Notes on the Completeness of Neural Net Models in a Modal Language

Step 0. Find a paper I enjoy, and read it. Try to understand its ideas, with an eye towards extending it/altering it.

This paper is inspired by Hannes Leitgeb's Nonmonotonic Reasoning by Inhibition Nets, which proves completeness for the neuro-symbolic interface suggested by Balkenius and Gärdenfors' Nonmonotonic Inferences in Neural Networks.

Step 1. Look for an extension/open problem that makes me think "What the fuck? That's still open? No way, this shit is low-hanging fruit, free paper here I come." i.e. something *easy* and *straightforward*, *without complications*.

Hannes Leitgeb showed that feed-forward neural networks are complete with respect to certain conditional laws of $\Rightarrow$. But $\varphi \Rightarrow \psi$ just reads "$\psi \subseteq \mathsf{Prop}(\varphi)$" (i.e. ψ is in the propagation of the signal φ), which we can re-write in modal language as $\mathbf{T}\varphi \rightarrow \psi$. In the same way that Hannes shows that feed-forward nets and preferential-conditional models are equivalent, it shouldn't be too hard at all to show that feed-forwards nets and neighborhood models are equivalent. (Note that it is well-known that neighborhood models are a generalization of preferential models.)

I also think I should be able to throw in a $\mathbf{K}$ modality (graph-reachability) in there, almost for free.

Step 2. Follow-up question (only answer after Step 1): Is the extension *interesting* or *surprising*? What do we learn by extending the result?

Why bother with completeness?. In formal specifications (of AI agents, or otherwise), we're often content with just listing some sound rules or behaviors that the agent will always follow. And it's definitely cool to see that neural networks satisfy some sound logical axioms. But if we want to fundamentally bridge the gap between logic and neural networks, we should set our aim higher: Towards *complete* logical characterizations of neural networks.

A more practical reason: Completeness gives us model-building, i.e. given a specification Γ, we can *build* a neural network $\mathcal{N}$ satisfying Γ.

Why bother with this modal language?. Almost all of the previous work bridging logic and neural networks has focused on neural net models of *conditionals*. In some sense, doing this in modal language is just a re-write of this old work. But this previous work hasn't addressed how *learning* or *update* in neural networks can be cast in logical terms. This is not merely due to circumstance — integrating conditionals with update is a long-standing controversial issue. So instead, we believe that it is more natural to work with modalities (instead of conditionals), because

Modal language natively supports update.

In other words, our modal setting sets us up to easily cast update operators (e.g. neural network learning) as modal operators in our logic.

Also this gives me an excuse to title a paper *Neural Network Models à la Mode* :-) (This is a play on both modal logic and also bringing some old work back in style!)

And LOL I can name a section "Learning: The Cherry on Top"

Step 3. Two things to do at this point:

- **Make a new Texmacs file named "PAPERNAME-master-notes.tm". Transcribe the key definitions, examples, lemmas, and results from the paper. This makes it easier to later copy-paste parts of proofs, and also ensures that I don't reinvent the wheel later (it's tempting to redefine everything yourself!)**
- **Go to `https://www.connectedpapers.com/` and download any major nearby papers. Upload the papers to paperless-ngx and make a point to read them (understanding context helps a lot!).**

Related Papers:

[]

Step 4. Write up my new definitions & proof in the Texmacs file. Again, should be a *very* straightforward extension, and the proof (proofs are just unit-tests for definitions) shouldn't take up too much room at all (1-2 pages, including defs)

1 Interpreted Neural Nets

1.1 Basic Definitions

DEFINITION 1.1. An **interpreted ANN** (Artificial Neural Network) is a pointed directed graph $\mathcal{N} = \langle N, E, W, T, A, V\rangle$, where

- N is a finite nonempty set (the set of **neurons**)
- $E \subseteq N \times N$ (the set of **excitatory neurons**)
- $W: E \to \mathbb{R}$ (the **weight** of a given connection)
- A is a function which maps each $n \in N$ to $A^{(n)}: \mathbb{R}^{\mathrm{k}} \times \mathbb{R}^{\mathrm{k}} \to \mathbb{R}$ (the **activation function** for n, where k is the indegree of n)
- O is a function which maps each $n \in N$ to $O^{(n)}: \mathbb{R} \to \{0, 1\}$ (the **output function** for n)
- V: propositions $\cup$ nominals $\to \mathcal{P}(N)$ is an assignment of nominals to individual neurons (the **valuation function**). If i is a nominal, we require $|V(i)| = 1$, i.e. a singleton.

DEFINITION 1.2. A **BFNN** (Binary Feedforward Neural Network) is an interpreted ANN $\mathcal{N} = \langle N, E, W, T, A, V\rangle$ that is

- **Feed-forward**, i.e. E does not contain any cycles
- **Binary**, i.e. the output of each neuron is in $\{0, 1\}$
- $O^{(n)} \circ A^{(n)}$ is **zero at zero** in the first parameter, i.e.

$$O^{(n)}(A^{(n)}(\vec{0}, \vec{w})) = 0$$

- $O^{(n)} \circ A^{(n)}$ is **strictly monotonically increasing** in the second parameter, i.e. for all $\vec{x}, \vec{w}_1, \vec{w}_2 \in \mathbb{R}^k$, if $\vec{w}_1 < \vec{w}_2$ then $O^{(n)}(A^{(n)}(\vec{x}, \vec{w}_1)) < O^{(n)}(A^{(n)}(\vec{x}, \vec{w}_2))$. We will more often refer to the equivalent condition:

$$\vec{w}_1 \leqslant \vec{w}_2 \quad \text{iff} \quad O^{(n)}(A^{(n)}(\vec{x}, \vec{w}_1)) \leqslant O^{(n)}(A^{(n)}(\vec{x}, \vec{w}_2))$$

DEFINITION 1.3. Given a BFNN $\mathcal{N}$, $\mathsf{Set} = \mathcal{P}(N) = \{S \mid S \subseteq N\}$

DEFINITION 1.4. For $S \in \mathsf{Set}$, let $\chi_S : N \to \{0,1\}$ be given by $\chi_S = 1$ iff $n \in S$

1.2 Prop and Reach

DEFINITION 1.5. Let $\mathsf{Prop} : \mathsf{Set} \to \mathsf{Set}$ be defined recursively as follows: $n \in \mathsf{Prop}(S)$ iff either

Base Case. $n \in S$, or

Constructor. For those $m_1, \ldots, m_k$ such that $(m_i, n) \in E$ we have

$$O^{(n)}(A^{(n)}(\vec{\chi}_{\mathsf{Prop}(S)}(m_i), \vec{W}(m_i, n))) = 1$$

PROPOSITION 1.6. (LEITGEB) Let $\mathcal{N} \in \mathsf{Net}$. For all $S, S_1, S_2 \in \mathsf{Set}$, Prop satisfies

(Inclusion). $S \subseteq \mathsf{Prop}(S)$

(Idempotence). $\mathsf{Prop}(S) = \mathsf{Prop}(\mathsf{Prop}(S))$

(Cumulative). If $S_1 \subseteq S_2 \subseteq \mathsf{Prop}(S_1)$ then $\mathsf{Prop}(S_1) \subseteq \mathsf{Prop}(S_2)$

(Loop). If $S_1 \subseteq \mathsf{Prop}(S_0), \ldots, S_n \subseteq \mathsf{Prop}(S_{n-1})$ and $S_0 \subseteq \mathsf{Prop}(S_n)$, then $\mathsf{Prop}(S_i) = \mathsf{Prop}(S_j)$ for all $i, j \in \{0, \ldots, n\}$

DEFINITION 1.7. Let $\mathsf{Reach} : \mathsf{Set} \to \mathsf{Set}$ be defined recursively as follows: $n \in \mathsf{Reach}(S)$ iff either

Base Case. $n \in S$, or

Constructor. There is an $m \in \mathsf{Reach}(S)$ such that $(m, n) \in E$.

