Neural Net Semantics with Modal Operators

Step 0. Find a paper I enjoy, and read it. Try to understand its ideas, with an eye towards extending it/altering it.

This paper is inspired by Hannes Leitgeb's Nonmonotonic Reasoning by Inhibition Nets, which proves completeness for the neuro-symbolic interface suggested by Balkenius and Gärdenfors' Nonmonotonic Inferences in Neural Networks.

Step 1. Look for an extension/open problem that makes me think "What the fuck? That's still open? No way, this shit is low-hanging fruit, free paper here I come." i.e. something *easy* and *straightforward*, *without complications*.

Hannes Leitgeb showed that feed-forward neural networks are complete with respect to certain conditional laws of $\Rightarrow$. But $\varphi \Rightarrow \psi$ just reads "$\psi \subseteq \mathsf{Prop}(\varphi)$" (i.e. ψ is in the propagation of the signal φ), which we can re-write in modal language as $\mathbf{T}\varphi \rightarrow \psi$. In the same way that Hannes shows that feed-forward nets and preferential-conditional models are equivalent, it shouldn't be too hard at all to show that feed-forwards nets and neighborhood models are equivalent. (Note that it is well-known that neighborhood models are a generalization of preferential models.)

I also think I should be able to throw in a $\mathbf{K}$ modality (graph-reachability) in there, almost for free.

Step 2. Follow-up question (only answer after Step 1): Is the extension *interesting* or *surprising*? What do we learn by extending the result?

Why bother with completeness?. In formal specifications (of AI agents, or otherwise), we're often content with just listing some sound rules or behaviors that the agent will always follow. And it's definitely cool to see that neural networks satisfy some sound logical axioms. But if we want to fundamentally bridge the gap between logic and neural networks, we should set our aim higher: Towards *complete* logical characterizations of neural networks.

A more practical reason: Completeness gives us model-building, i.e. given a specification Γ, we can *build* a neural network $\mathcal{N}$ satisfying Γ.

Why bother with this modal language?. Almost all of the previous work bridging logic and neural networks has focused on neural net models of *conditionals*. In some sense, doing this in modal language is just a re-write of this old work. But this previous work hasn't addressed how *learning* or *update* in neural networks can be cast in logical terms. This is not merely due to circumstance — integrating conditionals with update is a long-standing controversial issue. So instead, we believe that it is more natural to work with modalities (instead of conditionals), because

Modal language natively supports update.

In other words, our modal setting sets us up to easily cast update operators (e.g. neural network learning) as modal operators in our logic.

Also this gives me an excuse to title a paper *Neural Network Models à la Mode* :-) (This is a play on both modal logic and also bringing some old work back in style!)

And LOL I can name a section "Learning: The Cherry on Top"

Step 3. Two things to do at this point:

- **Make a new Texmacs file named "PAPERNAME-master-notes.tm". Transcribe the key definitions, examples, lemmas, and results from the paper. This makes it easier to later copy-paste parts of proofs, and also ensures that I don't reinvent the wheel later (it's tempting to redefine everything yourself!)**

- **Go to `https://www.connectedpapers.com/` and download any major nearby papers. Upload the papers to paperless-ngx and make a point to read them (understanding context helps a lot!).**

Related Papers:

Neural Network Semantics / Semantic Encodings.

Classic Papers. [18] [12]

Conditional Logic (Feedforward Net). [2], [15], [16], [8] (soundness), [9] (model-building) [Any other relevant work by the Garcez lab?]

Description Logic w. Typicality. [10], [11] [Any other relevant work by the Giordano lab?]

Modal Logic w. Typicality. [14]

[Any other big trends I'm missing? See the new survey by Odense + Garcez!]

Miscellaneous. [5], [6]

Surveys. [19] [1], [21], [13], [17], [3], [22] (the first few sections are a great introduction to Neural Network Semantics)

Help with Technical Details.

Neighborhood Models. [20]

Temporal Logic Rules. [7]

Nominals (Hybrid Logic). [4]

Step 4. Write up my new definitions & proof in the Texmacs file. Again, should be a *very* straightforward extension, and the proof (proofs are just unit-tests for definitions) shouldn't take up too much room at all (1-2 pages, including defs)

1 Interpreted Neural Nets

1.1 Basic Definitions

DEFINITION 1.1. An **interpreted ANN** (Artificial Neural Network) is a pointed directed graph $\mathcal{N} = \langle N, E, W, A, O, V \rangle$, where

- N is a finite nonempty set (the set of **neurons**)
- $E \subseteq N \times N$ (the set of **excitatory neurons**)
- $W: E \to \mathbb{R}$ (the **weight** of a given connection)
- A is a function which maps each $n \in N$ to $A^{(n)}: \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ (the **activation function** for n, where k is the indegree of n)
- V: propositions $\to \mathcal{P}(N)$ is an assignment of propositions to sets of neurons (the **valuation function**). [rename: The 'interpretation' of the net]

DEFINITION 1.2. A **BFNN** (Binary Feedforward Neural Network) is an interpreted ANN $\mathcal{N} = \langle N, E, W, A, O, V \rangle$ that is

- **Feed-forward**: E does not contain any cycles
- **Binary**: the output of each neuron is in $\{0, 1\}$
- $A^{(n)}$ is **zero at zero** in the first parameter: $A^{(n)}(\vec{0}, \vec{w}) = 0$
- $A^{(n)}$ is **monotonically increasing** in the second parameter: for all $\vec{x}, \vec{w}_1, \vec{w}_2 \in \mathbb{R}^k$, if $\vec{w}_1 < \vec{w}_2$ then $A^{(n)}(\vec{x}, \vec{w}_1) < A^{(n)}(\vec{x}, \vec{w}_2)$. We will more often refer to the equivalent condition:

$$\vec{w}_1 \leqslant \vec{w}_2 \quad \text{iff} \quad A^{(n)}(\vec{x}, \vec{w}_1) \leqslant A^{(n)}(\vec{x}, \vec{w}_2)$$

DEFINITION 1.3. Given a BFNN $\mathcal{N}$, $\mathsf{Set} = \mathcal{P}(N) = \{S | S \subseteq N\}$

DEFINITION 1.4. For $S \in \mathsf{Set}$, let $\chi_S\colon N \to \{0,1\}$ be given by $\chi_S = 1$ iff $n \in S$

We write W_{ij} to mean $W(i,j)$ for $(i,j) \in E$. To keep the notation from getting really messy, I'll also define:

DEFINITION 1.5. Let $S \in \mathsf{Set}$, $\vec{m} = m_1, \ldots, m_k$ be a sequence where each $m_i \in N$, and let $n \in N$. Then:

$$\mathsf{Activates}_S(\vec{m}, n) = A^{(n)}((\chi_S(m_1), \ldots, \chi_S(m_k)); (W(m_1, n), \ldots, W(m_k, n)))$$

i.e. the $m_i \in S$ subsequently "activate" n.

PROPOSITION 1.6. Let $S_1, S_2 \in \mathsf{Set}$, $\vec{m} = m_1, \ldots, m_k$ be a sequence where each $m_i \in N$, and let $n \in N$. Suppose that S_1 and S_2 agree on all m_i, i.e. for all $1 \leq i \leq k$, $m_i \in S_1$ iff $m_i \in S_2$. Then

$$\mathsf{Activates}_{S_1}(\vec{m}, n) = \mathsf{Activates}_{S_2}(\vec{m}, n)$$

Proof. We have:

$$\begin{array}{rcl} \mathsf{Activates}_{S_1}(\vec{m}, n) & = & A^{(n)}((\chi_{S_1}(m_1), \ldots, \chi_{S_1}(m_k)); (W(m_1, n), \ldots, W(m_k, n))) \\ & = & A^{(n)}((\chi_{S_2}(m_1), \ldots, \chi_{S_2}(m_k)); (W(m_1, n), \ldots, W(m_k, n))) \\ & = & \mathsf{Activates}_{S_2}(\vec{m}, n) \end{array}$$

□

1.2 Prop and Reach

DEFINITION 1.7. (Adapted from [15, Definition 3.4]) Let $\mathsf{Prop}\colon \mathsf{Set} \to \mathsf{Set}$ be defined recursively as follows: $n \in \mathsf{Prop}(S)$ iff either

Base Case. $n \in S$, or

Constructor. For those $\vec{m} = m_1, \ldots, m_k$ such that $(m_i, n) \in E$, $\mathsf{Activates}_{\mathsf{Prop}(S)}(\vec{m}, n) = 1$.

DEFINITION 1.8. Let $\mathsf{Reach}\colon \mathsf{Set} \to \mathsf{Set}$ be defined recursively as follows: $n \in \mathsf{Reach}(S)$ iff either

Base Case. $n \in S$, or

Constructor. There is an $m \in \mathsf{Reach}(S)$ such that $(m, n) \in E$.

PROPOSITION 1.9. **(Alternate characteriz. of Reach)** Let $n \in N$, $S \in \mathsf{Set}$. Then $n \in \mathsf{Reach}(S)$ iff there is a path from some $m \in S$ to n in E.

Proof. [] □

PROPOSITION 1.10. Let $\mathcal{N} \in \mathsf{Net}$. For all $S, S_1, S_2 \in \mathsf{Set}$, $n, m \in N$, Reach is

(Inclusive). $S \subseteq \mathsf{Reach}(S)$

(Idempotent). $\mathsf{Reach}(S) = \mathsf{Reach}(\mathsf{Reach}(S))$

(Acyclic). If $S_1 \subseteq \mathsf{Reach}(S_2)$ and $S_2 \subseteq \mathsf{Reach}(S_1)$ then $S_1 = S_2$.

