ETERNITY: FUNCTIONS

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0.1 Introduction

0.1.1 Discription

In mathematics, the gamma function is one commonly used extension of the factorial function to complex numbers, is a meromorphic function defined in the complex range, usually written so that negative integers and 0 are its first order poles.There are various definitions of the gamma function, we selected three of them here:

- For any positive integer n,

$$\Gamma\left(n\right) = \left(n - 1\right)!$$

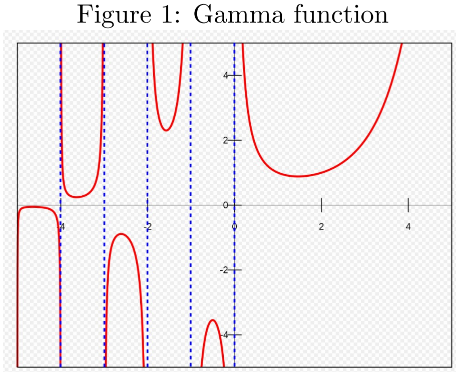
- Derived by Daniel Bernoulli, for complex numbers with a positive real part, the gamma function is defined via a convergent improper integral:

$$\Gamma\left(n\right) = \int_{0}^{+\infty} x^{z-1} e^{-x} \mathrm{d}x, \Re\left(z\right) > 0$$

- The gamma function on real number field is defined as:

$$\Gamma\left(x\right) = \int_{0}^{+\infty} t^{x-1} e^{-t} \mathrm{d}t\left(x > 0\right)$$

Figure 1: Gamma function



0.1.2 Stakeholders

- User1: Mathematicians and researchers in the fields of calculus, mathematical analysis, statistics.
- User2: Algorithm designers, researchers,engineers in the field of scientific computing in computer and software engineering.
- User3: People with low maths and programming background who need to obtain the values of gamma function.

0.1.3 Domain

All complex numbers except those whose real part are non-positive integers.

$$C/\{n \in Z, n \leq 0\}$$

0.1.4 Co-Domain

All real numbers excluding zero.

$$(-\infty, 0) \cup (0, +\infty)$$

0.1.5 Properties

One of the important functional equatuations for the gamma fucntion is Euler's reflection formula:

$$\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin \pi z}, z \notin Z$$

which implies,

0.1.6 Perticular Values

Including up to the first 20 digits after the decimal point, some of particular values of the gamma function are:

- $\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3} \approx +2.36327180120735470306$
- $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \approx -3.54490770181103205459$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \approx +1.77245385090551602729$
- $\Gamma(1) = 0! = +1$
- $\Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2} \approx +0.88622692545275801364$
- $\Gamma(2) = 1! = +1$
- $\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4} \approx +1.32934038817913702047$
- $\Gamma(3) = 2! \approx +2$
- $\Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8} \approx +3.32335097044784255118$
- $\Gamma(4) = 3! = +6$

0.2 Requirements

0.2.1 Functional Requirements

- FR1: We shall only consider the positive value as input value.
- FR2: We shall only consider the real number as input value, ignore the imaginary part.
- FR3: When the function value exceeds the maximum the value of a double type variable, the function will return NaN.

0.2.2 Assumptions

0.3 Algorithm

0.3.1 Integrals Method

Description

The first algorithm is based on the Gamma function formula derived by Daniel Bernoulli, using an approximating integrals method called trapezoidal rule. It is used for initial value problems. In calculus, the trapezoidal rule is a technique for approximation the definite integral. This algorithm involves two sub-functions. One is the power function, which given base and exponent as double type arguments, returns $base^{exponent}$. And the exp function, which given exponent as double type argument, returns $e^{exponent}$.

0.3.2 Lanczos Approximation

Description

The second algorithm is based on Lanczos approximation. In mathematics, the Lanczos approximation is a method for computing the gamma function numerically, published by Cornelius Lanczos in 1964. It is a practical alternative to the more popular Stirling's approximation for calculating the gamma function with fixed precision. This algorithm also involves power function, exp function, and sqrt function which given one double type argument, returns square root.

0.3.3 Pseudocode

Algorithm 1 Calculate Gamma function using integrals method

Input: Input value of x
Output: Output value of Gamma function

function $f_y(x)$
  return $s^{x-1} * e^{-s}$
end function

function $gamma(x)$
  $output \leftarrow Error$
  if $x < 0$ **then**
    $result \leftarrow Negative\ input\ is\ not\ allowed.$
  else if $x > 170$ **then**
    $result \leftarrow Infinity$
  else
    $result \leftarrow 0, intervalGap \leftarrow 10^{-3}, i \leftarrow 0$
    while $i < a\ particular\ value$ **do**
      $result \leftarrow result + \frac{1}{2} * intervalGap * (f_y\,(i) + y\,(i - intervalGap))$
      $i \leftarrow i + intervalGap$
    end while
    $output \leftarrow result$
  end if
  return $output$
end function

Algorithm 2 Calculate Gamma function using Lanczos approximation

Input: Array p as a coefficients, and a constant EPSILON
Output: Output value of Gamma function

function $gamma(x)$
    if $x < 0$ **then**
        return *Negative input is not allowed.*
    end if
    if $x < 0.5$ **then**
        return $\frac{\pi}{\sin(\pi * z) * gamma(1-z)}$
    else
        $x \leftarrow x - 1$
        $z \leftarrow 0.99999999999980993$
        for $(i, pval)\ in\ p$ **do**
            $z \leftarrow z + \frac{pval}{x+i+1}$
        end for
        $t \leftarrow x + length\,(p) - 0.5$
        $m \leftarrow \sqrt{2 * \pi} * pow\,(t, (x + 0.5)) * exp\,(-t) * z$
    end if
    return m
end function

0.3.4 Advantages and Disadvantages

Algorithm 1

Advantages:

1. Using the the integral method makes the algorithm more intuitive and easy to understand.
2. More precise calculations results can be obtained.
3. The entire algorithm involves loops without recurtion, it requires less memory from computing devices.

Disadvantages:

1. When the input value is large, it takes significantly longer to produce the result.

Algorithm 2

Advantages:

1. The algorithm makes computing the gamma function becomes a matter of evaluating only a small number of elementary functons and multiplying by stored constants.Simpler to implement.
2. Calculation time is almost independent of the input value.

Disadvantages:

1. Less precise calculations results than Algorithm 1.
2. The coefficients and Constants that we give in advance are not always precise and accurate.

0.3.5 Final Decision

The algorithm 1 is chosen for implementation. Because it could provide more precise results, although it is theoretically slower than the second one, there is no much difference in terms of actual execution time.

0.4 References

But I don't want my section to be numbered. $\text{function} = \Gamma\sqrt{1 - \frac{v^2}{c^2}}$ #LaTeX**Rules**

- Item 1.
- Item 2.
- ...

- Item n.

1. Item 1.
2. Item 2.
3. Item 3.