PROPOSITION 1.8. Let $\mathcal{N} \in \mathsf{Net}$. For all $S, S_1, S_2 \in \mathsf{Set}$, $n, m \in N$, Reach satisfies

(Inclusion). $S \subseteq \mathsf{Reach}(S)$

(Idempotence). $\mathsf{Reach}(S) = \mathsf{Reach}(\mathsf{Reach}(S))$

(Monotonicity). If $S_1 \subseteq S_2$ then $\mathsf{Reach}(S_1) \subseteq \mathsf{Reach}(S_2)$

(Containment). $\mathsf{Prop}(S) \subseteq \mathsf{Reach}(S)$

DEFINITION 1.9. For all $n \in N$, $\mathsf{Reach}^{-1}(n) = \bigcap_{n \notin \mathsf{Reach}(X)} X^{\mathsf{C}}$

PROPOSITION 1.10. For all $n \in N$, $\mathsf{Reach}^{-1}(n) = \{m \mid \text{there is an } E\text{-path from } m \text{ to } n\}$

PROPOSITION 1.11. Reach^{-1} is acyclic in the following sense: For $n_1, \ldots, n_k \in N$, if

$$n_1 \in \mathsf{Reach}^{-1}(n_2), \ldots, n_{k-1} \in \mathsf{Reach}^{-1}(n_k), n_k \in \mathsf{Reach}^{-1}(n_1)$$

Then each $n_i = n_j$.

PROPOSITION 1.12. **(Minimal Cause)** For all $n \in N$, if $n \in \mathsf{Prop}(S)$ then $n \in \mathsf{Prop}(S \cap \mathsf{Reach}^{-1}(n))$

1.3 Neural Network Semantics

DEFINITION 1.13. Formulas of our language $\mathcal{L}$ are given by

$$\varphi ::= i \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}\varphi \mid \mathbf{K}^{\leftarrow} i \mid \mathbf{T}\varphi \mid$$

where p is any propositional variable, and i is any nominal (denoting a neuron). Material implication $\varphi \to \psi$ is defined as $\neg\varphi \vee \psi$. We define $\bot, \vee, \leftrightarrow, \Leftrightarrow$, and the dual operators $\langle\mathbf{K}\rangle, \langle\mathbf{K}^{\leftarrow}\rangle, \langle\mathbf{T}\rangle$ in the usual way.

DEFINITION 1.14. Let $\mathcal{N} \in \mathsf{Net}$. The semantics $[\![\cdot]\!]: \mathcal{L} \to \mathsf{Set}$ for $\mathcal{L}$ are defined recursively as follows:

$[\![i]\!]$	$=$	$V(i) \in \mathsf{Set}$
$[\![p]\!]$	$=$	$V(p) \in \mathsf{Set}$
$[\![\neg\varphi]\!]$	$=$	$[\![\varphi]\!]^{\mathsf{C}}$
$[\![\varphi \wedge \psi]\!]$	$=$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$[\![\langle \mathbf{K} \rangle \varphi]\!]$	$=$	$\mathsf{Reach}([\![\varphi]\!])$
$[\![\langle \mathbf{K}^{\leftarrow} \rangle \varphi]\!]$	$=$	$\{n \mid \exists m \in [\![\varphi]\!]^{\mathsf{C}}$ such that $n \in \mathsf{Reach}^{-1}(m)\}$
$[\![\langle \mathbf{T} \rangle \varphi]\!]$	$=$	$\mathsf{Prop}([\![\varphi]\!])$

DEFINITION 1.15. **(Truth at a neuron)** $\mathcal{N}, n \Vdash \varphi$ iff $n \in [\![\varphi]\!]_{\mathcal{N}}$.

DEFINITION 1.16. **(Truth in a net)** $\mathcal{N} \models \varphi$ iff $\mathcal{N}, n \Vdash \varphi$ for all $n \in N$.

2 Neighborhood Models

2.1 Basic Definitions

DEFINITION 2.1. A **neighborhood frame** is a pair $\mathcal{F} = \langle W, f \rangle$, where W is a non-empty set of **worlds** and $f: W \to \mathcal{P}(\mathcal{P}(W))$ is a **neighborhood function**.

DEFINITION 2.2. A **multi-frame** is $\mathfrak{F} = \langle W, f, g \rangle$, where f and g are neighborhood functions.

DEFINITION 2.3. Let $\mathcal{F} = \langle W, f \rangle$ be a neighborhood frame, and let $w \in W$. The set $\bigcap_{X \in f(w)} X$ is called the **core** of $f(w)$. We often abbreviate this by $\cap f(w)$.

DEFINITION 2.4. Let $\mathcal{F} = \langle W, f \rangle, \mathcal{G} = \langle W, g \rangle$ be neighborhood frames with W nonempty.

- $\mathcal{F}$ is **closed under finite intersections** iff for all $w \in W$, if $X_1, \ldots, X_n \in f(w)$ then their intersection $\bigcap_{i=1}^{k} X_i \in f(w)$.
- $\mathcal{F}$ is **closed under supersets** iff for all $w \in W$, if $X \in f(w)$ and $X \subseteq Y \subseteq W$, then $Y \in f(w)$.
- $\mathcal{F}$ **contains the unit** iff $W \in f(w)$.
- $\mathcal{F}$ **contains the empty set** iff $\emptyset \in f(w)$.
- $\mathcal{F}$ is **reflexive** iff for all $w \in W$, $w \in \cap f(w)$
- $\mathcal{F}$ is **transitive** iff for all $w \in W$, if $X \in f(w)$ then $\{u \mid X \in f(u)\} \in f(w)$.
- $\mathcal{F}$ is **acyclic** iff for all $u_1, \ldots, u_n \in W$, if $u_1 \in \cap f(u_2), \ldots, u_{n-1} \in \cap f(u_n), u_n \in \cap f(u_1)$ then all $u_i = u_j$.
- $\mathcal{F}$ **guides** $\mathcal{G}$ iff for all $w \in W$, if $X \cup (\cap f(w))^{\mathsf{C}} \in g(w)$ then $X \in g(w)$.

