Indian Institute of Space Science and Technology, Thiruvananthapuram
Department of Aerospace Engineering

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Introduction to Differential Equations

Aim: To solve partial differential equations using FreeFem++ software with a focus on fluid and thermal problems.

Partial Differential Equations (PDEs): PDEs are mathematical equations that involve multiple variables and their partial derivatives. They are used to model and describe various phenomena in science, engineering, and mathematics. PDEs play a fundamental role in understanding dynamic systems and the behavior of complex processes that depend on multiple factors. In general, PDEs are classified into three types: *Elliptic*, *Parabolic*, and *Hyperbolic* based on underlying physics and mathematical properties.

Elliptic PDEs: These equations describe steady-state or equilibrium phenomena. They involve spatial derivatives but do not include time derivatives. Elliptic PDEs are fundamental for studying problems with no time evolution like steady-state diffusion.

Parabolic PDEs: Parabolic equations describe processes that evolve over time. They typically involve both spatial and time derivatives. These equations are well-suited for modeling transient behavior, such as heat conduction and diffusion phenomena.

Hyperbolic PDEs: Hyperbolic equations represent wave-like phenomena, where disturbances propagate through space and time. These PDEs include both spatial and time derivatives and are used to study problems involving waves, such as sound waves, electromagnetic waves, and fluid flow.

Order of PDEs: The highest order of partial derivatives present in the equation is called the order of a PDE. The One-Dimensional wave equation $\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0\right)$ is a first-order PDE. The heat conduction equation $\left(\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0\right)$ is a second-order PDE. If there are terms involving more than four derivatives, they are called Higher-order PDEs. The order of a PDE is crucial because it influences the number of boundary and initial conditions required to solve the problem. Higher-order PDEs are generally more difficult to solve analytically and often require advanced mathematical techniques or numerical methods for practical solutions.

Finite Element Method (FEM)

The Finite Element Method (FEM) is a versatile numerical technique used to solve various types of physical problems. It is highly adaptable and can handle complex geometries, finding extensive applications in structural analysis, fluid mechanics, heat transfer, electromagnetics, and various other engineering and scientific fields. Its capability to deal with irregular geometries and complex boundary conditions makes it a valuable tool for simulating real-life problems.

Formulation Approaches in FEM:

1. Direct Approach: The direct approach in the Finite Element Method is often referred to as the "direct stiffness method" in structural analysis. It is particularly suitable for beginners as it provides a straightforward understanding of FEM without delving deeply into complex mathematics. The primary advantage of the direct approach is its simplicity; however, it may not be adequate for solving more complex and challenging engineering problems.

2. Variational Approach: The variational approach involves rephrasing the original problem using a variational principle, often the principle of minimum potential energy. However, one major drawback of the variational approach is that it may not be applicable to all physical problems. Some problems, especially nonlinear ones, may lack a classical variational statement, making the variational approach unsuitable for those cases. Rayleigh-Ritz is an approximate technique based on the variational formulation.

3. Weighted Residual Method: The weighted residual method is a practical approach to handle complex physical problems by approximating the solution and minimizing the error between the approximate solution and the actual problem equation using weighting functions. The Galerkin weighted residual method simplifies PDEs by approximating the solution with a finite combination of basis functions, which are also used as weighting functions. This choice of functions reduces the residual error and leads to a system of equations that can be efficiently solved, providing an approximate but often accurate solution to the original problem. The Galerkin method is a fundamental approach in finite element analysis, a widely used numerical technique in engineering and applied sciences.

FreeFem++ Software: FreeFem++ is an open-source software package used for solving PDEs numerically. It is a powerful finite element tool with a large collection of predefined functions that help simulate and analyze a wide range of physical problems. Some applications include simulating fluid flow, solving Navier-Stokes equations, modeling turbulence in fluid mechanics, analyzing structural mechanics and deformation, modeling electromagnetic fields and wave propagation in electromagnetics, and investigating heat conduction, convection, and radiation in heat transfer problems.

Problem 1: A simple 1D heat diffusion problem with Dirichlet boundary conditions is simulated. The governing equations and boundary conditions are provided as follows.

$$\begin{gathered}\frac{d^2u}{dx^2} = 1, \qquad \forall 0 < x < 1 \\ u(x=0) = 0, \quad u(x=1) = 0\end{gathered} \tag{1}$$

Weak Formulation

Weak formulation is a way of representing the governing equation that is used commonly in finite element method for solving of partial differential equations. The method involves multiplying the original equation by a test function and integrating over the domain of interest.
The steady state diffusion on a domain of finite length is considered as in Equation 1. This equation is multiplied with a weight function (v) and is integrated over the domain.

$$\int_0^1 \frac{d^2u}{dx^2} \cdot v\, dx = \int_0^1 v dx \tag{2}$$

Equation 2 is simplified using chain rule and the integral can be written as,

$$\left(v\frac{du}{dx}\right)_{x=1} - \left(v\frac{du}{dx}\right)_{x=0} - \int_0^1 \frac{du}{dx} \cdot \frac{dv}{dx}\, dx = \int_0^1 v dx \tag{3}$$

Equation 3 is solved in FreeFem software for the boundary conditions defined in Equation 1.

