Multivariate Beta Distribution

Aishameriane Schmidt

14 de abril de 2017

Motivation

Uhlig (1997) propose a Bayesian method to a VAR model with stochastic volatility. For such, he uses results from the singular multivariate Beta distribution and Wishart distribution, proposed in Uhlig (1994). Before looking into them, we go a step back to understand their non-singular versions, described in Muirhead (2005).

Basic definitions

The folowing definitions and theorems can be found in Muirhead (2005), chapter 3.

Definition 1 (Wishart Distribution) If $A = Z'Z$, where the $n \times m$ matrix Z is $N(0, I_n \otimes \Sigma)$, then A is said to have the *Wishart Distribution* with n degrees of freedom and covariance matrix Σ. The notation is $A \sim \mathcal{W}_m(n, \Sigma)$, where m denotes the size of the matrix A.

The folowing definition applies when A from Definition 1 is not singular, i.e., $n \geq m$.

Theorem 1 (Wishart p.d.f.)

If A is $A \sim \mathcal{W}_m(n, \Sigma)$ with $n \geq m$ then the density of A is:

$$\frac{1}{2^{\frac{mn}{2}} \Gamma_m(\frac{1}{2}n) \, (det\Sigma)^{\frac{n}{2}}} \exp tr\left(-\frac{1}{2}\Sigma^{-1}A\right)(detA)^{\frac{n-m-1}{2}}, \quad A > 0 \tag{1}$$

Where $\Gamma_m(\cdot)$ denotes the multivariate gamma function given by:

$$\Gamma_m(a) = \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \Gamma\left[a - \frac{1}{2}(i-1)\right], \quad \left[Re(a) > \frac{1}{2}(m-1)\right] \tag{2}$$

And $Re(a)$ denotes the real part of a.

Theorem 2 (Multivariate Beta Distribution)

Let A and B be independent, $A \sim \mathcal{W}_m(n_1, \Sigma)$ and $B \sim \mathcal{W}_m(n_2, \Sigma)$m with $n_1 > m-1$ and $n_2 > m-1$. Put $A + B = T'T$, where T is an upper triangular $m \times n$ matrix with positive diagonal elements. Let U be the $m \times m$ symmetric matrix defined by $A = T'UT$. Then $A + B$ and U are independent; $A + B \sim \mathcal{W}_m(n_1 + n_2, \Sigma)$ and the density function of U is:

$$\frac{\Gamma_m\left[\frac{1}{2}(n_1+n_2)\right]}{\Gamma_m\left(\frac{1}{2}n_1\right)\Gamma_m\left(\frac{1}{2}n_2\right)}(detU)^{\frac{n_1-m-1}{2}} det(I_m - U)^{\frac{n_2-m-1}{2}} \quad (0 < U < I_m) \tag{3}$$

Where $(0 < U < I_m)$ means that U and $I_m - U$ are both positive definite.

Definition 2 (Multivariate Beta Distribution)

A matrix U like the one in Theorem 2 is said to have the *multivariate beta distribution* with parameters $\frac{1}{2}n_1$ and $\frac{1}{2}n_2$. We denote $U \sim \mathcal{B}_m(\frac{1}{2}n_1, \frac{1}{2}n_2)$.

Process

We want to generate random numbers from a Multivariate Beta distribution, denoted by $\mathcal{B}_m(\frac{1}{2}n_1, \frac{1}{2}n_2)$.

The first algorithm uses the Wishart p.d.f. already implemented in R. The second constructs the normal distributions needed for the Wishart.

Algorithm 1 - Sampling from the Wishart

Using theorem 2, we will generate A and B independently. We will use the `rWishart()` function.

