An Introduction to Abelian Categories

Final Paper and Presentation

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December 2010

§ 1. Introduction:

Abelian categories are the most general categories in which one can develop homological algebra. The notion of an abelian category derives its motivation from the category $\mathfrak{Ab}$ of abelian groups. Abelian categories extract the fundamental properties of abelian groups in the category $\mathfrak{Ab}$. The idea of an abelian category was first introduced by Saunders MacLane, who also coined the term 'abelian category.' However, the modern axiomatic definition of an abelian category was given by Alexander Grothendieck, in his famous Tôhoku paper that laid the foundations of homological algebra. Grothendieck used abelian categories in his work on algebraic geometry, in which the category of sheaves over schemes is an important example of an abelian category. This is an expository article on abelian categories that describes the necessary background to abelian categories, culminating in a definition of an abelian category. Knowledge of basic category theory, including limits and colimits, is assumed throughout. Apart from this, I have tried to keep this article as self-contained as possible.

§ 2. Zero Objects and Zero Morphisms:

Definition 2.1: Let $\mathscr{C}$ be a category. An object $Z \in \mathrm{ob}\,\mathscr{C}$ is called a ***zero object*** in $\mathscr{C}$ iff it is both an initial and a terminal object, i.e.,

$$\forall\ A \in \mathrm{ob}\ \mathscr{C},\ \exists!\ \phi_A:\ A \to Z \text{ and } \exists!\ \varphi_A:\ Z \to A$$

Note that a zero object, if it exists, is unique up to an isomorphism since initial and terminal objects are unique up to isomorphisms.

Examples 2.1:

- In the category $\mathfrak{Grp}$ of groups, a trivial group is a zero object. Similarly, in $\mathfrak{Ab}$, a trivial group is a zero object. Therefore, $\mathfrak{Grp}$ and $\mathfrak{Ab}$ have zero objects.
- Let $\mathfrak{pSet}$ denote the category of all pointed sets whose elements are ordered pairs (X, x) where X is a non-empty set and $x \in X$, and whose morphisms f: (X, x) → (Y, y) are functions f: X → Y such that $f(x) = y$. Then an ordered pair ({x}, x), where {x} is a singleton, is a zero object in $\mathfrak{pSet}$.
- In $\mathfrak{Sets}$, the category of sets, the empty set Ø is an initial object and singletons {x} are terminal objects. In fact, these are the only initial and terminal objects in $\mathfrak{Sets}$. Thus, $\mathfrak{Sets}$ does not have zero objects.

Definition 2.2: Let $\mathscr{C}$ be a category with zero objects. Then, ∀ objects A, B ∈ ob$\mathscr{C}$ and ∀ zero objects Z ∈ ob$\mathscr{C}$, a ***zero morphism*** from A → B, denoted $\mathbf{0^A{}_B: A \to B}$ is a morphism that makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[rr, "\mathbf{0^A{}_B}"] \arrow[dr, "\exists!\ \phi_A"'] & & B \\
& Z \arrow[ur, "\exists!\ \varphi_B"'] &
\end{tikzcd}$$

(Fig 1)

I.e., $0^A{}_B = \varphi_B \circ \phi_A$ where φ_B and ϕ_A are unique because Z is a zero object.

From this definition is would seem that a zero morphism from A → B depends not only on the choice of A and B, but also on the choice of our zero object Z. But, this is not the case. $0^A{}_B$ depends only the choice of A and B and is independent of Z (as proposition 2.3 will show), and so for any pair of objects in a category with zero objects, a zero morphism is unique. Hence, we can say that $0^A{}_B$ is ‘the’ zero morphism.

Examples 2.2:

- In $\mathfrak{Grp}$, for all groups G and H, $0^G{}_H$ is the homomorphism that maps every element of G to the identity element e_H in H. Similarly for $\mathfrak{Ab}$.
- Similarly, in $\mathfrak{pSet}$, for pointed sets (X, x) and (Y, y), $0^{(X, x)}{}_{(Y, y)}$ is the constant function that maps every element in X to the element $y \in Y$.

  Note: Every Group G is, in fact, a pointed set (G, e_G) where e_G is the identity in G. Every group homomorphism $\phi: G \to H$ maps $e_G \to e_H$. Thus, every group homomorphism $\phi: G \to H$ is a morphism of pointed sets $(G, e_G) \to (H, e_H)$.

