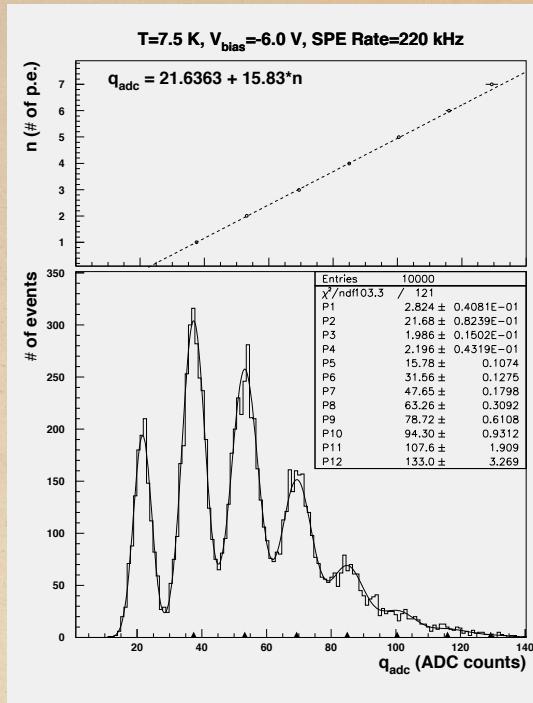
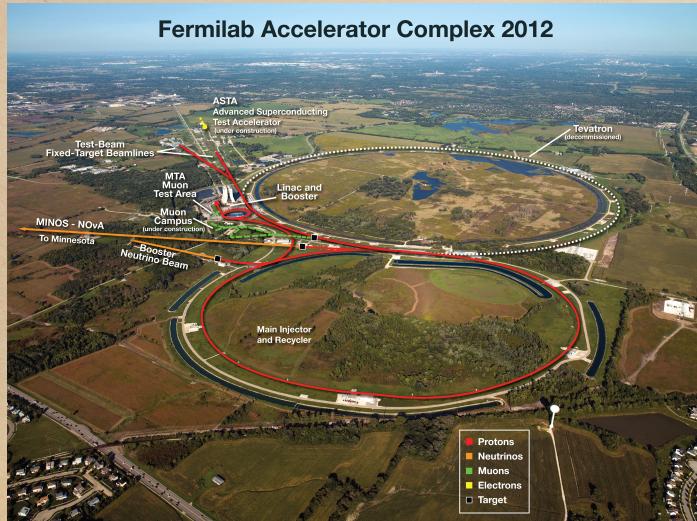
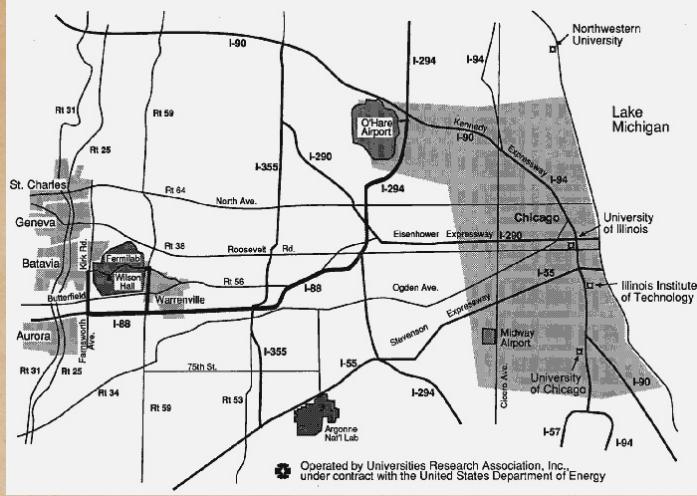
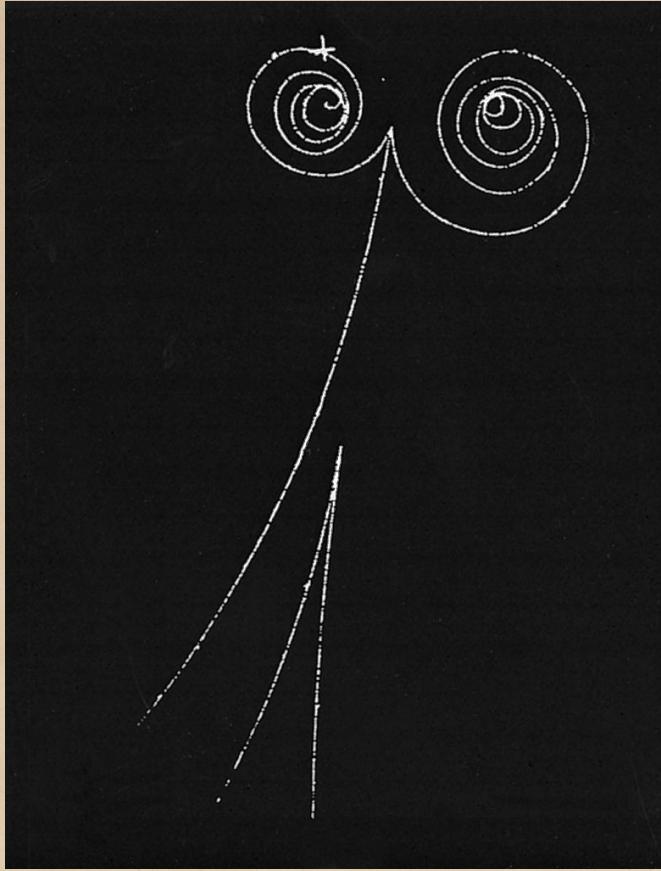


