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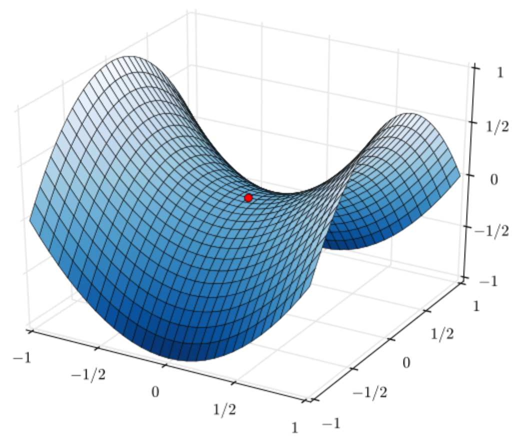
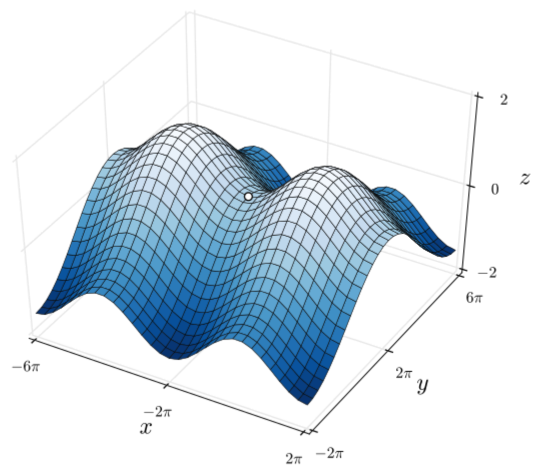
Saddle point

In mathematics, a **saddle point** or **minimax point**[1] is a point on the surface of the graph of a function where the slopes (derivatives) in orthogonal directions are both zero (a critical point), but which is not a local extremum of the function.[2] An example of a saddle point shown on the right is when there is a critical point with a relative minimum along one axial direction (between peaks) and at a relative maximum along the crossing axis. However, a saddle point need not be in this form. For example, the function $f(x,y) = x^2 + y^3$ has a critical point at $(0,0)$ that is a saddle point since it is neither a relative maximum nor relative minimum, but it does not have a relative maximum or relative minimum in the y-direction.



A saddle point (in red) on the graph of $z=x^2-y^2$ (hyperbolic paraboloid)

The name derives from the fact that the prototypical example in two dimensions is a surface that *curves up* in one direction, and *curves down* in a different direction, resembling a riding saddle or a mountain pass between two peaks forming a landform saddle. In terms of contour lines, a saddle point in two dimensions gives rise to a contour graph or trace in which the contour corresponding to the saddle point's value appears to intersect itself.



Saddle point between two hills (the intersection of the figure-eight z-contour)

Contents

Mathematical discussion

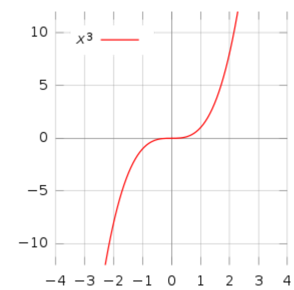
A simple criterion for checking if a given stationary point of a real-valued function $F(x,y)$ of two real variables is a saddle point is to compute the function's Hessian matrix at that point: if the Hessian is indefinite, then that point is a saddle point. For example, the Hessian matrix of the function $z = x^2 - y^2$ at the stationary point $(x,y,z) = (0,0,0)$ is the matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

which is indefinite. Therefore, this point is a saddle point. This criterion gives only a sufficient condition. For example, the point $(0, 0, 0)$ is a saddle point for the function $z = x^4 - y^4$, but the Hessian matrix of this function at the origin is the null matrix, which is not indefinite.

In the most general terms, a **saddle point** for a smooth function (whose graph is a curve, surface or hypersurface) is a stationary point such that the curve/surface/etc. in the neighborhood of that point is not entirely on any side of the tangent space at that point.

In a domain of one dimension, a saddle point is a point which is both a stationary point and a point of inflection. Since it is a point of inflection, it is not a local extremum.



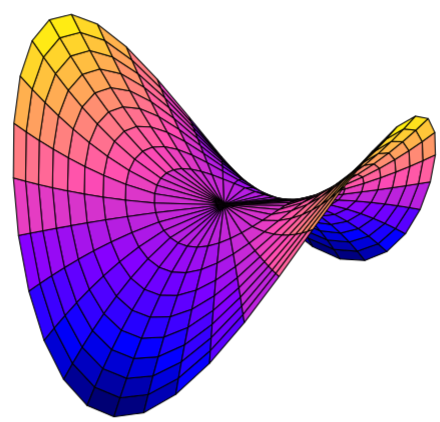
The plot of $y = x^3$ with a saddle point at 0

Saddle surface

A **saddle surface** is a smooth surface containing one or more saddle points.