Proposition 2.3: Let $\mathscr{C}$ be a category with zero objects. Then for objects $A, B \in \text{ob } \mathscr{C}$, the map $0^A{}_B$, as defined in Definition 2.2, is independent of the choice of our zero object Z.

Proof: Let Z, Y both be zero objects in $\mathscr{C}$. Then, by the definition of zero objects and by the definition of $0^A{}_B: A \to B$ we have the following diagram:

$$\begin{tikzcd}
B & & \\
& Z \arrow[ul, "\exists!\ \varphi_B"'] \arrow[r, "\exists!\ \varphi_Y"] & Y \arrow[ull, bend right, "\exists!\ \varphi_B'"'] \\
A \arrow[uu, "0^A{}_B"] \arrow[ur, "\exists!\ \phi_A"] \arrow[urr, bend right, "\exists!\ \phi_A'"'] & &
\end{tikzcd}$$

(Fig 2)

Now, $0^A{}_B = \varphi_B \circ \phi_A$. Since Y is a zero object, hence ϕ_A' and φ_B' are unique. Also, Z is a zero object and so φ_Y is unique. To show that $0^A{}_B$ is independent of the choice of zero objects, we need to show that $0^A{}_B = \varphi_B' \circ \phi_A'$.

By the uniqueness of ϕ_{A}' and φ_B we have:

$$\phi_{A}' = \varphi_Y \circ \phi_A \text{ and } \varphi_B = \varphi_B' \circ \varphi_Y.$$

Hence, $\varphi_B' \circ \phi_A' = \varphi_B' \circ (\varphi_Y \circ \phi_A) = (\varphi_B' \circ \varphi_Y) \circ \phi_A = \varphi_B \circ \phi_A = 0^A{}_B$. Thus $\varphi_B' \circ \phi_A' = \varphi_B \circ \phi_A$ and so $0^A{}_B$ is independent of the choice of our zero object.

□

Proposition 2.3 shows that $0^A{}_B$ is unique for objects A and B. Also, note that in a category $\mathscr{C}$ with zero objects, the zero morphism exists for any pair of objects (A, B) by its definition. Hence, $\forall$ A, B $\in$ ob $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B) \neq \varnothing$ since $0^A{}_B \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

We will now state some basic properties of zero morphisms in the following proposition.

Proposition 2.4: (We will not prove this proposition.) Let $\mathscr{C}$ be a category with zero objects. Then $\forall$ A, B, C, D $\in$ ob $\mathscr{C}$ and $\forall\, f \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, we have the following identities:

(i) $f \circ 0^A{}_B = 0^A{}_C$

(ii) $0^C{}_D \circ f = 0^B{}_D$

I.e., the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "0^A{}_B"] \arrow[dr, "0^A{}_C"'] & B \arrow[d, "f"] \arrow[dr, "0^B{}_D"] & \\
 & C \arrow[r, "0^C{}_D"'] & D
\end{tikzcd}$$

(Fig 3)

□

Thus, composition with zero morphisms yields zero morphisms.

From now on we will denote zero maps by just 0.

§ 3. Kernels and co-kernels:

Definition 3.1: Let $\mathscr{C}$ be a category with zero objects. Then, $\forall$ A, B $\in$ ob$\mathscr{C}$ and

$\forall f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$,

- a ***kernel*** of f: A → B is an object **ker*f*** $\in$ ob $\mathscr{C}$ with a map $\boldsymbol{k}$**: ker*f* → A** that equalizes the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

(Fig 4)

- a ***cokernel*** of f: A → B is an object **coker*f*** $\in$ ob $\mathscr{C}$ with a map $\boldsymbol{c}$**: B → coker*f*** that coequalizes the above diagram.

It is clear that kernels and cokernels are unique up to isomorphisms. Hence, the notations ker$f$ and coker$f$ make sense. Note that kernels and cokernels need not exist for all morphisms f: A → B.

As a result of Definition 3.1 we have:

$$f \circ k = 0 \text{ and } c \circ f = 0.$$

An alternate, but completely equivalent way to define kernels and cokernels (we will not show the equivalence of the two definitions) is the following:

Alternate definition of kernel and cokernel: Given a category $\mathscr{C}$ with zero objects,

- a ***kernel*** of f: A → B (if it exists) is a morphism k: A′ → A such that $f \circ k = 0$
- a ***cokernel*** of f: A → B (if it exists) is a morphism c: B → B′ such that $c \circ f = 0$.

If and when needed we will interchange between these two definitions.

