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**UNIT – III**  
**TWO DIMENSIONAL RANDOM VARIABLES**

**CHAPTER – I**

**INTRODUCTION TWO DIMENSIONAL RANDOM VARIABLES**

In the last unit, we introduced the concept of a single random variable. We observed that the various statistical averages or moments of the random variable like mean, variance, standard deviation, skewness give an idea about the characteristics of the random variable. But in many practical problems several random variables interact with each other and frequently we are interested in the joint behaviour of these random variables.

For example, to know the health condition of a person, doctors measure many parameters like height, weight, blood pressure, sugar level etc. We should now introduce techniques that help us to determine the joint statistical properties of several random variables. The concepts like distribution function, density function and moments that we defined for single random variable can be extended to multiple random variables also.

**Two Dimensional Random Variable**

Let  $S$  be the sample space associated with a random experiment  $E$ . Let  $X=X(s)$  and  $Y=Y(s)$  be two functions each assigning a real number to each outcomes  $s \in S$ . Then  $(X, Y)$  is called a two dimensional random variable.

**Note**

- (i) If the possible values of  $(X, Y)$  are finite or countable infinite,  $(X, Y)$  is called a two dimensional discrete random variable.
- (ii) If  $(X, Y)$  can assume all values in a specified region  $R$  in the  $xy$ -plane,  $(X, Y)$  is called a two dimensional continuous random variable.

**Joint probability Mass Function (Discrete Case)**

If  $(X,Y)$  is a two-dimensional discrete RV such that  $P(X = x_i, Y = y_j) = p_{ij}$ , then  $p_{ij}$  is called the probability mass function of  $(X, Y)$  provided  $p_{ij} \geq 0$  for all  $i$  and  $j$ .

$$(ii) \sum_i \sum_j p_{ij} = 1$$

**Joint probability Density Function (Continuous Case)**

If  $(X,Y)$  is a two-dimensional continuous RV such that

$$p \left\{ x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2} \text{ and } y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2} \right\} = f(x, y) dx dy,$$

then  $f(x,y)$  is called the joint pdf of  $(X,Y)$ , provided  $f(x,y)$  satisfies the following conditions.

(i)  $f(x, y) \geq 0$ , for all  $(x, y) \in R$ , where  $R$  is the range space

$$(ii) \iint_R f(x, y) dx dy = 1.$$

**Note**

$$P\{(X < Y) \in D\} = \iint_D f(x, y) dx dy. \text{ In particular}$$

$$P\{a \leq X \leq b, c \leq Y \leq d\} = \int_c^d \int_a^b f(x, y) dx dy$$

**Cumulative Distribution Function**

If  $(X, Y)$  is a two-dimensional RV (discrete or continuous), then  $F(x, y) = P\{X \leq x \text{ and } Y \leq y\}$  is called the cdf of  $(X,Y)$ . In the discrete case,

$$F(x, y) = \sum_j \sum_i p_{ij}$$

In the continuous case,

$$F(x, y) = \int_{-\infty}^y \int_{-\infty}^x f(x, y) dx dy$$

### Properties of F (x , y)

- (i)  $F(-\infty, y) = 0 = F(x, -\infty)$  and  $F(\infty, \infty) = 1$
- (ii)  $P\{a < X < b, Y \leq y\} = F(b, y) - F(a, y)$
- (iii)  $P\{X \leq x, c < Y < d\} = F(x, d) - F(x, c)$
- (iv)  $P\{a < X < b, c < Y < d\} = F(b, d) - F(a, d) - F(b, c) + F(a, c)$
- (v) At points of continuity of  $f(x, y)$

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$$

### Marginal Probability Distribution

$$P(X = x) = P\{(X = x_i \text{ and } Y = y_1) \text{ or } (X = x_i \text{ and } Y = y_2) \text{ or etc.}\}$$

$$= p_{i1} + p_{i2} + \dots = \sum_j p_{ij}$$

$$P_{i*} = P(X = x_i) = \sum_j p_{ij}$$

is called the marginal probability function of X. It is defined for  $X=x_1, x_2, \dots$  and denoted as  $P_{i*}$ . The collection of pairs  $\{X_i, P_{i*}\}$ ,  $i = 1, 2, 3, \dots$  is called the marginal probability distribution of X.

Similarly the collection of pairs  $\{Y_j, P_{*j}\}$ ,  $j = 1, 2, 3, \dots$  is called the marginal probability distribution of Y, where  $P_{*j} = \sum_i p_{ij} = P(Y = y_j)$ .

In the continuous case,

$$f_x(x) = \int_{-\infty}^{\infty} f(x, y) dy \text{ is called the marginal density of X.}$$

$$\text{Similarly, } f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \text{ is called the marginal density of Y}$$

### Note

$$P(a \leq X \leq b) = P(a \leq X \leq b, -\infty < Y < \infty)$$

$$= \int_{-\infty}^{\infty} \int_a^b f(x, y) dx dy$$

$$= \int_a^b \left[ \int_{-\infty}^{\infty} f(x, y) dy \right] dx$$

$$= \int_a^b f_x(x) dx$$

Similarly,

$$P(c \leq Y \leq d) = \int_c^d f_Y(y) dy$$

### Problems

**1 The joint probability mass function (PMF) of X and Y is**

P (x, y)	Y	0	1	2
X	0	0.1	0.04	0.02
	1	0.08	0.2	0.06
	2	0.06	0.14	0.3



2 The joint probability mass function of (X,Y) is given by

$$P(x, y) = k(2x + 3y); x = 0, 1, 2, y = 1, 2, 3.$$

Find the marginal probability distribution of X

	Y		
X	1	2	3
0	3K	6K	9K
1	5K	8K	11K
2	7K	10K	13K

**Answer**

∴ P(x,y) be the probability mass function, we have

$$\sum_{j=1}^3 \sum_{i=0}^2 P(x_i, y_j) = 1$$

$$\therefore 3K + 6K + 9K + 5K + 8K + 11K + 7K + 10K + 13K$$

$$= 1 \Rightarrow 72K = 1 \Rightarrow K = \frac{1}{72}$$

The marginal probability distribution of X :

	Y			$P_{i^*} = P(X = x_i)$
X	1	2	3	
0	3/72	6/72	9/72	$P(X = 0) = 18/72$
1	5/72	8/72	11/72	$P(X = 1) = 24/72$
2	7/72	10/72	13/72	$P(X = 2) = 30/72$

Hence the marginal probability distribution of X are given by

$$P(X = 0) = 18/72, P(X = 1) = 24/72 \text{ and } P(X = 2) = 30/72$$

**3 The following table represents the joint probability distribution of the discrete random variable (X, Y)**

$\begin{matrix} Y \\ X \end{matrix}$	1	2	3
1	$1/12$	$1/6$	0
2	0	$1/9$	$1/5$
3	$1/18$	$1/4$	$2/15$

- (i) Evaluate the marginal distribution of X and Y.  
(ii) Find the conditional distribution of X given  $Y = 2$ .  
(iii) Find the conditional distribution of Y given  $X = 3$ .  
(iv) Find  $P(X \leq 2, Y = 3)$ .  
(v) Find  $P(Y \leq 2), P(X + Y < 4)$ .

**Answer**

$\begin{matrix} X \\ Y \end{matrix}$	1	2	3	$p(y)$
1	$1/12$	$1/6$	0	$1/4$
2	0	$1/9$	$1/5$	$14/45$
3	$1/18$	$1/4$	$2/15$	$79/180$
$p(x)$	$5/36$	$19/36$	$1/3$	1

(i) Marginal distribution of X is

x	1	2	3
$p(x)$	$5/36$	$19/36$	$1/3$

Marginal distribution of Y is



y	1	2	3
p(y)	1/4	14/45	79/180

(ii) Conditional distribution of X given Y = 2 is given

$$P(X = x / Y = 2) = \frac{P(X = x \cap Y = 2)}{P(Y = 2)}, \text{ where } x = 1, 2, 3$$

$$P(X = 1 / Y = 2) = \frac{P(X = 1 \cap Y = 2)}{P(Y = 2)}$$

$$= \frac{0}{14/45}$$

$$= 0$$

$$P(X = 2 / Y = 2) = \frac{P(X = 2 \cap Y = 2)}{P(Y = 2)}$$

$$= \frac{1/9}{14/45}$$

$$= 5/14$$

$$P(X = 3 / Y = 2) = \frac{P(X = 3 \cap Y = 2)}{P(Y = 2)}$$

by

$$= \frac{1/5}{14/45}$$

$$= 9/14$$

(iii) Conditional distribution of Y given X = 3

$$P(Y = y / X = 3) = \frac{P(Y = y \cap X = 3)}{P(X = 3)}, \text{ where } y = 1, 2, 3$$

$$P(Y = 1 / X = 3) = \frac{P(Y = 1 \cap X = 3)}{P(X = 3)}$$

$$= \frac{0}{1/3}$$

$$= 0$$

$$P(Y = 2 / X = 3) = \frac{P(Y = 2 \cap X = 3)}{P(X = 3)}$$

$$= \frac{1/5}{1/3}$$

$$= 3/5$$

$$P(Y = 3 / X = 3) = \frac{P(Y = 3 \cap X = 3)}{P(X = 3)}$$

$$= \frac{2/15}{1/3}$$

$$= 2/5$$

(iv)  $P(X \leq 2, Y = 3)$  is given by

$$P(X \leq 2, Y = 3) = P(X = 1, Y = 3) + P(X = 2, Y = 3)$$

$$= \frac{1}{18} + \frac{1}{4}$$

$$= \frac{11}{36}$$

(v)  $P(Y \leq 2)$  is given by

$$P(Y \leq 2) = P(Y = 1) + P(Y = 2)$$

$$= \frac{1}{4} + \frac{14}{45}$$

$$= \frac{101}{80}$$

$P(X + Y < 4)$  is given by

$$P(X + Y < 4) = P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 2, Y = 1)$$

$$= \frac{1}{12} + 0 + \frac{1}{6}$$

$$= \frac{1}{4}$$

**4 The joint pdf of two random variables X and Y is given by**

$$f_{XY}(x, y) = \frac{1}{8}x(x - y); 0 < x < 2; -x < y < x \text{ and } 0, \text{ otherwise}$$

