APPLICATIONS OF INTEGRATION

Prepared by Dr.Hany El Deeb

In this chapter we explore some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, the length of a curve .

(1)Area between two curves

From the definition of definite integral of a positive function as the area under the curve and above the x-axis limited by the two vertical lines at the end of the interval, the area = $\int_a^b f(x)\,dx$

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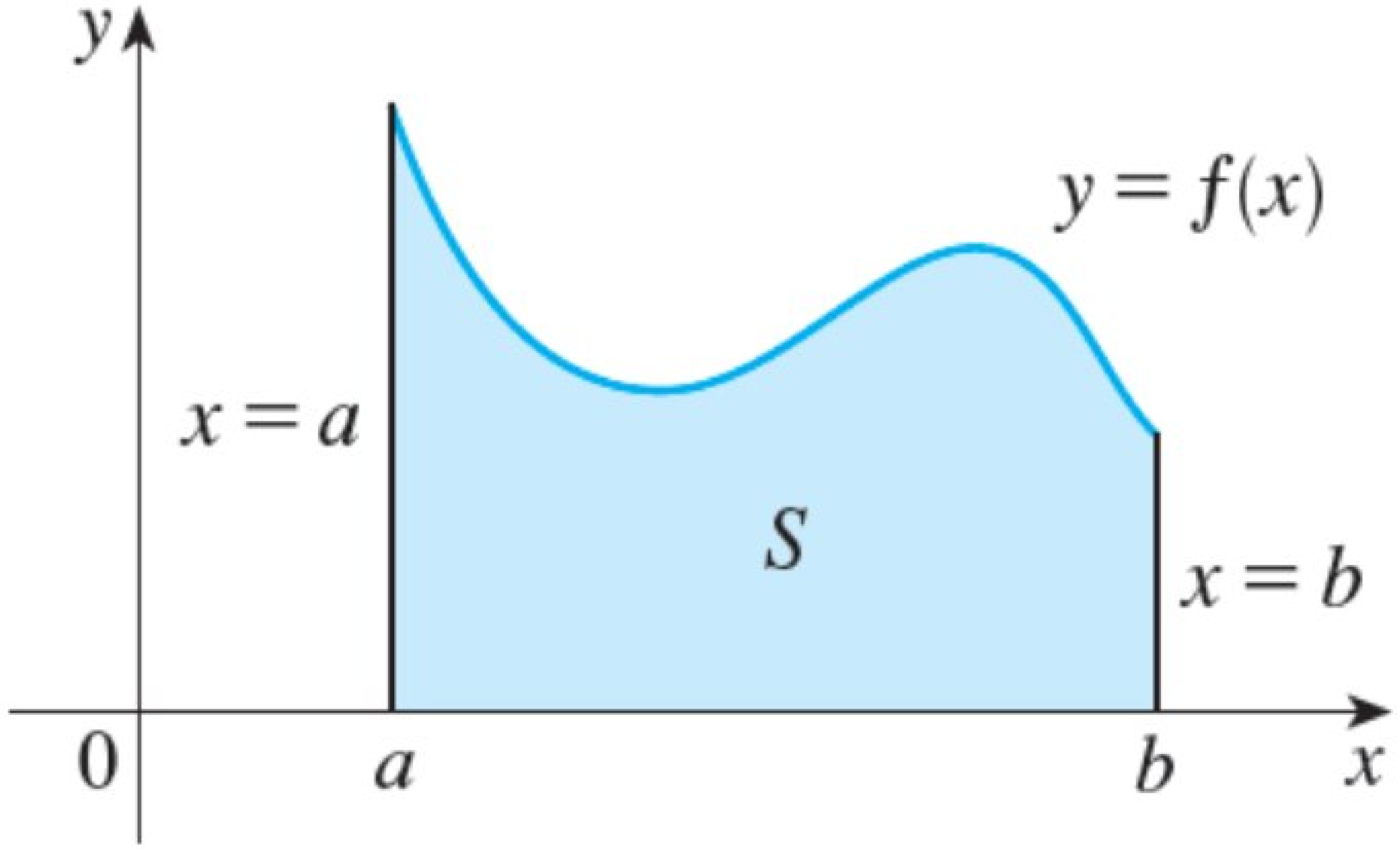

Example: **Find the area between the curve**

$y = 9 - x^2$ and $x - \text{axis}$ (above x-axis).

Solution **Let** $y = 9 - x^2 = 0 \Rightarrow x = \pm 3$

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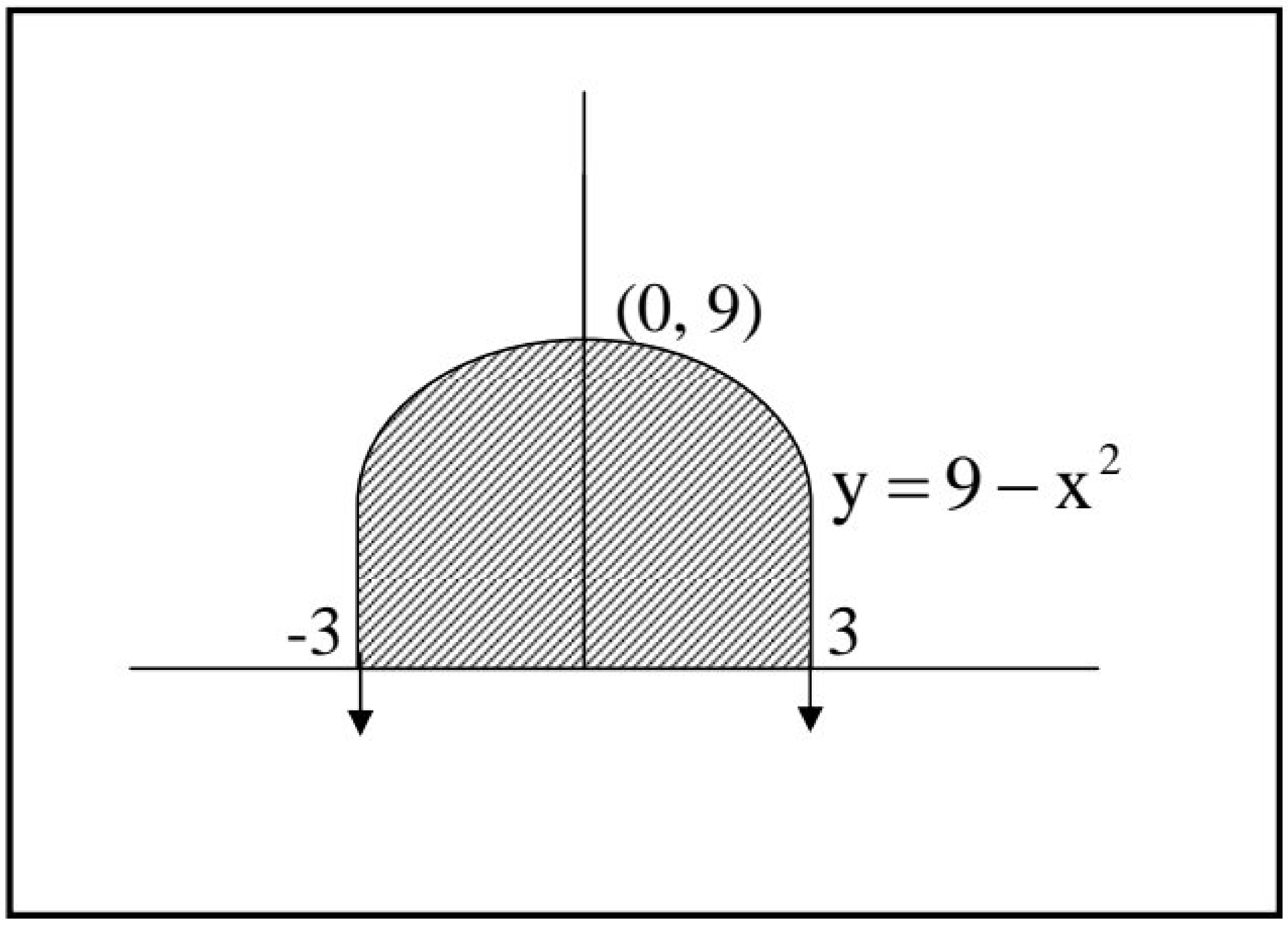