# *Stochastic analysis of quantum fields*







*Quantum mechanics*

+

*Special relativity*

=

*Quantum field theory*

V o l u m e I   F o u n d a t i o n s

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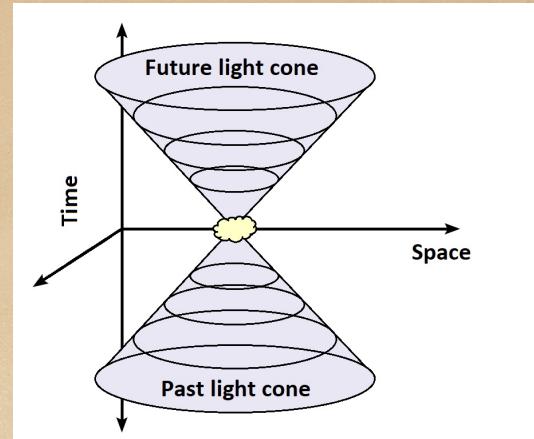
THE  
QUANTUM  
THEORY OF  
FIELDS

---

STEVEN WEINBERG

$$(\partial_t^2 - \Delta)\Phi(x) = F(\Phi(x)), \quad x \in \mathbb{R}^{3+1}$$

$$[\Phi(x), \dot{\Phi}(y)] = \delta(x - y)$$



$\Phi$  is an operator valued distribution on an Hilbert space:

$\Phi(f)$  is an unbounded self-adjoint operator for all test functions  $f: \mathbb{R}^{3+1} \rightarrow \mathbb{R}$

*Does a consistent theory of quantum fields exists?*

▷ Axiomatic approach ('60): Wightman, Haag-Kastler.

- i. Hilbert space  $\mathcal{H}$ ,
- ii. Positive energy representation  $(U(\Lambda, a))_{(\Lambda, a)}$  of Poincaré group  $G = \{(\Lambda, a)\}$
- iii. Fields  $\varphi(x)$  transforming as  $U(\Lambda, a)^* \varphi(x) U(\Lambda, a) = \varphi(\Lambda x + a)$ ,  $x \in \mathbb{R}^{3+1}$
- iv. Vacuum vector  $\Omega \in \mathcal{H}$ :  $U(\Lambda, a)\Omega = \Omega$

▷ Examples?

Progress in  $\mathbb{R}^{1+1}$  (1965-1976): Nelson, Glimm, Jaffe, Segal...

- ▷ Schwinger

$$t \rightarrow it \quad \square \longrightarrow \Delta$$

- ▷ Symanzik – functional integral representation of Euclidean correlation function
- ▷ Nelson – reconstruction of the Hilbert space from Euclidean data and spatial Markov property
- ▷ Schwinger functions

$$S_n(f_1 \otimes \cdots \otimes f_n) := \int_{\mathcal{S}'(\mathbb{R}^{d+1})} \varphi(f_1) \cdots \varphi(f_n) \nu(d\varphi).$$

**Distribution property** –  $\beta > 0$ ,

$$|S_n(f_1 \otimes \dots \otimes f_n)| \leq (n!)^\beta \prod_{i=1}^n \|f_i\|_s. \quad \forall n \geq 0, f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}_+^3).$$

