, d + \frac{1}{48}\, {A_1}^6\, d\, a\, b_3 + \frac{1}{12}\, {A_1}^4\, d\, b_2 \\ & +\frac{35}{165888}\, {A_1}^8\, b_4\, a^4 + \frac{5}{3456}\, {A_1}^6\, a^3\, b_3 + \frac{1}{96}\, {A_1}^4\, a^2\, b_2 + \frac{1}{12}\, {A_1}^2\, a\, b_1. \end{aligned} \tag{83}$$

Solution of Eq. (77) is periodic solution in the form

$$\begin{aligned} & y_6^{(w)}(t) = \pm \frac{3\, A_1\, \sqrt{d}}{\sqrt{9\wp(t - t_0,\, g_2,\, g_3) - 3\, a}}, \quad g_2 = \frac{4}{3}\, a^2 + 24\, d, \\ & \quad g_3 = -8\, a\, d - \frac{8}{27}\, a^3. \end{aligned} \tag{84}$$

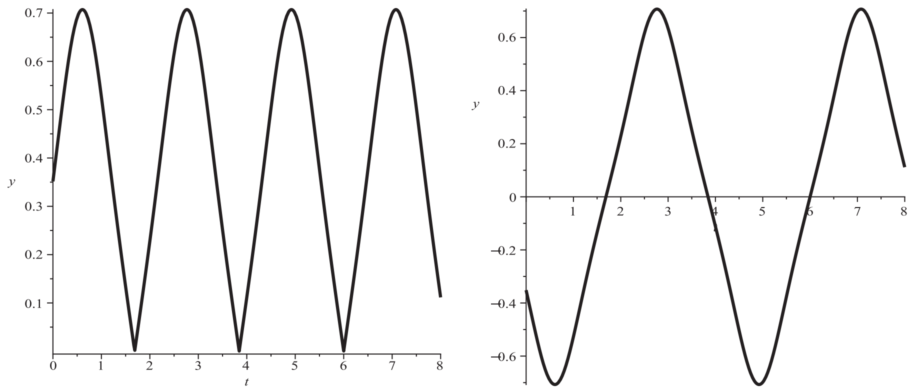

Fig. 5. Periodic oscillations of Eq. (77) with the the Weierstrass elliptic function (84) (on the left) and with the Jacobi elliptic function (85) (on the right) at $A_1 = 0.0,\ d = 0.5,\ a = 2.0$ and $z_0 = 6.0$.

or

$$y_6^{(s)}(t) = \pm \frac{2\,A_1\,d}{\sqrt{\chi\,d}}\,\mathrm{sn}\left\{\frac{\chi}{2}\,(t-t_0);\ K\right\},\qquad \chi = 2\sqrt{a^2+24\,d}-2\,a,$$
$$K = \frac{2\sqrt{6\,d\,(\chi\,a-24\,d)}}{\chi\,a-24\,d}. \tag{85}$$

If $d = 0$ the solution of Eq. (77) is the solitary pulse (66).

Periodic solutions (84) and (85) of Eq. (77) are demonstrated in Fig. 5. These solutions are similar but (84) is given at $y_6^{(w)}(t) \geq 0$ and is periodic compacton. Solution $y_6^{(w)}(t)$ is not smooth function, since this solution has a discontinuities of the derivatives in points x_l, where $\mathrm{sn}\{x_l, K\} = 0$. Solution (85) exist at $y(t) \geq 0$ and $y(t) \leq 0$ and is the smooth function.

Case $b_4 = b_3 = 0,\ b_2 \neq 0$**.**

At $b_4 = b_3 = 0$ but $b_2 \neq 0$ the generalized Duffing oscillator takes the form

$$a_4\,y_{8,t} + a_3\,y_{6,t} + a_2\,y_{4,t} + a_1\,y_{tt} + b_0\,y - b_1\,y^3 - b_2\,y^5 = 0. \tag{86}$$

Periodic solutions of Eq. (86) are expressed by formulas

$$y_7^{(w)}(t) = \pm \frac{3\,A_2\,d}{3\wp(t-t_0,\,g_2,\,g_3)-a},\qquad g_2 = \frac{4}{3}\,a^2+24\,d,$$
$$g_3 = -8\,a\,d - \frac{8}{27}\,a^3. \tag{87}$$

and

$$y_7^{(s)}(t) = \pm \frac{4\,A_1\,d}{\chi}\,\mathrm{sn}^2\left\{\frac{\chi}{2}\,(t-t_0);\ K\right\},\qquad \chi = 2\sqrt{a^2+24\,d}-2\,a,$$
$$K = \frac{2\sqrt{6\,d\,(\chi\,a-24\,d)}}{\chi\,a-24\,d}. \tag{88}$$