(Monotonic). If $S_1 \subseteq S_2$ then $\mathsf{Reach}(S_1) \subseteq \mathsf{Reach}(S_2)$

Proof. We check each in turn:

(Inclusive). If $n \in S$, then $n \in \mathsf{Reach}(S)$ by the base case of Reach.

(Idempotent). The $(\subseteq)$ direction is just Inclusion. As for $(\supseteq)$, let $n \in \mathsf{Reach}(\mathsf{Reach}(S))$, and proceed by induction on the outer Reach.

Base Step. $n \in \mathsf{Reach}(S)$, and so we are done.

Inductive Step. There is an $m \in \mathsf{Reach}(\mathsf{Reach}(S))$ such that $(m, n) \in E$. by inductive hypothesis, $m \in \mathsf{Reach}(S)$. And so by definition, $n \in \mathsf{Reach}(S)$.

(Acyclic). Suppose $S_1 \subseteq \mathsf{Reach}(S_2)$ and $S_2 \subseteq \mathsf{Reach}(S_1)$. We will show $S_1 \subseteq S_2$ (the other direction is similar). [Todo]

(Monotonic). Let $n \in \mathsf{Reach}(S_1)$. We proceed by induction on $\mathsf{Reach}(S_1)$.

Base Step. $n \in S_1$. So $n \in S_2 \subseteq \mathsf{Reach}(S_2)$.

Inductive Step. There is an $m \in \mathsf{Reach}(S_1)$ such that $(m, n) \in E$. By inductive hypothesis, $m \in \mathsf{Reach}(S_2)$. And so by definition, $n \in \mathsf{Reach}(S_2)$.

□

Proposition 1.11. (Adapted from [15, Remark 4]) Let $\mathcal{N} \in \mathsf{Net}$. For all $S, S_1, S_2 \in \mathsf{Set}$, Prop is

(Inclusive). $S \subseteq \mathsf{Prop}(S)$

(Idempotent). $\mathsf{Prop}(S) = \mathsf{Prop}(\mathsf{Prop}(S))$

(Contained in Reach). $\mathsf{Prop}(S) \subseteq \mathsf{Reach}(S)$

Proof. We check each in turn:

(Inclusive). Similar to the proof of Inclusion for Reach.

(Idempotent). The $(\subseteq)$ direction is just Inclusion. As for $(\supseteq)$, let $n \in \mathsf{Prop}(\mathsf{Prop}(S))$, and proceed by induction on $\mathsf{Prop}(\mathsf{Prop}(S))$.

Base Step. $n \in \mathsf{Prop}(S)$, and so we are done.

Inductive Step. For those $\vec{m} = m_1, \ldots, m_k$ such that $(m_i, n) \in E$,

$$\mathsf{Activates}_{\mathsf{Prop}(\mathsf{Prop}(S))}(\vec{m}, n) = 1$$

By inductive hypothesis, $m_i \in \mathsf{Prop}(\mathsf{Prop}(S))$ iff $m_i \in \mathsf{Prop}(S)$. By Proposition 1.6, $\mathsf{Activates}_{\mathsf{Prop}(S)}(\vec{m}, n) = 1$, and so $n \in \mathsf{Prop}(S)$.

(Contained in Reach). Let $n \in \mathsf{Prop}(S)$, and proceed by induction on Prop.

Base Step. $n \in S$. So $n \in \mathsf{Reach}(S)$.

Inductive Step. For those $\vec{m} = m_1, \ldots, m_k$ such that $(m_i, n) \in E$,

$$\mathsf{Activates}_{\mathsf{Prop}(S)}(\vec{m}, n) = 1$$

Since $A^{(n)}$ is zero at zero, we have $m_i \in \mathsf{Prop}(S)$ for *some* $m = m_i$. By inductive hypothesis, $m \in \mathsf{Reach}(S)$. And since $(m, n) \in E$, by definition of Reach, $n \in \mathsf{Reach}(S)$. □

Proposition 1.12. The Cumulative and Loop properties from [15] [The KLM Cumulative & Loop properties, actually], i.e.

(Cumulative). If $S_1 \subseteq S_2 \subseteq \mathsf{Prop}(S_1)$ then $\mathsf{Prop}(S_1) \subseteq \mathsf{Prop}(S_2)$

(Loop). If $S_1 \subseteq \mathsf{Prop}(S_0), \ldots, S_n \subseteq \mathsf{Prop}(S_{n-1})$ and $S_0 \subseteq \mathsf{Prop}(S_n)$,
then $\mathsf{Prop}(S_i) = \mathsf{Prop}(S_j)$ for all $i, j \in \{0, \ldots, n\}$

follow from the properties of Prop and Reach above.

Proof. [Todo – note that (Cumulative) actually follows from (Loop). Use acyclic property of Reach to get (Loop)] □

1.3 Neural Network Semantics

Definition 1.13. Formulas of our language $\mathcal{L}$ are given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}\varphi \mid \mathbf{T}\varphi$$

where p is any propositional variable, and i is any nominal (denoting a neuron). Material implication $\varphi \to \psi$ is defined as $\neg\varphi \vee \psi$. We define $\bot, \vee, \leftrightarrow, \Leftrightarrow,$and the dual operators $\langle \mathbf{K} \rangle, \langle \mathbf{T} \rangle$ in the usual way.

Definition 1.14. Let $\mathcal{N} \in \mathsf{Net}$. The semantics $[\![\cdot]\!]: \mathcal{L} \to \mathsf{Set}$ for $\mathcal{L}$ are defined recursively as follows:

$$\boxed{\begin{array}{lcl} [\![p]\!] & = & V(p) \in \mathsf{Set} \\ [\![\neg\varphi]\!] & = & [\![\varphi]\!]^{\mathsf{C}} \\ [\![\varphi \wedge \psi]\!] & = & [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\langle \mathbf{K} \rangle \varphi]\!] & = & \mathsf{Reach}([\![\varphi]\!]) \\ [\![\langle \mathbf{T} \rangle \varphi]\!] & = & \mathsf{Prop}([\![\varphi]\!]) \end{array}}$$

Definition 1.15. **(Truth at a neuron)** $\mathcal{N}, n \Vdash \varphi$ iff $n \in [\![\varphi]\!]_{\mathcal{N}}$.

Definition 1.16. **(Truth in a net)** $\mathcal{N} \models \varphi$ iff $\mathcal{N}, n \Vdash \varphi$ for all $n \in N$.

Definition 1.17. **(Entailment)** $\Gamma \models_{\mathrm{BFNN}} \varphi$ if for all BFNNs $\mathcal{N}$ for all neurons $n \in N$, if $\mathcal{N}, n \models \Gamma$ then $\mathcal{N}$, $n \models \varphi$.

2 Neighborhood Models

2.1 Basic Definitions

Definition 2.1. [20, Definition 1.9] A **neighborhood frame** is a pair $\mathcal{F} = \langle W, f \rangle$, where W is a non-empty set of **worlds** and $f: W \to \mathcal{P}(\mathcal{P}(W))$ is a **neighborhood function**. A **multi-frame** may have more than one neighborhood function, but to keep things simple I won't distinguish between frames and multi-frames.

Definition 2.2. [20, Section 1.1] Let $\mathcal{F} = \langle W, f \rangle$ be a neighborhood frame, and let $w \in W$. The set $\bigcap_{X \in f(w)} X$ is called the **core of $f(w)$**, abbreviated $\cap f(w)$.

Definition 2.3. [20, Definition 1.4] Let $\mathcal{F} = \langle W, f \rangle$ be a frame. $\mathcal{F}$ is a **proper filter** iff:

- f is **closed under finite intersections**: for all $w \in W$, if $X_1, \ldots, X_n \in f(w)$ then their intersection $\bigcap_{i=1}^{k} X_i \in f(w)$
- f is **closed under supersets**: for all $w \in W$, if $X \in f(w)$ and $X \subseteq Y \subseteq W$, then $Y \in f(w)$
- f **contains the unit**: iff $W \in f(w)$

Proposition 2.4. [20, Corollary 1.1] If $\mathcal{F} = \langle W, f \rangle$ is a filter, and W is finite, then $\mathcal{F}$ contains its core.

Proof. [Todo] □

Definition 2.5. Let $\mathcal{F} = \langle W, f, g \rangle$ be a frame. $\mathcal{F}$ is a **preferential filter** iff:

- W is finite
- $\langle W, f \rangle$ forms a proper filter, and g contains the unit
- f is **acyclic**: for all $u_1, \ldots, u_n \in W$, if $u_1 \in \cap f(u_2), \ldots, u_{n-1} \in \cap f(u_n), u_n \in \cap f(u_1)$ then all $u_i = u_j$.
- f, g are **reflexive**: for all $w \in W$, $w \in \cap f(w)$ (similarly for g)
- f, g are **transitive**: for all $w \in W$, if $X \in f(w)$ then $\{u | X \in f(u)\} \in f(w)$ (similarly for g)
- g contains f: for all $w \in W$, if $X \in f(w)$ then $X \in g(w)$.

Proposition 2.6. Let $\mathcal{F} = \langle W, f \rangle$ be a frame. Suppose f is reflexive, transitive, and **asymmetric**, i.e. $u_1 \in \cap f(u_2)$ and $u_2 \in \cap f(u_1)$ implies $u_1 = u_2$. Then f is acyclic.