DEFINITION 2.5. Let $\mathcal{F} = \langle W, f \rangle$ be a frame, and $\mathfrak{F} = \langle W, f, g \rangle$ be a multi-frame extending $\mathcal{F}$. We will focus on the following special classes of frames:

- $\mathcal{F}$ is a **proper filter** iff for all $w \in W$, $f(w)$ is closed under finite intersections, closed under supersets, contains the unit, and does not contain the empty set.
- $\mathfrak{F}$ is a **preferential multi-frame** iff
  - W is finite,
  - $\mathcal{F} = \langle W, f \rangle$ forms a reflexive, transitive, acyclic, proper filter,
  - $\mathcal{G} = \langle W, g \rangle$ is reflexive, transitive, and $\mathcal{F}$ guides $\mathcal{G}$.

PROPOSITION 2.6. (PACUIT) If $\mathcal{F} = \langle W, f \rangle$ is a filter, and W is finite, then $\mathcal{F}$ contains its core.

PROPOSITION 2.7. If $\mathcal{F} = \langle W, f \rangle$ is a proper filter, then for all $w \in W$, $Y^{\mathsf{C}} \in f(w)$ iff $Y \notin f(w)$.

2.2 Neighborhood Semantics

DEFINITION 2.8. Let $\mathcal{F} = \langle W, f \rangle$, $\mathcal{G} = \langle W, g \rangle$ be a neighborhood frame. A **neighborhood model** based on $\mathcal{F}$ and $\mathcal{G}$ is $\mathcal{M} = \langle W, f, g, V \rangle$, where $V: \mathcal{L} \to \mathcal{P}(W)$ is a valuation function.

DEFINITION 2.9. Let $\mathcal{M} = \langle W, f, g, V \rangle$ be a model based on two frames $\mathcal{F} = \langle W, f \rangle$, $\mathcal{G} = \langle W, g \rangle$. The (neighborhood) semantics for $\mathcal{L}$ are defined recursively as follows:

$\mathcal{M}, w \Vdash i$	iff	$V(i) = \{w\}$
$\mathcal{M}, w \Vdash p$	iff	$w \in V(p)$
$\mathcal{M}, w \Vdash \neg\varphi$	iff	$\mathcal{M}, w \nVdash \varphi$
$\mathcal{M}, w \Vdash \varphi \wedge \psi$	iff	$\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \mathbf{K}\varphi$	iff	$\{u \mid \mathcal{M}, u \Vdash \varphi\} \in f(w)$
$\mathcal{M}, w \Vdash \mathbf{K}^{\leftarrow}\varphi$	iff	$\forall u \in W$, if $w \in \cap f(u)$ then $\mathcal{M}, u \Vdash \varphi$
$\mathcal{M}, w \Vdash \mathbf{T}\varphi$	iff	$\{u \mid \mathcal{M}, u \Vdash \varphi\} \in g(w)$

In neighborhood semantics, the operators $\mathbf{K}$, $\mathbf{K}^{\leftarrow}$, and $\mathbf{T}$ are more natural to interpret. But when we gave our neural semantics, we instead interpreted the *duals* $\langle\mathbf{K}\rangle$, $\langle\mathbf{K}^{\leftarrow}\rangle$, and $\langle\mathbf{T}\rangle$. Since we need to relate the two, I'll write the explicit neighborhood semantics for the duals here:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash \langle\mathbf{K}\rangle\varphi & \text{iff} & \{u \mid \mathcal{M}, u \nVdash \varphi\} \notin f(w) \\
\mathcal{M}, w \Vdash \langle\mathbf{K}^{\leftarrow}\rangle\varphi & \text{iff} & \exists u \in W \text{ such that } w \in \cap f(u) \text{ and } \mathcal{M}, u \nVdash \varphi \\
\mathcal{M}, w \Vdash \langle\mathbf{T}\rangle\varphi & \text{iff} & \{u \mid \mathcal{M}, u \nVdash \varphi\} \notin g(w)
\end{array}
$$

DEFINITION 2.10. **(Truth in a model)** $\mathcal{M} \models \varphi$ iff $\mathcal{M}, w \Vdash \varphi$ for all $w \in W$.

3 From Nets to Frames

This is the easy ("soundness") direction!

DEFINITION 3.1. Given a BFNN $\mathcal{N}$, its **simulation frame** $\mathfrak{F}^{\bullet} = \langle W, f, g \rangle$ is given by:

- $W = N$
- $f(w) = \{S \subseteq W \mid w \notin \mathsf{Reach}(S^{\complement})\}$
- $g(w) = \{S \subseteq W \mid w \notin \mathsf{Prop}(S^{\complement})\}$

Moreover, the **simulation model** $\mathcal{M}^{\bullet} = \langle W, f, g, V \rangle$ based on $\mathfrak{F}^{\bullet}$ has:

- $V_{\mathcal{M}^{\bullet}}(p) = V_{\mathcal{N}}(p)$;
- $V_{\mathcal{M}^{\bullet}}(i) = V_{\mathcal{N}}(i)$

THEOREM 3.2. Let $\mathcal{N}$ be a BFNN, and let $\mathcal{M}^{\bullet}$ be the simulation model based on $\mathfrak{F}^{\bullet}$. Then for all $w \in W$,

$$\mathcal{M}^{\bullet}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{N}, w \Vdash \varphi$$

Proof. By induction on φ. The nominal, propositional, $\neg\varphi$, and $\varphi \wedge \psi$ cases are trivial.

$\langle\mathbf{K}\rangle\varphi$ case:

$$
\begin{array}{llll}
\mathcal{M}^{\bullet}, w \Vdash \langle\mathbf{K}\rangle\varphi & \text{iff} & \{u \mid \mathcal{M}^{\bullet}, w \nVdash \varphi\} \notin f(w) & \text{(by definition)} \\
& \text{iff} & \{u \mid u \notin [\![\varphi]\!]\} \notin f(w) & \text{(IH)} \\
& \text{iff} & [\![\varphi]\!]^{\complement} \notin f(w) & \\
& \text{iff} & w \in \mathsf{Reach}([\![(\varphi^{\complement})^{\complement}]\!]) & \text{(by choice of } f\text{)} \\
& \text{iff} & w \in \mathsf{Reach}([\![\varphi]\!]) & \\
& \text{iff} & w \in [\![\langle\mathbf{K}\rangle\varphi]\!] & \text{(by definition)} \\
& \text{iff} & \mathcal{N}, w \Vdash \langle\mathbf{K}\rangle\varphi & \text{(by definition)}
\end{array}
$$

$\langle \mathbf{K}^{\leftarrow}\rangle\varphi$ case:

$$\begin{array}{rll}
\mathcal{M}^{\bullet}, w \Vdash \langle \mathbf{K}^{\leftarrow}\rangle\varphi & \text{iff} \;\; \exists u \text{ such that } w \in \cap f(u) \text{ and } \mathcal{M}^{\bullet}, u \nVdash \varphi & \text{(by definition)} \\
& \text{iff} \;\; \exists u \text{ such that } w \in \cap f(u) \text{ and } u \notin [\![\varphi]\!] & \text{(IH)} \\
& \text{iff} \;\; \exists u \in [\![\varphi]\!]^{\mathsf{C}} \text{ such that } w \in \bigcap_{X \in f(u)} X & \\
& \text{iff} \;\; \exists u \in [\![\varphi]\!]^{\mathsf{C}} \text{ such that } w \in \bigcap_{u \notin \mathsf{Reach}(X^{\mathsf{C}})} X & \text{(by choice of } f\text{)} \\
& \text{iff} \;\; \exists u \in [\![\varphi]\!]^{\mathsf{C}} \text{ such that } w \in \mathsf{Reach}^{-1}(u) & \\
& \text{iff} \;\; \mathcal{N}, w \Vdash \langle \mathbf{K}^{\leftarrow}\rangle\varphi & \text{(by definition)}
\end{array}$$