**Euclidean invariance** –  $(a, R). f_n(x) = f_n(a + Rx)$ ,  $(a, R) \in \mathbb{R}^3 \times O(3)$

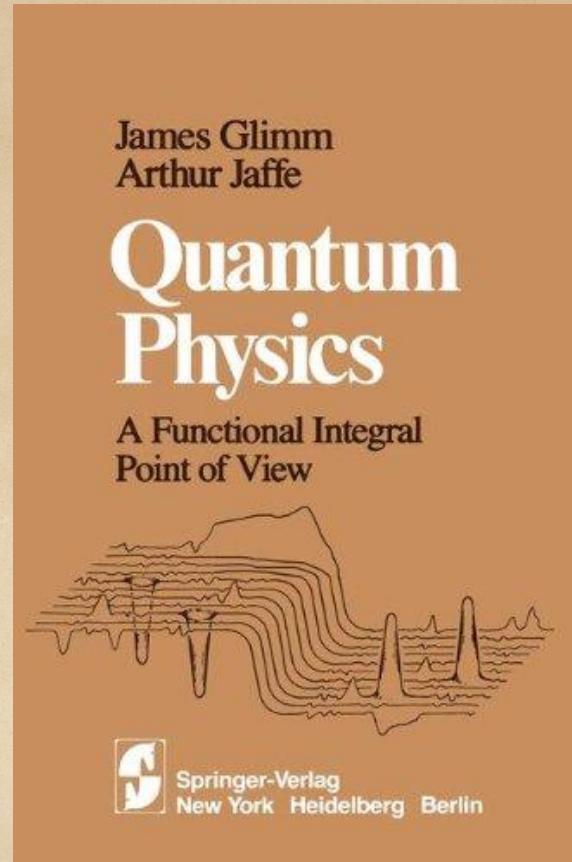
$$S_n((a, R). f_1 \otimes \dots \otimes (a, R). f_n) = S_n(f_1 \otimes \dots \otimes f_n),$$

**Reflection positivity** –  $(f_n \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}_+^{3n}))_{n \in \mathbb{N}_0}$  (with finitely many nonzero elements)

$$\sum_{n, m \in \mathbb{N}_0} S_{n+m}(\overline{\theta f_n} \otimes f_m) \geq 0,$$

**Symmetry** –  $S_n(f_1 \otimes \dots \otimes f_n) = S_n(f_{\pi(1)} \otimes \dots \otimes f_{\pi(n)})$

- ▷ ('70-'80) Glimm, Jaffe. Nelson. Segal. Guerra, Rosen, Simon, and many others...
- ▷ Construction of theories in  $2+1$  dimensions
- ▷  $(\Phi_3^4)_\Lambda$  Glimm ('69). Glimm, Jaffe. Feldman ('74), Y.M.Park ('75)
- ▷  $(\Phi_3^4)_{\mathbb{R}^3}$  Feldman, Osterwalder ('76). Magnen, Senéor ('76). Seiler, Simon ('76)
- ▷ *Other constructions of  $\Phi_3^4$ .* Benfatto, Cassandro, Gallavotti, Nicolò, Olivieri, Presutti, Scacciatelli ('80) Brydges, Fröhlich, Sokal ('83) Battle, Federbush ('83) Williamson ('87) Balaban ('83) Gawedzki, Kupiainen ('85) Watson ('89) Brydges, Dimock, Hurd ('95)



Ito and Doeblin want to study diffusion processes via their *sample paths*

<i>Measures</i>	<i>Samples</i>
$(\mu_t)_t \subseteq \Pi(S)$	$X: \Omega \rightarrow C(\mathbb{R}_+, S)$
$\mu_t(dy) = \int P_{t-s}(x, dy) \mu_s(dx)$	$dX_t = b(X_t)dt + dB_t$

### *Advantages*

- lower dimensional problem
- more tools (e.g. fixpoint theorems)
- more intuition
- ***canonical reference object***  $(B_t)_t$

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 45

# STOCHASTIC EQUATIONS IN INFINITE DIMENSIONS

GIUSEPPE DA PRATO  
& JERZY ZABCZYK

## Relation between a stochastic differential equation and a probability measure

(broadly speaking)