Proof. Let $u_1, \ldots, u_n \in W$, and suppose $u_1 \in \cap f(u_2), \ldots, u_{n-1} \in \cap f(u_n), u_n \in \cap f(u_1)$. WLOG we will show that $u_1 = u_n$. [Todo] □

2.2 Neighborhood Semantics

Definition 2.7. [20, Definition 1.11] Let $\mathcal{F}=\langle W,f,g\rangle$ be a neighborhood frame. A **neighborhood model** based on $\mathcal{F}$ is $\mathcal{M}=\langle W,f,g,V\rangle$, where $V:\mathcal{L}\to\mathcal{P}(W)$ is a valuation function.

Definition 2.8. [20, Definition 1.12] Let $\mathcal{M}=\langle W,f,g,V\rangle$ be a model based on $\mathcal{F}=\langle W,f,g\rangle$ The (neighborhood) semantics for $\mathcal{L}$ are defined recursively as follows:

$$\boxed{\begin{array}{lll}
\mathcal{M},w\Vdash p & \text{iff} & w\in V(p)\\
\mathcal{M},w\Vdash \neg\varphi & \text{iff} & \mathcal{M},w\nVdash\varphi\\
\mathcal{M},w\Vdash \varphi\wedge\psi & \text{iff} & \mathcal{M},w\Vdash\varphi \text{ and } \mathcal{M},w\Vdash\psi\\
\mathcal{M},w\Vdash \mathbf{K}\varphi & \text{iff} & \{u\mid\mathcal{M},u\Vdash\varphi\}\in f(w)\\
\mathcal{M},w\Vdash \mathbf{T}\varphi & \text{iff} & \{u\mid\mathcal{M},u\Vdash\varphi\}\in g(w)
\end{array}}$$

In neighborhood semantics, the operators **K**, and **T** are more natural to interpret. But when we gave our neural semantics, we instead interpreted the *duals* ⟨**K**⟩, and ⟨**T**⟩. Since we need to relate the two, I'll write the explicit neighborhood semantics for the duals here:

$$\begin{array}{lll}
\mathcal{M},w\Vdash \langle\mathbf{K}\rangle\varphi & \text{iff} & \{u\mid\mathcal{M},u\nVdash\varphi\}\notin f(w)\\
\mathcal{M},w\Vdash \langle\mathbf{T}\rangle\varphi & \text{iff} & \{u\mid\mathcal{M},u\nVdash\varphi\}\notin g(w)
\end{array}$$

Definition 2.9. [20, Definition 1.13] **(Truth in a model)** $\mathcal{M}\models\varphi$ iff $\mathcal{M},w\Vdash\varphi$ for all $w\in W$.

Definition 2.10. [20, Definition 2.32] **(Entailment)** Let F be a collection of neighborhood frames. $\Gamma\models_{\mathsf{F}}\varphi$ if for all models $\mathcal{M}$ based on a frame $\mathcal{F}\in\mathsf{F}$ and for all worlds $w\in W$, if $\mathcal{M},w\models\Gamma$ then $\mathcal{M},w\models\varphi$.

Note. This is the *local* consequence relation in modal logic.

3 From Nets to Frames

This is the easy ("soundness") direction!

Definition 3.1. Given a BFNN $\mathcal{N}$, its **simulation frame** $\mathcal{F}^\bullet=\langle W,f,g\rangle$ is given by:

- $W=N$
- $f(w)=\{S\subseteq W\mid w\notin\mathsf{Reach}(S^{\mathsf{C}})\}$
- $g(w)=\{S\subseteq W\mid w\notin\mathsf{Prop}(S^{\mathsf{C}})\}$

Moreover, the **simulation model** $\mathcal{M}^\bullet=\langle W,f,g,V\rangle$ based on $\mathcal{F}^\bullet$ has:

- $V_{\mathcal{M}^\bullet}(p)=V_{\mathcal{N}}(p)$

Theorem 3.2. Let $\mathcal{N}$ be a BFNN, and let $\mathcal{M}^\bullet$ be the simulation model based on $\mathcal{F}^\bullet$. Then for all $w\in W$,

$$\mathcal{M}^\bullet,w\Vdash\varphi \quad\text{iff}\quad \mathcal{N},w\Vdash\varphi$$

Proof. By induction on φ. The propositional, $\neg\varphi$, and $\varphi\wedge\psi$ cases are trivial.

⟨K⟩φ case:

$$\begin{array}{llll}
\mathcal{M}^\bullet,w\Vdash\langle\mathbf{K}\rangle\varphi & \text{iff} & \{u\mid\mathcal{M}^\bullet,w\nVdash\varphi\}\notin f(w) & \text{(by definition)}\\
& \text{iff} & \{u\mid u\notin[\![\varphi]\!]\}\notin f(w) & \text{(IH)}\\
& \text{iff} & [\![\varphi]\!]^{\mathsf{C}}\notin f(w) & \\
& \text{iff} & w\in\mathsf{Reach}([\![(\varphi^{\mathsf{C}})^{\mathsf{C}}]\!]) & \text{(by choice of } f)\\
& \text{iff} & w\in\mathsf{Reach}([\![\varphi]\!]) & \\
& \text{iff} & w\in[\![\langle\mathbf{K}\rangle\varphi]\!] & \text{(by definition)}\\
& \text{iff} & \mathcal{N},w\Vdash\langle\mathbf{K}\rangle\varphi & \text{(by definition)}
\end{array}$$

$\langle \mathbf{T} \rangle \varphi$ case:

$$\begin{array}{rlll}
\mathcal{M}^\bullet, w \Vdash \langle \mathbf{T} \rangle \varphi & \text{iff} & \{u \mid \mathcal{M}^\bullet, w \nVdash \varphi\} \notin g(w) & \text{(by definition)} \\
& \text{iff} & \{u \mid u \notin [\![\varphi]\!]\} \notin g(w) & \text{(IH)} \\
& \text{iff} & [\![\varphi]\!]^{\mathsf{C}} \notin g(w) & \\
& \text{iff} & w \in \mathsf{Prop}([\![(\varphi^{\mathsf{C}})^{\mathsf{C}}]\!]) & \text{(by choice of } g\text{)} \\
& \text{iff} & w \in \mathsf{Prop}([\![\varphi]\!]) & \\
& \text{iff} & w \in [\![\langle \mathbf{T} \rangle \varphi]\!] & \text{(by definition)} \\
& \text{iff} & \mathcal{N}, w \Vdash \langle \mathbf{T} \rangle \varphi & \text{(by definition)}
\end{array}$$

□

COROLLARY 3.3. $\mathcal{M}^\bullet \models \varphi$ iff $\mathcal{N} \models \varphi$.

THEOREM 3.4. $\mathcal{F}^\bullet$ is a preferential filter.

Proof. We show each in turn:

***W* is finite.** This holds because our BFNN is finite.

***f* is closed under finite intersection.** Suppose $X_1,\ldots,X_n \in f(w)$. By definition of f, $w \notin \bigcup_i \mathsf{Reach}(X_i^{\mathsf{C}})$ for all i. Since Reach is monotonic, [Make this a lemma!] we have $\bigcup_i \mathsf{Reach}(X_i^{\mathsf{C}}) = \mathsf{Reach}(\bigcup_i X_i^{\mathsf{C}}) = \mathsf{Reach}((\bigcap_i X_i)^{\mathsf{C}})$. So $w \notin \mathsf{Reach}((\bigcap_i X_i)^{\mathsf{C}})$. But this means that $\bigcap_i X_i \in f(w)$.

***f* is closed under superset.** Suppose $X \in f(w), X \subseteq Y$. By definition of f, $w \notin \mathsf{Reach}(X^{\mathsf{C}})$. Note that $Y^{\mathsf{C}} \subseteq X^{\mathsf{C}}$, and so by monotonicity of Reach we have $w \notin \mathsf{Reach}(Y^{\mathsf{C}})$. But this means $Y \in f(w)$, so we are done.

***f* contains the unit.** Note that for all $w \in W$, $w \notin \mathsf{Reach}(\emptyset) = \mathsf{Reach}(W^{\mathsf{C}})$. So $W \in f(w)$.

***g* contains the unit.** Same as the proof for f, except that we use the fact that for all w, $w \notin \mathsf{Prop}(\emptyset)$

***f* is acyclic.** Suppose $u_1,\ldots,u_n \in W$, with $u_1 \in \cap f(u_2),\ldots,u_{n-1} \in \cap f(u_n), u_n \in \cap f(u_1)$. That is, each $u_i \in \bigcap_{X \in f(u_{i+1})} X$. By choice of f, each $u_i \in \bigcap_{u_{i+1} \notin \mathsf{Reach}(X^{\mathsf{C}})} X$. Substituting X^{C} for X we get $u_i \in \bigcap_{u_{i+1} \notin \mathsf{Reach}(X)} X^{\mathsf{C}}$. In other words, $u_1 \in \mathsf{Reach}^{-1}(u_2),\ldots,u_{n-1} \in \mathsf{Reach}^{-1}(n), u_n \in \mathsf{Reach}^{-1}(u_1)$. [Update!] By Proposition ?, each $u_i = u_j$.

***f* is reflexive.** We want to show that $w \in \cap f(w)$. Well, suppose $X \in f(w)$, i.e. $w \notin \mathsf{Reach}(X^{\mathsf{C}})$ (by definition of f). Since for all S, $S \subseteq \mathsf{Reach}(S)$, we have $w \notin X^{\mathsf{C}}$. But this means $w \in X$, and we are done.

