

Curie-Weiss model

This is a model of interacting components where it is possible to use large deviation theory to understand the collective behavior of the system. Fix $\beta > 0$ and let $\mathcal{A}_n = \{-1, 1\}^n$ and consider the probability measure P_n over \mathcal{A}_n given by

$$P_n(\sigma_1, \dots, \sigma_n) = \frac{1}{2^n} \frac{e^{H_n(\sigma)}}{Z_n}, \quad (\sigma_1, \dots, \sigma_n) \in \mathcal{A}_n$$

where

$$H_n(\sigma) = \frac{1}{n} \sum_{i,j=1}^n \sigma_i \sigma_j$$

and where Z_n is a normalization constant such that

$$P_n(\mathcal{A}_n) = \sum_{(\sigma_1, \dots, \sigma_n) \in \mathcal{A}_n} P_n(\sigma_1, \dots, \sigma_n) = 1$$

that is

$$Z_n = \sum_{(\sigma_1, \dots, \sigma_n) \in \mathcal{A}_n} \frac{\exp(H_n(\sigma))}{2^n}.$$

Observe that $H_n(\sigma) = n F(M_n(\sigma))$ where $F(x) = x^2$ and $M_n(\sigma) = (\sigma_1 + \dots + \sigma_n)/n$ is the empirical mean of the random variables $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$. Let \mathbb{P}_n be the law of M_n under the probability measure P_n . Observe that $M_n: \mathcal{A}_n \rightarrow [-1, 1]$ so that \mathbb{P}_n is a probability over $[-1, 1]$ for all $n \geq 1$ and that the law \mathbb{P}_n is defined by

$$\int_{[-1, 1]} f(x) \mathbb{P}_n(dx) = \sum_{(\sigma_1, \dots, \sigma_n) \in \mathcal{A}_n} f(M_n(\sigma)) P_n(\sigma) = \sum_{(\sigma_1, \dots, \sigma_n) \in \mathcal{A}_n} f((\sigma_1 + \dots + \sigma_n)/n) \frac{\exp(H_n(\sigma))}{2^n Z_n}$$

- a) Prove that the family $\{\mathbb{P}_n\}_{n \geq 1}$ satisfy a large deviation principle on $[-1, 1]$ with rate function

$$I(x) = \frac{1+x}{2} \log(1+x) + \frac{1-x}{2} \log(1-x) + x^2 C$$

where the constant C is chosen such that $\inf_{x \in [-1, 1]} I(x) = 0$. Hint: Observe that the measure \mathbb{P}_n^0 can be written as

$$\int_{[-1, 1]} f(x) \mathbb{P}_n^0(dx) = \int_{[-1, 1]} f(x) \frac{e^{-nF(x)}}{Z_n} \mathbb{P}_n^0(dx)$$

where \mathbb{P}_n^0 is the law of M_n under P_n when $\beta = 0$, that is when each $\sigma_1, \dots, \sigma_n$ is an independent random variable such that $P_n(\sigma_i = 1) = 1/2$. So that by Cramers theorem $\{\mathbb{P}_n^0\}_{n \geq 1}$ satisfy a large deviation principle with suitable rate function. Use then Theorem 21 (Change of measure) of Poly 3.

- b) Prove that for $x \in]0, 1[$ the unique minimizer of the function $I: [-1, 1] \rightarrow \mathbb{R}_+$ is 0 and that if $\beta > 1$ then there exists a function $m(\beta)$ such that the function I has exactly two minima at the points $\pm m(\beta)$.

- c) Use the above results to prove that, when $\alpha < 1$ the family $\mathcal{P}_{q,q}$ weakly converges to the Dirac mass in 0 and that when $\alpha > 1$ it weakly converge to the probability measure

$$\frac{1}{2} \delta_{m(\cdot)} + \frac{1}{2} \delta_{-m(\cdot)}$$

that is the discrete measure that assign probability 1/2 to the points $\pm m(\cdot)$. Hint: when $\alpha < 1$ consider the probability $\mathcal{P}_{q,q}([-\epsilon, \epsilon])$ for some $\epsilon > 0$ and use the large deviation estimate to show that it goes to zero as $n \rightarrow \infty$. Use a similar strategy in the case $\alpha > 1$.

To understand fully this problem it is a good idea to give a look at the paper

<http://www.math.umass.edu/~rsellis/pdf-files/Les-Houches-lectures.pdf>

in particular Chapter 9.1.