Notes on the Transitive Reduction $\Box$ of $\Box^*$

Step 0. Find a paper I enjoy, and read it. Try to understand its ideas, with an eye towards extending it/altering it.

This idea is inspired by Larry Moss' paper that discusses $\mathbf{K}+\Box^*$, the Original Source on Segerberg axioms, and the concept of a Transitive Reduction (see the original paper).

Step 1. Look for an extension/open problem that makes me think "What the fuck? That's still open? No way, this shit is low-hanging fruit, free paper here I come." i.e. something *easy* and *straightforward*, *without complications*.

$\Box^*$ denotes the transitive-reflexive closure of $\Box$. We know that completeness for $K\Box^*$ follows from completeness for just K (i.e. just $\Box$) by adding only the axioms:

- **(Mix)** $\Box^*\varphi \to (\varphi \wedge \Box\Box^*\varphi)$
- **(Induction)** $(\varphi \wedge \Box^*(\varphi \to \Box\varphi)) \to \Box^*\varphi$

But we should easily have the other way around — i.e. given a normal modal logic L with $\Box$ satisfying (Mix) and (Induction), we should be able to get completeness for $L\Box^-$, where $\Box^-$ is the *transitive reduction* of L's $\Box$. Try it with $K\Box^*$!

Completeness for $\Box^*$ is mostly a matter of extending the accessibility relation for $\Box$ to be reflexive and transitive. Similarly, I expect completeness for $\Box^-$ to be a matter of taking the transitive-reflexive *reduction* of the graph for *its* $\Box$.

I *may* have to consider Hybrid Logic (i.e. modal logic in which we can name states) in case there are properties of $\Box^-$ I can't express, but that's an ordinary thing to expect.

Step 2. Follow-up question (only answer after Step 1): Is the extension *interesting* or *surprising*? What do we learn by extending the result?

Modal logic is really just an elegant language for reasoning about states in a graph. $\Box^*$ lets us reason about states that are arbitrarily far away, whereas $\Box^-$ lets us reason about *the very next state(s)*. This would absolutely be interesting to a Logic In Computer Science (LICS) or Knowledge Representation (KR) audience. Here are some examples of what this lets us express:

Example 1. (The next best states) Let's give $\Box$ a preferential reading. Let the accessibility relation for $\Box$ be xRy iff x is at least as good to the agent as y (note that this is a transitive and reflexive preference relation). So $\Box\varphi$ reads "φ holds in all states less preferable." Then $\Box^-\varphi$ reads "φ holds in all states immediately less preferable than this one," i.e. "φ holds in all the next best states."

Example 2. (What we learn in one step) Now suppose we have a modality $[\varphi]$ that indicates that an agent *learns* φ. The accessibility relation is functional: $xR_\varphi y$ iff $y = f(x,\varphi)$, where f is the agent's learning function. In this context, $[\varphi]^*\psi$ reads "In the limit of learning φ, ψ holds." We can then read $[\varphi]^-\psi$ as "after a *single step* of learning, ψ holds."

Step 3. Two things to do at this point:

- **Make a new Texmacs file named "PAPERNAME-master-notes.tm". Transcribe the key definitions, examples, lemmas, and results from the paper. This makes it easier to later copy-paste parts of proofs, and also ensures that I don't reinvent the wheel later (it's tempting to redefine everything yourself!)**
- **Go to `https://www.connectedpapers.com/` and download any major nearby papers. Upload the papers to paperless-ngx and make a point to read them (understanding context helps a lot!).**

Related Papers:

- The Transitive Reduction of a Directed Graph (1972)
  Introduced the concept of transitive reduction, proves important properties
- An Elementary Proof of the Completeness of PDL (1981)
  Proves completeness of PDL, making use of the (Mix) and (Induction) axioms
- Finite Models Constructed from Canonical Formulas (2005)
  Generalizes the proof from the 1981 paper for modal logics in general — this is the proof I will be adapting here.
- Internalizing Labeled Deduction (Blackburn, 2000)
  Defines irreflexivity and anti-transitivity using hybrid modal language
- Representation, Reasoning, and Relational Structures: a Hybrid Logic Manifesto
  The best source on Hybrid logic
- Modal Expressiveness of Graph Properties (2008)

It's actually pretty difficult to find *any* paper at all combining both modal logic and transitive reductions — even though modal logic with transitive closure is everywhere! (Note: Look into syllogistic logics with transitive closure.)