$\langle \mathbf{T}\rangle\varphi$ case:

$$\begin{array}{rll}
\mathcal{M}^{\bullet}, w \Vdash \langle \mathbf{T}\rangle\varphi & \text{iff} \;\; \{u \mid \mathcal{M}^{\bullet}, w \nVdash \varphi\} \notin g(w) & \text{(by definition)} \\
& \text{iff} \;\; \{u \mid u \notin [\![\varphi]\!]\} \notin g(w) & \text{(IH)} \\
& \text{iff} \;\; [\![\varphi]\!]^{\mathsf{C}} \notin g(w) & \\
& \text{iff} \;\; w \in \mathsf{Prop}([\![(\varphi^{\mathsf{C}})^{\mathsf{C}}]\!]) & \text{(by choice of } g\text{)} \\
& \text{iff} \;\; w \in \mathsf{Prop}([\![\varphi]\!]) & \\
& \text{iff} \;\; w \in [\![\langle \mathbf{T}\rangle\varphi]\!] & \text{(by definition)} \\
& \text{iff} \;\; \mathcal{N}, w \Vdash \langle \mathbf{T}\rangle\varphi & \text{(by definition)}
\end{array}$$

□

COROLLARY 3.3. $\mathcal{M}^{\bullet} \models \varphi$ iff $\mathcal{N} \models \varphi$.

THEOREM 3.4. $\mathfrak{F}^{\bullet}$ is a preferential multi-frame.

Proof. We show each in turn:

- ***W* is finite:** This holds because our BFNN is finite.
- **$\mathcal{F}$ is closed under finite intersection:** Suppose $X_1, \ldots, X_n \in f(w)$. By definition of f, $w \notin \bigcup_i \mathsf{Reach}(X_i^{\mathsf{C}})$ for all i. Since Reach is monotonic, [Make this a lemma!] we have $\bigcup_i \mathsf{Reach}(X_i^{\mathsf{C}}) = \mathsf{Reach}(\bigcup_i X_i^{\mathsf{C}}) = \mathsf{Reach}((\bigcap_i X_i)^{\mathsf{C}})$. So $w \notin \mathsf{Reach}((\bigcap_i X_i)^{\mathsf{C}})$. But this means that $\bigcap_i X_i \in f(w)$.
- **$\mathcal{F}$ is closed under superset:** Suppose $X \in f(w), X \subseteq Y$. By definition of f, $w \notin \mathsf{Reach}(X^{\mathsf{C}})$. Note that $Y^{\mathsf{C}} \subseteq X^{\mathsf{C}}$, and so by monotonicity of Reach we have $w \notin \mathsf{Reach}(Y^{\mathsf{C}})$. But this means $Y \in f(w)$, so we are done.
- **$\mathcal{F}$ contains the unit:** Note that for all $w \in W$, $w \notin \mathsf{Reach}(\emptyset) = \mathsf{Reach}(W^{\mathsf{C}})$. So $W \in f(w)$.
- **$\mathcal{F}$ is reflexive:** We want to show that $w \in \cap f(w)$. Well, suppose $X \in f(w)$, i.e. $w \notin \mathsf{Reach}(X^{\mathsf{C}})$ (by definition of f). Since for all S, $S \subseteq \mathsf{Reach}(S)$, we have $w \notin X^{\mathsf{C}}$. But this means $w \in X$, and we are done.
- **$\mathcal{F}$ is transitive:** Suppose $X \in f(w)$, i.e. $w \notin \mathsf{Reach}(X^{\mathsf{C}})$. Well,

$$\begin{array}{rcll}
\mathsf{Reach}(X^{\mathsf{C}}) & = & \mathsf{Reach}(\mathsf{Reach}(X^{\mathsf{C}})) & \text{(by Idempotence of Reach)} \\
& = & \mathsf{Reach}(\{u \mid u \in \mathsf{Reach}(X^{\mathsf{C}})\}) & \\
& = & \mathsf{Reach}(\{u \mid u \notin \mathsf{Reach}(X^{\mathsf{C}})\}^{\mathsf{C}}) & \\
& = & \mathsf{Reach}(\{u \mid X \in f(u)\}^{\mathsf{C}}) & \text{(by definition of } f\text{)}
\end{array}$$

  So by definition of f, $\{u \mid X \in f(u)\} \in f(w)$.
- **$\mathcal{F}$ is acyclic:** Suppose $u_1, \ldots, u_n \in W$, with $u_1 \in \cap f(u_2), \ldots, u_{n-1} \in \cap f(u_n), u_n \in \cap f(u_1)$. That is, each $u_i \in \bigcap_{X \in f(u_{i+1})} X$. By choice of f, each $u_i \in \bigcap_{u_{i+1} \notin \mathsf{Reach}(X^{\mathsf{C}})} X$. Substituting X^{C} for X we get $u_i \in \bigcap_{u_{i+1} \notin \mathsf{Reach}(X)} X^{\mathsf{C}}$. In other words, $u_1 \in \mathsf{Reach}^{-1}(u_2), \ldots, u_{n-1} \in \mathsf{Reach}^{-1}(n), u_n \in \mathsf{Reach}^{-1}(u_1)$. By Proposition 1.11, each $u_i = u_j$.

- $\mathcal{G}$ **is reflexive:** Follows similarly, since $X \subseteq \mathsf{Prop}(X)$ by (Inclusion).
- $\mathcal{G}$ **is transitive:** Follows similarly, since $\mathsf{Prop}(X) = \mathsf{Prop}(\mathsf{Prop}(X))$ by (Idempotence).
- $\mathcal{F}$ **guides** $\mathcal{G}$**:** Suppose $X \cup (\cap f(w))^{\mathsf{C}} \in g(w)$. By choice of g, $w \notin \mathsf{Prop}([X \cup (\cap f(w))^{\mathsf{C}}]^{\mathsf{C}})$. Distributing the outer complement, we have $w \notin \mathsf{Prop}(X^{\mathsf{C}} \cap (\cap f(w)))$, i.e. $w \notin \mathsf{Prop}(X^{\mathsf{C}} \cap (\bigcap_{Y \in f(w)} Y))$. By choice of f, $w \notin \mathsf{Prop}(X^{\mathsf{C}} \cap (\bigcap_{w \notin \mathsf{Reach}(Y^{\mathsf{C}})} Y))$. Substituting Y^{C} for Y, we get $w \notin \mathsf{Prop}(X^{\mathsf{C}} \cap (\bigcap_{w \notin \mathsf{Reach}(Y)} Y^{\mathsf{C}}))$. By definition of Reach^{-1}, $w \notin \mathsf{Prop}(X^{\mathsf{C}} \cap \mathsf{Reach}^{-1}(w))$. From (Minimal Cause), we conclude that $w \notin \mathsf{Prop}(X^{\mathsf{C}})$, i.e. $X \in g(w)$.

