C241: Discrete Mathematics

Proofs and Examples

An example using the Well-Ordering Principle

Proposition. *For all natural numbers $n \in \mathbb{N}$,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Prove this using the well-ordering principle.

Proof. By contradiction. Suppose there is some number for which the formula isn't true. Take the set of all of those numbers, C. C is a subset of $\mathbb{N}$, and C is nonempty. So by the Well-Ordering Principle, C has a smallest element.

Call that smallest element m. m can't be 0, this means we can consider $m - 1$. So that equation has to be true for $m - 1$. So we have:

$$1 + 2 + 3 + \cdots + (m-1) = \frac{(m-1)((m-1)+1)}{2}$$

Do some simplification:

$$1 + 2 + 3 + \cdots + (m-1) = \frac{(m-1)m}{2}$$

So we have:

$$\begin{aligned} 1 + 2 + 3 + \cdots + (m-1) + m &= \frac{(m-1)m}{2} + m \\ &= \frac{(m-1)m + 2m}{2} \\ &= \frac{m^2 - m + 2m}{2} \\ &= \frac{m^2 + m}{2} \\ &= \frac{m(m+1)}{2} \end{aligned}$$

Oops, the equation is true for m! This contradicts our assumption that m is the smallest counterexample. $\square$

An example using Induction

Proposition. *For all natural numbers $n \in \mathbb{N}$,*

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Prove this using the well-ordering principle.

Proof. (Using induction.)

Base Step. Our goal is to prove that the equation holds for $n=1$. The left-hand side evaluates to 1. The right-hand side evaluates to

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Inductive Step. Suppose the equation holds for some $n \geq 1$. Our Inductive Hypothesis says:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Now we need to show that the equation is true for $n+1$.

$$\begin{aligned}
1+2+3+\cdots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{n^2+n+2n+2}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

[Optional:] We proved that the equation holds for $n=1$, and if it holds for $n \geq 1$ then it holds for $n+1$. So we conclude by induction that the equation holds for all $n \geq 1$.

□

An example using Strong Induction

Proposition. *Every integer greater than or equal to* 14 *can be made using* 5¢ *and* 3¢ *coins.*

$$n = 5x + 3y \text{ for some } x, y \in \mathbb{N}$$

Proof. By strong induction.

Base Step. We need to show that $n = 5x + 3y$ for $n = 14, 15, 16, 17, 18$.

$$\begin{aligned} 14 &= 5+3+3+3 \\ 15 &= 5+5+5 \\ 16 &= 5+5+3+3 \\ 17 &= 5+3+3+3+3 \\ 18 &= 3+3+3+3+3+3 \end{aligned}$$

Inductive Step. Suppose $k = 5x + 3y$ for some x, y, for **all** $14 \leq k \leq n$, for some $n \geq 18$. Now we want to prove the same for $n+1$. Consider $n-4$. By our inductive hypothesis,

$$n - 4 = 5x + 3y$$

Well,

$$\begin{aligned} n+1 &= (n-4)+5 \\ &= 5x+3y+5 \\ &= 5(x+1)+3y \end{aligned}$$

So $n+1$ can be made out of 3¢ and 5¢ coins. □

$\sqrt{2}$ is irrational

Proposition. *If a^2 is divisible by 2, so is a.*

Proof. The contrapositive of this statement is: *If a is not even, then a^2 is not even.* So suppose a is not even. So a is odd. By definition of odd, $a = 2k+1$ for some $k \in \mathbb{Z}$. Now square it: $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. $2k^2 + 2k \in \mathbb{Z}$, so by the definition of an odd number, a^2 is odd. So a^2 is not even. □

Proposition. *If a^2 is divisible by 3, so is a.*

Proof. The contrapositive of this statement is: *If a is not divisible by 3, then a^2 is not divisible by 3.* So suppose a is not divisible by 3. Either the remainder of a divided by 3 is 1 or it's 2. We have these two cases:

1. The remainder is 1. So $a = 3k+1$. So $a^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. So a^2 is not divisible by 3.

2. The remainder is 2. So $a = 3k+2$. So $a^2 = (3k+2)^2 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1) + 1$. So a^2 is not divisible by 3.

In either case, a^2 is not divisible by 3. $\square$

Proposition. *$\sqrt{2}$ is irrational.*

Proof. Suppose for contradiction that $\sqrt{2}$ is actually rational. By definition of a rational number,

$$\sqrt{2} = \frac{a}{b} \text{ where } a, b \in \mathbb{Z} \text{ and } b \neq 0$$

Assume (without loss of generality) that $\frac{a}{b}$ is a simplified fraction, i.e., a, b have no common factors.

Take $\sqrt{2} = \frac{a}{b}$, square both sides: $2 = \frac{a^2}{b^2}$. So $2b^2 = a^2$. So a^2 is even (divisible by 2). So a is even. By definition of an even number, $a = 2k$ for some integer $k \in \mathbb{Z}$. Plug that back into the equation above:

$$2b^2 = (2k)^2 = 4k^2$$

Divide both sides by 2:

$$b^2 = 2k^2$$

So b^2 is even. But that means b is even. So a and b have a common factor of 2. Whoops, a contradiction! $\square$

Proposition. *Prove that for all sets A, B, $A \cup B = B \cup A$.*

Proof. First show $x \in A \cup B$ implies $x \in B \cup A$: Suppose $x \in A \cup B$. By definition of $\cup$, $x \in A$ or $x \in B$. So $x \in B$ or $x \in A$. So $x \in B \cup A$.