The curve intersect x-axis in ± 3. Then

$$\underset{\text{area}}{\underset{\downarrow}{A}} = \int_{-3}^{3} \left(9 - x^2\right) dx = 2\int_{o}^{3} \left(9 - x^2\right) dx = 2\left[9x - \frac{x^3}{x}\right]_o^3$$

$$= 2\left[27 - \frac{27}{3} - 0\right] = 36 \text{ square units.}$$

Example: **Find the area of the region bounded by the curves $y = e^{-x} + 1$, $x - \text{axis}$ and the two lines $x = 0, x = 2$.**

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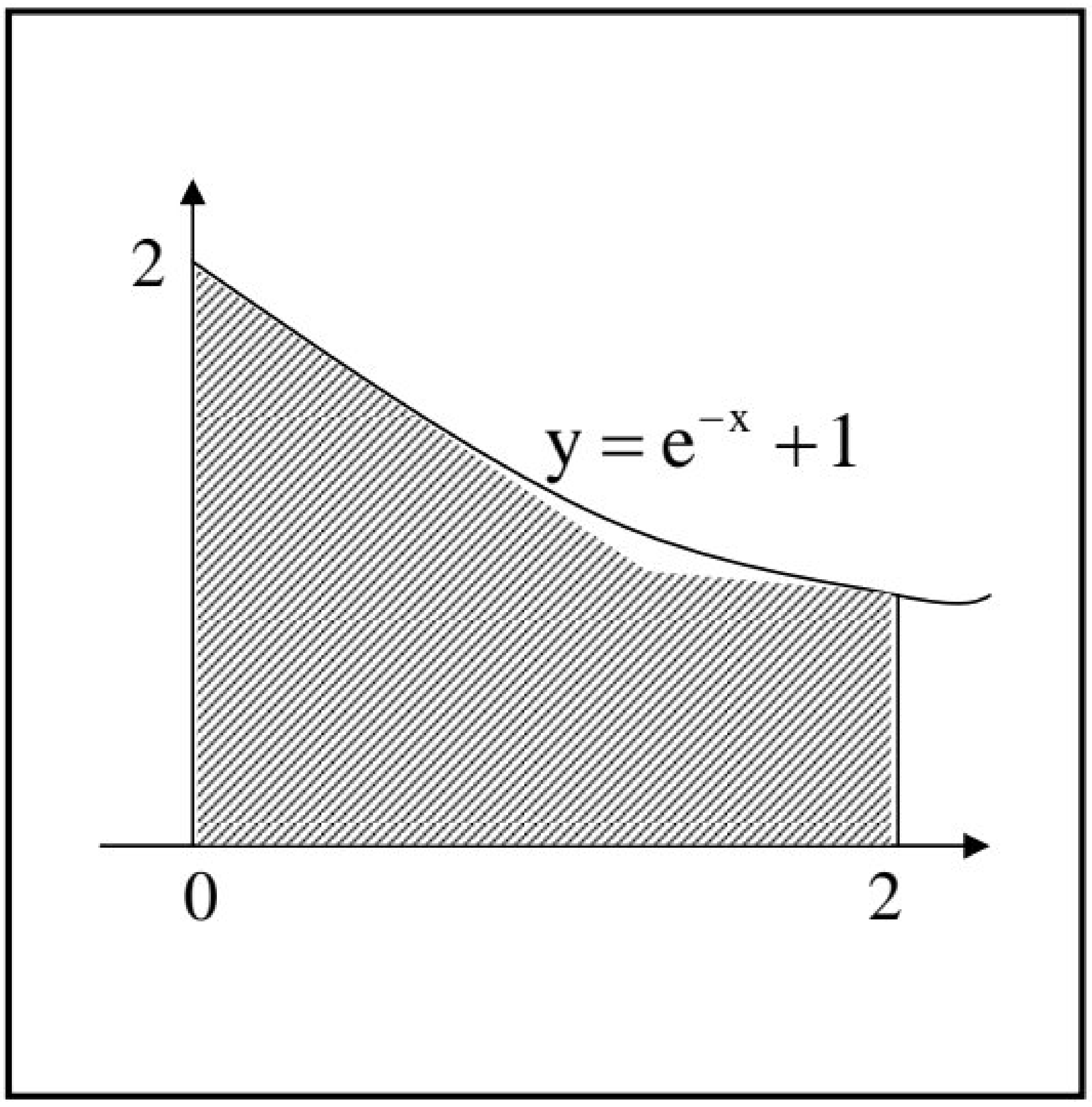

Solution:

$$\underset{\substack{\downarrow \\ \text{area}}}{A} = \int_{o}^{2} \left(e^{-x} + 1\right) dx$$

$$= \left[-e^{-x} + x\right]_{o}^{2}.$$

Example: **Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0, x = 1$.**

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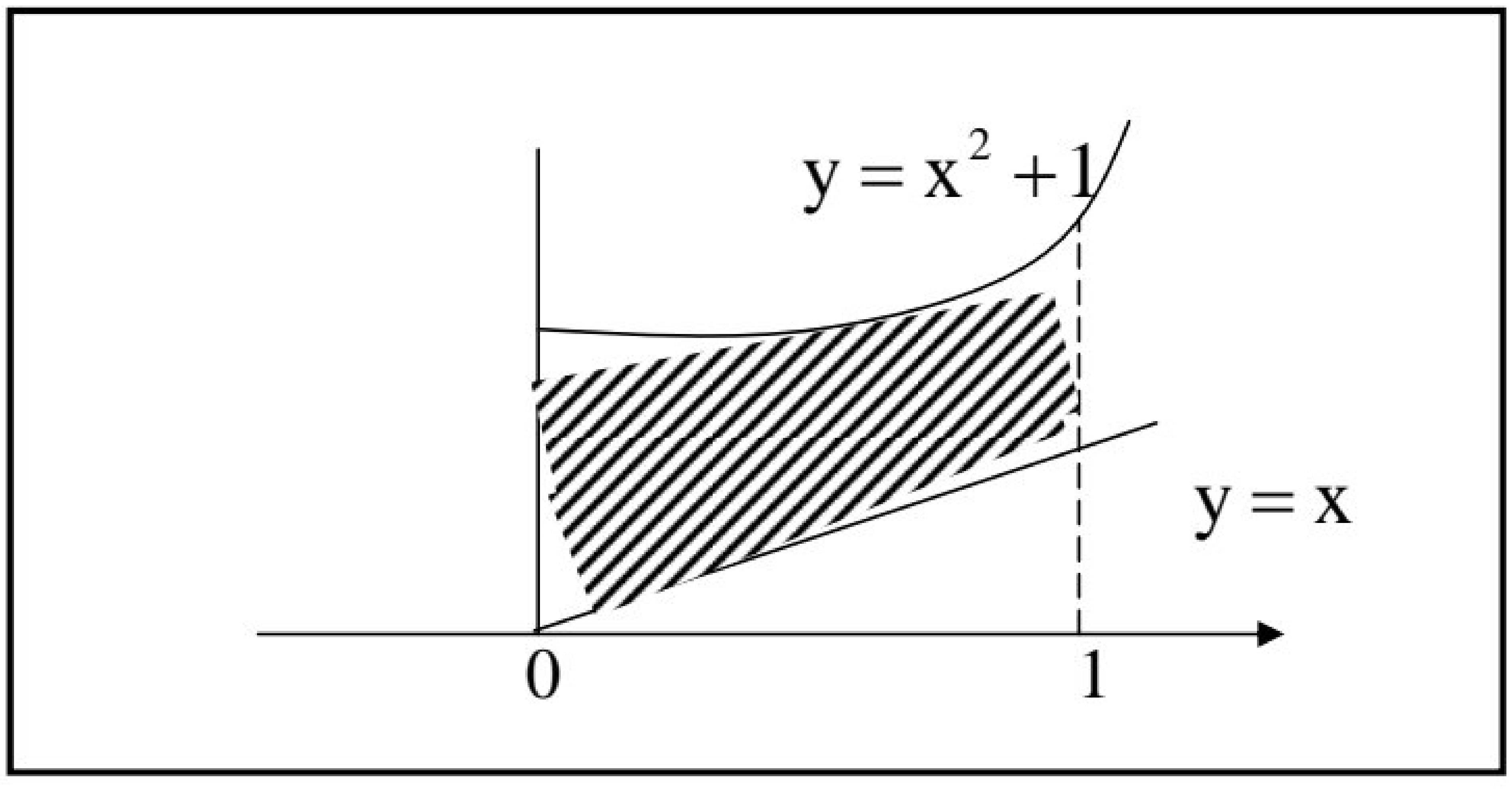
Solution: $A = \int_{o}^{1} \left[\left(x^2 + 1\right) - x\right] dx = \left[\frac{x^3}{3} + x - \frac{x^2}{2}\right]_{o}^{1} = \frac{5}{6}$