□

4 From Frames to Nets

This is the harder ("completeness") direction!

DEFINITION 4.1. Suppose we have net $\mathcal{N}$ and node $n \in N$ with incoming nodes $m_1, \ldots, m_k$, $(m_i, n) \in E$. Let $\mathsf{hash}: \mathcal{P}(\{m_1, \ldots, m_k\}) \times \mathbb{N}^k \to \mathbb{N}$ be defined by

$$\mathsf{hash}(S, \vec{w}) = \prod_{m_i \in S} w_i$$

PROPOSITION 4.2. $\mathsf{hash}(S, \overrightarrow{W}(m_i, n)) : \mathcal{P}(\{m_1, \ldots, m_k\}) \to P_k$, where

$$P_k = \{n \in \mathbb{N} \mid n \text{ is the product of some subset of primes } \{p_1, \ldots, p_k\}\}$$

is bijective (and so has a well-defined inverse hash^{-1}).

DEFINITION 4.3. Let $\mathcal{M}$ be a model based on preferential multi-frame $\mathfrak{F} = \langle W, f, g \rangle$. Its **simulation net** $\mathcal{N}^{\bullet} = \langle N, E, W, A, O, V \rangle$ is the BFNN given by:

- $N = W$
- $(u, v) \in E$ iff $u \in \cap f(v)$

Now let $m_1, \ldots, m_k$ list those nodes such that $(m_i, n) \in E$.

- $W(m_i, n) = p_i$, the ith prime number.
- $A^{(n)}(\vec{x}, \vec{w}) = \mathsf{hash}(\{m_i \mid (m_i, n) \in E \text{ and } x_i = 1\}, \vec{w})$
- $O^{(w)}(x) = 1$ iff $(\mathsf{hash}^{-1}(x)[0])^{\mathsf{C}} \notin g(n)$
- $V_{\mathcal{N}^{\bullet}}(p) = V_{\mathcal{M}}(p)$

CLAIM 4.4. $\mathcal{N}^{\bullet}$ is a BFNN.

Proof. Clearly $\mathcal{N}^{\bullet}$ is a binary ANN. We check the rest of the conditions:

$\mathcal{N}^{\bullet}$ **is feed-forward.** Suppose for contradiction that E contains a cycle, i.e. distinct $u_1, \ldots, u_n \in N$ such that $u_1 E u_2, \ldots, u_{n-1} E u_n, u_n E u_1$. Then we have $u_1 \in \cap f(u_2), \ldots, u_{n-1} \in \cap f(u_{n-1}), u_n \in \cap f(u_1)$, which contradicts the fact that $\mathcal{F}$ is acyclic.

$O^{(n)} \circ A^{(n)}$ **is zero at zero.** Suppose for contradiction that $O^{(v)}(A^{(v)}(\vec{0}, \vec{w})) = 1$. Then $(\mathsf{hash}^{-1}(\mathsf{hash}(\emptyset)))^{\mathsf{C}} = \emptyset^{\mathsf{C}} = W \notin g(v)$, which contradicts the fact that $\mathcal{F}$ contains the unit.

$O^{(n)} \circ A^{(n)}$ **is monotonically increasing.** Let $\vec{w}_1, \vec{w}_2$ be such that O is well-defined (i.e. are vectors of prime numbers), and suppose $\vec{w}_1 < \vec{w}_2$. If $O^{(v)}(A^{(v)}(\vec{x}, \vec{w}_1)) = 1$, then $(\mathsf{hash}^{-1}(\mathsf{hash}(\vec{x}, \vec{w}_1))[0])^{\mathsf{C}} \notin g(v)$. But this just means $\{m_i \mid x_i = 1\}^{\mathsf{C}} \notin g(v)$. And so $(\mathsf{hash}^{-1}(\mathsf{hash}(\vec{x}, \vec{w}_2))[0])^{\mathsf{C}} \notin g(v)$. But then $O^{(n)}(A^{(n)}(\vec{x}, \vec{w}_2)) = 1$.

The main point here is just that $\vec{w}_1$ and $\vec{w}_2$ are just indexing the set in question, and their actual values don't affect the final output (we don't need the $\vec{w}_1 < \vec{w}_2$ hypothesis!). The real work happens within $g(v)$. □

LEMMA 4.5. $\mathsf{Reach}_{\mathcal{N}^\bullet}(S) = \{v | S^{\mathsf{C}} \notin f(v)\}$

Proof. For the ($\supseteq$) direction, let v be such that $S^{\mathsf{C}} \notin f(v)$. By Proposition 2.7 and the fact that $\mathcal{F}$ is a proper filter, $S \in f(v)$. By definition of core, $\cap f(v) \subseteq S$. $\mathcal{F}$ is reflexive, which means in particular that $v \in \cap f(v) \subseteq S$. By the base case of Reach, we have $v \in \mathsf{Reach}_{\mathcal{N}^\bullet}(S)$.

Now for the ($\subseteq$) direction. Suppose $v \in \mathsf{Reach}(S)$, and proceed by induction on Reach.

Base step. $v \in S$. Suppose for contradiction that $S^{\mathsf{C}} \in f(v)$. By definition of core, $\cap f(v) \subseteq S^{\mathsf{C}}$. But since $\mathcal{F}$ is reflexive, $v \in \cap f(v)$. So $v \in S^{\mathsf{C}}$, which contradicts $v \in S$.

Inductive step. There is $u \in \mathsf{Reach}_{\mathcal{N}^\bullet}(S)$ such that $(u, v) \in E$ (and so $u \in \cap f(v)$). By inductive hypothesis, $S^{\mathsf{C}} \notin f(u)$. Now suppose for contradiction that $S^{\mathsf{C}} \in f(v)$. Since $\mathcal{F}$ is transitive, $\{t | S^{\mathsf{C}} \in f(t)\} \in f(v)$. By definition of core, $\cap f(v) \subseteq \{t | S^{\mathsf{C}} \in f(t)\}$. Since $u \in \cap f(v)$, $S^{\mathsf{C}} \in f(u)$. But this contradicts $S^{\mathsf{C}} \notin f(u)$! □

LEMMA 4.6. $\mathsf{Prop}_{\mathcal{N}^\bullet}(S) = \{v | S^{\mathsf{C}} \notin g(v)\}$

Proof. For the ($\supseteq$) direction, suppose $S^{\mathsf{C}} \notin g(v)$. Since $\mathcal{F}$ guides $\mathcal{G}$, we have $S^{\mathsf{C}} \cup (\cap f(v))^{\mathsf{C}} \notin g(v)$, i.e. $[S \cap (\cap f(v))]^{\mathsf{C}} \notin g(v)$. But $S \cap (\cap f(v)) = \{u | u \in S \text{ and } (u, v) \in E\} = \mathsf{hash}^{-1}(\mathsf{hash}(\overrightarrow{\chi}_{\mathsf{Prop}_{\mathcal{N}^\bullet}(S)}(u), \vec{W}(u, v)))[0]$, and so

$$(\mathsf{hash}^{-1}(\mathsf{hash}(\overrightarrow{\chi}_{\mathsf{Prop}_{\mathcal{N}^\bullet}(S)}(u), \vec{W}(u, v)))[0])^{\mathsf{C}} \notin g(v)$$

i.e. $O^{(v)}(A^{(v)}(\overrightarrow{\chi}_{\mathsf{Prop}_{\mathcal{N}^\bullet}(S)}(u), \vec{W}(u, v))) = 1$, and we conclude that $v \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)$.