***g* is reflexive.** Same as the proof for f, except we use the fact that for all S, $S \subseteq \mathsf{Prop}(S)$.

***f* is transitive.** Suppose $X \in f(w)$, i.e. $w \notin \mathsf{Reach}(X^{\mathsf{C}})$. Well,

$$\begin{array}{rcll}
\mathsf{Reach}(X^{\mathsf{C}}) & = & \mathsf{Reach}(\mathsf{Reach}(X^{\mathsf{C}})) & \text{(by Idempotence of Reach)} \\
& = & \mathsf{Reach}(\{u \mid u \in \mathsf{Reach}(X^{\mathsf{C}})\}) & \\
& = & \mathsf{Reach}(\{u \mid u \notin \mathsf{Reach}(X^{\mathsf{C}})\}^{\mathsf{C}}) & \\
& = & \mathsf{Reach}(\{u \mid X \in f(u)\}^{\mathsf{C}}) & \text{(by definition of } f\text{)}
\end{array}$$

So by definition of f, $\{u \mid X \in f(u)\} \in f(w)$.

***g* is transitive.** Same as the proof for f, except we use the fact that Prop is idempotent.

***g* contains *f*.** Suppose $X \in f(w)$, i.e. $w \notin \mathsf{Reach}(X^{\mathsf{C}})$. Since for all S, $\mathsf{Prop}(S) \subseteq \mathsf{Reach}(S)$, we have $w \notin \mathsf{Prop}(X^{\mathsf{C}})$. And so $X \in g(w)$, and we are done.

□

4 From Frames to Nets

This is the harder ("completeness") direction!

DEFINITION 4.1. Let $\mathcal{M}$ be a model based on preferential filter $\mathcal{F} = \langle W, f, g \rangle$. Its **simulation net** $\mathcal{N}^\bullet = \langle N, E, W, A, O, V \rangle$ is the BFNN given by:

- $N = W$

- $(u,v)\in E$ iff $u\in\cap f(v)$

Now let $m_1,\dots,m_k$ list those nodes such that $(m_i,n)\in E$.

- $W(m_i,n)=$ [Todo: Currently, the weights are completely arbitrary! Notice that they aren't even being used in $A^{(n)}(\vec{x},\vec{w})$. Maybe this is the source of our problems?]
- $A^{(n)}(\vec{x},\vec{w})=1$ iff $\{m_i|x_i=1\}^{\mathsf{C}}\notin g(n)$
- $V_{\mathcal{N}^\bullet}(p)=V_{\mathcal{M}}(p)$

LEMMA 4.2. Let $\vec{m}=m_1,\dots,m_k$ be those nodes such that $(m_i,n)\in E$. Then

$$\mathsf{Activates}_S(\vec{m},n)=1 \quad\text{iff}\quad \{m_i|m_i\in S\}^{\mathsf{C}}\notin g(n)$$

Proof. $\mathsf{Activates}_S(\vec{m},n)=1$ iff:

$$\begin{aligned} & A^{(n)}((\chi_S(m_1),\dots,\chi_S(m_k));(W(m_1,n),\dots,W(m_k,n)))=1\\ \text{iff}\quad & \{m_i|\ \chi_S(m_i)=1\}^{\mathsf{C}}\notin g(n)\\ \text{iff}\quad & \{m_i|m_i\in S\}^{\mathsf{C}}\notin g(n) \end{aligned}$$

□

CLAIM 4.3. $\mathcal{N}^\bullet$ is a BFNN.

Proof. Clearly $\mathcal{N}^\bullet$ is a binary ANN. We check the rest of the conditions:

$\mathcal{N}^\bullet$ is feed-forward. Suppose for contradiction that E contains a cycle, i.e. distinct $u_1,\dots,u_n\in N$ such that $u_1Eu_2,\dots,u_{n-1}Eu_n,u_nEu_1$. Then we have $u_1\in\cap f(u_2),\dots,u_{n-1}\in\cap f(u_{n-1}),u_n\in\cap f(u_1)$, which contradicts the fact that f is acyclic.

$O^{(n)}\circ A^{(n)}$ is zero at zero. Suppose for contradiction that $A^{(v)}(\vec{0},\vec{w})=1$. Then $\emptyset^{\mathsf{C}}=W\notin g(v)$, which contradicts the fact that f contains the unit.

$O^{(n)}\circ A^{(n)}$ is monotonically increasing. Notice that the actual value of $A^{(v)}(\vec{x},\vec{w})$ does not depend at all on $\vec{w}$. So if we have $\vec{w}_1,\vec{w}_2$ such that $\vec{w}_1<\vec{w}_2$, then we trivially have $A^{(v)}(\vec{x},\vec{w}_1)<A^{(v)}(\vec{x},\vec{w}_2)$. □

LEMMA 4.4. $\mathsf{Reach}_{\mathcal{N}^\bullet}(S)=\{n|S^{\mathsf{C}}\notin f(n)\}$

Proof. For the $(\supseteq)$ direction, suppose $S^{\mathsf{C}}\notin f(n)$. We claim that $\cap f(n)\nsubseteq S^{\mathsf{C}}$. Why not? If $\cap f(n)\subseteq S^{\mathsf{C}}$, we have $\cap f(n)\in f(n)$ (since f is closed under finite intersection) and so $S^{\mathsf{C}}\in f(n)$ (since f is closed under superset). This would contradict $S^{\mathsf{C}}\notin f(n)$.

So $\cap f(n)\nsubseteq S^{\mathsf{C}}$. This means that there is some $m\in\cap f(n)$ such that $m\notin S^{\mathsf{C}}$. That is, $(m,n)\in E$ and $m\in S$. But then $m\in\mathsf{Reach}_{\mathcal{N}^\bullet}(S)$, and by the constructor of Reach, $n\in\mathsf{Reach}_{\mathcal{N}^\bullet}(S)$.

Now for the $(\subseteq)$ direction. Suppose $n\in\mathsf{Reach}(S)$, and proceed by induction on Reach.

Base step. $n\in S$. Suppose for contradiction that $S^{\mathsf{C}}\in f(n)$. By definition of core, $\cap f(n)\subseteq S^{\mathsf{C}}$. But since $\mathcal{F}$ is reflexive, $n\in\cap f(n)$. So $n\in S^{\mathsf{C}}$, which contradicts $n\in S$.

Inductive step. There is $m\in\mathsf{Reach}_{\mathcal{N}^\bullet}(S)$ such that $(m,n)\in E$ (and so $m\in\cap f(n)$). By inductive hypothesis, $S^{\mathsf{C}}\notin f(m)$. Now suppose for contradiction that $S^{\mathsf{C}}\in f(n)$. Since f is transitive, $\{t|S^{\mathsf{C}}\in f(t)\}\in f(n)$. By definition of core, $\cap f(n)\subseteq\{t|S^{\mathsf{C}}\in f(t)\}$. Since $m\in\cap f(n)$, $S^{\mathsf{C}}\in f(m)$. But this contradicts $S^{\mathsf{C}}\notin f(m)$! □

LEMMA 4.5. $\mathsf{Prop}_{\mathcal{N}^\bullet}(S)=\{n|S^{\mathsf{C}}\notin g(n)\}$

Proof. First, let's consider the $(\supseteq)$ direction. Since $\mathcal{N}^\bullet$ is feed-forward (i.e. acyclic), we can perform a topological sort on its nodes to get a strict ordering $\prec$ such that for all $n\neq m\in N$,

$$\text{If } (m,n)\in E \text{ then } m\prec n$$

Let $n \in N$ be such that $S^{\mathsf{C}} \notin g(n)$. We proceed by induction on the ordering $\prec$.

Base Step. There are no nodes m such that $m \prec n$, and hence no nodes $m \neq n$ with $(m,n) \in E$. We also have $S^{\mathsf{C}} \notin g(n)$ from before. Our goal is to show that in this case we have $n \in S$, and hence $n \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)$ by the base case of Prop.

Well, since $S^{\mathsf{C}} \notin g(n)$, $S^{\mathsf{C}} \notin f(n)$ (since g contains f). Since f is closed under superset and finite intersection, $\cap f(n) \nsubseteq S^{\mathsf{C}}$, i.e. there is some $u \in \cap f(n)$ such that $u \in S$. $u \in \cap f(n)$ gives us $(u,n) \in E$, but there are *no* $u \neq n$ with $(u,n) \in E$. But that means that $u = n$ — and so $n \in S$!

Inductive Step. Let $\vec{m} = m_1, \ldots, m_k$ be all those nodes $m_i \neq n$ such that $(m_i, n) \in E$. In particular, each $m_i \prec n$, and so we can apply our Inductive Hypothesis to each m_i:

Inductive Hypothesis. $m_i \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)$ iff $S^{\mathsf{C}} \notin g(n)$

Using our Inductive Hypothesis, we can see that the following sets are equal:

$$\begin{aligned}
\{u \mid S^{\mathsf{C}} \notin g(u)\} \cap (\cap f(n) &= \{u \mid S^{\mathsf{C}} \notin g(u) \text{ and } u \in \cap f(n)\} && \text{(using set-builder notation)}\\
&= \{u \mid S^{\mathsf{C}} \notin g(u) \text{ and } (u,n) \in E\} && \text{(by definition of } E\text{)}\\
&= \{m_i \mid S^{\mathsf{C}} \notin g(m_i)\} && \text{(by choice of } m_1, \ldots, m_k\text{)}\\
&= \{m_i \mid m_i \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)\} && \text{(by Inductive Hypothesis)}
\end{aligned}$$

Putting everything together, we have:

$$\begin{aligned}
S^{\mathsf{C}} \notin g(n) &\rightarrow [\{u \mid S^{\mathsf{C}} \notin g(u)\} \cap (\cap f(n))]^{\mathsf{C}} \notin g(n) && \text{[Skeleton/Minimal Cause]}\\
&\rightarrow \{m_i \mid m_i \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)\}^{\mathsf{C}} \notin g(n) && \text{(by the equality chain above)}\\
&\rightarrow \mathsf{Activates}_{\mathsf{Prop}_{\mathcal{N}^\bullet}(S)}(\vec{m}, n) = 1 && \text{(by Proposition 4.2)}\\
&\rightarrow n \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S) && \text{(by the constructor of Prop)}
\end{aligned}$$

As for the $(\subseteq)$ direction, suppose $n \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)$, and proceed by induction on Prop.