Example: **Find the area of the region bounded by the curves** $y = \sin x,\ y = \cos x,\ x = 0$ **and** $x = \frac{\pi}{2}$ **.**

Solution:

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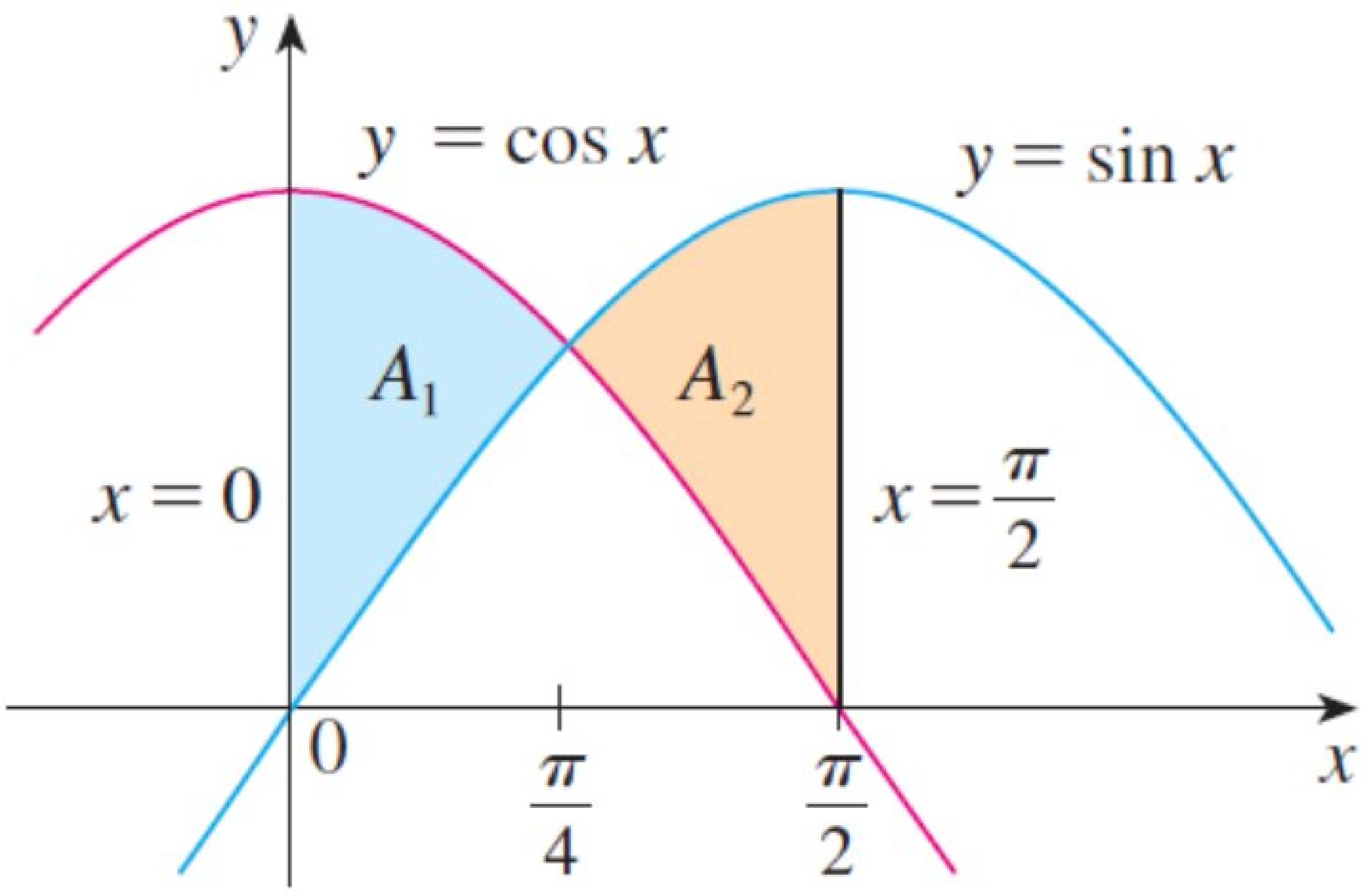

The two curves $y = \sin x,\ y = \cos x$ **intersect at** $x = \frac{\pi}{4}$

$$A = \int_{0}^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= \left[\sin x + \cos x\right]_{0}^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2} = 2\sqrt{2} - 2.$$

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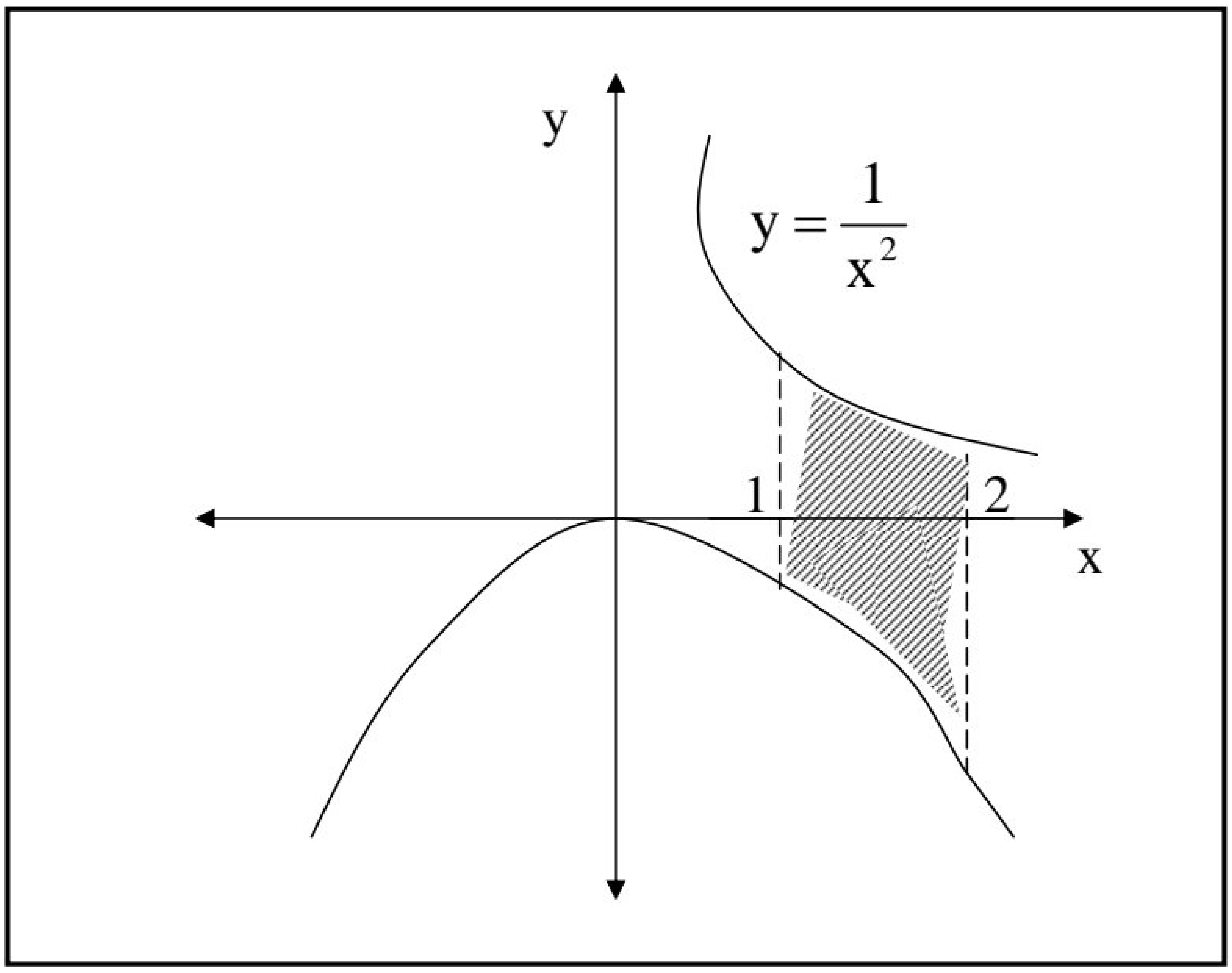
Example: **Find the area between curves**

$$y=\frac{1}{x^2} \quad , \quad y=-x^2 \quad , \quad x=1 \quad , \quad x=2$$

Solution:



