1 Producing antichain familes from suitable orders

Definition 1. Let $n \geq 1$. We write $[n]$ for $\{1, \ldots, n\}$. An *n-family* is a family of sets

$$S = (S_1, \ldots, S_n)$$

For a family S, we write $s_{i,j}$ for $|S_i \cup S_j|$. Note that $s_{i,j}$ is a number, not a set. We also write s_i for $s_{i,i}$. For a family T, we would of course use the notation $t_{i,j}$.

S is an *antichain family* if whenever $i \neq j$, S_i is not a subset of S_j.

The *size class* of (i, j) is $\{(a, b) : s_{a,b} = s_{i,j}\}$.

We write $\mathcal{C}$ for the set of all n-families S of sets

Definition 2. Let $n \geq 1$. We define sets Diag(n), Off-diag(n), and Pairs(n) as follows:

$$\begin{array}{lcl} \text{Diag}(n) & = & \{(i,j) \in [n]^2 : i = j\} \\ \text{Off-diag}(n) & = & \{(i,j) \in [n]^2 : i < j\} \\ \text{Pairs}(n) & = & \text{Diag}(n) \cup \text{Off-diag}(n) \\ m & = & |\text{Pairs}(n)| \end{array}$$

Observe that $|\text{Diag}(n)| = n$, $|\text{Off-diag}(n)| = \binom{n}{2}$, so that $m = |\text{Pairs}(n)| = \binom{n+1}{2}$.

Frequently we drop the n and just write Diag, Off-diag, and Pairs.

Definition 3. A *suitable linear pre-order* on Pairs(n) is a relation $\preceq$ such that

1. $\preceq$ is reflexive and transitive.
2. For all $(i, j), (k, l) \in \text{Pairs}(n)$, either $(i, j) \preceq (k, l)$ or $(k, l) \prec (i, j)$, where

$$(i,j) \prec (k,l) \quad \text{iff} \quad (i,j) \preceq (k,l) \text{ but not } (k,l) \preceq (i,j)$$

3. For $i < j$, $(i, i) \prec (i, j)$.

We write $(i, j) \equiv (k, l)$ if $(i, j) \preceq (k, l) \preceq (i, j)$.

Example 1. For any family S of sets, the relation $\preceq$ is suitable, where $\leq$ is defined by

$$(i,j) \preceq (k,l) \quad \text{iff} \quad s_{i,j} \leq s_{k,l}$$

Theorem 1. Let $\preceq$ be a suitable linear preorder. Then there is a family of sets S such that for all (i, j), $(k, l) \in \text{Pairs}(n)$,

$$(i,j) \preceq (k,l) \quad \text{iff} \quad s_{i,j} \leq s_{k,l}. \tag{1}$$

This representation theorem is tantamount to the completeness of the associated logical system.

1.1 The Clamp Construction on Families of Sets

Let S be a family, and let $(i,j) \in \mathrm{Pairs}(n)$. Let $r \in \omega$. We define a new family

$$\mathrm{Clamp}(S,i,j,r)$$

from S, i, j, and r, as follows: $*_1, \ldots, *_r$ be fresh points. For $a \in [n]$, let

$$\mathrm{Clamp}(S,i,j,p)_a \quad = \quad \begin{cases} S_a \cup \{*_1, \ldots, *_r\} & \text{if } a \neq i \text{ and } a \neq j \\ S_a & \text{if } a = i \text{ or } a = j \end{cases}$$

In words, we are adding r new points simultaneously to all sets S_a, except for S_i and S_j. In words, we clamp S_i and S_j, and raise all other sets by simultaneously adding points to them. Note that the clamping raises all unions, except for $S_i \cup S_i$, $S_j \cup S_j$, and $S_i \cup S_j$.

Proposition 1. Let S be a family on n, and fix $i,j \in [n]$ and $r \in \omega$. Write T for $\mathrm{Clamp}(S,i,j,r)$.

1. For $(a,b) \notin \{(i,j),(i,i),(j,j)\}$, $t_{a,b} = s_{a,b} + r$.
2. For $(a,b) \in \{(i,j),(i,i),(j,j)\}$, $t_{a,b} = s_{a,b}$.
3. For all $(a,b),(c,d)$ in
$$\mathrm{Pairs}(n) \setminus \{(i,j),(i,i),(j,j)\}$$
we have
$$s_{a,b} - s_{c,d} = t_{a,b} - t_{c,d}.$$
4. If $s_{k,l}, s_{m,n} > s_{i,j}$, then $s_{m,n} \leq s_{k,l}$ iff $t_{m,n} \leq t_{k,l}$.

