Incompressible Inviscid Flow using FreeFEM++

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1 Governing Equations

Incompressible inviscid flow is simulated by solving the govering equations in FreeFEM++ using artifical compressibility method. For 2D flow, the mesh is created in FreeFEM++, whereas in 3D flow, the mesh is created with GMSH and imported in FreeFEM++.

Continuity Equation

$$\nabla \cdot \vec{u} = 0$$

Momentum Equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla P$$

Note: The density (ρ) is taken as one for simplicity.

Boundary Conditions

Boundary	**Type**
Inlet	Dirchlet ($\vec{u} = [1\ 0\ 0]$)
Domain Walls	Free-slip
Cylinder Wall	Free-slip

2 Artificial Compressibility Method

The artificial compressibility method is used to solve incompressible flow equations with as hyperbolic equations in a time-marching fashion like compressible flow equations. It is achieved by introducing an artificial parameter in the continuity equation in terms of presssure related through a constant β.

Modified Continuity Equation

$$\beta \frac{\partial p}{\partial t} + \nabla \cdot \vec{u} = 0$$

Momentum Equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla P$$

The set of equations are solved until a steady solution is obtained. This steady solution is also the solution of the original govering equations as $\frac{\partial p}{\partial t}$ tends to 0 at steady state.

Weak Form

The time derivatives are written using Forward Euler and the weak form is derived for the above modified equations.

$$\int_\Omega \left(\frac{\vec{u}^{n+1} \cdot \vec{v}}{\Delta t} + (\vec{u}^{n+1} \cdot \nabla \vec{u}^{n+1}) \cdot \vec{v} - p \nabla \cdot \vec{v} - q \nabla \cdot \vec{u}^{n+1} - \frac{\beta p^{n+1} q}{\Delta t} \right) d\not{V} - \int_\Omega \left(\frac{\vec{u}^{n} \cdot \vec{v}}{\Delta t} - \frac{\beta p^{n} q}{\Delta t} \right) d\not{V} = 0$$

The non-linear term of the equation is computed using the inbuilt *convect* function of FreeFEM++.

Listing 1: Problem definition in FreeFEM++

```

 solve GE([u1,u2,p],[v1,v2,q])
=int2d(Th)
([u1,u2]'*[v1,v2]/dt
-div(v)*p
-div(u)*q
-Beta*p*q/dt
)
+int2d(Th)(
-(con(un,dt)'*[v1,v2])/dt
+Beta*pn*q/dt
)
+on(1, u1=1, u2=0)
+int1d(Th,qft=qf1pTlump,5)(1e10*(u1*normalappx+u2*normalappy)*(v1*normalappx+v2*normalappy))
+on(3, u2=0)
+on(4, u2=0);
```

3 Results of 2D Computation

The code is tested using the benchmark case of flow around a cylinder. The solution of the flow is already known from potential flow theory. β is a user-defined parameter whose value is varied along with the number of points on the cylinder boundary. An initial domain of [-10D,10D] in x and [-6D, 6D] in y-direction. The unstructured mesh used for computation is presented in **Figure 1**.

3.1 Number of points on cylinder

The number of points on the cylinder surface is increased from 50 to 150 in three steps. The variation in pressure distribution over the surface and residual is computed and plotted in **Figure 2**. The residual magnitude decreases with increase in the number of points but the trend remains unchanged. On increasing the number of points from 50 to 100, we see significant improvement in the pressure at top and bottom point on the surface whereas the increase from 100 to 150 has very little effect on pressure distribution. On the contrary, with increase in number of points little improvement is observed on the pressure at stagnation point.