Existing Definitions and Results:

Transitive-Reflexive Closure + Reduction

DEFINITION 1. Let R be a binary relation (graph) over vertices V. Then

- R^*, the **transitive-reflexive closure** of R, is that graph extending R with the minimum number of edges such that it is reflexive and transitive.
- R^-, the **transitive-reflexive reduction** of R, is the graph with the minimum number of edges such that $(R^-)^* = R^*$.

Note. If R is finite, then R^- exists and is a subset of R; if R is acyclic, then R^- is unique.

PROPOSITION 2. **(Characterizing R^*)** uR^*v iff there is a path from u to v in R.

Proof. ($\rightarrow$) [Todo – see proof wiki article]

($\leftarrow$) Suppose there is a path from u to v in R. By induction on the length l of this path:

Base Step. $l = 0$, i.e. $u = v$. By reflexivity, uR^*v.

Inductive Step. $l \geq 0$. Let x immediately precede v on the path, i.e. xRv. Since R^* extends R, xR^*v. Note that the path from u to x is of length $l - 1$; By Inductive Hypothesis, uR^*x. Since R^* is transitive, uR^*v. □

DEFINITION 3. We define the set of all subgraphs of R that share the same transitive-reflexive closure R^*:

$$S(R) = \{R_i | R_i^* = R^*\}$$

PROPOSITION 4. **(Characterizing R^-)** Suppose R^- is finite and acyclic. Then:

$$R^- = \bigcap_{R_i \in S(R)} R_i$$

Proof. The proof is in Aho, Garey, and Ullman's paper. It's a bit complicated, so I'm just going to take it for granted for now. □

The Logic K

DEFINITION 5. K is the smallest normal modal logic, i.e. the smallest logic containing

- **(K)** $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- **(Dual)** $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

and closed under **(Necessitation)**, i.e. if $\varphi \in K$ then $\Box\varphi \in K$.

DEFINITION 6. **(Proofs)** $K \vdash \varphi$ iff either $\varphi \in K$ is a tautology, or φ follows from $\psi_1, \ldots, \psi_k \in K$ via **(Modus Ponens)** and **(Necessitation)**. $\Gamma \vdash \varphi$ iff there exist $\psi_1, \ldots, \psi_k \in \Gamma$ such that $K \vdash \psi_1 \wedge \ldots \wedge \psi_k \to \varphi$

DEFINITION 7. A model is just a tuple $\mathcal{M} = \langle W, R, V \rangle$, where W is a set of worlds/states, $R: W \times W$ is an accessibility relation, and V: proposition $\to \mathcal{P}(W)$ is a valuation function mapping propositions to sets of states.

DEFINITION 8. **(Truth at a World)** We give the usual possible worlds interpretation. Given model $\mathcal{M} = \langle W, R, V \rangle$ and world $w \in W$, and given the transitive-reflexive closure R^* of R:

$$\begin{array}{lll} \mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\ \mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\ \mathcal{M}, w \Vdash \neg\varphi & \text{iff} & \mathcal{M}, w \nVdash \varphi \\ \mathcal{M}, w \Vdash \Box\varphi & \text{iff} & \mathcal{M}, u \Vdash \varphi \text{ for all } wRu \end{array}$$

DEFINITION 9. **(Truth in a Model)** $\mathcal{M} \models \varphi$ iff $\mathcal{M}, w \Vdash \varphi$ for all $w \in W$. If $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$, we just write $\models \varphi$ (φ is *valid*).

DEFINITION 10. **(Entailment)** $\Gamma \models \varphi$ if whenever $\mathcal{M}, w \Vdash \psi$ for all $\psi \in \Gamma$, it follows that $\mathcal{M}, w \Vdash \varphi$.