1.2 Equalizing the sizes of some pairs in a family

Lemma 2. Let S be a family. Let $k \geq 2$, and let

$$p_1 = (a_1,b_1), \ldots, p_k = (a_k,b_k)$$

be pairs, and assume that if $p_n = (a_n,b_n)$, then neither (a_n,a_n) nor (b_n,b_n) is not on the list $p_1,\ldots,p_k$. Then there is a family T such that

1. $t_{a_1,b_1} = t_{a_2,b_2} = \cdots = t_{a_k,b_k}$.
2. If (k,l) and (m,n) are any pairs with $s_{k,l}, s_{m,n} > \max_n s_{a_n,b_n}$, then
$$t_{k,l} \leq t_{m,n} \text{ if and only if } s_{k,l} \leq s_{m,n}.$$

Proof. Reorder the given pairs so that

$$s_{a_1,b_1} \leq s_{a_2,b_2} \leq \cdots \leq s_{a_k,b_k}.$$

Let

$$\begin{array}{rcl} T^1 & = & \mathrm{Clamp}(S, a_2, b_2, s_{a_2,b_2} - s_{a_1,b_1}) \\ T^2 & = & \mathrm{Clamp}(T^1, a_3, b_3, s_{a_3,b_3} - s_{a_2,b_2}) \\ & \vdots & \\ T^{k-1} & = & \mathrm{Clamp}(T^{k-2}, a_k, b_k, s_{a_k,b_k} - s_{a_{k-1},b_{k-1}}) \end{array}$$

Let $T = T^{k-1}$

To save on a lot of notation, let us write s_i for s_{a_i,b_i} and similarly for t_i^j.

An induction on $1 \leq i \leq k-1$ shows that for $1 \leq j \leq i+1$,

$$s_{i+1} = t_1^i = t_i^i = \cdots t_i^i = t_{i+1}^i. \tag{2}$$

For $i = 1$, recall that $s_1 \leq s_2$. Also, $t_1^1 = s_1 + (s_2 - s_1) = s_2$. Moreover, $t_2^1 = s_2$, since the definition of T^1 uses Clamp at p_2.

Assume (2) for i. Let $1 \leq j \leq i+1$. Then

$$t_j^{i+1} = s_{i+1} + (s_{i+2} - s_{i+1}) = s_{i+2}$$

Also, $t_{i+2}^{i+1} = s_{i+2}$ since T^{i+1} uses Clamp at p_{i+2}.

Taking $i = k-1$ in (2) proves our result. □

1.3 Making a list of pairs larger than all $\prec$-predecessors of it

Lemma 3. Let S be a family, and let $q_1 \preceq q_2 \preceq \cdots \preceq q_k$ be a sequence from Pairs(n). Then there is a family T such that

1. For $1 \leq i, j \leq k$, $s_i \leq s_j$ iff $t_i \leq t_j$.
2. for all pairs $p \prec q_1$, $t_p < t_1 = t_{q_i}$ for all i.

Proof. Let $m = \min_i s_i = \min_i s_{q_i}$. We call a pair p a *size competitor* if $p \prec q_1$ and $t_p \geq m$.

List the size competitors as $p_1, \ldots, p_k$. Note that if $p_i = (a_i, b_i)$, then none of the original points q_j are (a_i, a_i) or (b_i, b_i) or p_i. This is because $p_i \prec q_j$, and $(a_i, a_i), (b_i, b_i) \prec p_i$. (Recall Definition 3.) Let

$$\begin{array}{rcl} T^1 & = & \mathrm{Clamp}(S, p_1, s_{p_1} - m + 1) \\ T^2 & = & \mathrm{Clamp}(T^1, p_2, s_{p_2} - m + 1) \\ & \vdots & \\ T^k & = & \mathrm{Clamp}(T^{k-1}, p_k, s_{p_k} - m + 1) \end{array}$$

Let $T = T^k$. We claim that the original points q_j have $t_{q_j} > t_p$ for all $p \prec q_1$. The reason is that

$$t_{q_j} = s_{q_j} + (s_{p_1} - m + 1) + (s_{p_2} - m + 1) + \cdots + (s_{p_k} - m + 1)$$

