

Large deviations for random walks

Let $(X_n)_{n \geq 1}$ be a sequence of iid Bernoulli(p) random variables. Consider the process $S_n = X_1 + \dots + X_n$ with $S_0 = 0$. Define a continuous random function μ_n on $[0, 1]$ by

$$\mu_n(t) = \frac{S_k}{n} + \frac{(S_{k+1} - S_k)(t - k/n)}{n} \quad \text{for } k \leq t < k+1.$$

Let K be the subset of $C([0, 1])$ such that $f \in K$ if and only if $f(0) = 0$ and $|f(t) - f(s)| \leq |t - s|$ for all $0 \leq s < t \leq 1$. Observe that $\mu_n \in K$ and that using the notations of the section on large deviations for processes of the 4th lecture notes of the course we have

$$\mu_n(t) = \int_0^t F_n(X_n)(s) ds$$

where for every n the vectors $X_n = (X_1, \dots, X_n)$ are random elements in $\{0, 1\}^n$ and

$$F_n(x_1, \dots, x_n)(s) = \sum_{i=1}^n x_i 1_{[(i-1)/n, i/n)}(s) \in L^\infty([0, 1]).$$

Observe also that $\mu_n(t)$ is a piecewise linear function for which

$$\mu_n(k/n) = S_k/n.$$

Recall that J_p has been defined in Poly 4 (section on large deviations for processes) as

$$J_p(x) = H(\text{Ber}(x)/\text{Ber}(p)) = x \log \frac{x}{p} + (1-x) \log \frac{1-x}{1-p}$$

the relative entropy of a Bernoulli law of parameter x with respect to the Bernoulli law of parameter p .

- Prove that the set K with the norm of uniform convergence is compact.
- Prove that the laws μ_n of μ_n on K satisfy a large deviation principle with rate function

$$I(f) = \int_0^1 J_p(f(s)) ds$$

where $f'(s)$ is the derivative of $f \in K$ (which exists almost everywhere since f is Lipschitz). Hint: use Theorem 8 (Mogulskii theorem) of Poly 4 and the contraction principle (Thm 19 of Poly 3).

- With $p = 1/2$, use the result of point b) to prove that, if we set

$$\rho_n = \mathbb{P} \left(\frac{S_k}{n} \leq c \frac{k}{n} \right) \quad \text{for all } k = 1, \dots, n$$

then

$$(\rho_n)^{1/n} \sim \begin{cases} 1 & \text{for } c \in [0, 1/2] \\ \frac{1}{2^{c^2(1/2-c)^2}} & \text{for } c \in [1/2, 1] \end{cases}$$

Hint: use Corollary 9 of Poly 3 to convert the limit computation in a variational problem over the rate function, guess the shape of the extremal function and then use the fact that the function $J_{1/2}$ is convex and Jensen's inequality gives

$$\int_0^1 J_{1/2}(s) ds \leq J_{1/2}\left(\int_0^1 s ds\right).$$

To understand fully this problem it is a good idea to give a look at the paper

<http://www.math.tau.ac.il/~tsirel/Courses/LargeDev/lect5.pdf>

in particular Section 5a.