Base step. $n \in S$. Suppose for contradiction that $S^{\mathsf{C}} \in g(n)$. Since $\mathcal{G}$ is reflexive, $n \in \cap g(n)$. By definition of core, we have $\cap g(n) \subseteq S^{\mathsf{C}}$. But then $n \in \cap g(n) \subseteq S^{\mathsf{C}}$, i.e. $n \in S^{\mathsf{C}}$, which contradicts $n \in S$.

Inductive step. Let $\vec{m} = m_1, \ldots, m_k$ list those nodes such that $(u_i, v) \in E$. We have

$$\mathsf{Activates}_{\mathsf{Prop}_{\mathcal{N}^\bullet}(S)}(\vec{m}, n) = 1$$

By Lemma 4.2, this means that $\{m_i \mid m_i \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)\}^{\mathsf{C}} \notin g(n)$. But by our inductive hypothesis, $\{m_i \mid m_i \in \mathsf{Prop}_{\mathcal{N}^\bullet}(S)\} = \{m_i \mid S^{\mathsf{C}} \notin g(n)\}$. For convenience, let T be this latter set, i.e. $T = \{m_i \mid S^{\mathsf{C}} \notin g(n)\}$. So we have $T^{\mathsf{C}} \notin g(n)$.

We would like to show that $S^{\mathsf{C}} \notin g(n)$. Suppose for contradiction that $S^{\mathsf{C}} \in g(n)$. Notice that, by definition of T, $T^{\mathsf{C}} = \{u_i \mid S^{\mathsf{C}} \in g(u_i)\}$. Since $S^{\mathsf{C}} \in g(v)$ and $\mathcal{G}$ is transitive, $T^{\mathsf{C}} \in g(v)$, which contradicts $T^{\mathsf{C}} \notin g(v)$.

□

THEOREM 4.6. Let $\mathcal{M}$ be a model based on a preferential filter $\mathcal{F}$, and let $\mathcal{N}^\bullet$ be the corresponding simulation net. We have, for all $w \in W$,

$$\mathcal{M}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{N}^\bullet, w \Vdash \varphi$$

Proof. By induction on φ. Again, the propositional, $\neg\varphi$, and $\varphi \wedge \psi$ cases are trivial.

$\langle \mathbf{K} \rangle \boldsymbol{\varphi}$ case:

$$\begin{aligned}
\mathcal{M}, w \Vdash \langle \mathbf{K} \rangle \varphi \text{ iff } & \{u \mid \mathcal{M}, w \nVdash \varphi\} \notin f(w) && \text{(by definition)}\\
\text{iff } & \{u \mid u \notin [\![\varphi]\!]_{\mathcal{N}^\bullet}\} \notin f(w) && \text{(Inductive Hypothesis)}\\
\text{iff } & [\![\varphi]\!]^{\mathsf{C}}_{\mathcal{N}^\bullet} \notin g(w) && \\
\text{iff } & w \in \mathsf{Reach}_{\mathcal{N}^\bullet}([\![\varphi]\!]) && \text{(by Lemma 4.4)}\\
\text{iff } & w \in [\![\langle \mathbf{K} \rangle \varphi]\!]_{\mathcal{N}^\bullet} && \text{(by definition)}\\
\text{iff } & \mathcal{N}^\bullet, w \Vdash \langle \mathbf{K} \rangle \varphi && \text{(by definition)}
\end{aligned}$$

$\langle\mathbf{T}\rangle\varphi$ case:

$$\begin{array}{llll}
\mathcal{M}, w \Vdash \langle \mathbf{T}\rangle\varphi & \text{iff} & \{u \mid \mathcal{M}, u \nVdash \varphi\} \notin g(w) & \text{(by definition)} \\
 & \text{iff} & \{u \mid u \notin [\![\varphi]\!]_{\mathcal{N}^\bullet}\} \notin g(w) & \text{(Inductive Hypothesis)} \\
 & \text{iff} & [\![\varphi]\!]^{\complement}_{\mathcal{N}^\bullet} \notin g(w) & \\
 & \text{iff} & w \in \mathsf{Prop}_{\mathcal{N}^\bullet}([\![\varphi]\!]) & \text{(by Lemma 4.5)} \\
 & \text{iff} & w \in [\![\langle\mathbf{T}\rangle\varphi]\!]_{\mathcal{N}^\bullet} & \text{(by definition)} \\
 & \text{iff} & \mathcal{N}^\bullet, w \Vdash \langle\mathbf{T}\rangle\varphi & \text{(by definition)}
\end{array}$$

□

COROLLARY 4.7. $\mathcal{M} \models \varphi$ iff $\mathcal{N}^\bullet \models \varphi$.

5 Completeness

5.1 The Base Modal Logic

DEFINITION 5.1. Our logic $\mathbf{L}$ is the smallest set of formulas in $\mathcal{L}$ containing the axioms

(K). $\mathbf{K}(\varphi \to \psi) \to (\mathbf{K}\varphi \to \mathbf{K}\psi)$

($\mathbf{T_K}$). $\mathbf{K}\varphi \to \varphi$

($\mathbf{4_K}$). $\mathbf{K}\varphi \to \mathbf{K}\mathbf{K}\varphi$

(Grz). $\mathbf{K}(\mathbf{K}(\varphi \to \mathbf{K}\varphi) \to \varphi) \to \varphi$

($\mathbf{T_T}$). $\mathbf{T}\varphi \to \varphi$

($\mathbf{4_T}$). $\mathbf{T}\varphi \to \mathbf{T}\mathbf{T}\varphi$

(K-T). $\mathbf{K}\varphi \to \mathbf{T}\varphi$

that is closed under:

(Necessitation). If $\varphi \in \mathbf{L}$ then $\Box\varphi \in \mathbf{L}$ for $\Box \in \{\mathbf{K}, \mathbf{T}\}$

DEFINITION 5.2. [20, Definition 2.30] **(Deduction for L)** $\vdash \varphi$ iff either φ is an axiom, or φ follows from some previously obtained formula by one of the inference rules. If $\Gamma \subseteq \mathcal{L}$ is a set of formulas, $\Gamma \vdash \varphi$ whenever there are finitely many $\psi_1, \ldots, \psi_k \in \Gamma$ such that $\vdash \psi_1 \wedge \ldots \wedge \psi_k \to \varphi$.

DEFINITION 5.3. [20, Definition 2.36] Γ is **consistent** iff $\Gamma \nvdash \bot$. Γ is **maximally consistent** if Γ is consistent and for all $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma$ or $\varphi \notin \Gamma$.

LEMMA 5.4. [20, Lemma 2.19] ("Lindenbaum's Lemma") We can extend any set Γ to a maximally consistent set $\Delta \supseteq \Gamma$.

DEFINITION 5.5. [20, Definition 2.36] **(Proof Set)** $|\varphi|_{\mathbf{L}} = \{\Delta \mid \Delta \text{ is maximally consistent and } \varphi \in \Delta\}$

PROPOSITION 5.6. Let Δ be maximally consistent, and let $\Box \in \{\mathbf{K}, \mathbf{T}\}$. We have $\Box\varphi \in \Delta$ iff

$$\forall \Sigma \text{ maximally consistent, if } \forall \psi, \Box\psi \in \Delta \text{ implies } \psi \in \Sigma, \text{ then } \varphi \in \Sigma$$

Proof. The $(\to)$ direction is straightforward. As for the $(\leftarrow)$ direction, suppose contrapositively that $\Box\varphi \notin \Delta$, and let $\Sigma = \{\psi \mid \Box\psi \in \Delta\}$ [why is Σ maximally consistent?]. Then by construction, for all ψ $\Box\psi \in \Delta$ implies $\psi \in \Sigma$, but $\varphi \notin \Sigma$ (since $\Box\varphi \notin \Delta$). □

5.2 Soundness

THEOREM 5.7. **(Soundness)** If $\Gamma \vdash \varphi$ then $\Gamma \models_{\mathrm{BFNN}} \varphi$

Proof. Suppose $\Gamma \vdash \varphi$, and let $\mathcal{N}, n \models \Gamma$ We just need to check that each of the axioms and rules of inference are sound, from which we can conclude that $\mathcal{N}, n \models \varphi$. We can do this either by the semantics of BFNNs, or instead by checking them in an equivalent preferential frame $\mathcal{M}^{\bullet} = \langle W, f, g, V \rangle$:

To show soundness of:	**Use:**	**Alternative:**
(K)	Monotonicity of Reach	$\langle W, f \rangle$ forms a filter
($\mathrm{T}_{\mathbf{K}}$)	Inclusion of Reach	Reflexivity of f
($4_{\mathbf{K}}$)	Idempotence of Reach	Transitivity of f
(Grz)	Proposition ?[Check! – and update, since the def changed]	f is acyclic [Check!]
($\mathrm{T}_{\mathbf{T}}$)	Inclusion of Prop	Reflexivity of g
($4_{\mathbf{T}}$)	Idempotence of Prop	Transitivity of g
(**T–K**)	Reach contains Prop	g contains f
(Necessitation)	$\forall w, w \notin \mathsf{Reach}(\emptyset), \mathsf{Prop}(\emptyset)$	f, g contain the unit

□

5.3 Model Building

Given a set $\Gamma \subseteq \mathcal{L}$, I will show that we can build a net $\mathcal{N}$ that models Γ. Since preferential filters are equivalent to BFNNs (over $\mathcal{L}$), I will focus instead on building a preferential filter $\mathcal{F}$. This is the same strategy taken by [15], who constructs KLM cumulative-ordered models in order to build a neural net.