THEOREM 11. **(Soundness, K)** K is sound w.r.t. the class of all models, i.e.

$$\text{If } K \vdash \varphi \text{ then } \models \varphi$$

Proof. We just need to show that **(K)**, **(Dual)**, **(Necessitation)**, and **(Modus Ponens)** are valid. (The validity of any φ provable in K then follows by induction.) Let $\mathcal{M} = \langle W, R, V\rangle$ be a model, and $w \in W$ be a world.

⊧(K). Suppose $\mathcal{M}, w \Vdash \Box(\varphi \to \psi)$, and suppose $\mathcal{M}, w \Vdash \Box\varphi$. So for all wRu, we have $\mathcal{M}, u \Vdash \varphi \to \psi$ and $\mathcal{M}, u \Vdash \varphi$. Let u be an arbitrary world such that wRu. By English if-then we have $\mathcal{M}, u \Vdash \psi$. Since u is arbitrary, we get $\mathcal{M}, w \Vdash \Box\psi$.

⊧(Dual). Valid by by definition of $\diamond$.

⊧(Nec). Suppose $\mathcal{M}, u \Vdash \varphi$ *for all* $u \in W$. Then in particular $\mathcal{M}, v \Vdash \varphi$ for that v such that wRv, and so $\mathcal{M}, w \Vdash \Box\varphi$. Since w is arbitrary, $\mathcal{M}, u \Vdash \Box\varphi$ for all $u \in W$.

⊧(MP). Valid by definition of $\to$. □

THEOREM 12. **(Completeness, K)** K is complete w.r.t. the class of all models, i.e.

$$\text{If } \models \varphi \text{ then } K \vdash \varphi$$

Proof. [Todo] □

The Logic $K\Box^*$

DEFINITION 13. $K\Box^*$ is the smallest logic extending K with the axiom schemas:

- **(Mix)** $\Box^*\varphi \to (\varphi \wedge \Box\Box^*\varphi)$
- **(Induction)** $(\varphi \wedge \Box^*(\varphi \to \Box\varphi)) \to \Box^*\varphi$

again closed under **(Necessitation)**, i.e. if $\varphi \in K$ then $\Box^*\varphi \in K$.

DEFINITION 14. Let $\mathcal{M} = \langle W, R, V\rangle$, $w \in W$, and let R^* be the transitive-reflexive closure of R. We interpret $\Box^*$ by:

$$\mathcal{M}, w \models \Box^*\varphi \qquad \text{iff} \qquad \mathcal{M}, u \models \varphi \text{ for all } wR^*u$$

THEOREM 15. **(Soundness, $K\Box^*$)** If $K\Box^* \vdash \varphi$ then $\models \varphi$.

Proof. We just need to check that **(Mix)** and **(Induction)** are valid — the validity of K and any φ derivable from $K\Box^*$ follows by induction. Let $\mathcal{M} = \langle W, R, V\rangle$ be a model, and $w \in W$ be a world.

⊧(Mix). Suppose $\mathcal{M}, w \Vdash \Box^*\varphi$. So $\mathcal{M}, u \models \varphi$ for all worlds wR^*u. We have two things to show:

$\mathcal{M}, w \Vdash \varphi$. Since R^* is reflexive, wR^*w. So $\mathcal{M}, u \models \varphi$.

$\mathcal{M}, w \Vdash \Box\Box^*\varphi$. Let u be an arbitrary world with wRu, and v be an arbitrary world with uR^*v. By reflexivity of R^*, we have wR^*u. By transitivity, we have wR^*v. By our earlier hypothesis, this means that $\mathcal{M}, v \models \varphi$, i.e. that $\mathcal{M}, w \models \Box\Box^*\varphi$.