**Find  $f_{Y/X}\left(\frac{y}{x}\right)$**

**Answer**

$$f\left(\frac{y}{x}\right) = \frac{f_{XY}(x, y)}{f_X(x)}, \text{ where } f_X(x) \text{ is the marginal density function of } X$$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy = \int_{-x}^x \frac{1}{8} x(x-y) dy \\ &= \frac{1}{8} \left( x^2 y - x \frac{y^2}{2} \right)_{-x}^x \\ &= \frac{1}{8} \left( x^3 - \frac{x^3}{2} + x^3 - \frac{x^3}{2} \right) \\ &= \frac{x^3}{8}, 0 < x < 2 \end{aligned}$$

$$\begin{aligned} f_{Y/X}\left(\frac{y}{x}\right) &= \frac{\frac{1}{8} x(x-y)}{\frac{1}{8} x^3}, 0 < x < 2 \text{ and } -x < y < x \\ &= 0, \text{ otherwise} \end{aligned}$$

$$\text{i.e., } f\left(\frac{y}{x}\right) = \begin{cases} \frac{x-y}{x^2}, & -x < y < x \\ 0 & , \text{ otherwise} \end{cases}$$

**5 The joint p.d.f of a two dimensional R.V. (X,Y) is given by,**

$$f(x, y) = \begin{cases} 4xy, & 0 < x < 1, 0 < y < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

**Find P (X+Y<1).**

**Answer**

$$\begin{aligned}
P(X+Y < 1) &= \int_0^1 \int_0^{1-x} f(x, y) dy dx \\
&= \int_0^1 \int_0^{1-x} 4xy dy dx \\
&= 4 \int_0^1 x \left[ \frac{y^2}{2} \right]_0^{1-x} dx \\
&= 2 \int_0^1 x(1-x)^2 dx = 2 \int_0^1 x(1-2x+x^2) dx \\
&= 2 \int_0^1 (x-2x^2+x^3) dx \\
&= 2 \left[ \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 \\
&= 2 \left[ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = 2 \left[ \frac{6-8+3}{12} \right] = \frac{1}{6}
\end{aligned}$$

**6 The joint p.d.f of the RV (X, Y) is given by**

$$f(x, y) = Kxy e^{-(x^2+y^2)} \quad ; x > 0, y > 0.$$

**Find the value of K and prove also that X and Y are independent.**

**Solution :**

Here the range space is the entire first quadrant of the  $xy$  - plane

By the property of the joint. p. d.f., we have,

$$\int_{x>0} \int_{y>0} Kxy e^{-(x^2+y^2)} dx dy = 1$$

$$K \int_0^{\infty} \int_0^{\infty} ye^{-y^2} .xe^{-x^2} dx dy = 1$$

Put

$$x^2 = t \Rightarrow 2x dx = dt$$

$$\text{Then } \frac{K}{2} \int_0^{\infty} ye^{-y^2} \int_0^{\infty} e^{-t} dt .dy = 1$$

$$\Rightarrow \frac{K}{2} \int_0^{\infty} ye^{-y^2} \left( \frac{e^{-t}}{-1} \right)_0^{\infty} dy = 1$$

$$\Rightarrow \frac{K}{2} \int_0^{\infty} ye^{-y^2} (1) dy = 1$$

$$\Rightarrow \frac{K}{2} \int_0^{\infty} ye^{-y^2} dy = 1$$

$$\text{Put } y^2 = v \Rightarrow 2y dy = dv$$

$$\Rightarrow \frac{K}{2} \frac{1}{2} \int_0^{\infty} e^{-v} dv = 1$$

$$\Rightarrow \frac{K}{2} \frac{1}{2} \left( \frac{e^{-v}}{-1} \right)_0^{\infty} = 1$$

$$\Rightarrow \frac{K}{2} \frac{1}{2} (1) = 1$$

$$\Rightarrow \frac{K}{4} = 1 \Rightarrow K = 4$$

The marginal density of X is given by

$$\begin{aligned}f_x(x) &= \int_0^{\infty} f(x, y) dy \\&= 4x \int_0^{\infty} ye^{-(x^2+y^2)} dy \\&= 4xe^{-x^2} \int_0^{\infty} ye^{-(y^2)} dy \\&= 4xe^{-x^2} \int_0^{\infty} e^{-t} \frac{1}{2} dt \quad [\because \text{Put } y^2 = t \Rightarrow 2ydy = dt] \\&= 2xe^{-x^2} \left[ \frac{e^{-t}}{-1} \right]_0^{\infty} \\&= 2xe^{-x^2}, x > 0\end{aligned}$$

||| ly the marginal density of Y is given by

$$\begin{aligned}
 f_Y(y) &= \int_0^{\infty} f(x, y) dx \\
 &= 4ye^{-y^2} \int_0^{\infty} xe^{-x^2} dx \\
 &= 4ye^{-y^2} \int_0^{\infty} e^{-t} \frac{1}{2} dt \quad [\because \text{Put } x^2 = t \Rightarrow 2xdx = dt] \\
 &= 2ye^{-y^2} \left[ \frac{e^{-t}}{-1} \right]_0^{\infty} \\
 &= 2ye^{-y^2}, y > 0
 \end{aligned}$$

Consider,

$$f(x) \cdot f(y) = 4xy e^{-(x^2+y^2)} = f(x, y)$$

$\Rightarrow$  X and Y are independent RVs.

**7 If the joint pdf of (X,Y) is  $f(x,y) = \begin{cases} \frac{1}{4}, 0 \leq x, y \leq 2, \\ 0, \text{otherwise} \end{cases}$**

**Find P (x+y ≤ 1)**

**Solution :**

$$P(x+y \leq 1) = \int_0^1 \int_0^{1-y} f(x, y) dx dy$$



$$= \int_0^1 \int_0^{1-y} \frac{1}{4} dx dy$$

$$= \frac{1}{4} \int_0^1 [x]_0^{1-y} dy$$

$$= \frac{1}{4} \int_0^1 (1-y) dy$$

$$= \frac{1}{4} \left[ y - \frac{y^2}{2} \right]_0^1$$

$$= \frac{1}{4} \left[ \left( 1 - \frac{1}{2} \right) - (0 - 0) \right]$$

$$= \frac{1}{4} \left( \frac{1}{2} \right) = \frac{1}{8}.$$

**8 X and Y are two random variables having joint density function**

$$F(x,y) = \frac{1}{8}(6-x-y), 0 < x < 2, 2 < y < 4,$$

$$= 0, \quad \text{otherwise}$$

**Find (i)  $P(X < 1 \cap Y < 3)$  (ii)  $P(X + Y < 3)$  (iii)  $P(X < 1 / Y < 3)$**

**Solution :**

$$(i) P(X < 1 \cap Y < 3) = \int_{-\infty}^1 \int_{-\infty}^3 f(x, y) dx dy$$

$$\begin{aligned} &= \int_0^1 \int_2^3 \frac{1}{8} (6 - x - y) dx dy \\ &= \frac{1}{8} \int_0^1 \left( 6x - \frac{x^2}{2} - xy \right)_2^3 dy \\ &= \frac{1}{8} \int_0^1 \left( 18 - \frac{9}{2} - 3y - 12 + 2 + 2y \right) dy \\ &= \frac{1}{8} \int_0^1 \left( \frac{7}{2} - y \right) dy \\ &= \frac{1}{8} \left[ \frac{7}{2} y - \frac{y^2}{2} \right]_0^1 \\ &= \frac{1}{8} \left[ \frac{7}{2} - \frac{1}{2} \right] \\ &= \frac{3}{8} \end{aligned}$$

(ii)  $P(X + Y < 3)$

$$= \int_2^3 \int_0^{3-y} f(x, y) dx dy$$

$$= \int_2^3 \int_0^{3-y} \frac{1}{8} (6 - x - y) dx dy$$

$$= \frac{1}{8} \int_2^3 \left[ 6x - \frac{x^2}{2} - xy \right]_0^{3-y} dy$$

$$= \frac{1}{8} \int_2^3 \left[ 6(3-y) - \frac{(3-y)^2}{2} - y(3-y) \right] dy$$

$$= \frac{1}{8} \int_2^3 \left[ 18 - 6y - \frac{(3-y)^2}{2} - 3y + y^2 \right] dy$$

$$= \frac{1}{8} \int_2^3 \left[ 18 - 6y - \frac{1}{2} [9 - 6y + y^2] - 3y + y^2 \right] dy$$

$$= \frac{1}{8} \left[ 18y - \frac{6y^2}{2} - \frac{9}{2}y + \frac{6y^2}{4} - \frac{y^3}{6} - \frac{3y^2}{2} + \frac{y^3}{3} \right]_2^3$$

$$= \frac{1}{8} \left[ 54 - \frac{54}{2} - \frac{27}{2} + \frac{54}{4} - \frac{27}{6} - \frac{27}{2} + \frac{27}{3} \right]$$

$$- \frac{1}{8} \left[ 36 - \frac{24}{2} - \frac{18}{2} + \frac{24}{4} - \frac{8}{6} - \frac{12}{2} + \frac{8}{3} \right]$$

$$= \frac{5}{24}$$

(iii) To find  $P(X < 1 / Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)}$