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$$A=\int_1^2\left[\frac{1}{x^2}-\left(-x^2\right)\right]dx=\left[-\frac{1}{x}+\frac{x^3}{3}\right]_1^2=\frac{25}{6} \text{ units of area.}$$

Example: **Find the area of the circle using Integration.**

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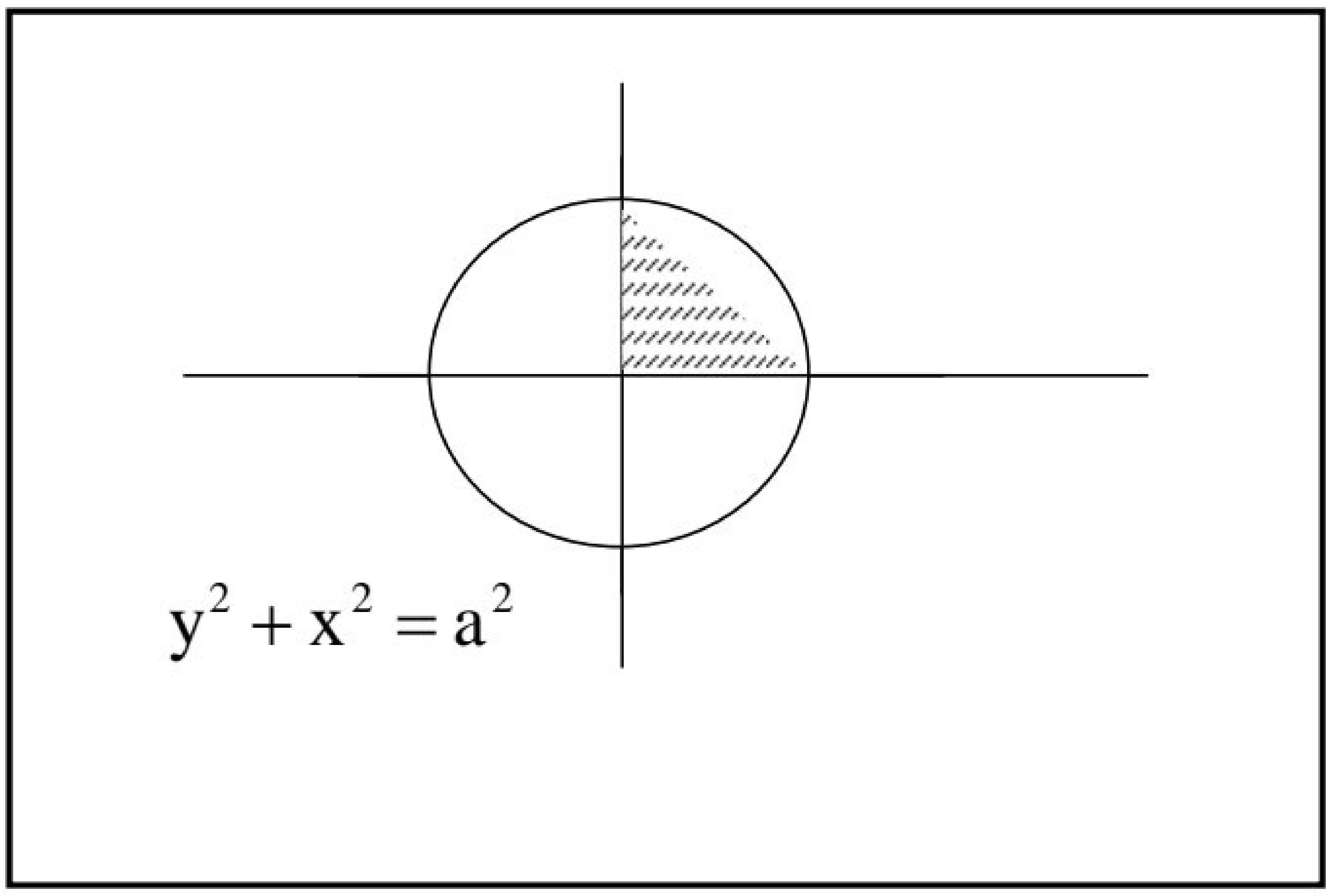
$$A = 4\int_{o}^{a} y\,dx = 4\int_{o}^{a} \sqrt{a^2 - x^2}\,dx$$



Let $\quad x = a\sin\theta \qquad x : 0 \to a$

$$dx = a\cos\theta \qquad \theta : 0 \to \frac{\pi}{2}$$

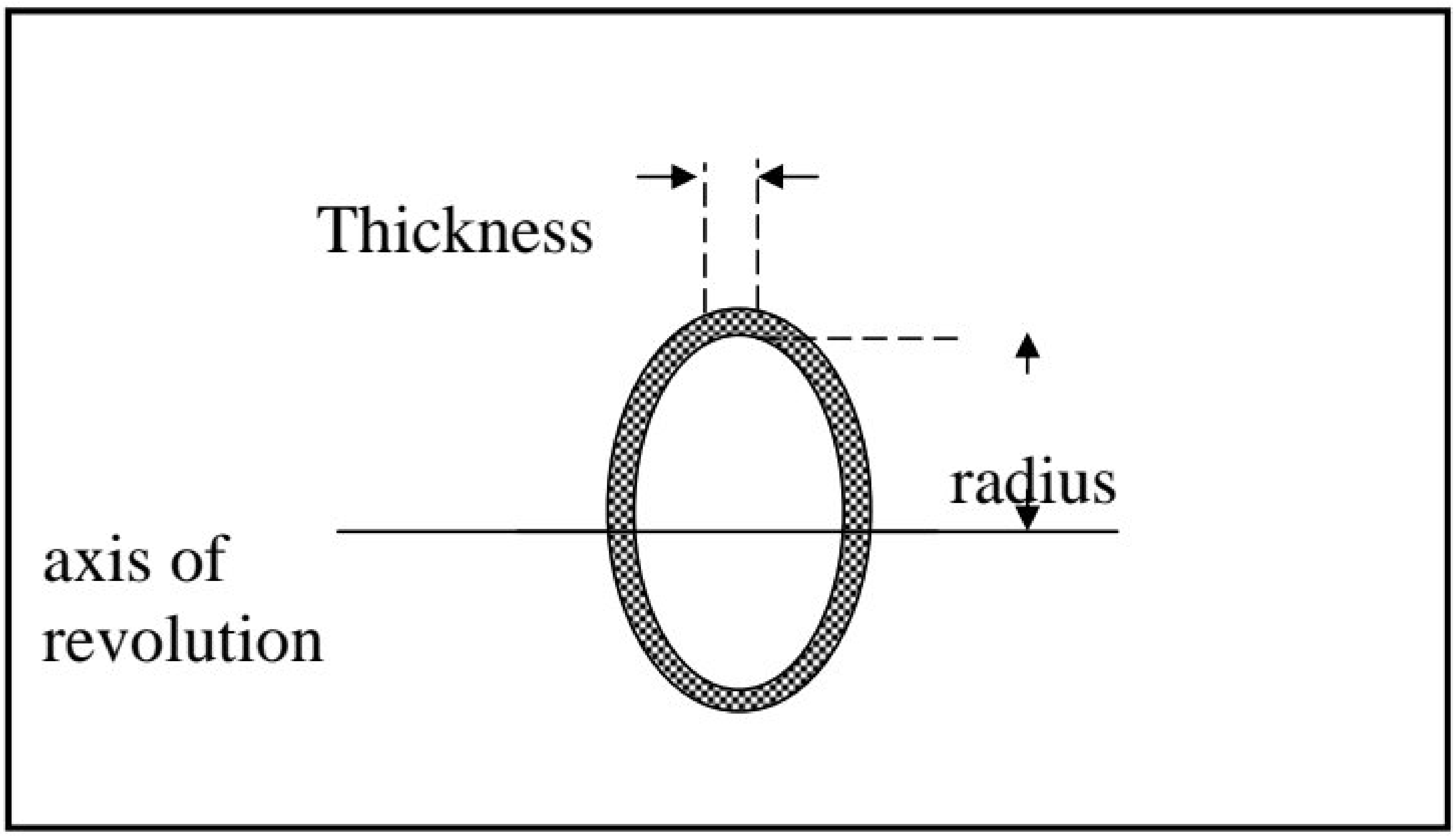
$$A = 4\int_{o}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta)d\theta = 4a^2\int_{o}^{\pi/2} \cos^2\theta d\theta$$