⊧(Induction). Suppose $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \Box^*(\varphi \to \Box\varphi)$. We will now show that $\mathcal{M}, w \models \Box^*\varphi$. Let u be some world with wR^*u. By our second hypothesis, $\mathcal{M}, u \models \varphi \to \Box\varphi$. In addition, we have a path from w to u in R. By induction on the length of this path:

Base Step. $u = w$. [Todo]

Inductive Step. [Todo: take x immediately preceding u and apply IH]

[TODO: Old, inelegant proof] By our second hypothesis, $\mathcal{M}, u \models \varphi \to \Box\varphi$. Since R^* is reflexive, w is such a u, and in particular we have $\mathcal{M}, w \models \varphi \to \Box\varphi$. Since $\mathcal{M}, w \models \varphi$, $\mathcal{M}, w \models \Box\varphi$, i.e. $\mathcal{M}, v \models \varphi$ for all wRv. But wRv implies wR^*v.

This means that for our u before, $\mathcal{M}, u \models \varphi$. So $\mathcal{M}, w \models \Box^*\varphi$. □

THEOREM 16. **(Completeness, $K\Box^*$)** If $\models \varphi$ then $K\Box^* \vdash \varphi$.

Proof. [Todo] □

Step 4. Write up my new definitions & proof in the Texmacs file. Again, should be a *very* straightforward extension, and the proof (proofs are just unit-tests for definitions) shouldn't take up too much room at all (1-2 pages, including defs)

My Own Definitions and Results

Properties of Transitive-Reflexive Reduction

PROPOSITION 1. **(Algebraic Characterization of R^-)** Suppose R is acyclic. R^- is the only subgraph of R that is irreflexive and anti-transitive.

Proof. First, we show that R^- is irreflexive and anti-transitive. Then, we show that it is unique.

R^- is irreflexive. Suppose for contradiction that uR^-u. Let R' be a graph constructed by removing this (u,u) edge from R^-. Note that $(R')^* = R^*$, since taking the transitive-reflexive closure just re-constructs the missing (u,u) edge. This contradicts the fact that R^- is the smallest graph such that $(R^-)^* = R^*$.

R^- is anti-transitive. Suppose uR^-v, vR^-w, but for contradiction uR^-w. Again, let R' be constructed by removing this (u,w) edge from R^-. And again, $(R')^* = R^*$, since taking the transitive-reflexive closure just re-constructs the missing (u,w) edge. This contradicts the fact that R^- is the smallest such graph.

R^- is the unique such graph. Let $R_k \in S(R)$ be some subgraph of R that shares the same transitive-reflexive closure, i.e. $R_k^* = R^*$, and suppose R_k is irreflexive and anti-transitive. We need to show that $R_k = R^-$.

($\supseteq$) This is the easy direction. $R_k \in S(R)$, and so $R^- = \bigcap_{R_i \in S(R)} R_i \subseteq R_k$.

($\subseteq$) Suppose for contradiction that $R_k \nsubseteq R^-$. So there is some $(u,v) \in R_k$ such that $(u,v) \notin R^-$. Since $R_k \subseteq R_k^* = (R^-)^*$, we have $(u,v) \in (R^-)^*$. That is, we have a path from u to v in R^-. By Well-Ordering, suppose this is the minimal such path. We have two cases depending on the length l of this path:

Base Step. $l = 0$, i.e. $u = v$. But $(u,v) \in R_k$, which contradicts the fact that R_k is irreflexive.

Inductive Step. $l > 0$. So there is some x with $u \neq x$, $v \neq x$ with $u(R^-)^*x$ and xR^-v. But by the previous ($\supseteq$) direction, $R^- \subseteq R_k$, and so $u(R_k)^*x$ and xR_kv. So we have edge uR_kv and a different path from u to v in R_k, which contradicts the fact that R_k is anti-transitive.

□

COROLLARY 2. Suppose R is acyclic. If R is also irreflexive and anti-transitive, then $R = R^-$.