First let us find the marginal density function of Y

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^2 \frac{1}{8} (6 - x - y) dx$$

$$= \frac{1}{8} \left[ 6x - \frac{x^2}{2} - xy \right]_0^2$$

$$= \frac{1}{8} [12 - 2 - 2y]$$

$$= \frac{1}{4} (5 - y)$$

$$P(Y < 3) = \int_2^3 f(y) dy$$

$$= \int_2^3 \frac{1}{4} (5 - y) dy$$

$$= \frac{1}{4} \left[ 5y - \frac{y^2}{2} \right]_2^3$$

$$= \frac{1}{4} \left[ 5(3 - 2) - \frac{9}{2} + 2 \right]$$

$$= \frac{1}{4} \left[ 7 - \frac{9}{2} \right] = \frac{5}{8}$$

$$P(X < 1 / Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$$

**9 If X and Y have the joint p.d.f.**

$$f(x,y) = \begin{cases} \frac{3}{4} + xy, & 0 < x < 1, 0 < y < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

**Find  $f\left(\frac{y}{x}\right)$  and  $P(Y > 1/2 / X = 1/2)$**

**Solution**

$$\text{Given } f(x,y) = \begin{cases} \frac{3}{4} + xy, & 0 < x < 1, 0 < y < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

W.K.T.  $f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$  where  $f(x)$  is the marginal density of X.

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dy$$

$$= \int_0^1 \left( \frac{3}{4} + xy \right) dy$$

$$= \left[ \frac{3}{4} y + \frac{xy^2}{2} \right]_0^1$$

$$= \frac{3}{4} + \frac{x}{2}, 0 < x < 1$$

$$\text{Now } f\left(\frac{y}{x}\right) = \frac{\frac{3}{4} + xy}{\frac{3}{4} + \frac{x}{2}} = \frac{3 + 4xy}{3 + 2x}$$

$$P(Y > 1/2 / X = 1/2) = \int_{1/2}^1 \left[ f\left(\frac{y}{x}\right) \right]_{x=1/2} dy$$

$$= \int_{1/2}^1 \left[ \frac{3+4xy}{3+2x} \right]_{x=1/2} dy$$

$$= \int_{1/2}^1 \frac{3+2y}{3+1} dy$$

$$= \frac{1}{4} [3y + y^2]_{1/2}^1$$

$$= \frac{1}{4} \left[ 4 - \frac{3}{2} - \frac{1}{4} \right]$$

$$= \frac{1}{16} (16 - 6 - 1)$$

$$= \frac{9}{16}$$

**10 Suppose the point probability density function (PDF) is given by**

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2); & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & , \text{ otherwise} \end{cases}$$

**Obtain the marginal PDF of X and that of Y. Hence find  $P\left[\frac{1}{4} \leq y \leq \frac{3}{4}\right]$ .**

**Solution :**

$$\text{Given that } f(x, y) = \begin{cases} \frac{6}{5}(x + y^2); & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & , \text{ otherwise} \end{cases}$$

The marginal pdf of X is

$$\begin{aligned}
 f(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\
 &= \frac{6}{5} \int_0^1 (x + y^2) dy \\
 &= \frac{6}{5} \left[ xy + \frac{y^3}{3} \right]_0^1 \\
 &= \frac{6}{5} \left( x + \frac{1}{3} \right), 0 < x \leq 1
 \end{aligned}$$

The marginal pdf of Y is

$$\begin{aligned}
 f(y) &= \int_{-\infty}^{\infty} f(x, y) dx \\
 &= \frac{6}{5} \int_0^1 (x + y^2) dx \\
 &= \frac{6}{5} \left[ \frac{x^2}{2} + xy^2 \right]_0^1 \\
 &= \frac{6}{5} \left( y^2 + \frac{1}{2} \right), 0 \leq y \leq 1.
 \end{aligned}$$

$$\begin{aligned}
 P\left[\frac{1}{4} \leq y \leq \frac{3}{4}\right] &= \int_{1/4}^{3/4} f(y) dy \\
 &= \int_{1/4}^{3/4} \left[ \frac{6}{5} \left( y^2 + \frac{1}{2} \right) \right] dy
 \end{aligned}$$

$$\begin{aligned}
&= \frac{6}{5} \left[ \frac{y^3}{3} + \frac{y}{2} \right]_{1/4}^{3/4} \\
&= \frac{6}{5} \left[ \frac{2y^3 + 3y}{6} \right]_{1/4}^{3/4} \\
&= \frac{1}{5} [2y^3 + 3y]_{1/4}^{3/4} \\
&= \frac{1}{5} \left[ 2 \left( \frac{27}{64} \right) + \frac{9}{4} - \left( 2 \left( \frac{1}{64} \right) + \frac{3}{4} \right) \right] \\
&= \frac{1}{5} \left[ \frac{27}{32} + \frac{9}{4} - \frac{1}{32} - \frac{3}{4} \right] \\
&= \frac{1}{5} \left[ \frac{26}{32} + \frac{6}{4} \right] \\
&= \frac{1}{5} \left[ \frac{13}{16} + \frac{3}{2} \right] \\
&= \frac{1}{5} \left[ \frac{13 + 24}{16} \right] \\
&= \frac{1}{80} [37] \\
&= \frac{37}{80} \\
&= 0.4625
\end{aligned}$$



**11 If joint density function of the two RVs 'X' and 'Y' be**

$$f(x, y) = \begin{cases} e^{-(x+y)}, & x \geq 0, y \geq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

**Find (i) P(X<1) and (ii) P(X+Y<1)**

**Solution**

To find the marginal density function of X

Let the marginal density function of X be g(x) and it is defined as

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$= \int_0^{\infty} e^{-(x+y)} dy$$

$$= e^{-x} \int_0^{\infty} e^{-y} dy$$

$$= e^{-x} \left[ -e^{-y} \right]_0^{\infty}$$

$$= e^{-x}, x \geq 0$$

$$\therefore g(x) = e^{-x}, x \geq 0$$

(i) To find P(X < 1)

$$P(X < 1) = \int_0^1 g(x) dx$$

$$= \int_0^1 e^{-x} dx$$

$$= -e^{-x} \Big|_0^1$$

$$= -[e^{-1} - 1]$$

$$= 1 - e^{-1}$$

(ii) To find  $P(X+Y < 1)$

'x' varies from '0' to  $(1-y)$  and 'y' varies from '0' to '1'

$$\therefore P(X+Y < 1) = \int_0^1 \int_0^{1-y} e^{-(x+y)} dx dy = \int_0^1 e^{-y} [-e^x]_0^{1-y} dy$$

$$= \int_0^1 e^{-y} (1 - e^{-(1-y)}) dy$$

$$= \int_0^1 [e^{-y} - e^{-1}] dy$$

$$= [-e^{-y} - e^{-1}y]_0^1$$

$$= [e^{-1} + e^{-1} - 1]$$

$$= 1 - 2e^{-1}$$

**12 Given**  $f_{xy}(x, y) = Cx(x - y), 0 < x < 2, -x < y < x$  **and 0 elsewhere**

**(a) Evaluate C (b) Find  $f_X(x)$  (c)  $f_{Y/X}\left(\frac{y}{x}\right)$  and (d)  $f_Y(y)$ .**

**Solution :**

By the property of joint p.d.f, we have,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$\int_0^2 \int_{-x}^x Cx(x - y) dy dx = 1$$

$$C \int_0^2 x \left( xy - \frac{y^2}{2} \right)_{-x}^x dx = 1$$

$$C \int_0^2 x \left[ \left( x^2 - \frac{x^2}{2} \right) - \left( -x^2 - \frac{x^2}{2} \right) \right] dx = 1$$

$$C \int_0^2 x \left( \frac{x^2}{2} + \frac{3x^2}{2} \right) dx = 1$$

$$\Rightarrow C \int_0^2 2x^3 dx = 1$$

$$\Rightarrow 2C \left[ \frac{x^4}{4} \right]_0^2 = 1$$

$$\Rightarrow 2C \left[ \frac{16}{4} \right] = 1$$

$$\Rightarrow C = \frac{1}{8}$$

(b) The marginal density of X is given by,

$$f_X(x) = f(x) = \int_{-x}^x f(x, y) dy = \frac{1}{8} \int_{-x}^x x(x-y) dy$$

$$= \frac{x}{8} \left[ xy - \frac{y^2}{2} \right]_{-x}^x$$

$$= \frac{x}{8} \left[ \left( x^2 - \frac{x^2}{2} \right) - \left( -x^2 - \frac{x^2}{2} \right) \right]$$

$$= \frac{x}{8} \left[ \frac{x^2}{2} + \frac{3x^2}{2} \right]$$

$$= \frac{x}{8} (2x^2)$$

$$= \frac{x^3}{4}, 0 < x < 2$$

$$(c) \quad f_{Y/X} \left( \frac{y}{x} \right) = \frac{f(x, y)}{f(x)} = \frac{\frac{1}{8} x(x-y)}{\frac{x^3}{4}} = \frac{1}{2x^2} (x-y), -x < y < x$$