As for the ($\subseteq$) direction, suppose $v \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)$, and proceed by induction on Prop.

Base step. $v \in S$. Suppose for contradiction that $S^{\mathsf{C}} \in g(v)$. Since $\mathcal{G}$ is reflexive, $v \in \cap g(v)$. By definition of core, we have $\cap g(v) \subseteq S^{\mathsf{C}}$. But then $v \in \cap g(v) \subseteq S^{\mathsf{C}}$, i.e. $v \in S^{\mathsf{C}}$, which contradicts $v \in S$.

Inductive step. Let $u_1, \ldots, u_k$ list those nodes such that $(u_i, v) \in E$. We have

$$O^{(v)}(A^{(v)}(\overrightarrow{\chi}_{\mathsf{Prop}_{\mathcal{N}^\bullet}(S)}(u_i), \vec{W}(u_i, v))) = 1$$

Let $T = \{u_i | S^{\mathsf{C}} \notin g(u_i)\}$. By our inductive hypothesis,

$$O^{(v)}(A^{(v)}(\overrightarrow{\chi}_T(u_i), \vec{W}(u_i, v))) = 1$$

By choice of O and A,

$$(\mathsf{hash}^{-1}(\mathsf{hash}(\overrightarrow{\chi}_T(u_i), \vec{W}(u_i, v)))[0])^{\mathsf{C}} \notin g(v)$$

i.e. $T^{\mathsf{C}} \notin g(v)$. We would like to show that $S^{\mathsf{C}} \notin g(v)$. Suppose for contradiction that $S^{\mathsf{C}} \in g(v)$. Recall that $T = \{u_i | S^{\mathsf{C}} \notin g(u_i)\}$, i.e. $T^{\mathsf{C}} = \{u_i | S^{\mathsf{C}} \in g(u_i)\}$. Since $S^{\mathsf{C}} \in g(v)$ and $\mathcal{G}$ is transitive, $T^{\mathsf{C}} \in g(v)$, which contradicts $T^{\mathsf{C}} \notin g(v)$.

□

THEOREM 4.7. Let $\mathcal{M}$ be a model based on a preferential multi-frame $\mathfrak{F}$, and let $\mathcal{N}^\bullet$ be the corresponding simulation net. We have, for all $w \in W$,

$$\mathcal{M}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{N}^\bullet, w \Vdash \varphi$$

Proof. By induction on φ. Again, the nominal, propositional, $\neg\varphi$, and $\varphi \wedge \psi$ cases are trivial.

$\langle \mathbf{K} \rangle \varphi$ case:

$$\begin{array}{rlll} \mathcal{M}, w \Vdash \langle \mathbf{K} \rangle \varphi & \text{iff} & \{u \mid \mathcal{M}, w \nVdash \varphi\} \notin f(w) & \text{(by definition)} \\ & \text{iff} & \{u \mid u \notin [\![\varphi]\!]_{\mathcal{N}^\bullet}\} \notin f(w) & \text{(Inductive Hypothesis)} \\ & \text{iff} & [\![\varphi]\!]^{\mathsf{C}}_{\mathcal{N}^\bullet} \notin g(w) & \\ & \text{iff} & w \in \mathsf{Reach}_{\mathcal{N}^\bullet}([\![\varphi]\!]) & \text{(by Lemma 4.5)} \\ & \text{iff} & w \in [\![\langle \mathbf{K} \rangle \varphi]\!]_{\mathcal{N}^\bullet} & \text{(by definition)} \\ & \text{iff} & \mathcal{N}^\bullet, w \Vdash \langle \mathbf{K} \rangle \varphi & \text{(by definition)} \end{array}$$

$\langle \mathbf{K}^{\leftarrow} \rangle \varphi$ case:

$$\begin{array}{rlll} \mathcal{M}, w \Vdash \langle \mathbf{K}^{\leftarrow} \rangle \varphi & \text{iff} & \exists u \text{ such that } w \in \cap f(u) \text{ and } \mathcal{M}, u \nVdash \varphi & \text{(by definition)} \\ & \text{iff} & \exists u \text{ such that } w \in \cap f(u) \text{ and } u \notin [\![\varphi]\!]_{\mathcal{N}^\bullet} & \text{(IH)} \\ & \text{iff} & \exists u \in [\![\varphi]\!]^{\mathsf{C}}_{\mathcal{N}^\bullet} \text{ such that } w \in \bigcap\limits_{X \in f(u)} X & \\ & & \exists u \in [\![\varphi]\!]^{\mathsf{C}}_{\mathcal{N}^\bullet} \text{ such that } w \in \bigcap\limits_{u \notin \mathsf{Reach}_{\mathcal{N}^\bullet}(X^{\mathsf{C}})} X & \text{(by Lemma 4.5)} \\ & & \exists u \in [\![\varphi]\!]^{\mathsf{C}}_{\mathcal{N}^\bullet} \text{ such that } w \in \mathsf{Reach}^{-1}_{\mathcal{N}^\bullet}(u) & \\ & \text{iff} & \mathcal{N}^\bullet, w \Vdash \langle \mathbf{K}^{\leftarrow} \rangle \varphi & \text{(by definition)} \end{array}$$

$\langle \mathbf{T} \rangle \varphi$ case:

$$\begin{array}{rlll} \mathcal{M}, w \Vdash \langle \mathbf{T} \rangle \varphi & \text{iff} & \{u \mid \mathcal{M}, u \nVdash \varphi\} \notin g(w) & \text{(by definition)} \\ & \text{iff} & \{u \mid u \notin [\![\varphi]\!]_{\mathcal{N}^\bullet}\} \notin g(w) & \text{(Inductive Hypothesis)} \\ & \text{iff} & [\![\varphi]\!]^{\mathsf{C}}_{\mathcal{N}^\bullet} \notin g(w) & \\ & \text{iff} & w \in \mathsf{Prop}_{\mathcal{N}^\bullet}([\![\varphi]\!]) & \text{(by Lemma 4.6)} \\ & \text{iff} & w \in [\![\langle \mathbf{T} \rangle \varphi]\!]_{\mathcal{N}^\bullet} & \text{(by definition)} \\ & \text{iff} & \mathcal{N}^\bullet, w \Vdash \langle \mathbf{T} \rangle \varphi & \text{(by definition)} \end{array}$$

□

COROLLARY 4.8. $\mathcal{M} \models \varphi$ iff $\mathcal{N}^\bullet \models \varphi$.