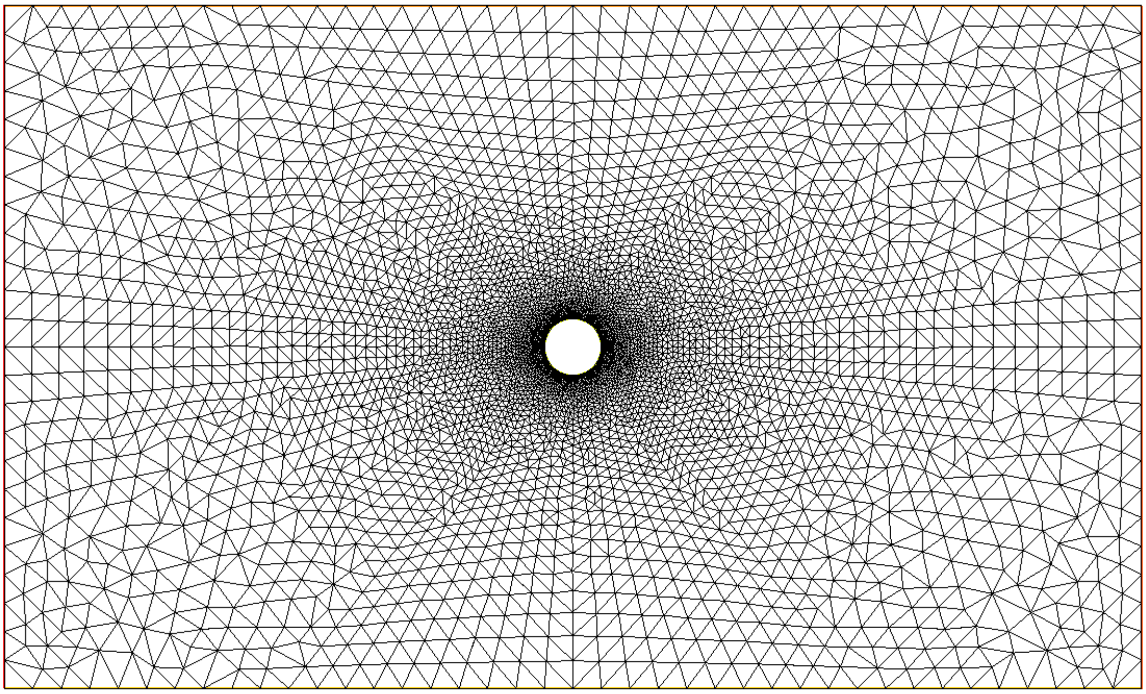

Figure 1: Unstructured mesh contructed in FreeFEM++ with 100 points on the cylinder body

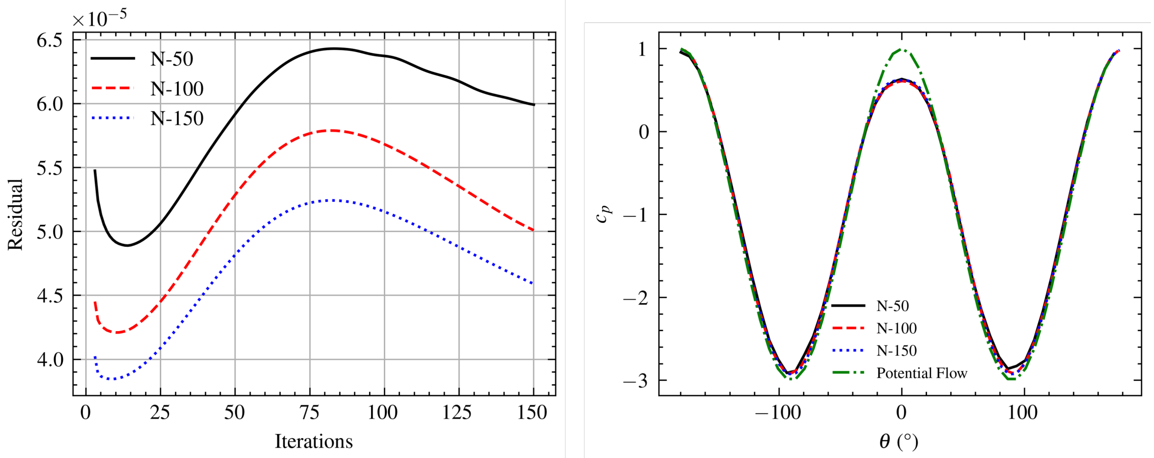

Figure 2: Variation in the residual and pressure distribution over the surface with time for different number of points on the cylinder boundary