(d) The marginal density of Y is given by ,

$$\begin{aligned}f_Y(y) &= \int_0^2 \frac{1}{8} x(x-y) dx = \frac{1}{8} \int_0^2 (x^2 - xy) dx \\&= \frac{1}{8} \left[ \frac{x^3}{3} - \frac{x^2 y}{2} \right]_0^2 \\&= \frac{1}{8} \left[ \frac{8}{3} - 2y \right] \\&= \frac{1}{24} (8 - 6y) \\&= \frac{1}{24} 2(4 - 3y) \\&= \frac{(4 - 3y)}{12}, -x < y < x\end{aligned}$$

**EXERCISE****PART-A****I Choose the best answer**

1  $F_{XY}(\infty, \infty) = \dots\dots\dots$

- (a) 1                      (b) 0                      (c)
- $\infty$
- (d) none

2  $E(XY) = \dots\dots\dots$  if X and Y are independent

- (a)
- $XE(Y)$
- (b)
- $YE(X)$
- (c)
- $E(X)E(Y)$
- (d) none

3  $F(-\infty, y) = \dots\dots\dots$

- (a) 1                      (b) 0                      (c)
- $\infty$
- (d)
- $-\infty$

**Answers**

- (1) 1                      (2)
- $E(X)E(Y)$
- (3) 0

**II Fill in the blanks**

1  $F_{XY}(x, -\infty) = \dots\dots\dots$

2  $P(a < X < b, Y < c) = \dots\dots\dots$

3 If  $f(x, y)$  is the joint pdf of X and Y then  $f(y) = \dots\dots\dots$

**Answers**

(2)  $F(b, c) - F(a, c)$                       (3)  $f(y) = \int_{-\infty}^{\infty} f(x, y) dx$

**III Say True or False**

1 The joint cdf of two random variables is non – increasing.

2  $0 \leq F(x, y) \leq 1$ .

3  $f(y/x) = f(x, y)/f(y)$ .

**Answers**

- (1) False                      (2) True                      (3) False

**IV Two mark questions**

1 Define joint cumulative distribution function.

2 Define joint density function.

3 What is marginal probability density function.

4 State any two properties of joint pdf.

5 Define conditional probability for two dimensional random variables.

6 Check whether the following density function is valid or not

$$f(x) = xy, 0 < x < 3, 0 < y < 5$$

### PART-B

1 For the following bivariate distribution, find (i)  $P(X \leq 2, Y = 2)$

(ii)  $P(Y = 3)$  (iii)  $F(3)$  (iv)  $P(X < 3, Y \leq 4)$

X/Y	1	2	3	4
1	0.1	0	0.2	0.1
2	0.05	0.12	0.08	0.01
3	0.1	0.05	0.1	0.09

2 For the following bivariate distribution

X/Y	0	1	2
0	0.1	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Compute the marginal pmfs,  $P(X \leq 1, Y \leq 1)$ , check if X and Y are independent.

3 The joint pdf of X and Y is given by

$$f(x,y) = (x+y)/21, x = 1,2,3; y = 1,2.$$

Find the marginal distributions and conditional distributions.

4 The joint distribution function of X and Y is given by

$$F(x,y) = \begin{cases} (1 - e^{-x})(1 - e^{-y}), & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the marginal density function of X and Y.

(ii) Are X and Y independent?

(iii) Find the value of  $P[1 < X < 3, 1 < Y < 2]$

5 The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 9e^{-3x} e^{-3y}, & x, y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

(i) Find the marginal density function of X and Y.

(ii) Are X and Y independent?

(iii) Find the value of  $P[0 < X < 2 / Y = 2]$

6 The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2(2x+3y)/5, & 0 \leq x, y \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

(i) Find the marginal density function of X and Y.

(ii) Are X and Y independent?

(iii) Find the value of  $P[1 < Y < 3 / X = 2]$



## CHAPTER-II

### Expected values of a two dimensional random variable

If  $(X, Y)$  is a two dimensional discrete random variable with joint probability mass function  $p_{ij}$ , then 
$$E[g(X, Y)] = \sum_j \sum_i g(x_i, y_j) p_{ij}$$

If  $(X, Y)$  is a two dimensional continuous random variable with joint probability density function  $f(x, y)$ , then 
$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

### Properties

(i)  $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$ , where  $f_x(x)$  is the marginal density of  $X$ .

(ii)  $E[h(Y)] = \int_{-\infty}^{\infty} h(y) f_y(y) dy$ , where  $f_y(y)$  is the marginal density of  $Y$ .

(iii)  $E(X + Y) = E(X) + E(Y)$

(iv)  $E(XY) = E(X) E(Y)$  if  $X$  and  $Y$  are independent.

### Covariance

The covariance between two random variables  $X$  and  $Y$  is defined as

$$\text{Cov}(X, Y) = E[(X - \bar{X})(Y - \bar{Y})]$$

### Correlation Coefficient

The correlation coefficient between two random variables  $X$  and  $Y$  is defined

as, 
$$\rho_{xy} = \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y}$$

**Note**

- (i) If  $\rho_{xy} = 0$  then X and Y are said to be uncorrelated.  
(ii) If  $E(XY) = 0$  then X and Y are said to be orthogonal.

**Note**

$$V(X + Y) = V(X) + V(Y) + 2 \text{Cov}(X, Y)$$

**Properties of covariance**

$$\text{Cov}(aX, bY) = ab \text{Cov}(X, Y)$$

$$\text{Cov}(X + a, Y + b) = \text{Cov}(X, Y)$$

$$\text{Var}(aX + bY + c) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y)$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

**Properties of correlation coefficient**

$$-1 \leq \rho(X, Y) \leq 1$$

Correlation coefficient is independent of change of origin and scale. i.e., If  $U = (X-a)/h$  and  $V = (Y-b)/k$  where  $h, k > 0$  then  
 $\rho(X, Y) = \rho(U, V)$ .

**Problems**

**1 Suppose the joint probability mass function of a RV (X,Y) is given by,**

$$P_{xy}(x, y) = \begin{cases} \frac{1}{3}, & \text{for } (0,1), (1,0), (2,2) \\ 0, & \text{otherwise} \end{cases}$$

**(i) Are X and Y independent ? (ii) Are X and Y uncorrelated ?**

Y / X	0	1	2	$P(y) = P_{*j}$
0	0	1/3	0	1/3
1	1/3	0	0	1/3
2	0	0	1/3	1/3
$P(x) = P_{i*}$	1/3	1/3	1/3	1

We have  $P_{01} = \frac{1}{3}$

$$P_{0*} = \frac{1}{3} \text{ and } P_{*1} = \frac{1}{3}$$

$$P_{0*} \times P_{*1} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \neq P_{01}$$

$\Rightarrow$  X and Y are not independent.

$$E(X) = \sum x_i P_i = (0 \times 1/3) + (1 \times 1/3) + (2 \times 1/3) = \frac{1}{3} + \frac{2}{3} = 1$$

$$E(Y) = \sum y_i P_i = (0 \times 1/3) + (1 \times 1/3) + (2 \times 1/3) = \frac{1}{3} + \frac{2}{3} = 1$$

$$E(X^2) = \sum x_i^2 P_i = (0 \times 1/3) + (1 \times 1/3) + (4 \times 1/3) = \frac{5}{3}$$

$$E(Y^2) = \sum y_i^2 P_i = (0 \times 1/3) + (1 \times 1/3) + (4 \times 1/3) = \frac{5}{3}$$

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{5}{3} - 1 = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= E(Y^2) - [E(Y)]^2 \\ &= \frac{5}{3} - 1 = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} E(XY) &= \sum_i \sum_j x_i y_j P_{ij} \\ &= (0.0.1/3) + (0.1.1/3) + (0.2.0) + (1.0.1/3) + (1.1.0) + (2.0.0) + (2.1.0) \\ &\quad + (2.2.1/3) \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} \therefore \text{COV}(X,Y) &= E(XY) - E(X) \cdot E(Y) \\ &= \frac{4}{3} - (1.1) = 1/3 \end{aligned}$$

$\Rightarrow$  the RVS X and Y are not uncorrelated.

