Multivariate Beta Distribution

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Motivation

Uhlig (1997) propose a Bayesian method to a VAR model with stochastic volatility. For such, he uses results from the singular multivariate Beta distribution and Wishart distribution, proposed in Uhlig (1994). Before looking into them, we go a step back to understand their non-singular versions, described in Muirhead (2005).

Basic definitions

The folowing definitions and theorems can be found in Muirhead (2005), chapter 3.

Definition 1 (Wishart Distribution) If $A = Z'Z$, where the $n \times m$ matrix Z is $N(0, I_n \otimes \Sigma)$, then A is said to have the *Wishart Distribution* with n degrees of freedom and covariance matrix Σ. The notation is $A \sim \mathcal{W}_m(n, \Sigma)$, where m denotes the size of the matrix A.

The folowing definition applies when A from Definition 1 is not singular, i.e., $n \geq m$.

Theorem 1 (Wishart p.d.f.)

If A is $A \sim \mathcal{W}_m(n, \Sigma)$ with $n \geq m$ then the density of A is:

$$\frac{1}{2^{\frac{mn}{2}} \Gamma_m(\frac{1}{2}n) \left(det\Sigma\right)^{\frac{n}{2}}} \exp tr \left(-\frac{1}{2}\Sigma^{-1}A\right) \left(detA\right)^{\frac{n-m-1}{2}}, \quad A > 0 \tag{1}$$

Where $\Gamma_m(\cdot)$ denotes the multivariate gamma function given by:

$$\Gamma_m(a) = \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \Gamma\left[a - \frac{1}{2}(i-1)\right], \quad \left[Re(a) > \frac{1}{2}(m-1)\right] \tag{2}$$

And $Re(a)$ denotes the real part of a.

Theorem 2 (Multivariate Beta Distribution)

Let A and B be independent, $A \sim \mathcal{W}_m(n_1, \Sigma)$ and $B \sim \mathcal{W}_m(n_2, \Sigma)$m with $n_1 > m-1$ and $n_2 > m-1$. Put $A + B = T'T$, where T is an upper triangular $m \times n$ matrix with positive diagonal elements. Let U be the $m \times m$ symmetric matrix defined by $A = T'UT$. Then $A+B$ and U are independent; $A+B \sim \mathcal{W}_m(n_1+n_2, \Sigma)$ and the density function of U is:

$$\frac{\Gamma_m\left[\frac{1}{2}(n_1+n_2)\right]}{\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m\left(\frac{1}{2}n_2\right)} (detU)^{\frac{n_1-m-1}{2}} det(I_m - U)^{\frac{n_2-m-1}{2}} \quad (0 < U < I_m) \tag{3}$$

Where $(0 < U < I_m)$ means that U and $I_m - U$ are both positive definite.

Definition 2 (Multivariate Beta Distribution)

A matrix U like the one in Theorem 2 is said to have the *multivariate beta distribution* with parameters $\frac{1}{2}n_1$ and $\frac{1}{2}n_2$. We denote $U \sim \mathcal{B}_m(\frac{1}{2}n_1, \frac{1}{2}n_2)$.

Process

We want to generate random numbers from a Multivariate Beta distribution, denoted by $\mathcal{B}_m(\frac{1}{2}n_1, \frac{1}{2}n_2)$.

The first algorithm uses the Wishart p.d.f. already implemented in R. The second constructs the normal distributions needed for the Wishart.

Algorithm 1 - Sampling from the Wishart

Using theorem 2, we will generate A and B independently. We will use the `rWishart()` function.

A single matrix (observation)

```
# rWishart(n, df, Sigma) generates n random matrices, distributed according
# to te Wishart distribution with parameters Sigma and df.

# For the first matrix
# Set the degrees of freedom
df1<-10

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_1 <- diag(4)

# Creates the Wishart first matrix
A<-rWishart(1, df1, sigma_1)
A<-matrix(A, nrow=nrow(sigma_1))

# For the second matrix
# Set the degrees of freedom
df2<-5

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_2 <- diag(4)

# Creates the Wishart first matrix
B<-rWishart(1, df2, sigma_2)
B<-matrix(B, nrow=nrow(sigma_2))

# Sums A and B
C<-A+B

# The chol() function gives T, an upper-triangular mxm matrix with positive diagonal elements
T<-chol(C)

# We now calculate U = T'^{-1}AT^{-1}
U <- solve(t(T))%*%A%*%solve(T)
U
```

```
##              [,1]        [,2]        [,3]        [,4]
## [1,]  0.42966176 -0.02075201 -0.21952180  0.01643146
```

```
## [2,] -0.02075201  0.47546973 -0.07881197 -0.26839396
## [3,] -0.21952180 -0.07881197  0.48816725 -0.17690593
## [4,]  0.01643146 -0.26839396 -0.17690593  0.75029014
```

General case

```r
# rWishart(n, df, Sigma) generates n random matrices, distributed according
# to te Wishart distribution with parameters Sigma and df.

# Set the number of matrices (samples)
n<-2

# For the first matrix
# Set the degrees of freedom
df1<-10

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_1 <- diag(4)

# Creates the Wishart first matrix
A<-rWishart(n, df1, sigma_1)

# For the second matrix
# Set the degrees of freedom
df2<-5

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_2 <- diag(4)

# Creates the Wishart first matrix
B<-rWishart(n, df2, sigma_2)

# Sums A and B
C<-A+B

# The chol() function gives T, an upper-triangular mxm matrix with positive diagonal elements
# Creates the T array with the same dimension of C
T<-C

# Populates T
for (i in 1:n) {
  T[,,i]<-chol(C[,,i])
}

# We now calculate U = T'^{-1}AT^{-1}
#Creates U
U<-T

for (i in 1:n) {
  U[,,i] <- solve(t(T[,,i]))%*%A[,,i]%*%solve(T[,,i])
}
```

```r
U
```

```
## , , 1
##
##            [,1]       [,2]        [,3]        [,4]
## [1,]  0.6178335 0.15352962 -0.16799284  0.05276430
## [2,]  0.1535296 0.73255973  0.17749605  0.05552717
## [3,] -0.1679928 0.17749605  0.75872548 -0.01367507
## [4,]  0.0527643 0.05552717 -0.01367507  0.85874354
##
## , , 2
##
##             [,1]         [,2]         [,3]        [,4]
## [1,]  0.59620347  0.053554114 -0.103624585  0.23969805
## [2,]  0.05355411  0.907531044 -0.006187045 -0.12232345
## [3,] -0.10362458 -0.006187045  0.438804345  0.05485229
## [4,]  0.23969805 -0.122323447  0.054852295  0.40923503
```

Algorithm 2 - Sampling from the Multivariate Normal to construct the Wisharts

(to be done by Future Aisha) See this function.

References

Muirhead, Robb John. 2005. *Aspects of Multivariate Statistical Theory*. Wiley-Interscience.

Uhlig, Harald. 1994. "On Singular Wishart and Singular Multivariate Beta Distributions." *The Annals of Statistics* 22 (1): 395–405. doi:10.1214/aos/1176325375.

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