We know that every equalizer is a monomorphism and every coequalizer is an epimorphism. Hence, kernels are monomorphisms and cokernels are epimorphisms. This fact will be useful later on in § 5 when we define the parallel morphism.

Examples 3.1:

- In $\mathfrak{Grp}$, every homomorphism $\phi: G \to H$ has a kernel, namely the kernel of the homomorphism $\ker\phi \subset G$ and the morphism associated with $\ker\phi$ is just the inclusion $\acute{\imath}: \ker\phi \to G$. We know that $\mathrm{im}\phi < H$. Let N be the smallest normal subgroup of H that contains $\mathrm{im}\phi$. Then, the cokernel of f is the quotient H/N and the associated morphism is just the projection $\pi: H \to H/N$ that maps $h \to hN$.
- In $\mathfrak{Ab}$, the kernel of an abelian group homomorphism $\varphi: G \to H$ is, again, just $\ker\varphi < G$. Since every subgroup of an abelian group is normal, hence, the cokernel is $H/\mathrm{im}\varphi$.

§ 4. Preadditive and Additive Categories:

Often in a category $\mathscr{C}$, the Hom-sets themselves have special structure. For example, the Hom-sets may have the structure of a group. A preadditive category is one in which the Hom-sets have the structure of an abelian group.

Definition 4.1: A category $\mathscr{C}$ is called a ***preadditive category*** iff $\forall$ A, B $\in$ ob $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ has the structure of an abelian group and composition of morphisms is bilinear, i.e.,
$\forall\ f, f' \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, $\forall\ \Upsilon \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, and $\forall\ \Upsilon' \in \mathrm{Hom}_{\mathscr{C}}(C, D)$,

$$(f + f') \circ \Upsilon = f \circ \Upsilon + f' \circ \Upsilon'$$

and

$$\Upsilon' \circ (f + f') = \Upsilon' \circ f + \Upsilon' \circ f'$$

Note the + comes from the abelian group structure on the Hom-sets.

Examples 4.1:

- $\mathfrak{Ab}$ is a preadditive category. The natural abelian group structure that can be given to $\mathrm{Hom}_{\mathfrak{Ab}}(A, B)$ is the following: $\forall\ f, f' \in \mathrm{Hom}_{\mathfrak{Ab}}(A, B)$, define $f + f': A \to B$ to be the morphism that maps $x \in A \to f(x) + f'(x)$. It can be easily verified that $f + f'$ is actually a group homomorphism, and that with this binary operation, $\mathrm{Hom}_{\mathfrak{Ab}}(A, B)$ is actually an abelian group. The identity of $\mathrm{Hom}_{\mathfrak{Ab}}(A, B)$ is just $0^A{}_B$. The bilinearity of composition follows from the definition of addition given above, and by the definition of a group homomorphism.

- The category $\mathcal{R}\text{-}\mathcal{mod}$ of left R-modules is, again, a pre-additive category. Since an R-module homomorphism is also a homomorphism of abelian groups, hence, a construction similar to the one given before gives the structure of an abelian group to the Hom-sets.
- The category $\mathfrak{Vect}_{\mathfrak{k}}$ is a preadditive category. It can be easily shown that $\mathrm{Hom}_{\mathfrak{Vect}\mathfrak{k}}(V,W)$ where V, W are vector spaces, is itself a vector space over $\mathfrak{k}$. Hence, in particular, $\mathrm{Hom}_{\mathfrak{Vect}\mathfrak{k}}(V,W)$ is an abelian group. Bilinearity of composition follows from the definition of linear maps and the definition of addition on the vector space $\mathrm{Hom}_{\mathfrak{Vect}\mathfrak{k}}(V,W)$.

Definition 4.2: A category $\mathscr{C}$ is called an ***additive category*** iff it satisfies the following axioms:

(1) $\mathscr{C}$ is a pre-additive category, i.e., every Hom-set has an abelian group structure and composition is bilinear.

(2) $\mathscr{C}$ has zero objects.

(3) $\mathscr{C}$ has finite products and co-products.

Note that both (1), (2) $\Rightarrow$ Every Hom-set is non-empty.

Notation: Instead of denoting the coproduct of A and B as $A \coprod B$, in an additive category the co-product is denoted as $A \oplus B$, which shows that it has connections with the notion of direct sum.