**2 If X ,Y and Z are uncorrelated RV's with zero mean and S.D. 5 , 12 and 9 respectively, and if U = X + Y and V = Y + Z , find the correlation coefficient between U and V.**

**Solution.**

Since X, Y and Z are uncorrelated,

$$\text{Cov}(X,Y) = 0 ; \text{Cov}(Y,Z) = 0 \text{ and } \text{Cov}(Z,X) = 0$$

$$\text{i.e., } E(XY) - E(X) E(Y) = 0;$$

$$E(YZ) - E(Y) E(Z) = 0 \text{ and}$$

$$E(ZX) - E(Z) E(X) = 0$$

$$\text{Given } E(X) = 0 = E(Y) = E(Z)$$

Also given S.D.of X = 5;

$$\text{S.D.of Y} = 12$$

$$\text{and S.D.of Z} = 9$$

$$\text{i.e., } \text{Var X} = 25 , \text{Var Y} = 144 \text{ and } \text{Var Z} = 81$$

$$E(X^2) = 25; E(Y^2) = 144 \text{ and } E(Z^2) = 81$$

$$\left( \because \text{Var X} = E(X^2) - [E(X)]^2 \right)$$

To find :  $r_{UV}$

$$r_{UV} = \frac{\text{Cov}(U,V)}{\sigma_U \cdot \sigma_V}$$

$$\text{Cov}(U,V) = E(UV) - E(U) E(V)$$

$$E(UV) = E((X+Y)(Y+Z))$$

$$= E(XY + XZ + Y^2 + YZ)$$

$$= E(X) E(Y) + E(X) E(Z) + E(Y^2) + E(Y) \cdot E(Z)$$

$$= E(Y^2) \quad [\text{since } E(X) = 0 = E(Y) = E(Z)]$$

$$E(U) \cdot E(V) = E(X+Y) \cdot E(Y+Z)$$

$$\begin{aligned}
 &= [E(X) + E(Y)] [E(Y) + E(Z)] \\
 &= E(X) E(Y) + E(X) E(Z) + (E(Y))^2 + E(Y) E(Z)
 \end{aligned}$$

$$\therefore \text{Cov}(U, V) = E(Y^2) - (E(Y))^2 = \text{Var } Y = 144$$

$$\begin{aligned}
 \sigma_U^2 &= \text{Var } U \\
 &= \text{Var}(X+Y) \\
 &= \text{Var } X + \text{Var } Y + 2 \text{Cov}(X, Y) \\
 &= 25 + 144 + 0 = 169
 \end{aligned}$$

$$\therefore \sigma_U = 13$$

$$\begin{aligned}
 \sigma_V^2 &= \text{Var } V \\
 &= \text{Var}(Y+Z) \\
 &= \text{Var } Y + \text{Var } Z + 2 \text{Cov}(Y, Z) \\
 &= 144 + 81 + 0 \\
 &= 225
 \end{aligned}$$

$$\therefore \sigma_V = 15$$

$$\therefore r_{UV} = \frac{144}{(13)(15)} = 0.7385$$

**3 Calculate the correlation Co-efficient for the following heights (in inches) of fathers X and their sons Y.**

<b>X:</b>	<b>65</b>	<b>66</b>	<b>67</b>	<b>67</b>	<b>68</b>	<b>69</b>	<b>70</b>	<b>72</b>
<b>Y:</b>	<b>67</b>	<b>68</b>	<b>65</b>	<b>68</b>	<b>72</b>	<b>72</b>	<b>69</b>	<b>71</b>

**Solution :**

<b>X</b>	<b>Y</b>	<b>XY</b>	<b>X<sup>2</sup></b>	<b>Y<sup>2</sup></b>
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
<b>544</b>	<b>552</b>	<b>37560</b>	<b>37028</b>	<b>38132</b>

$$\bar{X} = \frac{\sum X}{n} = \frac{544}{8} = 68;$$

$$\bar{Y} = \frac{\sum Y}{n} = \frac{552}{8} = 69$$

$$\bar{X}\bar{Y} = 68 \times 69 = 4692$$

$$\sigma_X = \sqrt{\frac{1}{n} \sum X^2 - \bar{X}^2} = \sqrt{\frac{1}{8} (37028) - 68^2} = \sqrt{4628.5 - 4624} = 2.121$$

$$\sigma_Y = \sqrt{\frac{1}{n} \sum Y^2 - \bar{Y}^2} = \sqrt{\frac{1}{8} (38132) - 69^2} = \sqrt{4766.5 - 4761} = 2.345$$

$$\begin{aligned} \text{Cov (X,Y)} &= \frac{1}{n} \sum XY - \bar{X}\bar{Y} = \frac{1}{8} (37560) - 68 \times 69 \\ &= 4695 - 4692 = 3 \end{aligned}$$

The correlation Co-efficient of X and Y is given by,

$$r(X, Y) \text{ or } \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{3}{(2.121)(2.345)} = \frac{3}{4.973} = 0.6032$$

**4 Two random variables X and Y have the joint density**

$$f(x, y) = \begin{cases} 2 - x - y, & 0 < x < 1, 0 < y < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

**Show that Cor (X, Y) =  $-\frac{1}{11}$**

**Solution :**

The marginal density function of X is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_0^1 (2 - x - y) dy \\ &= \left[ (2 - x)y - \frac{y^2}{2} \right]_0^1 \\ &= 2 - x - \frac{1}{2} \\ &= \frac{3}{2} - x, 0 < x < 1. \end{aligned}$$

Similarly, the marginal density function of Y is,

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^1 (2 - x - y) dx$$

$$= \left[ (2-y)x - \frac{x^2}{2} \right]_0^1$$

$$= 2 - y - \frac{1}{2}$$

$$= \frac{3}{2} - y, 0 < y < 1$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_0^1 x \left( \frac{3}{2} - x \right) dx$$

$$= \int_0^1 \left( \frac{3}{2}x - x^2 \right) dx$$

$$= \left[ \frac{3x^2}{4} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{3}{4} - \frac{1}{3}$$

$$= \frac{9-4}{12}$$

$$= \frac{5}{12}$$

Similarly,  $E(Y) = \frac{5}{12}$



$$\begin{aligned}
 E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_0^1 x^2 \left( \frac{3}{2} - x \right) dx \\
 &= \int_0^1 \left( \frac{3}{2} x^2 - x^3 \right) dx \\
 &= \left[ \frac{x^3}{2} - \frac{x^4}{4} \right]_0^1 \\
 &= \frac{1}{2} - \frac{1}{4} \\
 &= \frac{1}{4}
 \end{aligned}$$

Similarly  $E(Y^2) = \frac{1}{4}$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1}{4} - \left( \frac{5}{12} \right)^2$$

$$= \frac{1}{4} - \frac{25}{144}$$

$$= \frac{36 - 25}{144}$$

$$= \frac{11}{144}$$

Similarly,  $\text{Var}(Y) = \frac{11}{144}$

$$E(XY) = \int_0^1 \int_0^1 xyf(x, y) dx dy$$

$$= \int_0^1 \int_0^1 xy(2 - x - y) dx dy$$

$$= \int_0^1 \int_0^1 (2xy - x^2y - xy^2) dx dy$$

$$= \int_0^1 \left[ x^2y - \frac{x^3y}{3} - \frac{x^2y^2}{2} \right]_0^1 dy$$

$$= \int_0^1 \left( y - \frac{y}{3} - \frac{y^2}{2} \right) dy$$

$$= \int_0^1 \left( \frac{2y}{3} - \frac{y^2}{2} \right) dy$$

$$= \left[ \frac{y^2}{3} - \frac{y^3}{6} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{6}$$

$$= \frac{2-1}{6}$$

$$= \frac{1}{6}$$

$$\therefore \text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y)$$

$$= \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12}$$

$$= \frac{1}{6} - \frac{25}{144}$$

$$= \frac{24-25}{144}$$

$$= \frac{-1}{144}$$

$\therefore$  The correlation Co-efficient  $\gamma$  is

$$\begin{aligned}
 \rho_{xy} &= \frac{\text{Cov}(X,Y)}{\sigma_x \sigma_y} \\
 &= \frac{\left(\frac{-1}{144}\right)}{\left(\frac{\sqrt{11}}{12}\right)\left(\frac{\sqrt{11}}{12}\right)} \\
 &= \frac{-1}{11}
 \end{aligned}$$

**5 Suppose that the 2D RVs (X,Y) has the joint p.d.f.**

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

**Obtain the correlation co-efficient between X and Y.**

**Solution :**

The marginal density function of X is given by,

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$= \int_0^1 (x + y) dy$$

$$= \left[ xy + \frac{y^2}{2} \right]_0^1$$

$$= x + \frac{1}{2}, 0 < x < 1$$

The marginal density function of Y is given by,

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

$$= \int_0^1 (x + y) dx$$

$$= \left[ \frac{x^2}{2} + xy \right]_0^1$$

$$= y + \frac{1}{2}, 0 < y < 1$$

$$E(X) = \int_0^1 xf(x) dx$$

$$= \int_0^1 x \left( x + \frac{1}{2} \right) dx$$

$$= \int_0^1 \left( x^2 + \frac{x}{2} \right) dx$$

$$= \left[ \frac{x^3}{3} + \frac{x^2}{4} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E(Y) = \int_0^1 y f(y) dy = \int_0^1 y \left( y + \frac{1}{2} \right) dy$$

$$= \left[ \frac{y^3}{3} + \frac{y^2}{4} \right]_0^1 = \frac{7}{12}$$

$$E(X^2) = \int_0^1 x^2 f(x) dx = \int_0^1 x^2 \left( x + \frac{1}{2} \right) dx = \int_0^1 \left( x^3 + \frac{x^2}{2} \right) dx$$

$$= \left[ \frac{x^4}{4} + \frac{x^3}{6} \right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{3+2}{12} = \frac{5}{12}$$

$$\text{III ly } E(Y^2) = \frac{5}{12}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$= \frac{11}{144}$$

$$\therefore \sigma_X = \frac{\sqrt{11}}{12} \text{ and III ly } \sigma_Y = \frac{\sqrt{11}}{12}$$

$$E(XY) = \int_0^1 \int_0^1 (xy)(x+y) dx dy$$

$$= \int_0^1 \int_0^1 (x^2 y + xy^2) dx dy$$

$$= \int_0^1 \left[ \frac{x^3 y}{3} + \frac{x^2 y^2}{2} \right]_0^1 dy$$

$$= \int_0^1 \left( \frac{y}{3} + \frac{y^2}{2} \right) dy$$

$$= \left[ \frac{y^2}{6} + \frac{y^3}{6} \right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12}$$

$$= \frac{1}{3} - \frac{49}{144}$$

$$= \frac{48 - 49}{144}$$

$$= \frac{-1}{144}$$

$$\therefore r_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\frac{-1}{144}}{\frac{\sqrt{11}}{12} \cdot \frac{\sqrt{11}}{12}} = \frac{-1}{11} = -0.0909$$