Code

```
load "msh3"
int m=100;
meshL Th=segment(m,[x*1]);
real[int] xaxis(m+1),Uline(m+1);
ofstream file2("Results.dat");
fespace Vh(Th,P1);
Vh U,v;
solve Poisson(U,v)=int1d(Th)(dx(U)*dx(v))
                +int1d(Th)(v)    //weak form of the governing equation
                +on(1,2, U=0);   //Dirichlet boundary condition
plot(U, value=true);
for(int i=0;i<=m; i++)
    {
        xaxis[i]= i/m;
        Uline[i] =U[][i];
        file2<<xaxis[i]<<"\t"<<"\t"<< Uline[i]<<endl;
        cout<<xaxis[i]<<"\t"<<Uline[i]<<endl;
    }
```

Code Breakdown

1. Load a plugin("msh3") for using 3D mesh tools.
2. Define the number of segments 'm' for the 1D mesh.

3. Generate a 1D mesh called ’Th’ with ’m’ segments.
4. Declare arrays ’xaxis’ and ’Uline’ to store data values.
5. Create an output file named ’Results.dat’ for storing the data.
6. Define finite element space(”fespace”)’vh’ on the mesh ’Th’ using linear elements.
7. Declare variables ’U’ and ’v’ in the finite element space.
8-10. Solves the Poisson equation with weak formulation, including differentiation terms and boundary conditions. ”solve” function uses iterative methods (by default LU) to solve the linear systems of equations that arise from the finite element discretization of PDE’s.
11.Plot the solution ’U’ with values displayed on the plot.
12-18. Post-processing: In these lines we write the variation of U with evenly spaced values of x.

Results

A comparison between the results obtained from FreeFem++ and the analytical solution is illustrated in Figure 1.

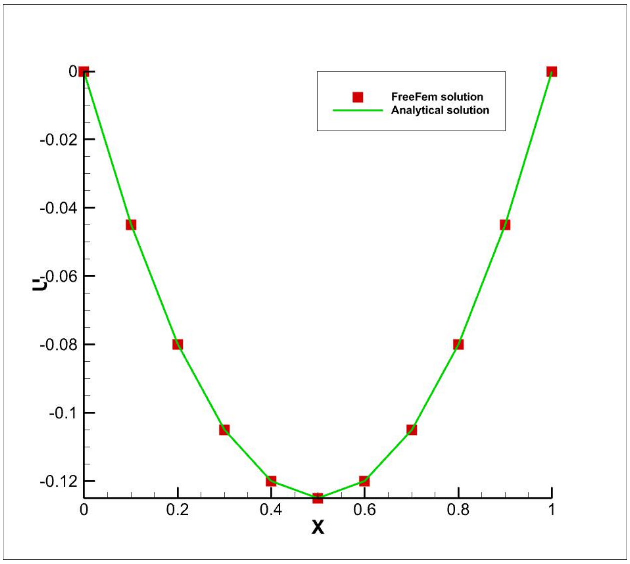

Figure 1: Variation of U with x

Exercise

1. In the current problem, the solution is obtained considering a Dirichlet boundary condition. Investigate the effect of a Neumann boundary condition (i.e., when the domain is insulated at both ends), by modifying the boundary conditions.

2. Perform a grid convergence study by varying the mesh size systematically. Observe how the numerical solution changes with finer grids and comment on the accuracy of the numerical method.

3. Implementing the problem using P2(quadratic) finite element space. Is there any significant differences in results when compared to P1 (linear) finite element space.

Assignments

Problem 2: Consider a 1mm diameter, 50mm long aluminium pin-fin. The fin's left end interfaces with a constant heat source at a temperature of $T_h = 300°C$, while its right end is thermally insulated. Heat is progressively released along the fin's length through convection, interacting with an ambient temperature of $T_\infty = 30°C$. The governing differential equation is given by:

$$k\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty) \tag{4}$$

In equation 4, the thermal conductivity (k) is 200 W/m·K and the convective heat transfer coefficient (h) is 20W/m²·K. P,A represents perimeter and area of cross-section of the fin. Establish a model for the temperature distribution along the fin under these specific conditions and compare with analytical results.

- How well does the solutions generated compares with Analytical results.
- For the same problem consider the right end of the fin is subjected to ambient Temperature of $T_\infty = 30°C$. Plot the temperature profile for this condition.

Problem 3: Consider a rectangular plate of dimensions $L_x \times L_y$, where one side is insulated, while the other three sides are maintained at constant temperatures. The temperature distribution$T(x,y)$ in the plate is governed by Laplace's equation:

$$\nabla^2 T = 0$$

Boundary Conditions:

1. At x = 0: $\frac{\partial T}{\partial x} = 0$
2. At x = $L_x : T(L_x, y) = 200°C$
3. At y = 0 : $T(x, 0) = 150°C$
4. At y = $L_y : (T(x, L_y) = 150°C$

Find the temperature distribution $T(x,y)$ within the plate.

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