if the following conditions are satisfied

$$a_4 = \frac{1}{470292480}\,b_2\,{A_2}^4,\qquad a_3 = -\frac{1}{3919104}\,b_2\,{A_2}^4\,a, \tag{89}$$

$$a_2 = \frac{43}{9797760}\,{A_2}^4\,b_2\,a^2 - \frac{1}{45360}\,{A_2}^4\,b_2\,d,\qquad a_1 = -\frac{5}{367416}\,{A_2}^4\,a^3\,b_2, \tag{90}$$

$$b_0 = -\frac{5}{378}\,{A_2}^4\,b_2\,d^2,\qquad b_1 = -\frac{4}{189}\,b_2\,{A_2}^2\,a^2 - \frac{11}{42}\,b_2\,{A_2}^2\,d. \tag{91}$$

Periodic solutions $y_7^{(w)}(t)$ and $y_7^{(s)}(t)$ of Eq. (86) are demonstrated in Fig. 6 in the left hand side. We see that solutions $y_7^{(w)}(t)$ and $y_7^{(s)}(t)$ in this case are the same. Solitary wave of Eq. (86) is given in the right hand side by the following formula

$$y_8(t) = \pm \frac{16\,A_2\,a^2\,e^{2\sqrt{a}(t-t_0)})}{\left(1+24\,a\,e^{2\sqrt{a}\,(t-t_0)}\right)^2} \tag{92}$$

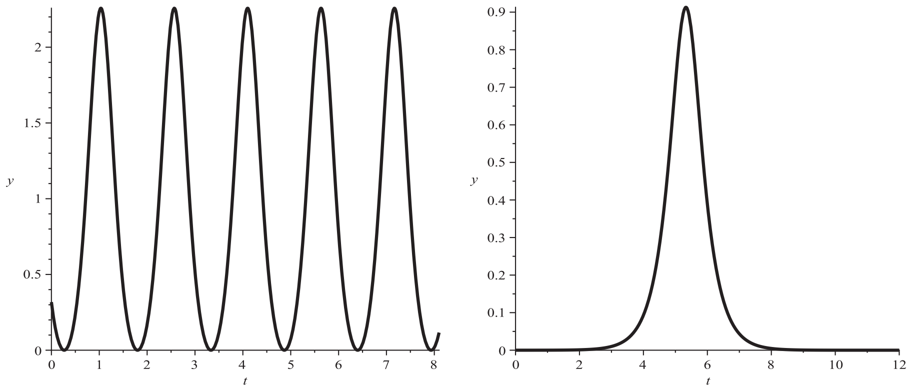

Fig. 6. Periodic oscillations of Eq. (86) with the the Weierstrass elliptic function $y_7^{(w)}(t)$ and with Jacobi elliptic function $y_7^{s}(t)$ (on the left) and solitary wave solution of (86) (on the right) at $A_1 = 0.0$, $A_3 = 2.0$, $a = 2.0$ and $z_0 = 6.0$.

Case $b_1 \neq 0,\ b_2 = b_3 = b_4 = 0$**.**

In the case $b_1 \neq 0$ and $b_2 = b_3 = b_4 = 0$ Eq. (77) is transformed to the following

$$a_4\, y_{8,t} + a_3\, y_{6,t} + a_2\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 = 0. \tag{93}$$

Exact solution of Eq. (93) can be found by the formula

$$y(t) = A_2\, R(t)^2 + A_4\, R(t)^4 \tag{94}$$

Substituting (94) into Eq. (77), we can find conditions for parameters of Eq. (77) too. These conditions at $d = 0$ are the following

$$a_4 = \frac{1}{22417274880}\, b_1\, {A_4}^2, \qquad a_3 = -\frac{97}{2802159360}\, b_1\, {A_4}^2\, a \tag{95}$$

$$a_2 = \frac{20791}{3113510400}\, b_1\, {A_4}^2\, a^2, \qquad a_1 = -\frac{8891}{27027000}\, b_1\, {A_4}^2\, a^3, \tag{96}$$

$$b_0 = \frac{5041}{2895750}\, b_1\, {A_4}^2\, a^4. \tag{97}$$

Solitary pulse of Eq. (93) is expressed by the formula

$$y_8(t) = \pm\frac{102\, A_4\, a^3\, e^{(2\sqrt{a}(t-t_0))}}{30\left(1 + 24\, a\, e^{(2\sqrt{a}\,(t-t_0))}\right)^2} \pm \frac{256\, a^4\, e^{(4\sqrt{a}(t-t_0))}}{\left(1 + 24\, a\, e^{(2\sqrt{a}\,(t-t_0))}\right)^4}. \tag{98}$$

In the case $d \neq 0$ we need to take into consideration $A_0 \neq 0$ and $A_2 \neq 0$ and we obtain the additional conditions for the parameters of Eq. (93). The algorithm of constructing the solution is similar, but the resulting formulas look more cumbersome than in case of constructing solution with the first-order pole. We do not give these formulas in this work, but we leave them for the interested reader in the home notes.