**6 The joint p.d.f. of R.V.s X and Y is given by**

$$f(x, y) = \begin{cases} 3(x + y), & 0 < x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

**Find (1) the marginal p.d.f. of X**

**(2)  $P[X + Y < \frac{1}{2}]$**

**(3)  $\text{cov}(X, Y)$**



**Solution :**

Given

$$f(x, y) = \begin{cases} 3(x + y), & 0 < x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0 & , \text{ otherwise} \end{cases}$$

The marginal p.d.f. of X is

$$\begin{aligned} g(x) &= \int_0^{1-x} 3(x + y) dy \\ &= 3 \left[ xy + \frac{y^2}{2} \right]_0^{1-x} \\ &= 3 \left[ x(1-x) + \frac{(1-x)^2}{2} \right] \\ &= 3 \left[ x - x^2 + \frac{(1-x)^2}{2} \right] \\ &= 3 \left[ x - x^2 + \frac{1+x^2-2x}{2} \right] \\ &= \frac{3}{2} [2x - 2x^2 + 1 + x^2 - 2x] \\ &= \frac{3}{2} [-x^2 + 1] \\ &= \frac{3}{2} [1 - x^2] \end{aligned}$$

$$\begin{aligned} P[X + Y < 1/2] &= \int_{x=0}^{1/2} \int_{y=0}^{\frac{1}{2}-x} 3(x + y) dy dx \\ &= 3 \int_{x=0}^{\frac{1}{2}} \left[ xy + \frac{y^2}{2} \right]_{y=0}^{y=\frac{1}{2}-x} dx \\ &= 3 \int_{x=0}^{1/2} \left[ x \left( \frac{1}{2} - x \right) + \frac{\left( \frac{1}{2} - x \right)^2}{2} \right] dx \\ &= 3 \int_{x=0}^{\frac{1}{2}} \left[ \frac{x}{2} - x^2 + \frac{\frac{1}{4} + x^2 - x}{2} \right] dx \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{2} \int_{x=0}^{\frac{1}{2}} \left[ x - 2x^2 + \frac{1}{4} + x^2 - x \right] dx \\
&= \frac{3}{2} \int_{x=0}^{1/2} \left( -x^2 + \frac{1}{4} \right) dx \\
&= \frac{3}{8} \int_{x=0}^{1/2} (1 - 4x^2) dx \\
&= \frac{3}{8} \left[ x - 4 \frac{x^3}{3} \right]_0^{1/2} \\
&= \frac{3}{8} \left[ \frac{1}{2} - \frac{4}{3} \left( \frac{1}{2} \right)^3 \right] \\
&= \frac{3}{8} \left[ \frac{1}{2} - \frac{4}{3} \times \frac{1}{8} \right] \\
&= \frac{3}{8} \left[ \frac{1}{2} - \frac{1}{6} \right] \\
&= \frac{3}{8} \left[ \frac{3-1}{6} \right] \\
&= \frac{3}{8} \left( \frac{2}{6} \right) \\
&= \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}
\end{aligned}$$

To find Cov (x,y)

$$E[XY] = \int_{x=0}^1 \int_{y=0}^{1-x} xy 3(x+y) dy dx$$

$$= 3 \int_0^1 \int_0^{1-x} [x^2 y + xy^2] dy dx$$

$$= 3 \int_0^1 \left[ x^2 \frac{y^2}{2} + \frac{xy^3}{3} \right]_0^{1-x} dx$$

$$= 3 \int_0^1 \left[ x^2 \frac{(1-x)^2}{2} + \frac{x}{3} (1-x)^3 \right] dx$$

$$= 3 \int_0^1 x(1-x)^2 \left[ \frac{x}{2} + \frac{1-x}{3} \right] dx$$

$$= 3 \int_0^1 x(1-x)^2 \left[ \frac{3x+2-2x}{6} \right] dx$$

$$= \frac{3}{6} \int_0^1 x(1-x)^2 [x+2] dx$$

$$= \frac{1}{2} \int_0^1 x(x+2) [1+x^2-2x] dx$$

$$= \frac{1}{2} \int_0^1 (x^2+2x)(1+x^2-2x) dx$$

$$= \frac{1}{2} \int_0^1 [x^2+x^4-2x^3+2x+2x^3-4x^2] dx$$

$$= \frac{1}{2} \int_0^1 [x^4-3x^2+2x] dx$$

$$= \frac{1}{2} \left[ \frac{x^5}{5} - 3\frac{x^3}{3} + 2\frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2} \left[ \frac{1}{5} - 1 + 1 \right]$$

$$= \frac{1}{2} \left[ \frac{1}{5} \right] = \frac{1}{10}$$

$$\begin{aligned}
 E[x] &= \int_0^1 x g(x) dx \\
 &= \int_0^1 x \left[ \frac{3}{2} (1 - x^2) \right] dx \\
 &= \frac{3}{2} \int_0^1 [x - x^3] dx \\
 &= \frac{3}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 \\
 &= \frac{3}{2} \left[ \frac{1}{2} - \frac{1}{4} \right] \\
 &= \frac{3}{2} \left[ \frac{1}{4} \right] = \frac{3}{8} \\
 \text{Similarly } E(y) &= \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{Cov}[x,y] &= E[xy] - E(x) E(y) \\
 &= \frac{1}{10} - \left( \frac{3}{8} \right) \left( \frac{3}{8} \right) \\
 &= \frac{1}{10} - \frac{9}{64} \\
 &= \frac{64-90}{640} = \frac{-26}{640} \\
 &= -13/320
 \end{aligned}$$

7 Find  $\text{cor}(x,y)$  for the following discrete bi-variate distribution

$\begin{matrix} X \\ Y \end{matrix}$	5	15
10	0.2	0.4
20	0.3	0.1

**Solution :**

$\begin{matrix} X \\ Y \end{matrix}$	5	15	$p(y)$
10	0.2	0.4	0.6
20	0.3	0.1	0.4
$p(x)$	0.5	0.5	1

$$E[X] = \sum xp(x) = 5 \left[ \frac{5}{10} \right] + 15 \left[ \frac{5}{10} \right] = \frac{25}{10} + \frac{75}{10} = \frac{100}{10} = 10$$

$$E[Y] = \sum yp(y) = 10 \left[ \frac{6}{10} \right] + 20 \left[ \frac{4}{10} \right] = \frac{60}{10} + \frac{80}{10} = \frac{140}{10} = 14$$

$$E[XY] =$$

$$\begin{aligned} & (10)(5)(0.2) + (10)(15)(0.4) + (20)(5)(0.3) + \\ & \quad (20)(15)(0.1) \\ & = 50 \times \frac{2}{10} + 150 \left( \frac{4}{10} \right) + 100 \left( \frac{3}{10} \right) + \frac{300}{10} \\ & = \frac{1300}{10} = 130 \end{aligned}$$

$$\begin{aligned} \text{Cov}(x,y) &= E[XY] - E[X] E[Y] \\ &= 130 - (10)(14) = 130 - 140 = -10 \end{aligned}$$

## Regression

Regression is a mathematical measure of the average relationship between two or more variables in terms of the original limits of the data.

## Lines of Regression

If the variables in a bivariate distribution are related we will find that the points in the scattered diagram will cluster around some curve called the curve. If regression of the curve is a straight line, it is called the line of regression between the variables, otherwise regression is said to be curvilinear.

The line of regression of y on x is given by,

$$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

The line of regression of x on y is given by,

$$x - \bar{x} = r \cdot \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

Where r is the correlation coefficient and  $\sigma_x, \sigma_y$  are standard deviations.

### Note

Both the lines of regression pass through  $(\bar{X}, \bar{Y})$ .

## Regression Coefficients

Regression coefficient of y on x is

$$b_{yx} = r \frac{\sigma_y}{\sigma_x}$$

Regression coefficient of y on x is

$$b_{xy} = r \frac{\sigma_x}{\sigma_y}$$

**Note**

$$\text{Correlation coefficient } r = \pm \sqrt{b_{xy} \times b_{yx}}$$

**Angle between two lines of regression**

The angle between two lines of regression is given by

$$\tan \theta = \frac{1-r^2}{r} \left( \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \right)$$

**Note**

When there is no linear regression between X and Y then the two lines of regression are at right angles

$$\text{i.e., if } r = 0 \text{ then } \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

**Problems on Regression**

**1** The regression lines between two random variables X and Y is given by  $3X + Y = 10$  and  $3X + 4Y = 12$ . Find the co-efficient of correlation between X and Y.