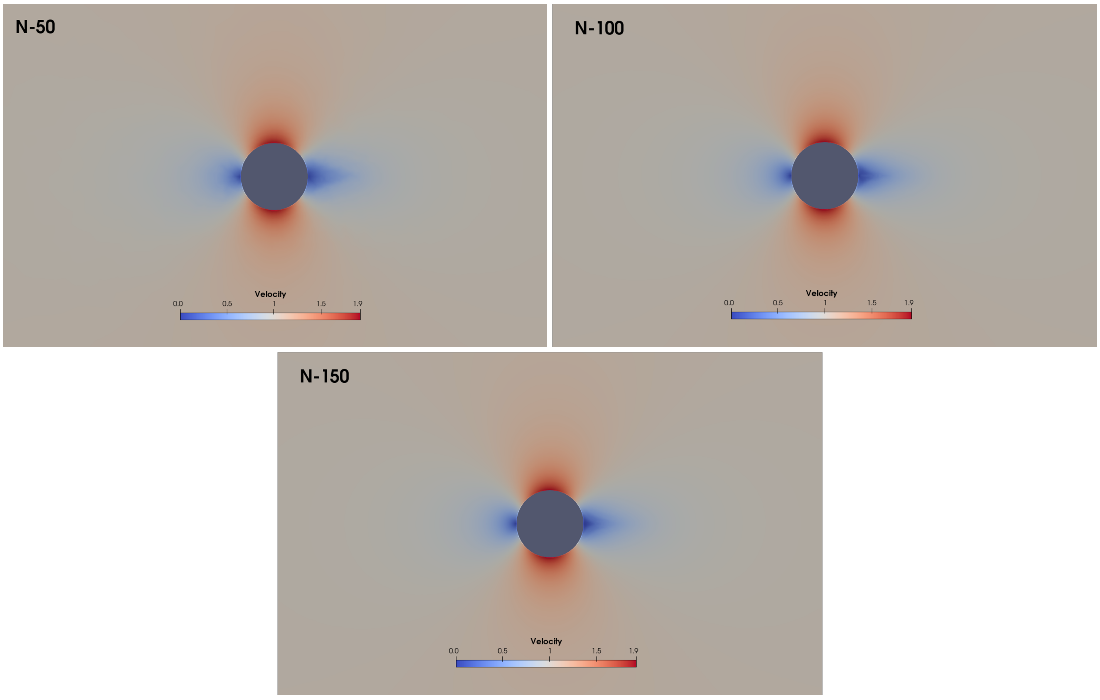

Figure 3: Velocity contour over the cylinder for the three cases.

3.2 Variation of β

β is the user-defined parameter to control the artifical compressibility introduced in the equation. In general, the value of β is choosen less than one and a smaller value helps in faster convergence. In our case, we test with three values of β and we observe that reducing the value from 10^{-5} to 10^{-7} had no effect on the residual but decreasing it further to 10^{-9} caused an increase in the magnitude of residual. The decrease in value of β has negligible effect on the pressure distribution as all the three curves approximately coincide with each other (**Figure 4**).

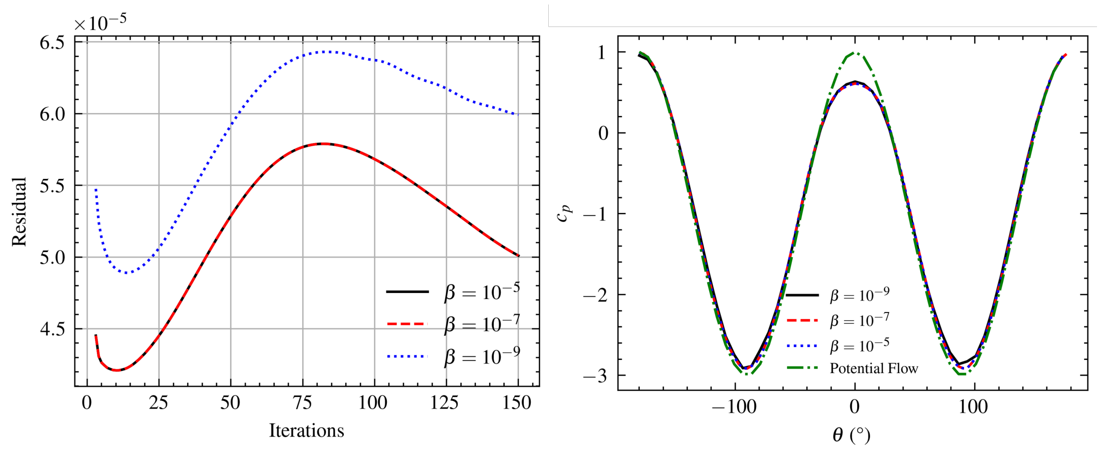

Figure 4: Variation in the residual and pressure distribution over the surface with time for different value of β.

