

Matrix Algebra: Table of Contents

Matrix Rank

This lesson introduces the concept of **matrix rank** and explains how the rank of a [matrix](#) is revealed by its [echelon form](#).

The Rank of a Matrix

You can think of an $r \times c$ matrix as a set of r row [vectors](#), each having c elements; or you can think of it as a set of c column vectors, each having r elements.

The **rank** of a matrix is defined as (a) the maximum number of [linearly independent column](#) vectors in the matrix or (b) the maximum number of linearly independent *row* vectors in the matrix. Both definitions are equivalent.

For an $r \times c$ matrix,

- If r is less than c , then the maximum rank of the matrix is r .
- If r is greater than c , then the maximum rank of the matrix is c .

The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.

How to Find Matrix Rank

In this section, we describe a method for finding the rank of any matrix. This method assumes familiarity with [echelon matrices](#) and [echelon transformations](#).

The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its [row echelon matrix](#). Therefore, to find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows.

Consider matrix **A** and its row echelon matrix, **A_{ref}**. Previously, we showed [how to find the row echelon form for matrix A](#).

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A **A_{ref}**

Because the row echelon form **A_{ref}** has two non-zero rows, we know that matrix **A** has two independent row vectors; and we know that the rank of matrix **A** is 2.

You can verify that this is correct. Row 1 and Row 2 of matrix **A** are linearly independent. However, Row 3 is a **linear combination** of Rows 1 and 2. Specifically, Row 3 = 3*(Row 1) + 2*(Row 2). Therefore, matrix **A** has only two independent row vectors.

Full Rank Matrices

When all of the **vectors** in a matrix are **linearly independent**, the matrix is said to be **full rank**. Consider the matrices **A** and **B** below.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Notice that row 2 of matrix **A** is a scalar multiple of row 1; that is, row 2 is equal to twice row 1. Therefore, rows 1 and 2 are linearly dependent. Matrix **A** has only one linearly independent row, so its rank is 1. Hence, matrix **A** is not full rank.

Now, look at matrix **B**. All of its rows are linearly independent, so the rank of matrix **B** is 3. Matrix **B** is full rank.

Test Your Understanding

Problem 1

Consider the matrix **X**, shown below.

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix}$$

What is its rank?

- (A) 0
- (B) 1
- (C) 2
- (D) 3
- (E) 4

Solution

The correct answer is (C). Since the matrix has more than zero elements, its rank must be greater than zero. And since it has fewer rows than columns, its maximum rank is equal to the maximum number of linearly independent rows. And because neither row is linearly dependent on the other row, the matrix has 2 linearly independent rows; so its rank is 2.

Problem 2

Consider the matrix **Y**, shown below.

$$\mathbf{Y} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 5 & 9 \end{bmatrix}$$

What is its rank?

- (A) 0
- (B) 1
- (C) 2
- (D) 3
- (E) 4

Solution

The correct answer is (C). Since the matrix has more than zero elements, its rank must be greater than zero. And since it has fewer columns than rows, its maximum rank is equal to the maximum number of linearly independent columns.

Columns 1 and 2 are independent, because neither can be derived as a scalar multiple of the other. However, column 3 is linearly dependent on columns 1 and 2, because column 3 is equal to column 1 plus column 2. That leaves the matrix with a maximum of two linearly independent columns; that is., column 1 and column 2. So the matrix rank is 2.

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