- ▷ Nelson and Parisi–Wu ('84) advocated the *constructive* use of stochastic partial differential equations (SPDEs) to realize a given Gibbs measure for the use of Euclidean quantum field theory (in particular gauge theories)
- ▷ Theoretical version of MCMC methods

$\Lambda = \text{finite set}, \mathbb{T}^d, \mathbb{R}^d$

equation

$$\partial_t \phi(t) = -\frac{\delta V(\phi(t))}{\delta \phi} + \sqrt{2} \xi(t), \quad \phi: \mathbb{R}_+ \times \Lambda \rightarrow \mathbb{R}$$

---

measure

$$\phi(t) \sim \nu(d\varphi) = \frac{e^{-V(\varphi)}}{Z} d\varphi \in \text{Prob}(\Lambda \rightarrow \mathbb{R})$$

- ▷ The measure  $\nu$  is described via *white noise*
- ▷ Markov process, invariant measures, ergodicity

# dynamic $\Phi_d^4$

$$V(\varphi) = \int \frac{1}{2} |\nabla \varphi|^2 + \frac{m^2 - \infty}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4.$$

$$\partial_t \varphi = \Delta \varphi - \lambda (\varphi^3 - \infty \varphi) - m^2 \varphi + \sqrt{2} \xi \quad \mathbb{R}^3 \times \mathbb{R}_+$$

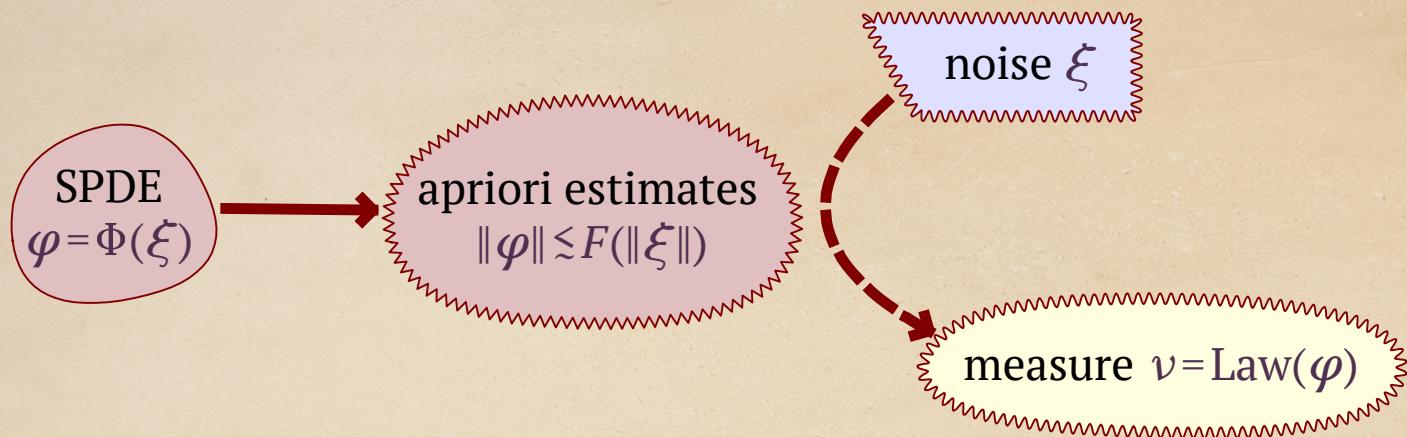
(d=2) Jona-Lasinio, P.K.Mitter ('85) Borkar, Chari, S.K.Mitter ('88) Albeverio, Röckner ('91) Da Prato, Debussche ('03) Mourrat, Weber ('17) Tsatsoulis, Weber ('16) Röckner, R.Zhu, X.Zhu ('17)

▷  $d=3$  is more singular: regularity structures (Hairer), paracontrolled distributions (G. Imkeller, Perkowski)

Hairer ('14) Kupiainen ('16) Catellier, Chouk ('17) Mourrat, Weber ('17) Hairer, Mattingly ('18) R.Zhu, X.Zhu ('18) Albeverio, Kusuoka ('18) G, Hofmanová ('18) Moinat, Weber ('18)

Reflection positivity + Euclidean invariance  $\Rightarrow$  singularities, infinite volume limit

G. Hofmanová ('18) – construction of  $\Phi_3^4$  on  $\mathbb{R}^3$  via stochastic quantisation and verification of (most of) the axioms.