**Solution :**

Assume that  $3X + Y = 10$  be the regression line of X and Y.

$$\therefore 3X = -Y + 10 \Rightarrow X = \frac{-Y}{3} + \frac{10}{3}$$

Hence the regression co-efficient of X on Y is given by,

$$b_{xy} = \frac{-1}{3}$$

Similarly assume that  $3X + 4Y = 12$  be the regression line of Y on X.

$$\therefore 4Y = -3X + 12 \Rightarrow Y = \frac{-3}{4}X + 3$$

Hence the regression co-efficient of Y on X is given by

$$b_{yx} = \frac{-3}{4}$$

$\therefore$  the correlation co-efficient between X and Y is given by,

$$r_{xy} = \sqrt{\left(\frac{-1}{3}\right)\left(\frac{-3}{4}\right)} = \frac{1}{2} = 0.5$$

**2 The regression equations of X on Y and Y on X are respectively**

**$5x - y = 22$  and  $64x - 45y = 24$ . Find the mean of X and Y.**

**Solution :**

Since both the regression equations pass thro'  $(\bar{x}, \bar{y})$ , we get

$$5\bar{x} - \bar{y} = 22 \quad \dots (1)$$

$$64\bar{x} - 45\bar{y} = 24 \quad \dots (2)$$

$$\times 45 \Rightarrow 225\bar{x} - 45\bar{y} = 990 \quad \dots (3)$$

$$(3) - (2) \Rightarrow 161\bar{x} = 966 \Rightarrow \bar{x} = \frac{966}{161} = 6$$

$\therefore$  The Mean value of X = 6

Put  $\bar{x} = 6$  in (1).

$$(1) \Rightarrow 5(6) - \bar{y} = 22$$

$$\bar{y} = 30 - 22 = 8$$

$\therefore$  The mean value of Y = 8



X	Y	X=x-69	Y = y-152	$X^2$	$Y^2$	XY
62	126	-7	-26	49	676	182
64	125	-5	-27	25	729	135
65	139	-4	-13	16	169	52
69	145	0	-7	0	49	0
70	165	1	13	1	169	13
71	152	2	0	4	0	0
72	180	3	28	9	784	84
74	208	5	56	25	3136	280
		-5	24	129	5712	746

**3 Find the coefficient of correlation & obtain the lines of regressions from the data given below :**

x :	62	64	65	79	70	71	72	74
y :	126	125	139	145	165	152	180	208

**Solution**

$$\bar{X} = \frac{\sum x}{n} = 68.375$$

$$\bar{Y} = \frac{\sum Y}{n} = 155$$

$$\sigma_x^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = \frac{129}{8} - \left(\frac{-5}{8}\right)^2 = \frac{129}{8} - \frac{25}{64} = \frac{1007}{64}$$

$$\therefore \sigma_x = 3.97$$

$$\sigma_y^2 = \frac{\sum Y^2}{n} - \left(\frac{\sum Y}{n}\right)^2 = \frac{5712}{8} - \left(\frac{24}{8}\right)^2 = 705$$

$$\therefore \sigma_y = 26.56$$

$$\text{Cov}(X,Y) = \frac{\sum XY}{n} - \bar{X}\bar{Y} = \frac{746}{8} - (68.375 \times 155)$$

$$= 95.125$$

$$\therefore r_{xy} = \frac{cov(X,Y)}{\sigma_x \sigma_y} = \frac{95.125}{3.97 \times 26.56} = 0.9$$

The regression line of Y on X is,

$$Y - \bar{Y} = r \cdot \frac{\sigma_y}{\sigma_x} (X - \bar{X})$$

$$Y - 155 = 0.9 \times \frac{26.56}{3.97} (X - 68.375)$$

$$= 6.02 (X - 68.375)$$

$$\therefore Y = 6.02X - 411.62 + 155$$

$$Y = 6.02X - 256.62$$

Similarly the regression line of X on Y is,

$$X - \bar{X} = r \cdot \frac{\sigma_x}{\sigma_y} (Y - \bar{Y})$$

$$X - 68.375 = 0.9 \times \frac{3.97}{26.56} (Y - 155)$$

$$= 0.13 (Y - 155)$$

$$\therefore X = 0.13Y + 68.375 - 20.15$$

$$X = 0.13Y + 48.23$$

**EXERCISE****PART-A****I Choose the best answer**

1 The random variables are uncorrelated with each other, if the correlation between them is equal to .....

- (a)  $\infty$               (b) 1              (c) 0              (d) none

2 The correlation coefficient ranges from ..... to .....

- (a) 0,1              (b) -1,1 (c) -1,0 (d) none

3 The correlation coefficient = .....

- (a)  $\text{cov}(x,y)/\sigma_x^2 \sigma_y^2$     (b)  $\text{cov}(x,y)/2\sigma_x^2 \sigma_y^2$     (c)  $\text{cov}(x,y)/\sigma_x \sigma_y$     (d) none

4 The angle between two regression lines is given by  $\tan \theta$  .....

- (a)  $\frac{(1-r^2)}{r} X \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$     (b)  $\frac{(1-r^2)}{r^2} X \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$     (c)  $\frac{(1-r^2)}{r} X \frac{\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}$

(d) none of these

**Answers**

- (1) 0              (2) -1,1              (3)  $\text{cov}(x,y)/\sigma_x \sigma_y$               (4)  $\frac{(1-r^2)}{r} X \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$

**II Fill in the blanks**

1 The line of regression of y on x is .....

2 The regression coefficient of x on y  $b_{yx} =$  .....

3 The two lines of regression are parallel to each other if  $\tan \theta = \dots$  or .....

4 The two lines of regression are perpendicular to each other if  $\tan \theta = \dots$

**Answer**

- (1)  $(y - \bar{y}) = r \cdot \frac{\sigma_y}{\sigma_x} (x - \bar{x})$     (2)  $r \cdot \frac{\sigma_x}{\sigma_y}$               (3) 0 or  $\pi$               (4)  $\pi/2$

**III Say True or False**

1 Correlation between the variables gives the degree of relationship between them.

2 Regression between the variables gives the degree of relationship between them.

- 3 Correlation between X and Y is different from Y and X.  
 4 Regression between X and Y is same as that between Y and X.  
 5 Correlation between X and Y can be  $\infty$ .

**Answers**

- (1) True      (2) True      (3) False      (4) False      (5) False

**IV Two mark questions**

- 1 Define correlation coefficient.  
 2 What are regression lines?  
 3 Define regression coefficient.  
 4 State the properties of regression lines.  
 5 Differentiate between correlation and regression.

**PART-B**

- 1 Derive the limits of regression.  
 2 Derive the angle between two lines of regression.  
 3 Find the correlation coefficient for the following data

X	42	44	58	55	89	98	66
Y	56	49	53	58	67	76	58

- 4 The joint pdf of X and Y is given by

$$f(x,y) = cx^2(1-y), 0 \leq x,y \leq 1$$

Find the correlation coefficient between X and Y.

- 5 For the following bivariate distribution

X/Y	0	1	2
0	0.1	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Find the Correlation coefficient between X and Y.

- 6 Obtain the Regression lines for the following data

X	1	2	3	4	5	6	7
Y	9	8	10	12	11	13	14

7 The equations of lines of regressions are given by,

$x + 2y - 5 = 0$  ,  $2x + 3y - 8 = 0$  , compute the correlation coefficient.

### CHAPTER-III

### TRANSFORMATION OF RANDOM VARIABLES

#### INTRODUCTION

Most of the times, we subject the signals to undergo transformation, in order to derive certain advantages of the signal. For example, a weak signal received may be amplified in order to increase the amplitude of the signal. An ac signal may be rectified to yield a dc signal. If the input is a random variable with a particular probability density or distribution function and when it undergoes a transformation yielding another random variable; a natural question arises as to what the probability density or distribution function of the output random variable is. In other words, what is the effect of transformation unit on the pdf of input random variable?

In general, transformation may be of different nature. A single random variable may be transformed into another single random variable or more than one random variables may be transformed into a single random variable or multiple random variables may be transformed into another set of multiple random variables.

#### Transformation of two dimensional random variables

Suppose that  $(X,Y)$  is a two dimensional random variable with joint pdf  $f(x,y)$ .

Let  $U = h_1(x,y)$  and  $V = h_2(x,y)$ . Then the joint pdf of  $(U, V)$  is given by  $g(u, v) = f(x,$

$$y) | J | \text{ where } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is called the jacobian of transformation.

