Indian Institute of Space Science and Technology, Thiruvananthapuram
Department of Aerospace Engineering

AE 342 : Modeling and Analysis Lab, Aug-Jan 2023

Session: Nov-2023

Aim: To solve partial differential equation using FreeFem++ software with a focus on fluid and thermal problems.

Numerical Analysis of Parabolic Equations

Weak Form

One dimensional transient heat conduction equation can be represented as in equation .

$$\frac{du}{dt} = \alpha \frac{d^2u}{dx^2} \tag{1}$$

This equation is multiplied with a weight function (v) and is integrated over the domain.

$$\frac{du}{dt} \cdot v = \alpha \frac{d^2u}{dx^2} \cdot v \tag{2}$$

Equation 2 incorporates the temporal term, which is discretized using Euler's forward difference method. Simultaneously, the diffusive term is simplified through the application of the chain rule, selectively eliminating terms that become zero at the boundaries. The resulting expression is presented in equation 3.

$$v\left(\frac{u^{n+1} - u^n}{dt}\right) = -\alpha \int_0^L \frac{du}{dx} \cdot \frac{dv}{dx}\, dx \tag{3}$$

Problem 1: A simple 1D transient heat conduction is simulated. The governing equations and boundary conditions are provided as follows.

$$\begin{gathered} \frac{du}{dx} = \frac{d^2u}{dx^2}, \quad \forall\ 0 < x < 1 \quad ,t > 0 \\ u(x,0) = x - x^2, \quad u(0,t) = u(1,t) = 0 \end{gathered} \tag{4}$$

1. Compare the obtained solution with analytical solution as in Equation 5

$$u(x,t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} e^{-n^2\pi^2 t} \sin(n\pi x) \tag{5}$$

2. In the current problem, the solution is obtained considering a Dirichlet boundary condition. Investigate the effect of a Neumann boundary condition (i.e., when the domain is insulated at both ends), by modifying the boundary conditions.

3. Perform a grid convergence and time step convergence study.

Problem 2: A combination of Dirichlet and Neumann boundary condition is considered for this problem. The governing equation, boundary conditions, and initial conditions are stated below

$$\begin{aligned} u_t &= u_{xx} \quad \text{for } 0 < x < 2, \quad t > 0 \\ u(x,0) &= \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \end{cases} \\ u(0,t) &= u_x(2,t) = 0 \end{aligned} \tag{6}$$

Compare the obtained solution with analytical solution

$$u(x,t) = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{\left(\sin \frac{(2n-1)\pi}{4} + \cos n\pi\right)}{(2n-1)^2} e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{2n-1}{4} \pi x$$

Problem 3: In practical problems, there can be instances when we encounter non-standard or non-homogeneous boundaries. Here, a non-homogeneous boundary condition problem is discussed.

$$u_t = u_{xx} \quad \text{for } 0 < x < 3, \quad t > 0 \tag{7}$$

$$u(x,0) = 4x - x^2, \quad u(0,t) = 0, \quad u(3,t) = 3 \tag{8}$$

Compare the obtained solution with analytical solution.

$$u(x,t) = x + \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x.$$

Problem 4: A problem similar to the previous one attempted. However in this problem a neumann condition of a constant gradient is given at the left boundary. The governing equations initial conditions and the boundary conditions are summarised below.

$$\begin{aligned} u_t &= u_{xx} \quad \text{for } 0 < x < 1, \quad t > 0 \\ u(x,0) &= 0, \quad u_x(0,t) = -1, \quad u(1,t) = 0 \end{aligned} \tag{9}$$

Compare the obtained solution with analytical solution.

$$u(x,t) = \frac{1}{2}(x^2 + 2t) - x + \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x.$$

Dr. Manu. K.V.,
Email: gvrmanu00@gmail.com.