The following are the standard canonical construction and facts for neighborhood models (see Eric Pacuit's book). Adapting these to our logic of $\mathbf{K}, \mathbf{K}^{\downarrow}, \mathbf{T}$ is a straightforward exercise in modal logic.

LEMMA 5.8. [20, Lemma 2.12 & Definition 2.37] We can build a **canonical** neighborhood model for $\mathbf{L}$, i.e. a model $\mathcal{M}^C = \langle W^C, f^C, g^C, V^C \rangle$ such that:

- $W^C = \{\Delta \mid \Delta$ is maximally consistent$\}$
- For each $\Delta \in W^C$ and each $\varphi \in \mathcal{L}$, $|\varphi|_{\mathbf{L}} \in f^C(\Delta)$ iff $\mathbf{K}\varphi \in \Delta$
- For each $\Delta \in W^C$ and each $\varphi \in \mathcal{L}$, $|\varphi|_{\mathbf{L}} \in g^C(\Delta)$ iff $\mathbf{T}\varphi \in \Delta$
- $V^C(p) = |p|_{\mathbf{L}}$

Note. This is where the Necessitation rules come into play — we need them in order to guarantee that we can actually build this model!

LEMMA 5.9. [20, Lemma 2.13] **(Truth Lemma)** We have, for canonical model $\mathcal{M}^C$,

$$\{\Delta \mid \mathcal{M}^C, \Delta \Vdash \varphi\} = |\varphi|_{\mathbf{L}}$$

Proof. By induction on φ. The propositional, and boolean cases are straightforward.

K case.

$$\begin{array}{llll}
\mathcal{M}^C, \Delta \Vdash \mathbf{K}\varphi & \text{iff} & \{u \mid \mathcal{M}^C, \Sigma \Vdash \varphi\} \in f^C(\Delta) & \text{(by definition)} \\
 & \text{iff} & |\varphi|_{\mathbf{L}} \in f^C(\Delta) & \text{(by IH)} \\
 & \text{iff} & \mathbf{K}\varphi \in \Delta & \text{(since } \mathcal{M}^C \text{ is canonical)} \\
 & \text{iff} & \Delta \in |\mathbf{K}\varphi|_{\mathbf{L}} & \text{(by definition)}
\end{array}$$

T case.

$$\begin{array}{llll}
\mathcal{M}^C, \Delta \Vdash \mathbf{T}\varphi & \text{iff} & \{u \mid \mathcal{M}^C, \Sigma \Vdash \varphi\} \in g^C(\Delta) & \text{(by definition)} \\
 & \text{iff} & |\varphi|_{\mathbf{L}} \in g^C(\Delta) & \text{(by IH)} \\
 & \text{iff} & \mathbf{T}\varphi \in \Delta & \text{(since } \mathcal{M}^C \text{ is canonical)} \\
 & \text{iff} & \Delta \in |\mathbf{T}\varphi|_{\mathbf{L}} & \text{(by definition)}
\end{array}$$

□

THEOREM 5.10. [State that our logic has the finite model property]

Proof. [Prove it by the usual filtration construction — the fact that the filtration is closed under $\cap, \subseteq$, reflexive, and transitive are all shown in Pacuit's book. So I just need to show that the same is true of the acyclic & skeleton properties.] □

PROPOSITION 5.11. If $\mathcal{M}$ is finite and satisfies the Truth Lemma, then $\mathcal{M}$ is a preferential filter.

Proof. W^C is finite by assumption. Since $\mathbf{L}$ contains all instances of **(K)**, **(T)**, **(4)**, **(T)**, **(4)** it follows that f^C is a reflexive, transitive, proper filter, and g^C is reflexive and transitive (this is another classical result, see Pacuit's book). The only things left to show are that f^C is acyclic and f^C is the skeleton of g^C.

***W^C* is finite.** Holds by assumption.

***f^C* is closed under finite intersection.** It's enough to show that f^C is closed under binary intersections. $\mathbf{L}$ contains all instances of **(K)**, from which we can derive all instances of:

$$\textbf{(C)} \quad \mathbf{K}\varphi \wedge \mathbf{K}\psi \rightarrow \mathbf{K}(\varphi \wedge \psi)$$

Suppose $|\varphi|_{\mathbf{L}}, |\psi|_{\mathbf{L}} \in f^C(\Delta)$. By definition of f^C, $\mathbf{K}\varphi \in \Delta$ and $\mathbf{K}\psi \in \Delta$. So $\mathbf{K}\varphi \wedge \mathbf{K}\psi \in \Delta$. Applying **(C)**, $\mathbf{K}(\varphi \wedge \psi) \in \Delta$. So $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}} \in \Delta$.

***f^C* is closed under superset.** $\mathbf{L}$ contains all instances of **(K)** and the necessitation rule, from which we can derive:

$$\textbf{(RM)} \quad \text{If } \varphi \rightarrow \psi \in \mathbf{L} \text{ then } \mathbf{K}\varphi \rightarrow \mathbf{K}\psi \in \mathbf{L}$$

Suppose $|\varphi|_{\mathbf{L}} \in f^C(\Delta)$, and $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$. The former fact gives us $\mathbf{K}\varphi \in \Delta$. The latter gives us, for all maximally consistent Δ, if $\varphi \in \Delta$ then $\psi \in \Delta$, i.e. $\varphi \rightarrow \psi \in \mathbf{L}$ [Is this correct? Probably not; we need to close the canonical model under superset]. By **(RM)**, we have $\mathbf{K}\psi \in \Delta$, i.e. $|\psi|_{\mathbf{L}} \in f^C(\Delta)$.

***f^C* contains the unit.** $\mathbf{L}$ is closed under necessitation for $\mathbf{K}$, from which we can derive:

$$\textbf{(N)} \quad \mathbf{K}\top$$

That is, $\mathbf{K}\top \in \Delta$ for all maximally consistent Δ. So $|\top|_{\mathbf{L}} \in f^C(\Delta)$, i.e. $W^C \in f^C(\Delta)$.

***f^C* is reflexive.** First, let $\Delta \in W^C$, and suppose $|\varphi|_{\mathbf{L}} \in f^C(\Delta)$. By definition of f^C, $\mathbf{K}\varphi \in \Delta$. By **($\mathbf{T_K}$)**, $\varphi \in \Delta$. Since φ was chosen arbitrarily, we have for all φ, if $|\varphi|_{\mathbf{L}} \in f^C(\Delta)$ then $\varphi \in \Delta$. In other words, $\Delta \in \bigcap_{|\varphi|_{\mathbf{L}} \in f^C(\Delta)} |\varphi|_{\mathbf{L}} = \cap f^C(\Delta)$.

***f^C* is transitive.** Suppose $|\varphi|_{\mathbf{L}} \in f^C(\Delta)$. By definition of f^C, $\mathbf{K}\varphi \in \Delta$. By the **($\mathbf{4_K}$)** axiom, $\mathbf{K}\mathbf{K}\varphi \in \Delta$. But this means that $|\mathbf{K}\varphi|_{\mathbf{L}} \in f^C(\Delta)$. By definition of proof set, we have $\{\Sigma \mid \mathbf{K}\varphi \in \Sigma\} \in f^C(\Delta)$. That is, $\{\Sigma \mid |\varphi|_{\mathbf{L}} \in f^C(\Sigma)\} \in f^C(\Delta)$, and we are done.

***f^C* is acyclic.** [Update! – no more nominals, acyclic rule has changed to Grz!] Since f^C is reflexive and transitive, by Proposition 2.6 it's enough to show that f^C is asymmetric. Suppose $\Delta_1 \in \cap f^C(\Delta_2)$ and $\Delta_2 \in \cap f^C(\Delta_1)$. By definition of core, $\Delta_1 \in \bigcap_{|\varphi|_{\mathbf{L}} \in f^C(\Delta_2)} |\varphi|_{\mathbf{L}}$ and $\Delta_2 \in \bigcap_{|\varphi|_{\mathbf{L}} \in f^C(\Delta_1)} |\varphi|_{\mathbf{L}}$, i.e. we have both of the following:

1. $\forall \varphi$, if $\mathbf{K}\varphi \in \Delta_2$ then $\varphi \in \Delta_1$
2. $\forall \varphi$, if $\mathbf{K}\varphi \in \Delta_1$ then $\varphi \in \Delta_2$

We want to show that $\Delta_1 = \Delta_2$. I'll show the $(\subseteq)$ direction (the other direction is similar). Suppose for contradiction that $\varphi \in \Delta_1$, but $\varphi \notin \Delta_2$ (i.e. $\neg\varphi \in \Delta_2$).

Since Δ_1 is named, some $i \in \Delta_1$. By **(Antisym)**, $\mathbf{K}(\langle\mathbf{K}\rangle i \rightarrow i) \in \Delta_1$. By (2), $\langle\mathbf{K}\rangle i \rightarrow i \in \Delta_2$. Rewriting, we get $\neg i \rightarrow \mathbf{K}\neg i \in \Delta_2$. [What next?]

***g^C* contains the unit.** Similar to the proof for f^C, but apply necessitation for $\mathbf{T}$ instead of $\mathbf{K}$.