- ▷ Much like Ito's approach to diffusions
- ▷ Markovianity does not play any role

equation

$$\left\{ \begin{array}{l} \partial_t \phi(t) = -\frac{\delta H(\phi(t), \dot{\phi}(t))}{\delta \dot{\phi}} \\ \partial_t \dot{\phi}(t) = -\underbrace{\frac{\delta H(\phi(t), \dot{\phi}(t))}{\delta \phi}}_{\text{Hamiltonian dynamics}} - \underbrace{\gamma \dot{\phi}(t) + \sqrt{2} \xi(t)}_{\text{linear Langevin dynamics}}, \end{array} \right. \quad \phi, \dot{\phi}: \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$$

$$H(\varphi, \dot{\varphi}) := V(\varphi) + \frac{\gamma}{2} \dot{\varphi}^2$$

measure

$$(\phi(t), \dot{\phi}(t)) \sim \nu(d\phi d\dot{\phi}) = \frac{e^{-H(\varphi, \dot{\varphi})}}{Z} d\varphi d\dot{\varphi} \in \text{Prob}(\Lambda \rightarrow \mathbb{R}^2)$$

- ▷ Introduced by Ryang, Saito and Shigemoto ('85).

# singular stochastic wave equations

For  $\Phi_d^4$ ,  $d=1, 2, 3$

$$V(\varphi) = \int \frac{1}{2} |\nabla \varphi|^2 + \frac{m^2 - \infty}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4,$$

$$\partial_t^2 \phi = \Delta \phi + (m^2 - \infty) \phi + \lambda \phi^3 - \gamma \partial_t \phi + \sqrt{2} \xi,$$

**Problem:** no Schauder estimates, scaling arguments less clear.

**Conjecture:** same renormalization constants of the static measure!

- ▷  $d=1$ . Tolomeo ('18) unique ergodicity.
- ▷  $d=2$ . G, Koch, Oh ('18) local well-posedness (any polynomial), G, Koch, Oh, Tolomeo (in preparation) global well-posedness.
- ▷  $d=3$ . G, Koch, Oh ('18) only quadratic nonlinearity.

# elliptic stochastic quantisation

equation

$$\Delta_z \phi(z) = -\frac{\delta V(\phi(z))}{\delta \phi} + \xi(z), \quad \phi: \mathbb{R}^2 \times \Lambda \rightarrow \mathbb{R}$$

measure

$$\phi(z) \sim \nu(d\varphi) = \frac{e^{-4\pi V(\varphi)}}{Z} d\varphi \in \text{Prob}(\Lambda \rightarrow \mathbb{R})$$

Discovered perturbatively by Imry, Ma ('75), Young ('77). Non-perturbative “proof” by Parisi and Sourlas ('79-'82) using *supersymmetry*

$$(\text{SPDE})_{d+2} \xrightarrow{\text{“Girsanov”}} (\text{SUSY EQFT})_{d+2} \xrightarrow{\text{dimensional reduction}} (\text{measure})_d$$

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 \quad \Delta_z \varphi(z) = -m^2\varphi(z) + \xi(z), \quad z \in \mathbb{R}^2$$

$$\varphi(z) = \int_{\mathbb{R}^d} \frac{e^{ik \cdot z}}{|k|^2 + m^2} \frac{\eta(dk)}{2\pi}$$

$$\mathbb{E}[\varphi(0)^2] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{dk}{(|k|^2 + m^2)^2} = \frac{1}{(2\pi)^2 m^2} \int_{\mathbb{R}^2} \frac{dk}{(|k|^2 + 1)^2} = \frac{1}{4\pi m^2} \int_0^\infty \frac{d\rho^2}{(\rho^2 + 1)^2} = \frac{1}{4\pi m^2}$$

$$\varphi(0) \sim e^{-4\pi \frac{m^2}{2}\phi^2} d\phi \sim e^{-4\pi V(\phi)} d\phi$$

- ▷ Rigorous proof of dimensional reduction by Klein, Landau and Perez ('84)
- ▷ Recently complete proof by Albeverio, G. and De Vecchi ('18). First for  $\Lambda$  finite dimensional + technical conditions. Then extended to (some) renormalized EQFT.