5 Completeness

Axioms:

(K). $\mathbf{K}(\varphi \to \psi) \to (\mathbf{K}\varphi \to \mathbf{K}\psi)$

(K←). $\mathbf{K}^{\leftarrow}(\varphi \to \psi) \to (\mathbf{K}^{\leftarrow}\varphi \to \mathbf{K}^{\leftarrow}\psi)$

(Back). $\varphi \to \mathbf{K}\langle \mathbf{K}^{\leftarrow} \rangle \varphi$

(Forth). $\varphi \to \mathbf{K}^{\leftarrow}\langle \mathbf{K} \rangle \varphi$

(T). $\mathbf{K}\varphi \to \varphi$

(4). $\mathbf{K}\varphi \to \mathbf{K}\mathbf{K}\varphi$

(Grz). $\mathbf{K}(\mathbf{K}(\varphi \to \mathbf{K}\varphi) \to \varphi) \to \varphi$

(Incl). $\mathbf{K}\varphi \to \mathbf{T}\varphi$

(Skel). $i \wedge \mathbf{T}(\langle \mathbf{K}^{\leftarrow} \rangle i \to \varphi) \to \mathbf{T}\varphi$

The first group of axioms say that $\mathbf{K}$ characterizes a monotonic, reflexive, transitive, acyclic graph. The second group are axioms relating $\mathbf{K}$ and $\mathbf{K}^{\leftarrow}$ — these are from the minimal Tense Logic in temporal logic ($\mathbf{K}$ is "looking into the future", $\mathbf{K}^{\leftarrow}$ is "looking into the past"). See the SEoP page for more details.

The third group characterize **T** in terms of how it interacts with **K** and $\mathbf{K}^{\leftarrow}$.

PROPOSITION 5.1. Let $\mathcal{M}^{\min}$ be the minimal canonical model based on frames $\mathcal{F} = \langle W, f \rangle, \mathcal{G} = \langle W, g \rangle$. Then $\mathcal{F}$ is a reflexive, transitive, acyclic, proper filter, $\mathcal{G}$ contains $\mathcal{F}$, and $\mathcal{F}$ guides $\mathcal{G}$.

Proof. [] $\square$

LEMMA 5.2. Our logic is complete w.r.t. preferential multi-frames. [State precisely!]

THEOREM 5.3. Our logic is complete w.r.t BFNNs. [State precisely!]

Appendix

Proof. (of Proposition 1.6) We prove each in turn:

(Inclusion). If $n \in S$, then $n \in \mathsf{Prop}(S)$ by the base case of Prop.

(Idempotence). The ($\subseteq$) direction is just Inclusion. As for ($\supseteq$), let $n \in \mathsf{Prop}(\mathsf{Prop}(S))$, and proceed by induction on $\mathsf{Prop}(\mathsf{Prop}(S))$.

Base Step. $n \in \mathsf{Prop}(S)$, and so we are done.

Inductive Step. For those $m_1, \ldots, m_k$ such that $(m_i, n) \in E$,

$$O^{(n)}(A^{(n)}(\overrightarrow{\chi}_{\mathsf{Prop}(\mathsf{Prop}(S))}(m_i), \overrightarrow{W}(m_i, n))) = 1$$

By inductive hypothesis, $\chi_{\mathsf{Prop}(\mathsf{Prop}(S))}(m_i) = \chi_{\mathsf{Prop}(S)}(m_i)$. By definition, $n \in \mathsf{Prop}(S)$.

(Cumulative). For the ($\subseteq$) direction, let $n \in \mathsf{Prop}(S_1)$. We proceed by induction on $\mathsf{Prop}(S_1)$.

Base Step. Suppose $n \in S_1$. Well, $S_1 \subseteq S_2 \subseteq \mathsf{Prop}(S_2)$, so $n \in \mathsf{Prop}(S_2)$.

Inductive Step. For those $m_1, \ldots, m_k$ such that $(m_i, n) \in E$,

$$O^{(n)}(A^{(n)}(\overrightarrow{\chi}_{\mathsf{Prop}(S_1)}(m_i), \overrightarrow{W}(m_i, n))) = 1$$

By inductive hypothesis, $\chi_{\mathsf{Prop}(S_1)}(m_i) = \chi_{\mathsf{Prop}(S_2)}(m_i)$. By definition, $n \in \mathsf{Prop}(S_2)$.

Now consider the ($\supseteq$) direction. The Inductive Step holds similarly (just swap S_1 and S_2). As for the Base Step, if $n \in S_2$ then since $S_2 \subseteq \mathsf{Prop}(S_1)$, $n \in S_1$.

(Loop). Let $n \geq 0$ and suppose the hypothesis. Our goal is to show that for each i, $\mathsf{Prop}(S_i) \subseteq \mathsf{Prop}(S_{i-1})$, and additionally $\mathsf{Prop}(S_0) \subseteq \mathsf{Prop}(S_n)$. This will show that all $\mathsf{Prop}(S_i)$ contain each other, and so are equal. Let $i \in \{0, \ldots, n\}$ (if $i = 0$ then $i - 1$ refers to n), and let $e \in \mathsf{Prop}(S_i)$. We proceed by induction on $\mathsf{Prop}(S_i)$.

Base Step. $e \in S_i$, and since $S_i \subseteq \mathsf{Prop}(S_{i-1})$ by assumption, $e \in \mathsf{Prop}(S_{i-1})$.

Inductive Step. For those $m_1, \ldots, m_k$ such that $(m_i, n) \in E$,

$$O^{(e)}(A^{(e)}(\overrightarrow{\chi}_{\mathsf{Prop}(S_i)}(m_i), \overrightarrow{W}(m_i, e))) = 1$$

By inductive hypothesis, $\chi_{\mathsf{Prop}(S_i)}(m_j) = \chi_{\mathsf{Prop}(S_{i-1})}(m_j)$. By definition, $n \in \mathsf{Prop}(S_{i-1})$. $\square$

Proof. (of Proposition 1.8) We check each in turn:

(Inclusion). Similar to the proof of Inclusion for Prop.

(Idempotence). Similar to the proof of Idempotence for Prop.

(Monotonicity). Let $n \in \mathsf{Reach}(S_1)$. We proceed by induction on $\mathsf{Reach}(S_1)$.

Base Step. $n \in S_1$. So $n \in S_2 \subseteq \mathsf{Reach}(S_2)$.

Inductive Step. There is an $m \in \mathsf{Reach}(S_1)$ such that $(m,n) \in E$. By inductive hypothesis, $m \in \mathsf{Reach}(S_2)$. And so by definition, $n \in \mathsf{Reach}(S_2)$.