7. Exact solutions of Eq. (3) at $N = 5$.

As the last example of constructing solutions to a generalized Duffing oscillator let us obtain the periodic oscillations and solitary pulses described by tenth-order differential Eq. (3) at $N = 5$ in the form

$$a_5\, y_{10,t} + a_4\, y_{8,t} + a_6\, y_{6,t} + a_3\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 - b_2\, y^5 - b_3\, y^7 - b_4\, y^9 - b_5\, y^{11} = 0. \tag{99}$$

Case $b_5 \neq 0$**.**

In the case of $b_5 \neq 0$ the general solution of Eq. (99) has the first-order pole and the solution of Eq. (99) can be searched using the formula (44). Using the same algorithm for finding exact solutions, we substitute (44) and its derivatives into Eq. (99). After that we have again the algebraic equation with respect to $R(t)$. Equating expressions at various powers of function $R(t)$ to zero, we obtain the following conditions for parameters of Eq. (99).

$$a_5 = -\frac{1}{169305292800}\, b_5\, {A_1}^{10}, \tag{100}$$

$$a_4 = \frac{19}{16930529280}\, {A_1}^{10}\, b_5\, a + \frac{1}{261273600}\, {A_1}^{8}\, b_4, \tag{101}$$

$$a_3 = -\frac{1}{622080} A_1{}^6 b_3 - \frac{2977}{42326323200} A_1{}^{10} a^2 b_5 - \frac{1}{2612736} A_1{}^8 a\, b_4 - \frac{37}{146966400} A_1{}^{10} b_5\, d, \tag{102}$$

$$a_2 = \frac{1}{2592} A_1{}^4 b_2 + \frac{11}{155520} A_1{}^6 a\, b_3 + \frac{11971}{7054387200} A_1{}^{10} a^3 b_5 + \frac{727}{65318400} A_1{}^8 a^2 b_4 + \frac{4729}{293932800} A_1{}^{10} a\, b_5\, d + \frac{43}{907200} A_1{}^8 b_4\, d, \tag{103}$$

$$a_1 = -\frac{1}{24} A_1{}^2 b_1 - \frac{757}{3265920} A_1{}^{10} b_5 a^2 d - \frac{1627}{117573120} A_1{}^{10} b_5 a^4 - \frac{49}{116640} A_1{}^{10} b_5 d^2 - \frac{1}{270} A_1{}^6 b_3\, d - \frac{37}{51840} A_1{}^6 b_3 a^2 - \frac{121}{113400} A_1{}^8 a\, b_4\, d - \frac{533}{5443200} A_1{}^8 a^3 b_4 - \frac{7}{1296} A_1{}^4 a\, b_2, \tag{104}$$

$$b_0 = \frac{1}{432} A_1{}^{10} b_5\, a\, d^2 + \frac{1}{6480} A_1{}^8 a^4 b_4 + \frac{1}{180} A_1{}^8 b_4 d^2 + \frac{1}{12} A_1{}^2 a b_1 + \frac{1}{108} A_1{}^4 a^2 b_2 + \frac{1}{1944} A_1{}^{10} b_5 a^3 d + \frac{1}{72} A_1{}^6 a\, b_3\, d + \frac{1}{360} A_1{}^8 a^2 b_4\, d + \frac{1}{864} A_1{}^6 a^3 b_3 + \frac{1}{46656} A_1{}^{10} b_5 a^5 + \frac{1}{18} A_1{}^4 b_2\, d. \tag{105}$$

We get the solution of Eq. (99) at conditions (100), (101), (102), (104) and (105) in the form

$$y_9^{(w)}(t) = \frac{3 A_1 \sqrt{d}}{\sqrt{9\wp(t-t_0, g_2, g_3) - 3a}}, \quad g_2 = \frac{4}{3} a^2 + 24\, d, \quad g_3 = -8\, a\, d - \frac{8}{27} a^3. \tag{106}$$

$$y_9^{(s)}(t) = \pm \frac{2 A_1 d}{\sqrt{\chi\, d}} \operatorname{sn}\left\{\frac{\chi}{2}(t-t_0);\; K\right\}, \qquad \chi = 2\sqrt{a^2 + 24\, d} - 2\, a, \quad K = \frac{2\sqrt{6\, d\, (\chi\, a - 24\, d)}}{\chi\, a - 24\, d}. \tag{107}$$