$$= 4a^2\int_{o}^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)d\theta = 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{o}^{\pi/2}$$

$$= 2a^2\left[\frac{\pi}{2} + 0 - 0\right] = \pi a^2$$

(2)Volume of solid of revolution.

The volume of an object plays an important role in many problems in the physical sciences, such as finding centers of mass and moments of inertia. Since it is difficult to determine the volume of an irregularly shaped object, we shall begin with objects that have simple shapes.

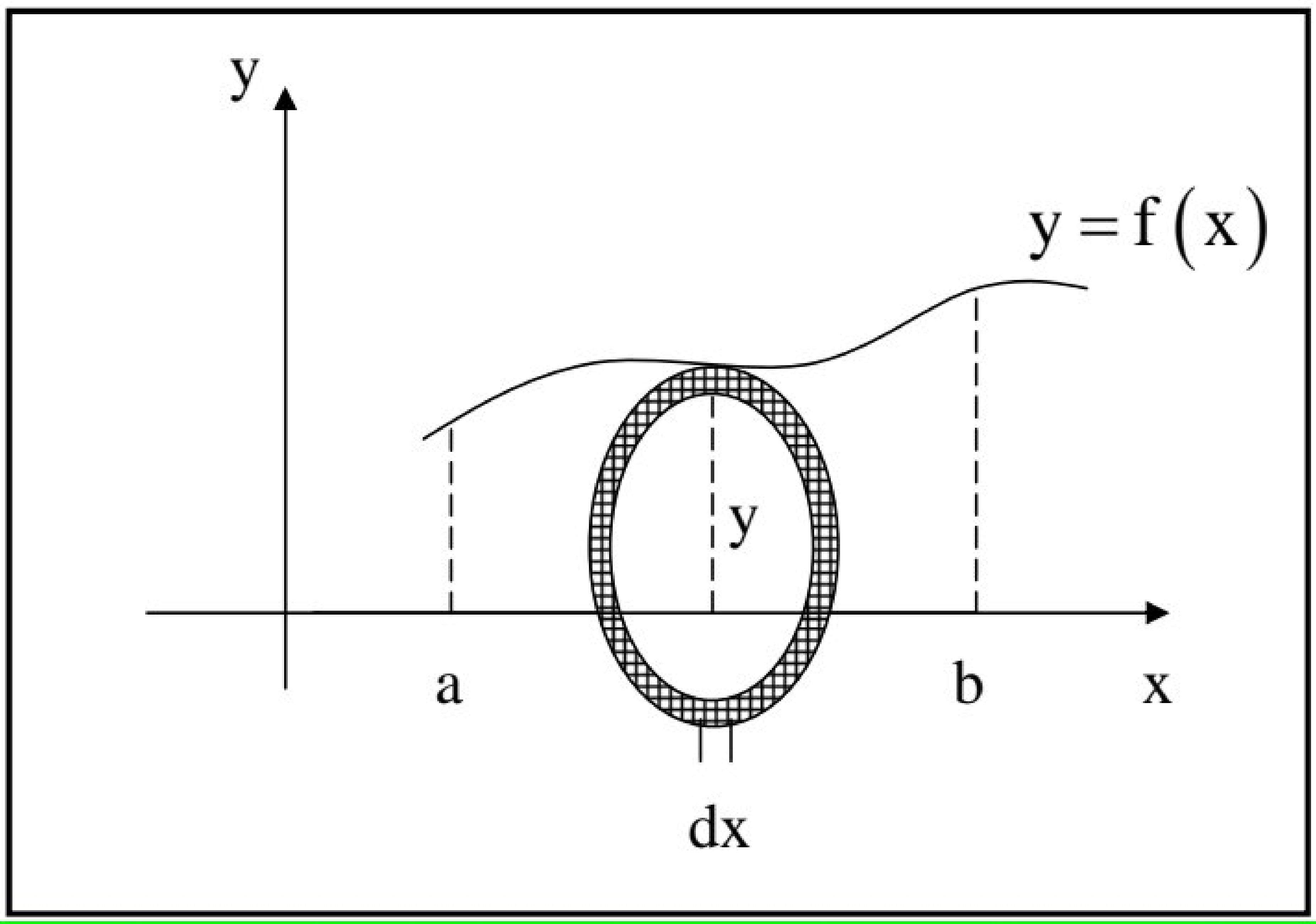


Volume V of a circular disk $V = \pi\left(\text{radius}\right)^2 .\left(\text{thickness}\right)$

The solid of revolution is the solid generated by the rotation of the plane area around an axis lies in its plane. $dv = \text{volume of the circular disk}\;\; dv = \pi y^2 .dx$ **By summing over the interval [a, b] we get** $\boxed{V = \int_{x=a}^{b} \pi y^2 dx}$

Similarly, we can show that if the revolution about y-axis, then

$$\boxed{V = \int_{y=a}^{b} \pi x^2 dy}$$



Example: **Show that the volume of a sphere of radius r is** $V = \frac{4}{3}\pi r^3$.

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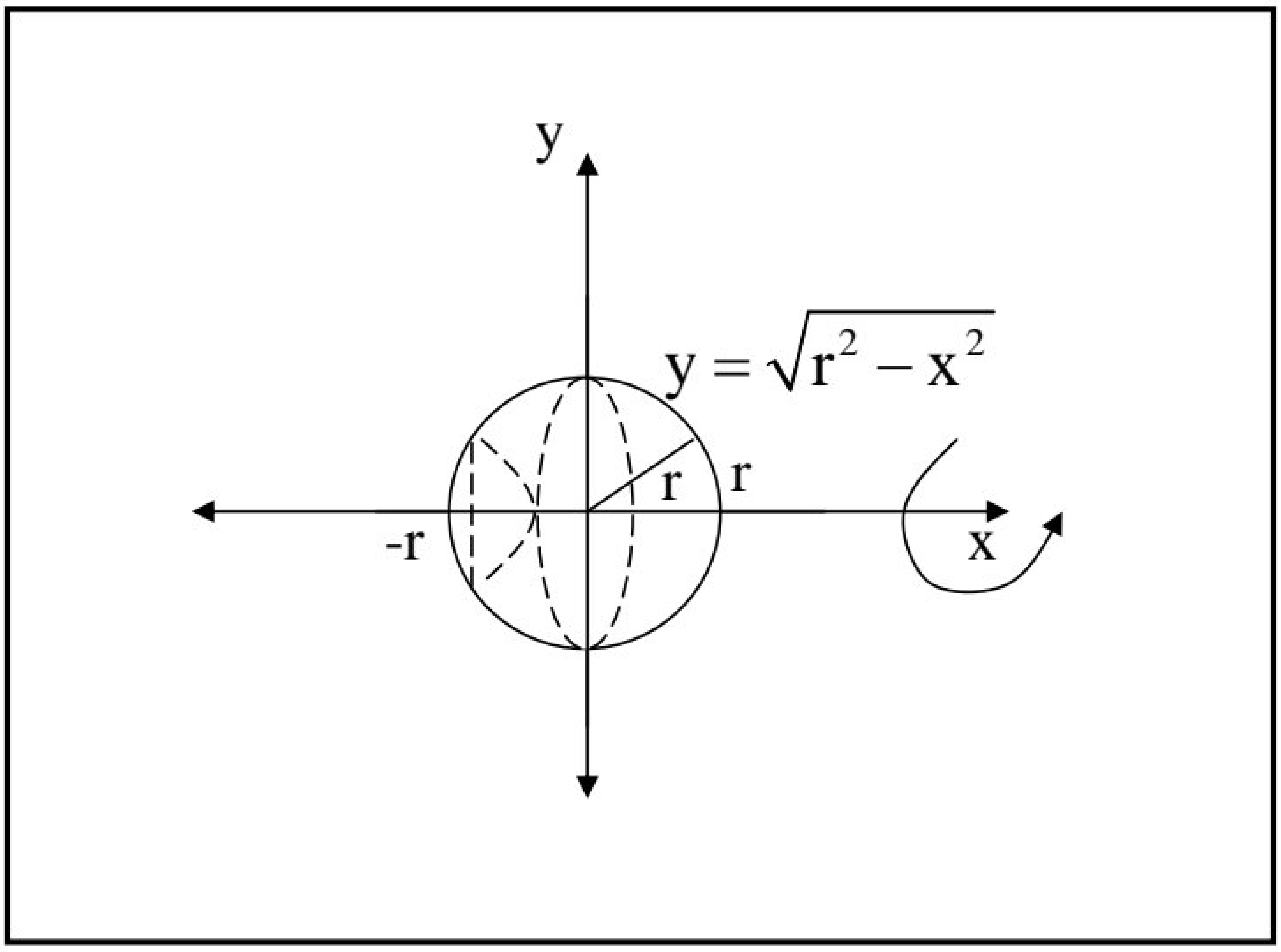
If we rotate the half circle about x-axis, we get a sphere of volume