Classical examples of two-dimensional saddle surfaces in the Euclidean space are second order surfaces, the hyperbolic paraboloid $z = x^2 - y^2$ (which is often referred to as "*the* saddle surface" or "the standard saddle surface") and the hyperboloid of one sheet. The Pringles potato chip or crisp is an everyday example of a hyperbolic paraboloid shape.

Saddle surfaces have negative Gaussian curvature which distinguish them from convex/elliptical surfaces which have positive Gaussian curvature. A classical third-order saddle surface is the monkey saddle.[3]


Hyperbolic paraboloid

Examples

In a two-player zero sum game defined on a continuous space, the equilibrium point is a saddle point.

For a second-order linear autonomous system, a critical point is a saddle point if the characteristic equation has one positive and one negative real eigenvalue.[4]

In optimization subject to equality constraints, the first-order conditions describe a saddle point of the Lagrangian.

Other uses

In dynamical systems, if the dynamic is given by a differentiable map f then a point is hyperbolic if and only if the differential of f^n (where n is the period of the point) has no eigenvalue on the (complex) unit circle when computed at the point. Then a *saddle point* is a hyperbolic periodic point whose stable and unstable manifolds have a dimension that is not zero.

A saddle point of a matrix is an element which is both the largest element in its column and the smallest element in its row.

See also

- Saddle-point method is an extension of Laplace's method for approximating integrals
- Extremum
- Derivative test
- Hyperbolic equilibrium point
- Minimax theorem
- Max–min inequality
- Monkey saddle

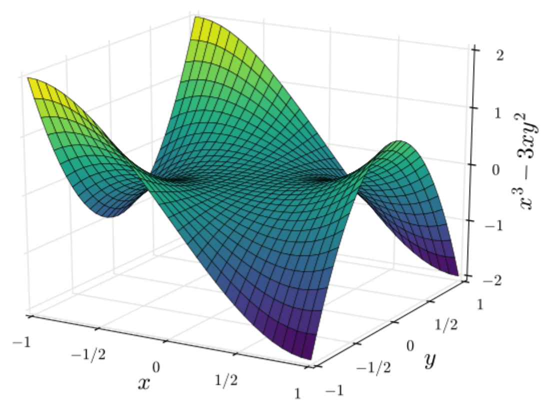
A model of an elliptic hyperboloid of one sheet

Notes

1. Howard Anton, Irl Bivens, Stephen Davis (2002): *Calculus, Multivariable Version*, p. 844
2. Chiang, Alpha C. (1984). *Fundamental Methods of Mathematical Economics* (3rd ed.). New York: McGraw-Hill. p. 312. ISBN 0-07-010813-7.
3. Buck, R. Creighton (2003). *Advanced Calculus* (https://books.google.com/books?id=7cYQAAAAQBAJ&pg=PA160) (Third ed.). Long Grove, IL: Waveland Press. p. 160. ISBN 1-57766-302-0.
4. von Petersdorff 2006

References

- Gray, Lawrence F.; Flanigan, Francis J.; Kazdan, Jerry L.; Frank, David H; Fristedt, Bert (1990), *Calculus two: linear and nonlinear functions*, Berlin: Springer-Verlag, p. 375, ISBN 0-387-97388-5
- Hilbert, David; Cohn-Vossen, Stephan (1952), *Geometry and the Imagination* (2nd ed.), New York: Chelsea, ISBN 978-0-8284-1087-8
- von Petersdorff, Tobias (2006), "Critical Points of Autonomous Systems", *Differential Equations for Scientists and Engineers (Math 246 lecture notes)* (http://www2.math.umd.edu/~petersd/246/stab.html)
- Widder, D. V. (1989), *Advanced calculus*, New York: Dover Publications, p. 128, ISBN 0-486-66103-2
- Agarwal, A., *Study on the Nash Equilibrium (Lecture Notes)* (http://www.cse.iitd.ernet.in/~rahul/cs905/lecture3/nash1.html#SECTION00041000000000000000)



A monkey saddle

Further reading

- Hilbert, David; Cohn-Vossen, Stephan (1952). *Geometry and the Imagination* (2nd ed.). Chelsea. ISBN 0-8284-1087-9.