***g^C* is reflexive.** Similar to the proof for f^C, but apply **($\mathbf{T_T}$)** instead of **($\mathbf{T_K}$)**.

***g^C* is transitive.** Similar to the proof for f^C, but apply **($\mathbf{4_T}$)** instead of **($\mathbf{4_K}$)**.

***g^C* contains *f^C*.** Suppose $|\varphi|_{\mathbf{L}} \in f^C(\Delta)$. By definition of f^C, $\mathbf{K}\varphi \in \Delta$. By the (**K–T**) axiom, $\mathbf{T}\varphi \in \Delta$. And so $|\varphi|_{\mathbf{L}} \in f^C(\Delta)$.

□

THEOREM 5.12. **(Model Building)** Given any consistent $\Gamma \subseteq \mathcal{L}$, we can construct a BFNN $\mathcal{N}$ and neuron $n \in N$ such that $\mathcal{N}, n \models \Gamma$.

Proof. Extend Γ to maximally consistent Δ using Lemma 5.4. Let $\mathcal{M}^C$ be a canonical model for **L** guaranteed by Lemma 5.8. By the Truth Lemma (Lemma 5.9), $\mathcal{M}^C, \Delta \models \Delta$. So in particular, $\mathcal{M}^C, \Delta \models \Gamma$.

By the Finite Model Property (Lemma 5.10), we can construct a finite model $\mathcal{M}'$ satisfying exactly the same formulas at all worlds. By Proposition 5.11, $\mathcal{M}'$ is a preferential filter.

From here, we can build our net $\mathcal{N}^\bullet$ as before, satisfying exactly the same formulas as $\mathcal{M}$ at all neurons (by Theorem 4.6). And so $\mathcal{N}^\bullet, \Delta \models \Gamma$. □

THEOREM 5.13. **(Completeness)** For all consistent $\Gamma \subseteq \mathcal{L}$, if $\Gamma \models_{\text{BFNN}} \varphi$ then $\Gamma \vdash \varphi$

Proof. Suppose contrapositively that $\Gamma \nvdash \varphi$. This means that $\Gamma \cup \{\neg\varphi\}$ is consistent, i.e. by Theorem 5.12 we can build a BFNN $\mathcal{N}$ and neuron n such that $\mathcal{N}, n \models \Gamma \cup \{\neg\varphi\}$. In particular, $\mathcal{N}, n \not\models \varphi$. But then we must have $\Gamma \not\models \varphi$. □

TODO:

- Double-check properties for canonical model & completeness
- Do filtration/finite model property
- Get bound on the size of the finite model.
- Think about complexity of decidability of the logic (but only if it seems easy)
- Copy-paste flipping $\wedge, \vee, \neg$ considerations
- Write up fuzzy network considerations (in a crisp (non-fuzzy) language) — fuzzy nets satisfy *exactly* the same crisp formulas as binary nets
- Make drawings in Tikz
- Make corrections Saul gave
- Close the canonical model under superset
- Put the page number/theorem number for each result
- Rename the axioms to something more readable (($\mathbf{T_T}$) is confusing as hell)

References

[1] Sebastian Bader and Pascal Hitzler. Dimensions of neural-symbolic integration-a structured survey. *ArXiv preprint cs/0511042*, 2005.

[2] Christian Balkenius and Peter Gärdenfors. Nonmonotonic Inferences in Neural Networks. In *KR*, pages 32–39. 1991.

[3] Vaishak Belle. Logic Meets Learning: From Aristotle to Neural Networks. In *Neuro-Symbolic Artificial Intelligence: The State of the Art*, pages 78–102. IOS Press, 2021.

[4] Patrick Blackburn, Maarten De Rijke, and Yde Venema. *Modal logic: graph. Darst*, volume 53. Cambridge University Press, 2001.

[5] Reinhard Blutner. Nonmonotonic inferences and neural networks. In *Information, Interaction and Agency*, pages 203–234. Springer, 2004.

[6] Antony Browne and Ron Sun. Connectionist inference models. *Neural Networks*, 14(10):1331–1355, 2001.

[7] Dov M Gabbay, Ian Hodkinson, and Mark A Reynolds. Temporal logic: mathematical foundations and computational aspects. 1994.

[8] Artur S d'Avila Garcez, Krysia Broda, and Dov M Gabbay. Symbolic knowledge extraction from trained neural networks: a sound approach. *Artificial Intelligence*, 125(1-2):155–207, 2001.

[9] Artur S d'Avila Garcez, Luis C Lamb, and Dov M Gabbay. *Neural-symbolic cognitive reasoning*. Springer Science Business Media , 2008.

[10] Laura Giordano, Valentina Gliozzi, and Daniele Theseider Dupré. From common sense reasoning to neural network models through multiple preferences: An overview. *CoRR*, abs/2107.04870, 2021.

[11] Laura Giordano, Valentina Gliozzi, and Daniele Theseider DuprÉ. A conditional, a fuzzy and a probabilistic interpretation of self-organizing maps. *Journal of Logic and Computation*, 32(2):178–205, 2022.

[12] Donald Hebb. *The organization of behavior: A neuropsychological theory*. Wiley, New York, 1949.

[13] The Third AI Summer, AAAI Robert S. Engelmore Memorial Award Lecture. AAAI, 2020.

[14] Caleb Kisby, Saúl Blanco, and Lawrence Moss. The logic of hebbian learning. In *The International FLAIRS Conference Proceedings*, volume 35. 2022.

[15] Hannes Leitgeb. Nonmonotonic reasoning by inhibition nets. *Artificial Intelligence*, 128(1-2):161–201, 2001.

[16] Hannes Leitgeb. Nonmonotonic reasoning by inhibition nets II. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 11(supp02):105–135, 2003.

[17] Hannes Leitgeb. Neural Network Models of Conditionals. In *Introduction to Formal Philosophy*, pages 147–176. Springer, 2018.

[18] Warren S McCulloch and Walter Pitts. A logical calculus of the ideas immanent in nervous activity. *The bulletin of mathematical biophysics*, 5(4):115–133, 1943.

[19] Simon Odense and Artur d'Avila Garcez. A semantic framework for neural-symbolic computing. *ArXiv preprint arXiv:2212.12050*, 2022.

[20] Eric Pacuit. *Neighborhood semantics for modal logic*. Springer, 2017.

[21] Md Kamruzzaman Sarker, Lu Zhou, Aaron Eberhart, and Pascal Hitzler. Neuro-symbolic artificial intelligence: current trends. *ArXiv preprint arXiv:2105.05330*, 2021.

[22] Dongran Yu, Bo Yang, Dayou Liu, and Hui Wang. A survey on neural-symbolic systems. *ArXiv preprint arXiv:2111.08164*, 2021.

Appendix A Helper Proofs

Proof. (of Proposition ?) To show that hash is injective, suppose $\mathsf{hash}(S_1) = \mathsf{hash}(S_2)$. So $\prod_{m_i \in S_1} p_i = \prod_{m_i \in S_2} p_i$, and since products of primes are unique, $\{p_i | m_i \in S_1\} = \{p_i | m_i \in S_2\}$. And so $S_1 = S_2$.

To show that hash is surjective, let $x \in \mathsf{P}_k$. Now let $S = \{m_i | \, p_i \text{ divides } x\}$. Then $\mathsf{hash}(S) = \prod_{m_i \in S} p_i = \prod_{(p_i \text{ divides } x)} p_i = x$. □

Step 5. Step away (for a few days). Come back and check the proof *slowly* to make sure there aren't any missing edge cases or conditions.

- **If it's all good — congratulations, you got a free paper!**
- **Usually there will be some idiotic mistake in the proof. It may seem like *you're* the idiot for trying it — but in fact, it's now your job to figure out *what conditions will make this naive proof work*!**

Step 10. Move on to the write-up stage. But otherwise, step away from the problem — there are too many other interesting things to spend all of your time on this one. Trust that one day a different solution will come to you.

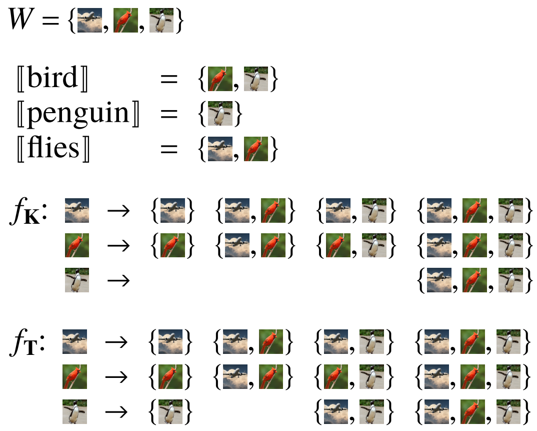
Talk Abstract

Artificial Intelligence is in the midst of a crisis. Its two main paradigms — connectionist neural networks and classical (logic) models — seem diametrically opposed, with no clear way to reconcile the two. Neural networks learn flexibly from unstructured data, and are more cognitively plausible. But it is difficult to interpret what a neural network *knows* or *has learned*. On the other hand, classical models *do* represent knowledge explicitly and transparently, but are notoriously rigid and are often criticized for being completely cognitively implausible. This talk is an introduction to a developing theory of *Neuro-Symbolic Artificial Intelligence* that aims to bridge the gap. The key insight of this theory is that neural networks and classical models are really two different representations of the same information; it is in principle possible to translate between the two. I will illustrate how this is possible by offering (1) a translation for feed-forward nets that is provably correct, and (2) a logical account for "naive" Hebbian learning. Although this talk is primarily from the point-of-view of AI, I hope to convince you that this is a deep result for cognitive science at large.