The above examples of constructing periodic solutions of the generalized Duffing oscillator allow us to determine the general laws for finding exact solutions of Eq. (3) at $b_N \neq 0$. It is clear that substituting expression (44) in Eq. (3), we obtain an algebraic equation for $R(t)$. The maximum degree of the function $R(t)$ is $N+1$, and the minimum degree of the function is 1. Moreover, there are no even degrees of the function $R(t)$. Equating the expressions at various degrees of the function $R(t)$ to zero, we obtain algebraic equations to determine the conditions for the parameters $a_n, a_{n-1}, \ldots, b_0$ of Eq. (3). These algebraic equations for parameters allow us to consistently find the values of the parameters of Eq. (3).

Case $b_1 \neq 0$ and $b_2 = b_3 = b_4 = b_5 = 0$.

The algorithm of finding exact solution becomes much more complex if the pole order of the general solution is not equal to one. In particular, for the general solution of Eq. (99) at $b_1 \neq 0$ and $b_2 = b_3 = b_4 = b_5 = 0$ we have the equation

$$a_5\, y_{10,t} + a_4\, y_{8,t} + a_6\, y_{6,t} + a_3\, y_{4,t} + a_1\, y_{tt} + b_0\, y - b_1\, y^3 = 0 \tag{108}$$

and we have to look for the fifth-order pole of the general solution of Eq. (108). The periodic solution of Eq. (108) is found very difficult but it is easy to find the solitary wave solution at $d = 0$.

For a example, using the formula

$$y(t) = A_3 R^3 + A_5 R^5 \tag{109}$$

we find at least three values for coefficients A_3

$$A_3^{(1)} = 0, \quad A_3^{(2)} = \frac{1}{6} a A_5, \quad A_3^{(3)} = \frac{11}{27} a A_5. \tag{110}$$

Taking into account the value $A_3^{(2)}$, we have the following solitary wave solution of Eq. (108)

$$y_{10}(t) = \pm \frac{32 A_5\, a^4\, e^{3\sqrt{a}(t-t_0)}}{3\left(1 + 24\, a\, e^{2\sqrt{a}(t-t_0)}\right)^3} \pm \frac{1024 A_5\, a^5\, e^{5\sqrt{a}(t-t_0)}}{\left(1 + 24\, a\, e^{2\sqrt{a}(t-t_0)}\right)^5} \tag{111}$$

with conditions for parameters of Eq. (108)

$$a_5 = -\frac{1}{75322043596800}\, b_1\, A_5{}^2, \qquad a_4 = \frac{137}{9415255449600}\, b_1\, a\, A_5{}^2, \tag{112}$$

$$a_3 = -\frac{89371}{18830510899200}\, b_1\, a^2\, A_5{}^2, \qquad a_2 = \frac{436291}{784604620800}\, b_1\, a^3\, A_5{}^2, \tag{113}$$

$$a_1 = -\frac{2894569}{130767436800}\, b_1\, A_5{}^2\, a^4, \qquad b_0 = \frac{13}{67200}\, b_1\, A_5{}^2\, a^5. \tag{114}$$

Note that we can use some some dimensionless variables in the original equation Eq. (3). For example, it is natural to assume that $b_M = 1$ and $a_N = 1$. In this case the first condition for a_N turns into an equation for finding A_N.

8. Application of the method for another hierarchy

The approach of this work can be used for more general equations than Eq. (3). We also can consider the hierarchy in the form

$$\sum_{n=1}^{N} a_n\, y_{2n,t} + c_0\, y - \sum_{m=1}^{M} c_m\, y^m = 0, \qquad y_{n,t} = \frac{d^n y}{dt^n}, \tag{115}$$

where c_m are parameters of equation. For example, let us demonstrate the application of the method for the tenth-order differential equation in the form

$$a_5\, y_{10,t} + a_4\, y_{8,t} + a_3\, y_{6,t} + a_2\, y_{4,t} + a_1\, y_{tt} + c_0\, y - c_1\, y^2 - c_2\, y^3 - c_3\, y^4 - c_4\, y^5 - c_5\, y^6 = 0. \tag{116}$$

General solution of Eq. (116) has the second order and we can look for exact solution of Eq. (116) at $c_5 \neq 0$ and $c_1 \neq 0$ in the form

$$y(t) = A_2\, R(t)^2, \tag{117}$$

where $R(t)$ is the solution of Eq. (29).

