

Gibbsian conditioning

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space on which we consider a sequence of iid discrete random variables $(X_n)_{n \geq 1}$ with law $\mathbb{P}(X_n = \pm 1) = 1/2$. Let $M_n = (X_1 + \dots + X_n)/n$ the empirical mean of the first n variables. Fix $\varepsilon > 0$ and $m \in]-\varepsilon, \varepsilon[$ and let B_n be the set $B_n = \{M_n \in [m, m + \varepsilon]\}$. We want to study the limit law as $n \rightarrow \infty$ of the k -ple (X_1, \dots, X_k) when (X_1, \dots, X_n) is conditioned on the event B_n . More precisely $k \geq 1$ and let \mathbb{P}_n^k be the law of (X_1, \dots, X_k) conditional on B_n :

$$\mathbb{P}_n^k(x_1, \dots, x_k) = \mathbb{P}(X_1 = x_1, \dots, X_k = x_k | B_n) = \frac{\mathbb{P}(X_1 = x_1, \dots, X_k = x_k, B_n)}{\mathbb{P}(B_n)}.$$

Our aim is to prove that the family $\{\mathbb{P}_n^k\}_{n \geq 1}$ converge weakly to the law μ^k on $\{\pm 1\}^k$ for which all the components are independent and

$$\mu^k(x_1, \dots, x_k) = \mu(x_1) \dots \mu(x_k)$$

where μ is the discrete probability on $\{\pm 1\}$ given by

$$\mu(x) = \frac{e^{-\beta x}}{e^{-\beta} + e^{\beta}} \quad \text{for } x = \pm 1$$

where β is a real number which is determined by the fact that the mean of the measure should be m :

$$m = \sum_{x=\pm 1} x \mu(x) = \frac{e^{-\beta} - e^{\beta}}{e^{-\beta} + e^{\beta}} = \tanh(-\beta).$$

To prove this weak convergence result you need to understand the discussion on Gibbsian conditioning in the Poly 4 of the lecture notes and proceed as follows:

- a) Start by proving the statement for $k = 1$. Note that for any continuous function

$$f(x_1) \mathbb{P}_n^1(x_1) = \frac{\mathbb{E}[f(X_1) 1_{B_n}]}{\mathbb{E}[1_{B_n}]} = \frac{\mathbb{E}[L_n(f) 1_{B_n}]}{\mathbb{E}[1_{B_n}]}$$

where $L_n(f) = \frac{1}{n} \sum_{i=1}^n f(X_i)$ is the mean of f with respect to the empirical measure of the random vector $(X_i)_{i=1}^n$. Observe also that $B_n = \{L_n(h) \in [m, m + \varepsilon]\}$ where $h: \{\pm 1, \dots, \pm 1\} \rightarrow \mathbb{R}$ is the identity function given by $h(x) = x$. Then

$$f(x_1) \mathbb{P}_n^1(x_1) = \frac{\mathbb{E}[L_n(f) 1_{L_n(h) \in [m, m + \varepsilon]}]}{\mathbb{E}[1_{L_n(h) \in [m, m + \varepsilon]}]}$$

Use Sanov theorem (Theorem 7 of Poly 4) and Proposition 1 and Corollary 2 of Poly 4 to deduce a large deviation principle for \mathbb{P}_n^1 . Conclude that $\mathbb{P}_n^1 \rightarrow \mu$.

- b) Follow the discussion on Gibbsian conditioning in the Poly 4 of the lecture notes to extend the argument to $k > 1$. For the purpose of the exam it is enough to prove the statement for $k = 2$: that is we want to prove that conditionally on B_n , the pair (X_1, X_2) converge weakly to a pair of independent variables each of them with law μ . (with depending on m).