$$V=\int_{-r}^{r}\pi y^2dx=2\pi\int_{o}^{r}\left(r^2-x^2\right)dx$$

$$=2\pi\left[r^2x-\frac{x^3}{3}\right]_o^r=2\pi\left[r^3-\frac{1}{3}r^3\right]=\frac{4}{3}\pi r^3.$$

Example: **Find the volume of the solid of revolution of the enclosed area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around x-axis.**

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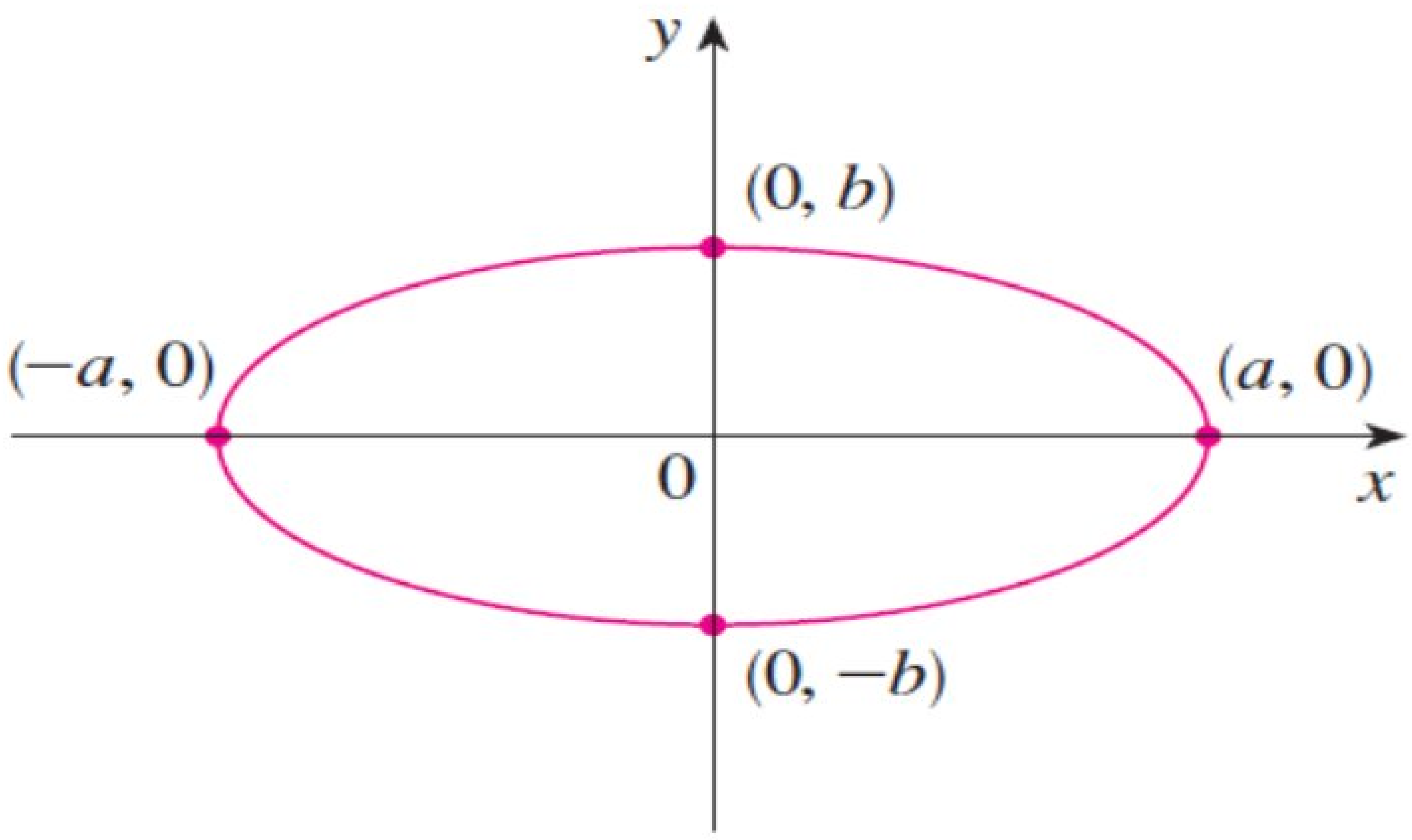

$$V=\int_{-a}^{a}\pi y^2dx=2\pi\int_{o}^{a}b^2\left(1-\frac{x^2}{a^2}\right)dx=2\pi b^2\left[x-\frac{x^3}{3a^2}\right]_o^a$$

$$=2\pi b^2\left[a-\frac{1}{3}\frac{a^3}{a^2}\right]=2\pi b^2\left(\frac{2}{3}a\right)=\frac{4}{3}\pi b^2a.$$

Example: **Find the volume of the solid obtained by rotating the region bounded by** $y = x^3, y = 8$ and $x = 0$ **about y-axis.**