**Problems on transformation of random variables**

**1 Consider two random variables U and V defined as  $U = X + Y$  and  $V = X - Y$  where X and Y are two random variables. Find the joint pdf of U and V.**

**Answer**

Given  $U = X + Y$  and  $V = X - Y$

$$U + V = 2X$$

$$\Rightarrow X = \frac{U + V}{2}$$

$$U - V = 2Y$$

$$\Rightarrow Y = \frac{U - V}{2}$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left(-\frac{1}{4}\right) \left(-\frac{1}{4}\right)$$

$$= \frac{1}{16}$$

$$\therefore g(u, v) = |J| f(x, y)$$

$$= \frac{1}{16} f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

**2 If X and Y are independent random variables each normally distributed with mean zero and variance  $\sigma^2$ , find the density function of**

$$R = \sqrt{X^2 + Y^2} \text{ and } \phi = \tan^{-1}\left(\frac{Y}{X}\right).$$

**Solution**

Since X and Y are independent normal random variables with mean zero and variance  $\sigma^2$ ,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-0)^2}{2\sigma^2}}, -\infty < x < \infty \text{ and}$$

$$f(y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(y-0)^2}{2\sigma^2}}, -\infty < y < \infty$$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}}, -\infty < x < \infty \text{ and}$$

$$f(y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{y^2}{2\sigma^2}}, -\infty < y < \infty$$

Since X and Y are independent,

$$f(x, y) = f(x) f(y)$$

$$= \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}, -\infty < x, y < \infty$$



Here  $R = \sqrt{X^2 + Y^2}$  and  $\phi = \tan^{-1}\left(\frac{Y}{X}\right)$

$$\text{i.e., } R^2 = X^2 + Y^2$$

The parametric equations are

$$X = R \cos \theta, Y = R \sin \theta$$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r \cos^2 \theta + r \sin^2 \theta$$

$$= r$$

$$\therefore g(r, \theta) = f(x, y) |J|$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} \cdot r$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \cdot r, \quad 0 < r < \infty, \quad 0 < \theta < 2\pi$$

$$g_R(r) = \int_{-\infty}^{\infty} g(r, \theta) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \cdot r d\theta$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} [\theta]_0^{2\pi}$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \cdot 2\pi$$

$$= \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \quad 0 < r < \infty$$

$$g_\theta(\theta) = \int_{-\infty}^{\infty} g(r, \theta) dr$$

$$= \int_0^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \cdot r dr$$

$$= \int_0^{\infty} \frac{1}{2\pi\sigma^2} e^{-t} \sigma^2 dt \quad \left[ \text{put } \frac{r^2}{2\sigma^2} = t \Rightarrow 2rdr = 2\sigma^2 dt \right]$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[ \frac{e^{-t}}{-1} \right]_0^\infty \\
&= -\frac{1}{2\pi} [e^{-\infty} - e^0] \\
&= -\frac{1}{2\pi} [0 - 1] \\
&= \frac{1}{2\pi}
\end{aligned}$$

**3 The joint pdf of (X, Y) is given by  $f(x, y) = x + y$ ,  $0 \leq x, y \leq 1$ . Find the pdf of  $U = XY$ .**

**Solution**

Given  $f(x, y) = x + y$ ,  $0 \leq x, y \leq 1$  and  $U = XY \rightarrow (1)$

Assume the auxiliary rv  $V = Y$ .

Then from (1),

$$U = XV \Rightarrow X = \frac{U}{V}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v} - 0 = \frac{1}{v}$$

$$\therefore g(u, v) = f(x, y)|J|$$

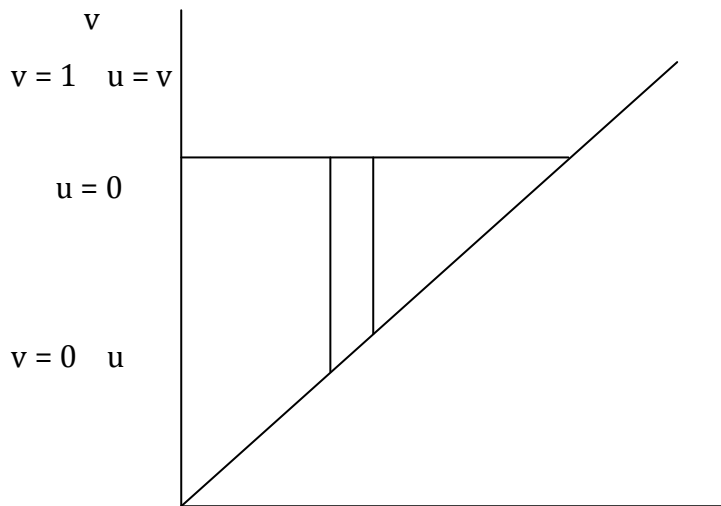
$$= (x+y) \cdot \frac{1}{v}$$

$$= \left( \frac{u}{v} + v \right) \cdot \frac{1}{v}$$

$$= \frac{u}{v^2} + 1, 0 \leq u \leq v, 0 \leq v \leq 1$$

$$[\because 0 \leq x \leq 1 \Rightarrow 0 \leq \frac{u}{v} \leq 1 \Rightarrow 0 \leq u \leq v]$$

$$[\text{and } 0 \leq y \leq 1 \Rightarrow 0 \leq v \leq 1]$$



The pdf of U is given by,

$$\begin{aligned}
g_u(u) &= \int_{-\infty}^{\infty} g(u, v) dv \\
&= \int_u^1 \left( \frac{u}{v^2} + 1 \right) dv \\
&= \left( -\frac{u}{v} + v \right)_u^1 \\
&= -u + 1 - (-1 + u) \\
&= -u + 1 + 1 - u \\
&= 2 - 2u \\
&= 2(1 - u), \quad 0 < u < 1
\end{aligned}$$

**4 If X and Y each follow exponential distribution with parameter 1 and are independent, find the pdf of U = X - Y.**

**Solution**

Since X and Y are independent random variables following exponential distribution with parameter 1,

$$\therefore f(x) = e^{-x}, x > 0 \text{ and } f(y) = e^{-y}, y > 0$$

Since X and Y are independent,

$$f(x, y) = f(x) f(y) = e^{-x} e^{-y} = e^{-(x+y)}, x > 0, y > 0$$

$$\text{Given } U = X - Y \quad \rightarrow (1)$$

Assume an auxiliary random variable V = Y.

Then from (1), U = X - V

$$\Rightarrow X = U + V$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1 - 0 = 1$$

The joint pdf of (U, V) is given by,

$$g(u, v) = f(x, y)|J|$$

$$= e^{-(x+y)} \cdot 1$$

$$= e^{-(u+v+v)}$$

$$[v > 0 \text{ and } -u < v]$$

$$= e^{-(u+2v)}$$

The pdf of U is given by,

$$g_u(u) = \int_{-\infty}^{\infty} g(u, v) dv$$

$$= \int_{-u}^{\infty} e^{-(u+2v)} dv$$

$$= e^{-u} \int_{-u}^{\infty} e^{-2v} dv$$

$$= e^{-u} \left( \frac{e^{-2v}}{-2} \right)_{-u}^{\infty}$$

$$= e^{-u} \left( 0 + \frac{e^{2u}}{2} \right)$$

$$= \frac{e^u}{2}, u < 0$$

**EXERCISE****PART-A****I Choose the best answer**

1 If  $f(x) = 3x$  and  $Y = 9X$ , then the pdf of  $Y$  is .....

- (a)  $x/4$       (b)  $x/3$       (c)  $x/5$       (d) none

2 If  $f(x) = e^{-x}$  and  $Y = X^{1/2}$ , then the pdf of  $Y =$  .....

- (a)  $2ye^{-y^2}$       (b)  $2ye^{-y}$       (c)  $2ye^y$       (d) none

3 If  $x = uv$  and  $y = v$  then  $J =$  .....

- (a) 1      (b)  $1-v$       (c)  $v$       (d) none

4 If  $x = v$  and  $y = u/v$  then  $J =$  .....

- (a)  $1/v$       (b)  $-1/v$       (c)  $v$       (d) none

**Answers**

- (1)  $x/3$  (2)  $2ye^{-y^2}$       (3)  $v$       (4)  $-1/v$

**II Fill in the blanks**

1 If  $U = X+Y$  and  $V = X-Y$  then  $J =$  .....

2 If  $U = XY$  and  $V = X^2$  then  $J =$  .....

3 If  $X$  and  $Y$  are two rvs then the pdf of  $Z = XY$  is.....

**Answers**

- (1) -2      (2)  $2v$       (3)  $f_Z(z) = \int_{-\infty}^{\infty} \left| \frac{1}{y} \right| f_X\left(\frac{z}{y}\right) f_Y(y) dy$

**III Say True or False**

1 A linear transformation of random variables affect the type of input probability density function.

2 If  $X$  is uniformly distributed in  $(0,1)$  and  $Y = 3X + 9$ , the range of  $Y$  is  $(9,12)$ .



**Answers**

(1)False      (2)True

**IV Two mark questions**

- 1 What is the relationship between the probability density function of X and  $Y=T(X)$ ?
- 2 If the density function of X is  $1/2$  ,  $-1 < x < 1$ , what is the density function of  $Y = -6 + 8x$ ?
- 3 A continuous random variable X has a pdf  $f(x) = e^{-x}$  ,  $x > 0$  .Find the pdf of  $Y = X^3$ .
- 4 If X is a random variable and  $Y = 1/X^2$ , find the distribution function of Y interms of X.

**PART-B**

- 1 Find the pdf of  $Y = e^X$  where X is a normal random variable with mean 0 and variance  $\sigma^2$ .
- 2 If X is a normal random variable with mean zero and variance 2, find the pdf of  $Y = X^{1/2}$ .
- 3 If X is exponentially distributed with parameter a, find the pdf of  $Y = \log X$ .
- 4 Let (X,Y) be a two dimensional random variable having the density function

$$f(x, y) = 4xye^{-(x^2+y^2)}, x \geq 0, y \geq 0$$

Find the density function of  $U = (X^2 + Y^2)$ .

- 5 The joint pdf of two rvs is given by

$$f(x, y) = e^{-(x+y)}, x > 0, y > 0$$

Find the pdf of  $U = (X + Y)/2$ .