

Large deviations for random walks

Let $(X_n)_{n \geq 1}$ be a sequence of iid Bernoulli(p) random variables. Consider the process $S_n = X_1 + \dots + X_n$ with $S_0 = 0$. Define a continuous random function ω_n on $[0, 1]$ by

$$\omega_n(t) = \frac{S_k}{n} + \frac{(S_{k+1} - S_k)(t - k/n)}{1/n} \quad \text{for } k/n \leq t < (k+1)/n.$$

Let K be the subset of $C([0, 1])$ such that $f \in K$ if and only if $f(0) = 0$ and $|f(t) - f(s)| \leq |t - s|$ for all $0 \leq s < t \leq 1$. Observe that $\omega_n \in K$ and that using the notations of the section Large deviations for processes of the 4th lecture notes of the course we have

$$\omega_n(t) = \int_0^t F_n(\omega_n)(s) ds$$

where for every n the vectors $X_n = (X_1, \dots, X_n)$ are random elements in $\{0, 1\}^n$ and

$$F_n(x_1, \dots, x_n)(s) = \sum_{i=1}^n x_i 1_{[(i-1)/n, i/n)}(s) \in L^\infty([0, 1]).$$

Observe also that $\omega_n(t)$ is a piecewise linear function for which

$$\omega_n(k/n) = S_k/n.$$

Recall that J_p has been defined in Poly 4 (section Large deviations for processes) as

$$J_p(x) = H(\text{Ber}(x)/\text{Ber}(p)) = x \log \frac{x}{p} + (1-x) \log \frac{1-x}{1-p}$$

the relative entropy of a Bernoulli law of parameter x with respect to the Bernoulli law of parameter p .

- Prove that the set K with the norm of uniform convergence is compact.
- Prove that the laws \mathbb{P}_n of ω_n on K satisfy a large deviation principle with rate function

$$I(f) = \int_0^1 J_p(f'(s)) ds$$

where $f'(s)$ is the derivative of $f \in K$ (which exists almost everywhere since f is Lipschitz). Hint: use Theorem 8 (Mogulskii theorem) of Poly 4 and the contraction principle (Thm 19 of Poly 3).

- With $p = 1/2$, use the result of point b) to prove that, if we set

$$\rho_n = \mathbb{P} \left(\frac{S_k}{n} \leq \frac{1}{2} - \frac{k}{n} \right)^2 \quad \text{for all } k = 1, \dots, n$$

then

$$\limsup_{n \rightarrow +\infty} (\rho_n)^{1/n} \leq \frac{e}{2} \quad \text{as } 0 < \epsilon < 1.$$

Hint: use Proposition 10 of Poly 3.

To understand fully this problem it is a good idea to give a look at the paper

<http://www.math.tau.ac.il/~tsirel/Courses/LargeDev/lect5.pdf>

in particular Section 5a.