Stochastic quantisation of Liouville action up to the critical value of  $\sigma^2 < 8\pi$  in  $\Lambda = \mathbb{T}^2$

$$V(\varphi) = \int_{\mathbb{T}^2} \frac{1}{2} |\nabla \varphi|^2 + \alpha e^{\sigma \varphi - \sigma^2 \infty}$$

# variational approach for $\Phi_3^4$

An alternative approach to induce an equation for  $\Phi_3^4$  in  $\mathbb{T}^3$  [Barashkov, G. ('18)]

$$e^{-\mathcal{W}(f)} = \int_{\mathcal{S}'(\mathbb{T}^3)} e^{\langle f, \phi \rangle - \lambda \int \phi^4} \underbrace{e^{-\frac{1}{2} \int |\nabla \phi|^2 + a \phi^2} d\phi}_{\text{Gaussian free field}} = \lim_{T \rightarrow \infty} \mathbb{E}[e^{\langle f, W_T \rangle - \lambda \int (W_T^4 - a_T W_T^2)}]$$

▷  $W_T$  is a regularisation of the Gaussian free field at scale  $T$ :

$$\mathbb{E}[W_T(f)W_S(f)] = (T \wedge S) \langle f, \rho((-\Delta)^{1/2}/S)(1-\Delta)^{-1}g \rangle$$

$$W_t = \int_0^t J_t dX_t$$

with  $(X_t)_t$  cylindrical Wiener process in  $L^2(\mathbb{T}^3)$ .

# stochastic control problem

Take  $V_T(\phi) = \lambda \int (\phi^4 - a_T \phi^2 - b_T)$

$$\mathcal{W}_T(f) = -\log \mathbb{E}[e^{\langle f, W_T \rangle - V_T(W_T)}]$$

$$= \inf_u \mathbb{E} \left[ \langle f, W_T + I_T(u) \rangle - V_T(W_T + I_T(u)) + \frac{1}{2} \int_0^\infty \|u_s^2\|_{L^2(\mathbb{T}^3)} ds \right]$$

where

$$I_T(u) = \int_0^T J_t u_t dt$$

and the infimum is over processes  $u$  adapted to the Wiener filtration.

# renormalized control problem

$\exists a_T, b_T$  with  $a_T, b_T \rightarrow \infty$  as  $T \rightarrow \infty$ , such that

$$\begin{aligned}\mathcal{W}_T(f) &= \mathbb{E} \left[ -\langle f, W_T + I_T(u) \rangle + \lambda V_T(W_T + I_T(u)) + \frac{1}{2} \|u\|_{\mathcal{H}}^2 \right] \\ &= \mathbb{E} \left[ -f(W_T + I_T(u)) + \Phi_T(W, u) + \lambda \int (I_T(u))^4 + \frac{1}{2} \|l^T(u)\|_{\mathcal{H}}^2 \right]\end{aligned}$$

$$l_t^T(u) := u_t + \lambda \mathbb{1}_{t \leq T} J_t \mathbb{W}_t^3 + \lambda \mathbb{1}_{t \leq T} J_t (\mathbb{W}_t^2 \succ I_t^\flat(u)), \quad \mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{T}^3),$$

$$|\mathbb{E} \Phi_T(W, u)| \leq \mathbb{E} Q(W) + \frac{1}{4} (\lambda \|I_T(u)\|_{L^4}^4 + \|l^T(u)\|_{\mathcal{H}}^2)$$

# a description of $\Phi_3^4$

Barashkov, G. ('18)

## Theorem

$$\begin{aligned}\mathcal{W}_\infty(f) &:= \liminf_{T \rightarrow \infty} \mathbb{E} \left[ -f(W_T + I_T(u)) + \Phi_T(W, u) + \lambda \int (I_T(u))^4 + \frac{1}{2} \|l^T(u)\|_{\mathcal{H}}^2 \right] \\ &= \inf_{u \in \mathbb{H}_a^{-1/2-\varepsilon}} \mathbb{E} \left[ -f(W_\infty + I_\infty(u)) + \Phi_\infty(W, u) + \lambda \int (I_\infty(u))^4 + \frac{1}{2} \|l^\infty(u)\|_{\mathcal{H}}^2 \right]\end{aligned}$$

$$\int e^f d\Phi_3^4 = e^{\mathcal{W}_\infty(0) - \mathcal{W}_\infty(f)} = \lim_{T \rightarrow \infty} \int e^f d\Phi_{3,T}^4.$$

- ▷ First “explicit” description of  $\Phi_3^4$

*Thanks.*