Then show $x \in B \cup A$ implies $x \in A \cup B$. Similar. $\square$

Expect to be able to prove stuff about: $x \in \overline{A \cup (B \cap C)}$:

$$\begin{aligned}
x \in \overline{A \cup (B \cap C)} \quad & \text{iff} \quad \neg(x \in A \cup (B \cap C)) \\
& \text{iff} \quad \neg(x \in A \text{ or } x \in B \cap C) \\
& \text{iff} \quad \neg(x \in A \text{ or } (x \in B \text{ and } x \in C)) \\
& \text{iff} \quad \ldots
\end{aligned}$$

Examples from Homework 3

Prove that the sentence "It is an absolute truth that no truth is absolute" is false.

Proof. Suppose for contradiction that "It is an absolute truth that no truth is absolute" is true. Then it *is* an absolute truth that no truth is absolute. But then that means no truth is absolute, which contradicts us saying that the previous statement was an absolute truth. □

Proposition. *There are two irrational numbers, a and b such that a^b is rational.*

Proof. Let's consider $\sqrt{2}^{\sqrt{2}}$. Let's consider two cases:

1. $\sqrt{2}^{\sqrt{2}}$ is rational. In this case, let $a=\sqrt{2}$, and $b=\sqrt{2}$. So a, b are both irrational, and we assumed a^b is rational.
2. $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a=\sqrt{2}^{\sqrt{2}}, b=\sqrt{2}$. So

$$a^b=\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}=\sqrt{2}^{\sqrt{2}\sqrt{2}}=\sqrt{2}^{2^{\frac{1}{2}}2^{\frac{1}{2}}}=\sqrt{2}^{2^1}=\sqrt{2}^2=2$$

and 2 is rational. □

Proposition. $\log_{12}18$ *is irrational.*

Proof. For contradiction, suppose $\log_{12}18$ is rational, i.e. $\log_{12}18=\frac{a}{b}$ for integers a, b, where $b\neq 0$. So

$$12^{\log_{12}18}=12^{\frac{a}{b}}$$

So

$$(12^{\log_{12}18})^b=12^a$$

Cancelling the log,

$$18^b=12^a$$

Okay, now consider the prime factors of 18 and 12:

$$(3\times 3\times 2)^b=(2\times 2\times 3)^a$$

So

$$3^{2b}\times 2^b=2^{2a}\times 3^a$$

Moving the powers of 3 to one side, powers of 2 to the other side:

$$2^{b-2a}=3^{a-2b}$$

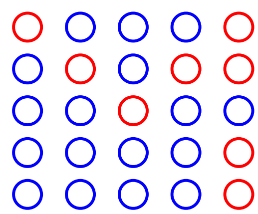
There are lots of contradictions we can already see. Any of the following work:

- The left-hand side doesn't have 3 as a factor, but the right-hand side does

- The left-hand side is a power of 2, which is even, but the right-hand side is a power of 3, which is odd

- Derive 18=12 from here (see the solution) □

Proposition 1. *Suppose you have a rectangular array of pebbles, where each pebble is either red or blue.*
Suppose that for every way of choosing one pebble from each column, there exists a red pebble among
the chosen ones.Prove that there must exist an all-red column. Hint: either use proof by contradiction
or directly prove the contrapositive.



Proof. Suppose for contradiction that there is no all-red column. So there is a blue pebble in each column. This means there is a way of picking the pebbles so that they're all blue. This contradicts our assumption, that every way of picking a pebble from each column has a red pebble. □

$$\underbrace{(\overline{P} \text{ OR } Q)}_{\textit{clause (1)}} \text{ AND } \underbrace{(\overline{Q} \text{ OR } R)}_{\textit{clause (2)}} \text{ AND } \underbrace{(\overline{R} \text{ OR } S)}_{\textit{clause (3)}} \text{ AND } \underbrace{(\overline{S} \text{ OR } P)}_{\textit{clause (4)}} \text{ AND } M \text{ AND } \overline{N}$$

Proposition 2. *This formula has exactly two satisfying assignments.*

Proof. To see that it has 2: (1) P:T, Q:T, R:T, S:T, M:T, N:F (2) P:F, Q:F, R:F, S:F, M:T, N:F

To see that it has only those two, consider that we have two cases:

1. P is true. In order for $\neg P \vee Q$ to be true, Q has be true. But then for $\neg Q \vee R$ to be true, R has to be true, and so on (everything's true.)

2. P is false. In order for $\neg S \vee P$ to be true, S has to be false, ... (you get the idea lol) □

$p \rightarrow q$

contrapositive: $\neg q \rightarrow \neg p$

Proposition 3. *If r is irrational, then $r^{\frac{1}{5}}$ is irrational. (Hint: prove the contrapositive)*

Proof. The contrapositive is: If $r^{\frac{1}{5}}$ is rational, then r is rational. Now, let's prove it. Suppose $r^{\frac{1}{5}}$ is rational. So

$$r^{\frac{1}{5}} = \frac{a}{b} \text{ for integers } a, b \text{ with } b \neq 0$$

Raise both sides to the power of 5:

$$r = \left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}$$

a^5 is an integer. So is b^5. $b^5 \neq 0$. So this number r is rational. □

Test Prep Guide

Test 2. Truth tables, equivalences, sets, $\sqrt{2}$ is irrational, prove stuff about rational numbers, prove stuff about odd & even numbers **(Hw 2–4)**

Test 3. Functions, relations (injections, surjections, etc.), first-order logic, well-ordering principle. **(Hw 5–7)**

Test 4. Induction, Strong Induction, Well-Ordering Principle **(Hw 8–9)**

Final. Cumulative, includes content on directed graphs, undirected graphs, graph colorings. **(Practice Exam)**