Examples 4.2:

- The category $\mathfrak{Ab}$ of abelian groups is an additive category. We have already shown that $\mathfrak{Ab}$ is preadditive. $\mathfrak{Ab}$ has zero objects, namely the trivial group. Also, products and co-products in $\mathfrak{Ab}$ coincide. They are just the direct product of groups.
- $\mathcal{R}\text{-}\mathcal{mod}$ is an additive category.

- $\mathfrak{Vect}_k$ is an additive category.

§ 5. Parallel morphism:

Let $\mathscr{C}$ be a category with zero objects and in which every morphism has a kernel and a cokernel.

Let f: A → B be a morphism in $\mathscr{C}$. Then, we have the following morphisms:

$$\begin{tikzcd}
\ker f \arrow[r, "k"] & A \arrow[r, "f"] \arrow[d, "c'"'] & B \arrow[r, "c"] & \operatorname{coker} f \\
& \operatorname{coker} k & \ker c \arrow[u, "k'"'] &
\end{tikzcd}$$

(Fig 5)

Define

coim$f$:= coker k and **im$f$:= ker c**

So, we have

$$\begin{tikzcd}
\ker f \arrow[r, "k"] & A \arrow[r, "f"] \arrow[d, "c'"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "c"] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[ur, "\exists!\, \tilde{f}"] \arrow[r, dashed, "\exists!\, \phi"'] & \operatorname{im} f \arrow[u, "k'"'] \arrow[ul, phantom, "\circlearrowright"] &
\end{tikzcd}$$

(Fig 6)

(The explanation for $\tilde{f}$ and ϕ is given on the next page.)

Now, $f \circ k = 0$ (by definition of the kernel). Thus, by the universal property of the cokernel, $\exists!\ \tilde{f}$: coim$f \to$ B such that $f = \tilde{f} \circ c'$. Now, c': A $\to$ coim$f$ is a coequalizer. Hence, by discussion in § 3, page 5, c': A $\to$ coim$f$ is an epimorphism. By definition of the cokernel, $c \circ f = 0 \quad \Rightarrow c \circ (\tilde{f} \circ c') = 0 \quad \Rightarrow (c \circ \tilde{f}) \circ c' = 0 = 0 \circ c'$ (By Proposition 2.4) $\quad \Rightarrow \quad c \circ \tilde{f} = 0$ (since c' is an epimorphism). Thus, by the universal property of the kernel, $\exists!\ \phi$: coim$f \to$ im$f$ such that $\tilde{f} = k' \circ \phi$. The morphism ϕ is defined as the ***morphism parallel to f*** since one can see in Fig 6 that the morphisms f and ϕ are literally parallel to each other.

§ 6. Abelian Categories:

Definition 6.1: A category $\mathscr{C}$ is called an ***abelian category*** iff it satisfies the following axioms:

(AI) $\mathscr{C}$ is an additive category.

(AII) Every morphism f: A $\to$ B in $\mathscr{C}$ as a kernel and a cokernel.

(AIII) Given any morphism f: A $\to$ B, the morphism ϕ: coim$f \to$ im$f$ parallel to f (as defined in § 5) is an isomorphism, i.e., coim$f \cong$ im$f$. Hence,

coker(ker(f: A $\to$ B)) $\cong$ ker(coker(f: A $\to$ B)).

Examples 6.1:

- $\mathfrak{Ab}$ is an abelian category. It has already been shown that $\mathfrak{Ab}$ satisfies (AI). It was also shown in example 3.1 that $\mathfrak{Ab}$ satisfies (AII). The kernel of any homomorphism ϕ^*: G $\to$ H is just ker$\phi^* <$ G with associated morphism ι: ker$\phi^* \to$ G (the inclusion map) and the cokernel is H/imϕ^* with associated morphism, the natural projection π: H $\to$ H/imϕ^*. The coimϕ^* is G/imι = G/kerϕ^* with natural projection π^*: G $\to$ G/kerϕ^*. The imϕ^* is kerπ = imϕ^* with the inclusion map ι: im$\phi^* \to$ H. The first isomorphism theorem in group theory states that $\exists!$ isomorphism ϕ: G/ker$\phi^* \to$ imϕ^*, which is our parallel morphism from coim$\phi^* \to$ imϕ^* as can be easily verified from the definition of the parallel morphism given in § 5. Thus, $\mathfrak{Ab}$ satisfies (AIII). Hence, $\mathfrak{Ab}$ is an abelian category.
- $\mathcal{R}$-$\mathcal{mod}$ is an abelian category.
- $\mathfrak{Vect}_{\mathfrak{k}}$ is an abelian category.

Sources:
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