A single matrix (observation)

```
# rWishart(n, df, Sigma) generates n random matrices, distributed according
# to te Wishart distribution with parameters Sigma and df.

# For the first matrix
# Set the degrees of freedom
df1<-10

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_1 <- diag(4)

# Creates the Wishart first matrix
A<-rWishart(1, df1, sigma_1)
A<-matrix(A, nrow=nrow(sigma_1))

# For the second matrix
# Set the degrees of freedom
df2<-5

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_2 <- diag(4)

# Creates the Wishart first matrix
B<-rWishart(1, df2, sigma_2)
B<-matrix(B, nrow=nrow(sigma_2))

# Sums A and B
C<-A+B
C
```

```
##             [,1]        [,2]        [,3]        [,4]
## [1,] 11.45136135  0.38271409 -0.33414170 -0.05750305
## [2,]  0.38271409 10.66732389 -0.01937527 -2.40956696
## [3,] -0.33414170 -0.01937527 23.99000220 -2.48827911
## [4,] -0.05750305 -2.40956696 -2.48827911  8.56846337
```

```
# The chol() function gives T, an upper-triangular mxm matrix with positive diagonal elements
T<-chol(C)
T
```

```
##          [,1]      [,2]         [,3]       [,4]
## [1,] 3.383986 0.1130956 -0.098742045 -0.0169927
## [2,] 0.000000 3.2641283 -0.002514599 -0.7376074
## [3,] 0.000000 0.0000000  4.896962925 -0.5088484
## [4,] 0.000000 0.0000000  0.000000000  2.7866078
```

```
# We now calculate U = T'^{-1}AT^{-1}

U <- solve(t(T))%*%A%*%solve(T)
U
```

```
##             [,1]         [,2]        [,3]         [,4]
## [1,]  0.45042761 -0.154212659 -0.08667557  0.138415176
## [2,] -0.15421266  0.906547386  0.03722988 -0.008801199
## [3,] -0.08667557  0.037229876  0.79607397  0.145584454
## [4,]  0.13841518 -0.008801199  0.14558445  0.817546195
```

General case (Still in progress)

```
# rWishart(n, df, Sigma) generates n random matrices, distributed according
# to te Wishart distribution with parameters Sigma and df.

# Set the number of matrices (samples)
n<-2

# For the first matrix
# Set the degrees of freedom
df1<-10

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_1 <- diag(4)

# Creates the Wishart first matrix
A<-rWishart(1, df1, sigma_1)

## This could be discarted?
A<-matrix(A, nrow=nrow(sigma_1))

# For the second matrix
# Set the degrees of freedom
df2<-5

# Set the covariance matrix
# The diag(n) creates the nxn identity matrix
sigma_2 <- diag(4)

# Creates the Wishart first matrix
B<-rWishart(1, df2, sigma_2)
B<-matrix(B, nrow=nrow(sigma_2))

# Sums A and B
C<-A+B
C
```

```
##             [,1]        [,2]        [,3]      [,4]
## [1,] 11.4966607  5.76618706 -0.28332222  3.829482
## [2,]  5.7661871 10.97116903 -0.01557176  4.125010
## [3,] -0.2833222 -0.01557176  9.56042384 -1.591802
## [4,]  3.8294824  4.12501045 -1.59180173 20.133570
```

```
# The chol() function gives T, an upper-triangular mxm matrix with positive diagonal elements
# There will be an error problem right here because C is not a matrix. Maybe iterate only here?
T<-chol(C)
T
```

```
##          [,1]     [,2]        [,3]       [,4]
## [1,] 3.390673 1.700603 -0.08355930  1.1294167
## [2,] 0.000000 2.842379  0.04451531  0.7755198
## [3,] 0.000000 0.000000  3.09054365 -0.4956898
## [4,] 0.000000 0.000000  0.00000000  4.2439190
```

```
# We now calculate U = T'^{-1}AT^{-1}
# Need to think more carefully about U, A and B because of the dimensions
U <- solve(t(T))%*%A%*%solve(T)
U
```

```
##               [,1]          [,2]        [,3]        [,4]
## [1,]  0.8532415301 -0.0003893415  0.01420916 -0.03259425
## [2,] -0.0003893415  0.9588721133 -0.05161901  0.08451241
## [3,]  0.0142091575 -0.0516190135  0.39410015 -0.05033338
## [4,] -0.0325942545  0.0845124120 -0.05033338  0.44786953
```

Algorithm 2 - Sampling from the Multivariate Normal to construct the Wisharts

See this function.

References

Muirhead, Robb John. 2005. *Aspects of Multivariate Statistical Theory*. Wiley-Interscience.

Uhlig, Harald. 1994. "On Singular Wishart and Singular Multivariate Beta Distributions." *The Annals of Statistics* 22 (1): 395–405. doi:10.1214/aos/1176325375.

———. 1997. "Bayesian Vector Autoregressions with Stochastic Volatility." *Econometrica* 65 (1): 59. doi:10.2307/2171813.