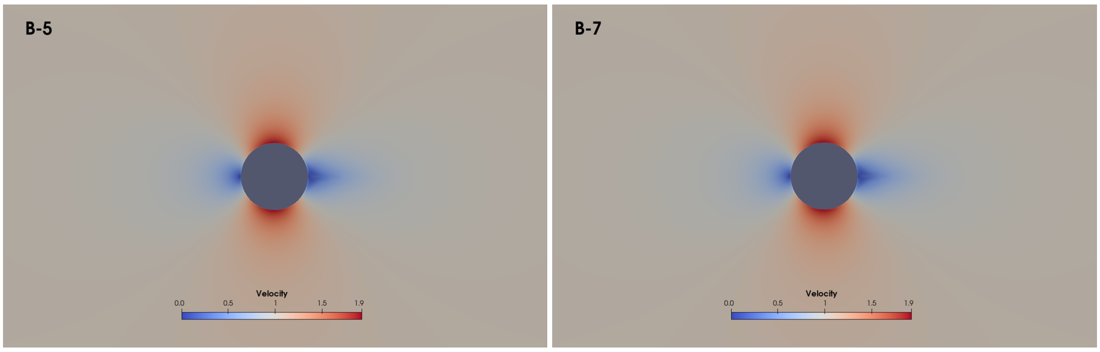

Figure 5: Velocity contour over the cylinder for the three cases.

3.3 Variation with Domain

The dependence of solution with domain size is tested by reducing the domain to [-5D,5D] in x and [-4D, 4D] in y from the initial size and then increasing it to [-15D, 15D] in x and [-10D, 10D] in y. The cell size on domain boundaries and cylinder surface is kept constant for all the domains. For the largest domain of 30×16, a sharp decrease in residual is observed with the least residual compared to other domain sizes. The pressure distribution is approximately similar in all the domains as the plots coincide with each other (**Figure 6**).

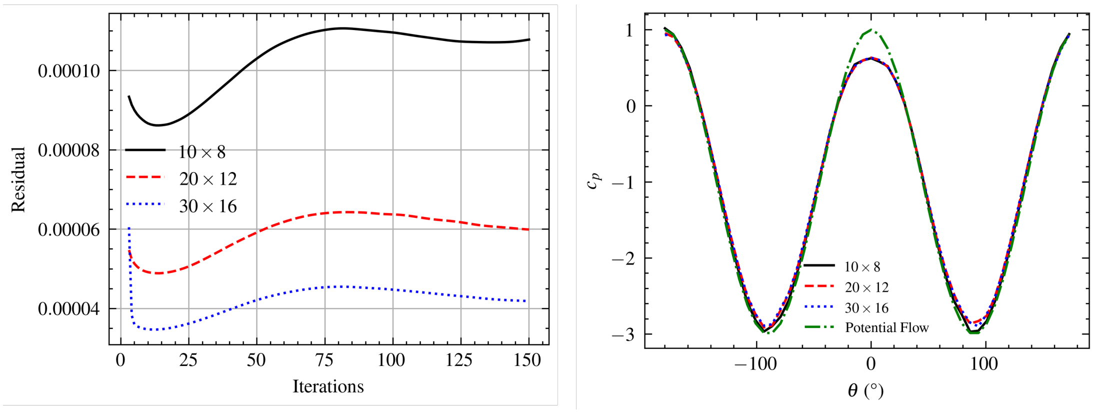

Figure 6: Variation in the residual and pressure distribution over the surface with time for domain.

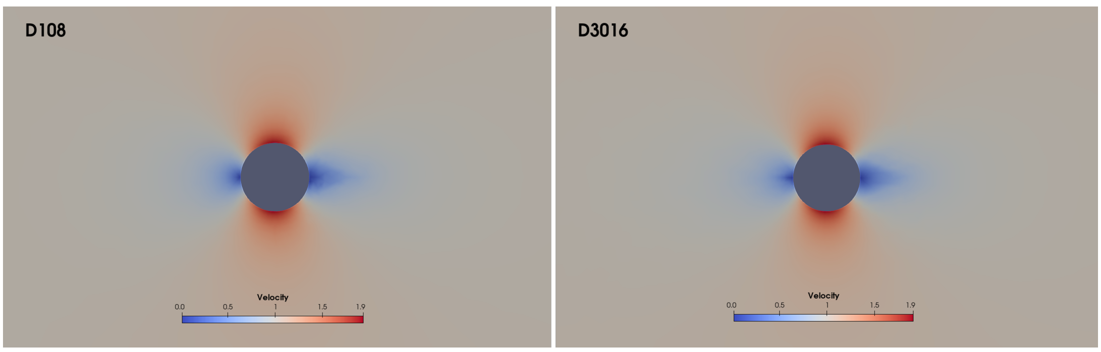

Figure 7: Velocity contour over the cylinder for the three cases.