(Containment). Let $n \in \mathsf{Prop}(S)$. We proceed by induction on $\mathsf{Prop}(S)$.

base step. $n \in S$. So $n \in \mathsf{Reach}(S)$.

inductive step. For those $m_1, \ldots, m_k$ such that $(m_i, n) \in E$,

$$O^{(n)}(A^{(n)}(\overrightarrow{\chi}_{\mathsf{Prop}(S)}(m_i), \overrightarrow{W}(m_i,n))) = 1$$

Note that one of these m_i must be in $\mathsf{Prop}(S)$, i.e. $\chi_{\mathsf{Prop}(S)}(m_i) = 1$, since otherwise

$$O^{(n)}(A^{(n)}(\overrightarrow{\chi}_{\mathsf{Prop}(S)}(m_i), \overrightarrow{W}(m_i,n))) = O^{(n)}(A^{(n)}(\vec{0}, \overrightarrow{W}(m_i,n))) = 0$$

For this m_i, our inductive hypothesis gives us $\chi_{\mathsf{Reach}(S)}(m_i) = 1$. So $m_i \in \mathsf{Reach}(S)$, and $(m_i, n) \in E$, so by definition $n \in \mathsf{Reach}(S)$. □

Proof. (of Proposition 1.10) $(\rightarrow)$ Suppose $u \in \mathsf{Reach}^{-1}(n)$, i.e. for all X such that $n \notin \mathsf{Reach}(X)$, $u \in X^{\mathsf{C}}$. Consider in particular

$$X = \{m | \text{there is an } E\text{-path from } m \text{ to } n\}^{\mathsf{C}}$$

Notice that $n \notin \mathsf{Reach}(X)$. And so $u \in X^{\mathsf{C}}$, i.e. there *is* an E-path from u to n.

$(\leftarrow)$ Suppose there is an E-path from u to n, and let X be such that $n \notin \mathsf{Reach}(X)$. By definition of Reach, there is no $m \in X$ with an E-path from m to n. So in particular, $u \notin X$, i.e. $u \in X^{\mathsf{C}}$. So $u \in \bigcap_{n \notin \mathsf{Reach}(X)} X^{\mathsf{C}} = \mathsf{Reach}^{-1}(n)$. □

Proof. (of Proposition 1.11) Suppose $n_1 \in \mathsf{Reach}^{-1}(n_2), \ldots, n_{k-1} \in \mathsf{Reach}^{-1}(n_k), n_k \in \mathsf{Reach}^{-1}(n_1)$. By Proposition 1.10, there is an E-path from each n_i to n_{i+1}, and an E-path from n_k to n_1. But since E is acyclic, each $n_i = n_j$. □

Proof. (of Proposition 1.12) Let $n \in \mathsf{Prop}(S)$. We proceed by induction on $\mathsf{Prop}(S)$.

Base Step. $n \in S$. Our plan is to show $n \in \bigcap_{n \notin \mathsf{Reach}(X)} X^{\mathsf{C}} = \mathsf{Reach}^{-1}(n)$ (so $n \in S \cap \mathsf{Reach}^{-1}(n)$), which will give us our conclusion by the base case of Prop. Let X be any set where $n \notin \mathsf{Reach}(X)$. So $n \notin X$ (since $X \subseteq \mathsf{Reach}(X)$), i.e. $n \in X^{\mathsf{C}}$. But this is what we needed to show.

Inductive Step. Suppose $n \in \mathsf{Prop}(S)$ via its constructor, i.e. for those $m_1, \ldots, m_k$ such that $(m_i, n) \in E$,

$$O^{(n)}(A^{(n)}(\overrightarrow{\chi}_{\mathsf{Prop}(S)}(m_i), \overrightarrow{W}(m_i,n))) = 1$$

By inductive hypothesis,

$$\chi_{\mathsf{Prop}(S)}(m_i) = \chi_{\mathsf{Prop}(S \cap (\bigcap_{n \notin \mathsf{Reach}(X)} X^{\mathsf{C}}))}(m_i)$$

So we can substitute the latter for the former. By definition, $n \in \mathsf{Prop}(S \cap (\bigcap_{n \notin \mathsf{Reach}(X)} X^{\mathsf{C}}))$. □

Proof. (of Proposition 2.7) $(\rightarrow)$ Suppose for contradiction that $Y^{\mathsf{C}} \in f(w)$ and $Y \in f(w)$. Since $\mathcal{F}$ is closed under intersection, $Y^{\mathsf{C}} \cap Y = \emptyset \in f(w)$, which contradicts the fact that $\mathcal{F}$ is proper.

$(\leftarrow)$ Suppose for contradiction that $Y \notin f(w)$, yet $Y^{\mathsf{C}} \notin f(w)$. Since $\mathcal{F}$ is closed under intersection, $\cap f(w) \in f(w)$. Moreover, since $\mathcal{F}$ is closed under superset we must have $\cap f(w) \nsubseteq Y$ and $\cap f(w) \nsubseteq Y^{\mathsf{C}}$. But this means $\cap f(w) \nsubseteq Y \cap Y^{\mathsf{C}} = \emptyset$, i.e. there is some $x \in \cap f(w)$ such that $x \in \emptyset$. This contradicts the definition of the empty set. □

Proof. (of Proposition 4.2) To show that hash is injective, suppose $\mathsf{hash}(S_1) = \mathsf{hash}(S_2)$. So $\prod_{m_i \in S_1} p_i = \prod_{m_i \in S_2} p_i$, and since products of primes are unique, $\{p_i | m_i \in S_1\} = \{p_i | m_i \in S_2\}$. And so $S_1 = S_2$.

To show that hash is surjective, let $x \in P_k$. Now let $S = \{m_i | p_i \text{ divides } x\}$. Then $\mathsf{hash}(S) = \prod_{m_i \in S} p_i = \prod_{(p_i \text{ divides } x)} p_i = x$. □

Step 5. Step away (for a few days). Come back and check the proof *slowly* to make sure there aren't any missing edge cases or conditions.

- **If it's all good — congratulations, you got a free paper!**
- **Usually there will be some idiotic mistake in the proof. It may seem like *you're* the idiot for trying it — but in fact, it's now your job to figure out *what conditions will make this naive proof work*!**

Step 6. Write a computer program/simulation to collect statistics on the objects/models. Ask: *How unusual* is it for the models to fail the proof scenario? What about this lemma? This other lemma? Am I looking for a weird exception here, or is it very common? Make the simulation as *visual* as possible so that I can *picture* the condition/failure.

Step 7. If the condition is rare, try to modify the proof to account for the exceptions (they may satisfy the theorem but fail just this proof). Think: "is there a simple thing I can add to the system that will help the proof go through?"

Otherwise, sit down and try to define *exactly* that condition the proof doesn't fuck up at that step. Use the generated examples for help. Prove the claim for models satisfying Condition.

Step 8. Prove (i.e. unit-test/sanity-check) general properties of models satisfying Condition. Build up a theory of how Condition behaves — what is it like? What algebra does it follow? What is it similar to? What does it mean?

Step 9. Consider whether this partial result is still *interesting* enough to be published.

Is it meaningful to everyone in the field? ⟶ Submit it to a top-tier conference

Is it meaningful to this niche sub-field? ⟶ Submit it to the main conference for the sub-field

Is it meaningful as a technical lemma? ⟶ Submit it to a conference specifically for technical results

None of the above? ⟶ It's okay to not publish for now, and wait until you see the whole proof.

Step 10. Move on to the write-up stage. But otherwise, step away from the problem — there are too many other interesting things to spend all of your time on this one. Trust that one day a different solution will come to you.