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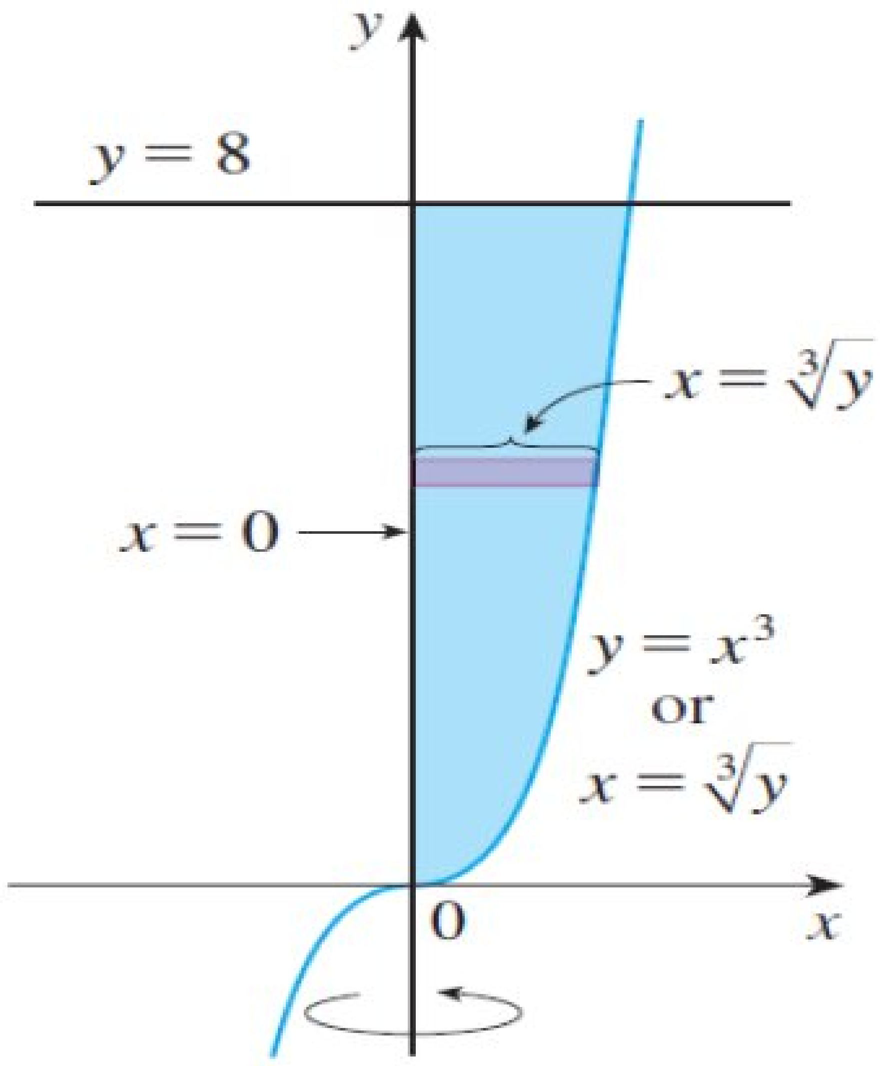
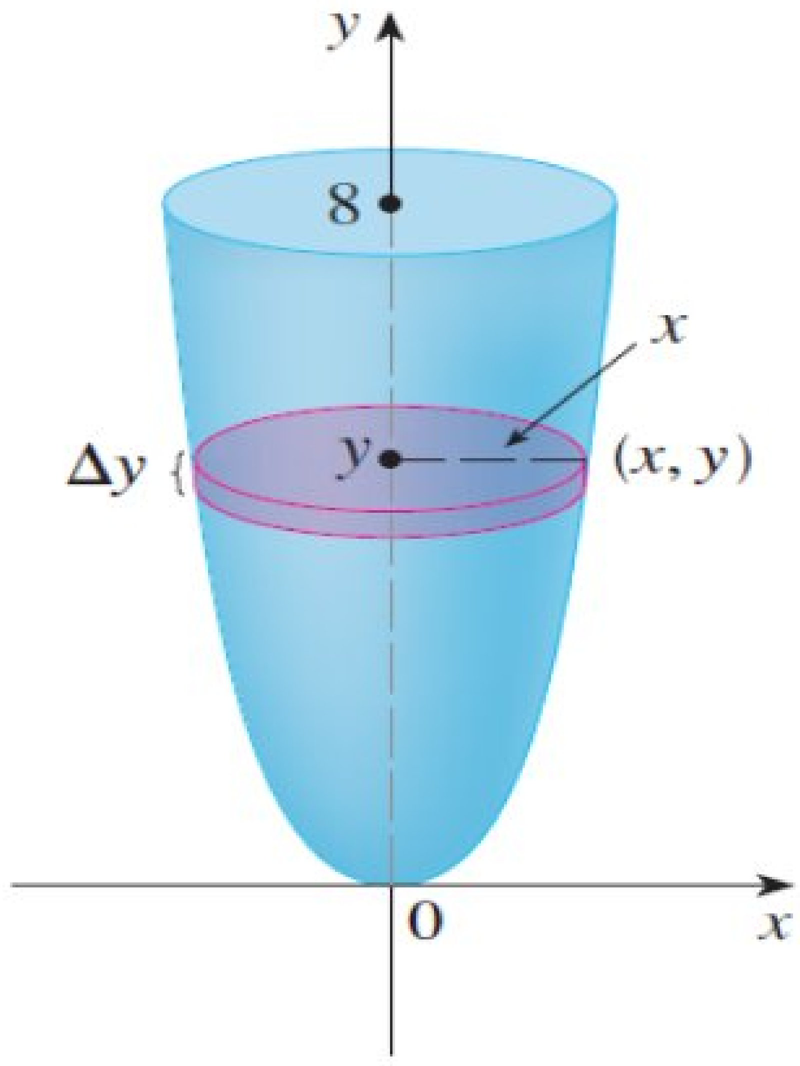
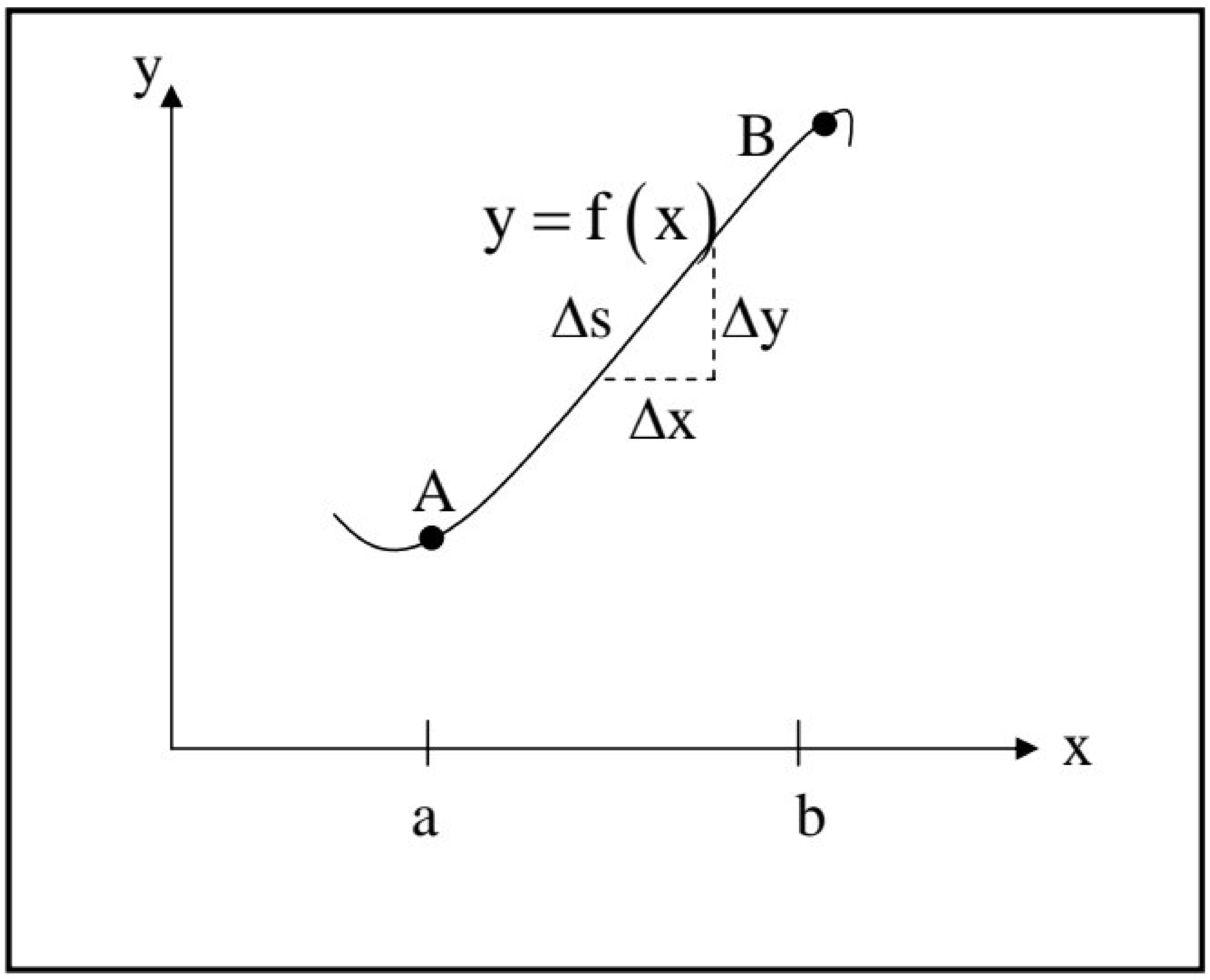




$$V = \int_{o}^{8} \pi x^2 dy = \pi \int_{y=o}^{8} y^{2/3} dy = \pi \left[\frac{y^{5/3}}{5/3} \right]_{y=o}^{8} = \frac{96\pi}{5}.$$

(3)The arc length of a graph.