Bio:

Caleb Schultz Kisby is a fifth-year Computer Science PhD student at Indiana University, co-advised by Saúl Blanco and Larry Moss. He is interested in combining neural and symbolic accounts of reasoning, especially for systems that learn (which has long been neglected by Neuro-Symbolic AI). Before joining IU, he received his B.S. in Computer Science and Mathematics at the University of South Carolina.

Figures for the talk:



$\mathcal{M} = \langle W, f_{\mathbf{K}}, f_{\mathbf{T}}, \llbracket p \rrbracket \rangle$

$\mathcal{N} = \langle N, E, W, A \rangle$

$p \quad \bot \quad \neg A \quad A \to B \quad \mathbf{K}A \quad \mathbf{T}A$

$$(A \to B) \wedge (B \to C) \to (A \to C)$$
$$(\neg A \to \bot) \to A$$
$$(A \wedge \neg A) \to B$$

$$\frac{A \qquad A \to B}{B}$$

$$\mathbf{K}A \to A$$
$$\mathbf{K}A \to \mathbf{KK}A$$
$$\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$$

$$\mathbf{T}A \to A$$
$$\mathbf{T}A \to \mathbf{TT}A$$

- E is feed-forward (no cycles)
- A is monotonically increasing
- A is binary
- $f_{\mathbf{K}}$ is reflexive, transitive, acyclic, monotonic
- $f_{\mathbf{T}}$ is reflexive, transitive, **not** monotonic
- $f_{\mathbf{K}}$ is a skeleton for $f_{\mathbf{T}}$

Knowledge Base:

$\neg(\text{airplane} \rightarrow \text{bird})$
$\mathbf{T}(\text{airplane} \rightarrow \text{flies})$
$\text{penguin} \rightarrow \text{bird}$
$\mathbf{T}(\text{bird}) \rightarrow \text{flies}$
$\neg(\text{penguin} \rightarrow \text{flies})$

$\mathcal{M}, w \models \mathbf{K}A$ iff $\{u \mid \mathcal{M}, u \models A\} \in f_{\mathbf{K}}(w)$
$\mathcal{M}, w \models \mathbf{T}A$ iff $\{u \mid \mathcal{M}, u \models A\} \in f_{\mathbf{T}}(w)$

$$\begin{array}{lcl}
[A]^*p & \leftrightarrow & p \\
[A]^*\neg B & \leftrightarrow & \neg[A]^*B \\
[A]^*(B \rightarrow C) & \leftrightarrow & [A]^*B \rightarrow [A]^*C \\
[A]^*\mathbf{K}B & \leftrightarrow & \mathbf{K}[A]^*B \\
 & & \\
[A]^*\mathbf{T}B & \leftrightarrow & [(\mathbf{T}A \vee \mathbf{T}B \leftrightarrow \bot) \rightarrow \mathbf{T}[A]^*B] \\
 & & \vee \quad [\neg(\mathbf{T}A \vee \mathbf{T}B \leftrightarrow \bot) \rightarrow \mathbf{T}[A]^*B \wedge (\mathbf{T}A \vee \mathbf{K}B)]
\end{array}$$

(K). $\mathbf{K}(\varphi \rightarrow \psi) \rightarrow (\mathbf{K}\varphi \rightarrow \mathbf{K}\psi)$
(T$_{\mathbf{K}}$). $\mathbf{K}\varphi \rightarrow \varphi$
(4$_{\mathbf{K}}$). $\mathbf{K}\varphi \rightarrow \mathbf{K}\mathbf{K}\varphi$
(Grz). $\mathbf{K}(\mathbf{K}(\varphi \rightarrow \mathbf{K}\varphi) \rightarrow \varphi) \rightarrow \varphi$

(T$_{\mathbf{T}}$). $\mathbf{T}\varphi \rightarrow \varphi$
(4$_{\mathbf{T}}$). $\mathbf{T}\varphi \rightarrow \mathbf{T}\mathbf{T}\varphi$
(K-T). $\mathbf{K}\varphi \rightarrow \mathbf{T}\varphi$

that is closed under:

(Necessitation). If $\varphi \in \mathbf{L}$ then $\Box\varphi \in \mathbf{L}$ for $\Box \in \{\mathbf{K}, \mathbf{T}\}$

$$\begin{array}{l}
\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B) \\
\mathbf{K}A \rightarrow A \\
\mathbf{K}A \rightarrow \mathbf{K}\mathbf{K}A \\
\mathbf{K}(\mathbf{K}(A \rightarrow \mathbf{K}A) \rightarrow A) \rightarrow A \\
 \\
\mathbf{K}^{\downarrow}(A \rightarrow B) \rightarrow (\mathbf{K}^{\downarrow}A \rightarrow \mathbf{K}^{\downarrow}B) \\
A \rightarrow \mathbf{K}\langle\mathbf{K}^{\downarrow}\rangle A \\
A \rightarrow \mathbf{K}^{\downarrow}\langle\mathbf{K}\rangle A \\
 \\
\mathbf{T}A \rightarrow A \\
\mathbf{T}A \rightarrow \mathbf{T}\mathbf{T}A \\
 \\
(\mathbf{T}A \rightarrow \mathbf{K}^{\downarrow}B) \leftrightarrow (\mathbf{T}(\mathbf{T}A \vee \mathbf{K}^{\downarrow}B) \rightarrow \mathbf{K}^{\downarrow}B)
\end{array}$$

$\mathsf{Reach} : \mathsf{Set} \to \mathsf{Set}$
$\mathsf{Reach}(S) = \text{The set of neurons graph-reachable from } S$
$[\![\mathbf{K}A]\!] = \mathsf{Reach}([\![A]\!])$

$\mathsf{Reach}^{\downarrow} : \mathsf{Set} \to \mathsf{Set}$
$\mathsf{Reach}^{\downarrow}(S) = \text{The set of neurons that graph-reach some } n \text{ in } S$
$[\![\mathbf{K}^{\downarrow}A]\!] = \mathsf{Reach}^{\downarrow}([\![A]\!])$

$\mathsf{Prop} : \mathsf{Set} \to \mathsf{Set}$
$\mathsf{Prop}(N) = \text{The set of neurons activated by } S$
$[\![\mathbf{T}A]\!] = \mathsf{Prop}([\![A]\!])$

$\mathsf{Inc}^{*} : \mathsf{Net} \times \mathsf{Set} \to \mathsf{Set}$
$\mathsf{Inc}^{*}(\mathcal{N}, S) =$ The net obtained by **maximally** strengthening all weights within $\mathsf{Prop}(S)$
$[\![[A]^{*}B]\!]_{\mathcal{N}} = [\![B]\!]_{\mathsf{Inc}(\mathcal{N},[\![A]\!])}$

$$\begin{aligned}
[\![p]\!] &= \text{some } S_p \text{ in } \mathsf{Set} \\
[\![\neg A]\!] &= [\![A]\!]^{\mathsf{C}} \\
[\![A \to B]\!] &= \text{“}[\![A]\!] \supseteq [\![B]\!]\text{”} \\
\\
[\![\mathbf{K}A]\!] &= \mathsf{op}([\![A]\!])
\end{aligned}$$

$^{*}\mathsf{Set} = \mathcal{P}(N)$

*Officially, $[\![A \to B]\!] = [\![A]\!]^{\mathsf{C}} \cap [\![B]\!]$

$$\mathcal{N} \longrightarrow \mathcal{M}$$

$$\langle N, E, W, A, [\![p]\!]_{\mathcal{N}} \rangle \longrightarrow \langle W, f_{\mathbf{K}}, f_{\mathbf{T}}, [\![p]\!]_{\mathcal{M}} \rangle$$

$$\begin{aligned}
&W = N \\
&f_{\mathbf{K}}(w) = \{ S \subseteq W \mid w \in \mathsf{Reach}(S) \} \\
&f_{\mathbf{T}}(w) = \{ S \subseteq W \mid w \in \mathsf{Prop}(S) \} \\
&[\![p]\!]_{\mathcal{M}} = [\![p]\!]_{\mathcal{N}}
\end{aligned}$$

$$\mathcal{M} \longrightarrow \mathcal{N}$$

$$\langle W, f_{\mathbf{K}}, f_{\mathbf{T}}, [\![p]\!]_{\mathcal{M}} \rangle \longrightarrow \langle N, E, W, A, [\![p]\!]_{\mathcal{N}} \rangle$$

$$\begin{aligned}
&N = W \\
&(m_i, n) \in E \text{ iff } n \in \bigcap_{X \in f(m)} X \\
&W(m_i, n) = \text{arbitrary} \\
&A^{(n)}(\vec{x}, \vec{w}) = 1 \text{ iff } \{ m_i \mid x_i = 1 \} \in g(n) \\
&[\![p]\!]_{\mathcal{N}} = [\![p]\!]_{\mathcal{M}}
\end{aligned}$$

DEFINITION 1. Let $\mathcal{M}$ be a model based on preferential filter $\mathcal{F} = \langle W, f, g \rangle$. Its **simulation net** $\mathcal{N}^{\bullet} = \langle N, E, W, A, O, V \rangle$ is the BFNN given by:

- $N = W$
- $(u, v) \in E$ iff $u \in \cap f(v)$

Now let $m_1, \ldots, m_k$ list those nodes such that $(m_i, n) \in E$.

- $W(m_i, n) =$ [Todo: Currently, the weights are completely arbitrary! Notice that they aren't even being used in $A^{(n)}(\vec{x}, \vec{w})$. Maybe this is the source of our problems?]
- $A^{(n)}(\vec{x}, \vec{w}) = 1$ iff $\{m_i | x_i = 1\}^{\complement} \notin g(n)$
- $V_{\mathcal{N}^{\bullet}}(p) = V_{\mathcal{M}}(p)$