On the other hand, for one of the size competitors, say p_i

$$\begin{array}{rcl} t_{p_i} & = & s_{p_i} + (s_{p_1} - m + 1) + \cdots + (s_{p_{i-1}} - m + 1) + (s_{p_{i+1}} - m + 1) + \cdots + (s_{p_k} - m + 1) \\ & > & \end{array}$$

That is, p_i is clamped as we move from T^{i-1} to T^i. The upshot is that

$$t_{q_j} - t_{p_i} \geq s_{q_j} + (s_{p_i} - m + 1) - s_{p_i} = s_{q_j} - m + 1 \geq s_{q_j} - s_{q_i} + 1 = 1.$$

The reason that the first $\geq$ is not an equals sign $=$ is that it may be the case that p_i is of the form (a, a) and some other $p_{i'}$ is (a, b) for some b. At the end, we used the fact that $m \leq s_{q_i}$. And so $t_{q_j} > t_{p_i}$.

For all $p \prec q_1$ which are not size competitors, the calculations are easier. For such p, $s_p < s_{q_j}$ for all j. So we get $t_{q_j} - t_p \geq s_{q_j} - s_p > 0$.

This completes the proof. □

2 Algorithm

We prove Theorem 1 by designing an algorithm which represents a suitable linear preorder $\preceq$ by a family of sets. We construct the family using Lemmas 2 and 3 on each of the size classes of $\preceq$.

In more detail, consider the given orderng $\preceq$. A *size class* is a set of $\equiv$ pairs p. We list

2.1 Example

Let $n = 9$, and let $\prec$ have size classes as shown in lists below:

$[(5,5), (6,6)]$
$[(5,6), (4,4), (7,7)]$
$[(7,4), (4,5), (2,2), (1,1), (0,0), (8,8), (3,3)],$
$[(3,2), (2,1), (3,1), (7,0), (3,0), (2,0)]$
$[(1,0), (4,0), (7,1), (7,2), (8,2), (8,1), (8,3), (8,7)]$
$[(7,3), (7,5), (7,6), (4,1), (4,2), (4,3), (8,6)]$
$[(6,0), (6,1), (6,2), (6,3), (5,1), (8,5), (8,4)]$
$[(6,4), (5,0), (5,2), (5,3)]$

We illustrate with step 6. We begin with a family S with cardinalities as shown below.

$$\begin{array}{l}
|S[0]| = 65 \\
|S[1]| = 63 \\
|S[2]| = 63 \\
|S[3]| = 64 \\
|S[4]| = 64 \\
|S[5]| = 70 \\
|S[6]| = 71 \\
|S[7]| = 60 \\
|S[8]| = 68 \\
\hline
|S[5] \cup S[5]| = 70 \\
|S[6] \cup S[6]| = 71
\end{array}$$

$$\begin{array}{l}
|S[5] \cup S[6]| = 79 \\
|S[4] \cup S[4]| = 64 \\
|S[7] \cup S[7]| = 60
\end{array}$$

$$\begin{array}{l}
|S[7] \cup S[4]| = 79 \\
|S[4] \cup S[5]| = 79 \\
|S[2] \cup S[2]| = 63 \\
|S[1] \cup S[1]| = 63 \\
|S[0] \cup S[0]| = 65 \\
|S[8] \cup S[8]| = 68 \\
|S[3] \cup S[3]| = 64
\end{array}$$

$$\begin{array}{l}
|S[3] \cup S[2]| = 80 \\
|S[2] \cup S[1]| = 80 \\
|S[3] \cup S[1]| = 80 \\
|S[7] \cup S[0]| = 80 \\
|S[3] \cup S[0]| = 80 \\
|S[2] \cup S[0]| = 80
\end{array}$$

$$\begin{array}{l}
|S[1] \cup S[0]| = 81 \\
|S[4] \cup S[0]| = 81 \\
|S[7] \cup S[1]| = 81 \\
|S[7] \cup S[2]| = 81 \\
|S[8] \cup S[2]| = 81 \\
|S[8] \cup S[1]| = 81 \\
|S[8] \cup S[3]| = 81 \\
|S[8] \cup S[7]| = 81
\end{array}$$