Proof. This follows from the fact that R^- is the unique such subgraph of R. □

The Logic $K\Box^*\Box^-$

DEFINITION 3. In order to reason about the transitive-reflexive reduction, we need to expand our language. Let PROP $= \{p, q, \ldots\}$ denote finitely many propositional variables, and NOM $= \{i, j, \ldots\}$ denote finitely many nominal variables.

$$\varphi := i \mid p \mid \neg\varphi \mid \varphi \to \psi \mid \Box\varphi \mid \Box^*\varphi \mid \Box^-\varphi \mid @_i\varphi$$

The new additions are i, $@_i\varphi$, and $\Box^-\varphi$. i is a formula from hybrid logic that is true exactly at the world denoted by i. $@_i\varphi$ is also from hybrid logic, and is read "φ holds at world i." We will interpret $\Box^-\varphi$ as the transitive-reflexive reduction of $\Box$.

DEFINITION 4. $K\Box^*\Box^-$ is the smallest hybrid logic with the axiom schemas:

- **(K)** $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- **(Dual)** $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

- **(Mix)** $\Box^*\varphi \to (\varphi \wedge \Box\Box^*\varphi)$
- **(Induction)** $(\varphi \wedge \Box^*(\varphi \to \Box\varphi)) \to \Box^*\varphi$

- **(Acyclic)** $i \to \Box^*()$ [Not quite right…]
- **(Re-Mix)** $\neg\Box i \to (i \wedge \Box\Box\neg i)$
- **(Re-Duction)** $\Box^-\Box^*\varphi \leftrightarrow \Box\Box^*\varphi$

again closed under **(Necessitation)** for each of the modalities, e.g. if $\varphi \in K$ then $\Box^-\varphi \in K$.

DEFINITION 5. Let $\mathcal{M} = \langle W, R, V\rangle$, $w \in W$, R^- be the transitive-reflexive reduction of R. We interpret $\Box^-$ by:

$$\mathcal{M}, w \models \Box^-\varphi \qquad \text{iff} \qquad \mathcal{M}, u \models \varphi \text{ for all } wR^-u$$

THEOREM 6. **(Soundness, $K\Box^*\Box^-$)** If $K\Box^*\Box^- \vdash \varphi$ then $\models \varphi$.

Proof. [] $\Box$

THEOREM 7. **(Completeness, $K\Box^*\Box^-$)** If $\models \varphi$ then $K\Box^*\Box^- \vdash \varphi$.

Proof. [] $\Box$

Step 5. Step away (for a few days). Come back and check the proof *slowly* to make sure there aren't any missing edge cases or conditions.

- **If it's all good — congratulations, you got a free paper!**
- **Usually there will be some idiotic mistake in the proof. It may seem like *you're* the idiot for trying it — but in fact, it's now your job to figure out *what conditions will make this naive proof work*!**

Step 6. Write a computer program/simulation to collect statistics on the objects/models. Ask: *How unusual* is it for the models to fail the proof scenario? What about this lemma? This other lemma? Am I looking for a weird exception here, or is it very common? Make the simulation as *visual* as possible so that I can *picture* the condition/failure.

Step 7. If the condition is rare, try to modify the proof to account for the exceptions (they may satisfy the theorem but fail just this proof). Think: "is there a simple thing I can add to the system that will help the proof go through?"

Otherwise, sit down and try to define *exactly* that condition the proof doesn't fuck up at that step. Use the generated examples for help. Prove the claim for models satisfying Condition.

Step 8. Prove (i.e. unit-test/sanity-check) general properties of models satisfying Condition. Build up a theory of how Condition behaves — what is it like? What algebra does it follow? What is it similar to? What does it mean?

Step 9. Consider whether this partial result is still *interesting* enough to be published.

Is it meaningful to everyone in the field? ⟶ Submit it to a top-tier conference

Is it meaningful to this niche sub-field? ⟶ Submit it to the main conference for the sub-field

Is it meaningful as a technical lemma? ⟶ Submit it to a conference specifically for technical results

None of the above? ⟶ It's okay to not publish for now, and wait until you see the whole proof.

Step 10. Move on to the write-up stage. But otherwise, step away from the problem — there are too many other interesting things to spend all of your time on this one. Trust that one day a different solution will come to you.