Substituting dependencies $y(t)$ and its derivatives of $R(t)$ into Eq. (116), we get the equation for $R(t)$. Equating all coefficients at $R(t)$ to zero, we obtain the conditions on parameters in the following form

$$a_5 = -\frac{1}{1345036492800}\, c_5 A_2{}^5, \tag{118}$$

$$a_4 = \frac{5}{26900729856}\, c_5 A_2{}^5\, a + \frac{1}{1724405760}\, A_2{}^4\, c_4, \tag{119}$$

$$\begin{aligned} a_3 = &-\frac{5377}{336259123200}\, A_2{}^5\, c_5\, a^2 - \frac{31}{583783200}\, A_2{}^5\, c_5\, d \\ &-\frac{1}{12317184}\, A_2{}^4\, a\, c_4 - \frac{1}{3265920}\, A_2{}^3\, c_3, \end{aligned} \tag{120}$$

$$\begin{aligned} a_2 = &\frac{31841}{56043187200}\, A_2{}^5\, c_5\, a^3 + \frac{11819}{2335132800}\, A_2{}^5\, c_5\, a\, d \\ &+\frac{509}{143700480}\, A_2{}^4\, a^2\, c_4 + \frac{17}{816480}\, A_2{}^3\, a\, c_3 + \frac{83}{5987520}\, A_2{}^4\, c_4\, d \\ &+\frac{1}{10080}\, A_2{}^2\, c_2, \end{aligned} \tag{121}$$

$$\begin{aligned} a_1 = &-\frac{2683}{934053120}\, A_2{}^5\, c_5 a^4 - \frac{2809}{77837760}\, A_2{}^5 c_5 a^2 d - \frac{221}{11975040}\, A_2{}^4\, c_4\, a^3 \\ &-\frac{4}{31185}\, A_2{}^4\, c_4\, a\, d - \frac{19}{540540}\, A_2{}^5\, c_5\, d^2 - \frac{1}{4536}\, A_2{}^3\, c_3\, d \\ &-\frac{31}{272160}\, A_2{}^3\, c_3\, a^2 - \frac{1}{1680}\, A_2{}^2\, a\, c_2, \end{aligned} \tag{122}$$

$$\begin{aligned} c_0 = &\frac{1}{132}\, A_2{}^4 c_4 d^2 + \frac{1}{14}\, c_2\, A_2{}^2\, d + \frac{1}{468}\, c_5\, A_2{}^5\, a\, d^2 + \frac{1}{108}\, A_2{}^3\, a\, c_3\, d \\ &+\frac{1}{5616}\, c_5\, A_2{}^5\, a^3\, d + \frac{1}{792}\, A_2{}^4\, a^2\, c_4\, d, \end{aligned} \tag{123}$$

$$\begin{aligned} c_1 = &-\frac{5}{468}\, A_2{}^4\, c_5\, d^2 - \frac{5}{54}\, A_2{}^2\, c_3\, d - \frac{5}{936}\, A_2{}^4\, c_5\, a^2\, d - \frac{5}{324}\, A_2{}^2\, c_3\, a^2 \\ &-\frac{5}{198}\, A_2{}^3\, c_4\, a\, d - \frac{5}{42}\, A_2\, a\, c_2 - \frac{5}{16848}\, A_2{}^4\, c_5\, a^4 - \frac{5}{2376}\, A_2{}^3\, c_4\, a^3. \end{aligned} \tag{124}$$

Exact solution

$$y_{11}(t) = \frac{3\,A_2\,d}{3\wp(t-t_0,\,g_2,\,g_3) - a}. \tag{125}$$

of Eq. (116) at additional conditions (118)-(124) on parameters describes the periodic oscillators. At $d = 0$ formulas (118)-(124) are simplified and we obtain solution of Eq. (116) in the form of the solitary pulse.

9. Conclusion

Let us briefly formulate the results of this work. We have considered a generalized Duffing oscillator that takes into account high-order derivatives and power dependencies of nonlinearities. The main point of work is the question for the integrability of this mathematical model. Using the Painlevé test, we have shown that the generalized Duffing oscillator does not have general solution except of the classical Duffing oscillator. However, we have obtained that there are two arbitrary constants in the expansion of the general solution into the Laurent series for a generalized Duffing oscillator. As a consequence, we have found exact solutions to a generalized Duffing oscillator with two arbitrary constants. In this paper, we have proposed the algorithm for finding exact solutions to a generalized Duffing oscillator. Solutions of the generalized Duffing oscillator for the fourth, sixth, seventh, and tenth orders have been found in the form of solitary and periodic waves.

Declaration of Competing Interest

None.

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