$$\begin{array}{l}
|S[7] \cup S[3]| = 82 \\
|S[7] \cup S[5]| = 82 \\
|S[7] \cup S[6]| = 82 \\
|S[4] \cup S[1]| = 82 \\
|S[4] \cup S[2]| = 82 \\
|S[4] \cup S[3]| = 82 \\
|S[8] \cup S[6]| = 82
\end{array}$$

$$\begin{array}{l}
|S[6] \cup S[0]| = 83 \\
|S[6] \cup S[1]| = 83 \\
|S[6] \cup S[2]| = 83 \\
|S[6] \cup S[3]| = 83 \\
|S[5] \cup S[1]| = 83 \\
|S[8] \cup S[5]| = 83 \\
|S[8] \cup S[4]| = 83
\end{array}$$

$$\begin{array}{l}
|S[6] \cup S[4]| = 84 \\
|S[5] \cup S[0]| = 84 \\
|S[5] \cup S[2]| = 84 \\
|S[5] \cup S[3]| = 84
\end{array}$$

Step 6 concerns the sixth size class, starting from the highest one. So we are working on the size class

$$(7,4),(4,5),(2,2),(1,1),(0,0),(8,8),(3,3)$$

The first step is to equalize the sizes in this class, using Lemma 2. We reorder this in size order in our family above, obtaining

$$(1,1),(2,2),(3,3),(0,0),(8,8),(7,4),(4,5)$$

We therefore calculate:

$$\begin{array}{rcl}
T^1 & = & \mathrm{Clamp}(S,(2,2),63-63) \\
T^2 & = & \mathrm{Clamp}(T^1,(3,3),64-63) \\
T^3 & = & \mathrm{Clamp}(T^2,(0,0),65-64) \\
T^4 & = & \mathrm{Clamp}(T^3,(8,8),68-65) \\
T^5 & = & \mathrm{Clamp}(T^4,(7,4),79-69) \\
T^6 & = & \mathrm{Clamp}(T^5,(4,5),79-79)
\end{array}$$

We use T^6.

After equalizing, we get

$$
\begin{aligned}
|S[0]| &= 103 \\
|S[1]| &= 103 \\
|S[2]| &= 103 \\
|S[3]| &= 103 \\
|S[4]| &= 72 \\
|S[5]| &= 94 \\
|S[6]| &= 111 \\
|S[7]| &= 84 \\
|S[8]| &= 103
\end{aligned}
$$

$$
\begin{aligned}
|S[5] \cup S[5]| &= 94 \\
|S[6] \cup S[6]| &= 111
\end{aligned}
$$

$$
\begin{aligned}
|S[5] \cup S[6]| &= 119 \\
|S[4] \cup S[4]| &= 72 \\
|S[7] \cup S[7]| &= 84
\end{aligned}
$$

$$
\begin{aligned}
|S[7] \cup S[4]| &= 103 \\
|S[4] \cup S[5]| &= 103 \\
|S[2] \cup S[2]| &= 103 \\
|S[1] \cup S[1]| &= 103 \\
|S[0] \cup S[0]| &= 103 \\
|S[8] \cup S[8]| &= 103 \\
|S[3] \cup S[3]| &= 103
\end{aligned}
$$

$$
\begin{aligned}
|S[3] \cup S[2]| &= 120 \\
|S[2] \cup S[1]| &= 120 \\
|S[3] \cup S[1]| &= 120 \\
|S[7] \cup S[0]| &= 120 \\
|S[3] \cup S[0]| &= 120 \\
|S[2] \cup S[0]| &= 120
\end{aligned}
$$

$$
\begin{aligned}
|S[1] \cup S[0]| &= 121 \\
|S[4] \cup S[0]| &= 121 \\
|S[7] \cup S[1]| &= 121 \\
|S[7] \cup S[2]| &= 121 \\
|S[8] \cup S[2]| &= 121 \\
|S[8] \cup S[1]| &= 121 \\
|S[8] \cup S[3]| &= 121 \\
|S[8] \cup S[7]| &= 121
\end{aligned}
$$

$$
\begin{aligned}
|S[7] \cup S[3]| &= 122 \\
|S[7] \cup S[5]| &= 122 \\
|S[7] \cup S[6]| &= 122 \\
|S[4] \cup S[1]| &= 122 \\
|S[4] \cup S[2]| &= 122 \\
|S[4] \cup S[3]| &= 122 \\
|S[8] \cup S[6]| &= 122
\end{aligned}
$$