If we want to evaluate the length of the graph $y = f(x)$ from A to B. At the first let us divide the curve into a small segments say ΔS. It seems to be a straight line then summing these lengths as $\Delta S \to 0$. From the figure

$$(\Delta S)^2 = (\Delta x)^2 + (\Delta y)^2 \xrightarrow{\div(\Delta x)^2}$$

$$\left(\frac{\Delta S}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

So the length of the small segment ΔS is

$$\Delta S = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x \ \text{, when } \Delta x \to 0$$

$$dS = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Taking infinite sum for both sides, we get

$$\boxed{S = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx}$$

By a similar way we can prove that

(2) If $x = g(y)$ then

$$\boxed{S = \int_{y=y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy}$$

(3) In case of parametric equations $x = x(t), y = y(t)$

$$\boxed{S = \int_{t=t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.}$$

<u>Example:</u> **If** $f(x) = 3x^{2/3} - 10$**, find the arc length of the graph of f from the point** $A(8,2)$ **to** $B(27,17)$**.**

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<u>Solution:</u> $y = 3x^{2/3} - 10,\quad \frac{dy}{dx} = 2x^{-1/3} = \frac{2}{\sqrt[3]{x}}$

$$\ell = \int_{x=8}^{27} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_{8}^{27} \sqrt{1 + \frac{4}{x^{2/3}}}\ dx$$

$$= \int_{8}^{27} \sqrt{\frac{x^{2/3} + 4}{x^{2/3}}} dx = \frac{3}{2}\int_{8}^{27} \left(\frac{2}{3}x^{-1/3}\right).\left(x^{1/3} + 4\right)^{1/2} dx$$

$$= \left[\frac{3}{2}\frac{\left(x^{2/3} + 4\right)^{3/2}}{3/2}\right]_8^{27} \approx 24.2$$

<u>Example:</u> **Find the length of the arc of the curve** $y^2 = x^3$ **between the points** $(1,1)$ **and** $(4,8)$**.**

<u>Solution</u> $y^2 = x^3 \quad \Rightarrow \quad y = x^{3/2}\ ,\quad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$

$$\ell = \int_{x=1}^{4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx = \int_{1}^{4} \sqrt{1+\frac{9}{4}x}\ dx = \left[\frac{4}{9}\frac{\left(1+\frac{9}{4}x\right)^{3/2}}{3/2}\right]_1^4$$

$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right).$$

Example: **Find the length of the curve**

$x = 3(\theta - \sin\theta)$, $y = 3(1-\cos\theta)$ **From** $\theta = 0$ to $\theta = \pi.$

Solution

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$$\frac{dx}{d\theta} = 3(1-\cos\theta) \quad , \quad \frac{dy}{d\theta} = 3\sin\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9\left(1-2\cos\theta+\cos^2\theta\right)+9\sin^2\theta$$

$$= 9 - 18\cos\theta + 9 = 18(1-\cos\theta)$$

$$\therefore \quad \ell = \int_{\theta=0}^{\pi} \sqrt{18(1-\cos\theta)}\ d\theta$$

$$= \int_{\theta=0}^{\pi} \sqrt{18.2.\sin^2\frac{\theta}{2}}\quad d\theta = \int_{\theta=0}^{\pi} 6\sin\frac{\theta}{2}d\theta$$

$$=12\int_{\theta=0}^{\pi}\tfrac{1}{2}\sin\left(\frac{\theta}{2}\right)d\theta=12\left[-\cos\left(\frac{\theta}{2}\right)\right]_{\theta=0}^{\pi}$$

$$=12\left[-\text{zero}+1\right]=12.$$

Example: **Find the circumference of the circle whose radius a.**

Solution: $x^2+y^2=a^2$

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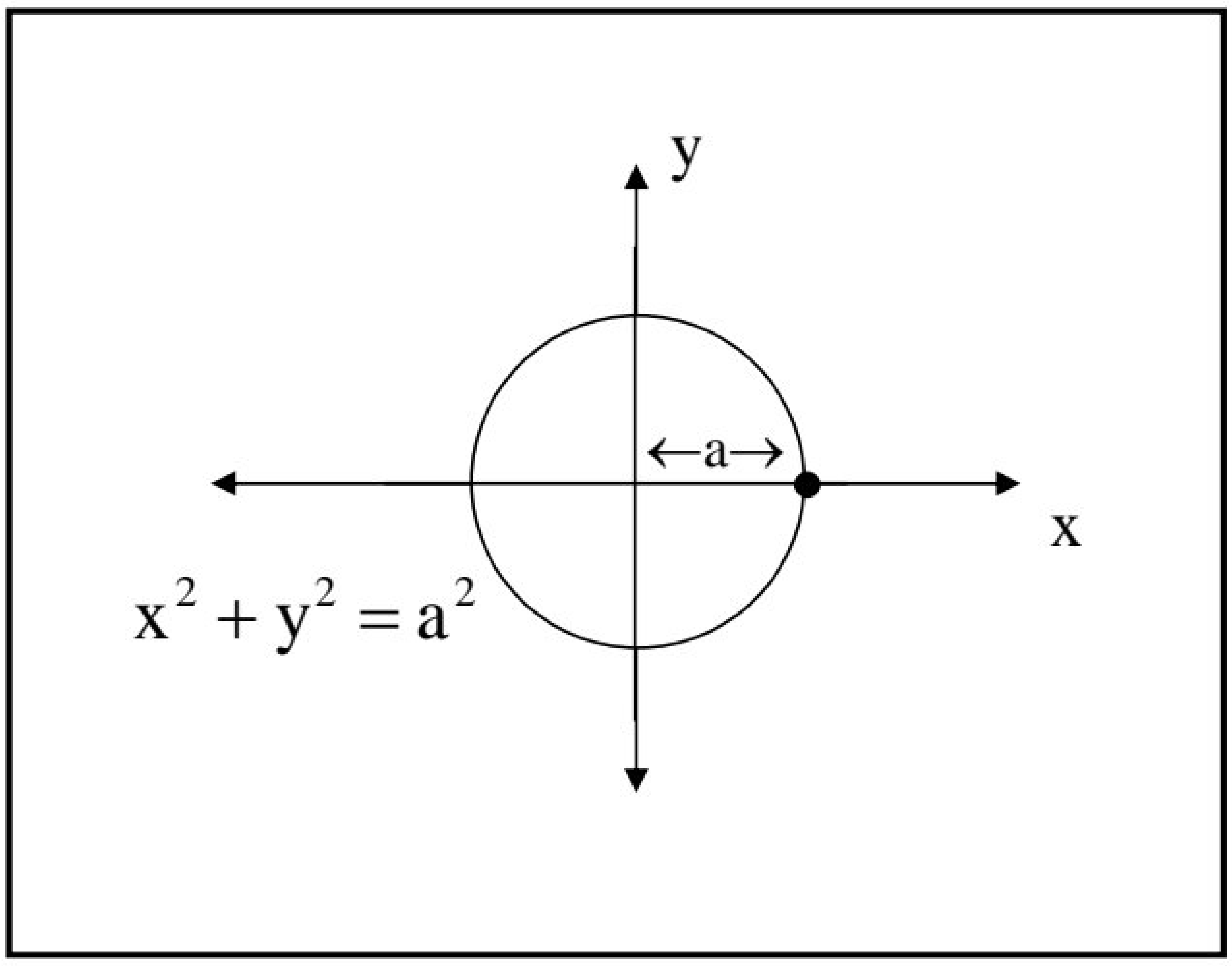
$$\ell=4\int_{x=0}^{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

If $x^2+y^2=a^2$ **then** $2x+2y\dfrac{dy}{dx}=0$



$$\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{\sqrt{a^2-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2=\frac{x^2}{a^2-x^2}$$

$$\therefore \ell=4\int_o^a\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx=4\int_0^a\sqrt{\frac{a^2}{a^2-x^2}}\,dx$$

$$=4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a=4a\left[\sin^{-1}(1)-\sin^{-1}(0)\right]=\boxed{2\pi a}$$

With my best wishes

Dr.Hany El Deeb