$$
\begin{aligned}
|S[6] \cup S[0]| &= 123 \\
|S[6] \cup S[1]| &= 123 \\
|S[6] \cup S[2]| &= 123 \\
|S[6] \cup S[3]| &= 123 \\
|S[5] \cup S[1]| &= 123 \\
|S[8] \cup S[5]| &= 123 \\
|S[8] \cup S[4]| &= 123
\end{aligned}
$$

$$
\begin{aligned}
|S[6] \cup S[4]| &= 124 \\
|S[5] \cup S[0]| &= 124 \\
|S[5] \cup S[2]| &= 124 \\
|S[5] \cup S[3]| &= 124
\end{aligned}
$$

Note that for classes above the classes of interest in this step, the sizes stay larger during the equalization.

At this point, the size competitors are $(5,6)$ and $(6,6)$. We want to make the sizes of the sets in our current size class larger than the sizes of $(5,6)$ and $(6,6)$. So we use Lemma 3. That is, we clamp $(5,6)$ and $(5,5)$, increasing all sets by one more than the difference of the sizes of those sets with 103, We get

$$\begin{array}{l}
|S[0]| = 120 \\
|S[1]| = 120 \\
|S[2]| = 120 \\
|S[3]| = 120 \\
|S[4]| = 89 \\
|S[5]| = 94 \\
|S[6]| = 111 \\
|S[7]| = 101 \\
|S[8]| = 120 \\
\hline
|S[5] \cup S[5]| = 94 \\
|S[6] \cup S[6]| = 111
\end{array}$$

$$\begin{array}{l}
|S[5] \cup S[6]| = 119 \\
|S[4] \cup S[4]| = 89 \\
|S[7] \cup S[7]| = 101
\end{array}$$

$$\begin{array}{l}
|S[7] \cup S[4]| = 120 \\
|S[4] \cup S[5]| = 120 \\
|S[2] \cup S[2]| = 120 \\
|S[1] \cup S[1]| = 120 \\
|S[0] \cup S[0]| = 120 \\
|S[8] \cup S[8]| = 120 \\
|S[3] \cup S[3]| = 120
\end{array}$$

$$\begin{array}{l}
|S[3] \cup S[2]| = 137 \\
|S[2] \cup S[1]| = 137 \\
|S[3] \cup S[1]| = 137 \\
|S[7] \cup S[0]| = 137 \\
|S[3] \cup S[0]| = 137 \\
|S[2] \cup S[0]| = 137
\end{array}$$

$$\begin{array}{l}
|S[1] \cup S[0]| = 138 \\
|S[4] \cup S[0]| = 138 \\
|S[7] \cup S[1]| = 138 \\
|S[7] \cup S[2]| = 138 \\
|S[8] \cup S[2]| = 138 \\
|S[8] \cup S[1]| = 138 \\
|S[8] \cup S[3]| = 138 \\
|S[8] \cup S[7]| = 138
\end{array}$$

$$\begin{array}{l}
|S[7] \cup S[3]| = 139 \\
|S[7] \cup S[5]| = 139 \\
|S[7] \cup S[6]| = 139 \\
|S[4] \cup S[1]| = 139 \\
|S[4] \cup S[2]| = 139 \\
|S[4] \cup S[3]| = 139 \\
|S[8] \cup S[6]| = 139
\end{array}$$

$$\begin{array}{l}
|S[6] \cup S[0]| = 140 \\
|S[6] \cup S[1]| = 140 \\
|S[6] \cup S[2]| = 140 \\
|S[6] \cup S[3]| = 140 \\
|S[5] \cup S[1]| = 140 \\
|S[8] \cup S[5]| = 140 \\
|S[8] \cup S[4]| = 140
\end{array}$$

$$\begin{array}{l}
|S[6] \cup S[4]| = 141 \\
|S[5] \cup S[0]| = 141 \\
|S[5] \cup S[2]| = 141 \\
|S[5] \cup S[3]| = 141
\end{array}$$

Note that it wasn't really necessary to clamp $(5,6)$ after we clamped $(5,5)$. So our algorithm does a bit of work that is not necessary. It could be elaborated to produce slightly smaller sets in the end. But it is correct.