Portfolio Optimization using Machine Learning

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This report presents an empirical study of allocation stability on the historical dataset of index futures. Various optimization objectives, constraint sets and moments estimation approaches are considered, including the Ledoit-Wolf shrinkage and the Black-Litterman model. An application of machine learning classifiers for view generation is studied.

Keywords: portfolio optimization, Black-Litterman model, shrinkage estimators, robust asset allocation, machine learning

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Porthos: He thinks he can challenge the mighty Porthos with a sword...
D'Artagnan: The mighty who?
Porthos: Don't tell me you've never heard of me.
D'Artagnan: The world's biggest windbag?
Porthos: Little pimple... meet me behind the Luxembourg at 1 o'clock and bring a long wooden box.

Alexandre Dumas

Introduction

The problem of optimal asset allocation is one of the most important topics in quantitative finance, attracting attention from academics and practitioners for several decades, if not centuries. The extensive study of the subject in a formal mathematical setting gave rise to the mean-variance optimization framework (MVO), pioneered by Markowitz, (Markowitz 1970), which over the course of years was enriched with numerous alternative approaches and heuristics.

Another significant breakthrough was achieved by Black and Litterman, (Black & Litterman 1992), who applied the Bayesian reasoning to incorporate one's views into the estimation procedure. With the relentless interest of the audience towards machine learning in the recent years, many studies suggest various settings to generate return forecasts, yet there is no apparent consensus on the practical usability.

We aim to conduct a comprehensive study of portfolio optimization techniques with focus on allocation robustness and sensitivity towards various inputs. The applicability of machine learning for view generation in the Black-Litterman model is also of interest.

The study is organized as follows. First, we discuss technical details and provide instructions on how to reproduce the study. Then we introduce key concepts and list a brief description of problems solved. The allocation study contains results of a one-period estimation and analyses stability of solutions. Finally, we shift to the backtesting performance perspective and search for the best performing dynamic strategies.

Implementation details and reproducibility

To cover all algorithms — optimization, moments estimates, backtesting — in a transparent and convenient way, we accompany this report with an R package `porthos`[1]. The package implements all key stages mentioned in this research:

- Constrained and unconstrained optimization for linear, quadratic and nonlinear problems;
- A set of objective functions to optimize for;
- Moments estimation, including the Ledoit-Wolf shrinkage;
- Full coverage of the Black-Litterman model;
- Calculation of the efficient frontier;
- Stability metrics for asset allocations;
- Backtesting framework and performance analysis;
- View generation via machine learning.

Existing R packages were used, and these algorithms are out of scope of the `porthos` package:

- PCA;
- Optimization solvers;
- Machine learning classifiers;
- Miscellaneous time series utilities.

To ensure the correctness of the installation, first install all necessary dependencies from CRAN. The full list is

- `magrittr`, `tidyr`, `zoo`, `xts` (time series manipulation)
- `ROI` (generic optimization infrastructure)
- `ROI.plugin.glpk` (linear solver)
- `quadprog`, `ROI.plugin.quadprog` (quadratic solver)
- `nloptr`, `ROI.plugin.nloptr` (nonlinear solver)
- `nnet`, `e1071`, `randomForest` (machine learning classifiers)
- `PortfolioAnalytics`, `PerformanceAnalytics`, `Hmisc` (utilities, quality control)
- `ggplot2`, `ggrepel` (plotting instruments)
- `xtable`, `ggpubr`, `stringr`, `rmarkdown`, `knitr` (utilities used for report generation)

The package source code is provided as a `.tar.gz` file, and the installation is typical:

```
install.packages("path/to/porthos_1.0.0.tar.gz", repos = NULL)
```

The package functionality is documented, the help is available similarly to any other R package:

```
?porthos::backtest_portfolio
```

Apart from occasional sanity checks described in this report, quality control of the `porthos` package is done on the development level, using unit tests that verify optimization solutions (whether they are indeed optimal, the constraints are not violated, etc.). Benchmark solutions

[1] An acronym for “Portfolio High-Level Optimization Suite”.

were obtained independently, using packages `PortfolioAnalytics` (Peterson & Carl 2015), and `ROI` (Theußl et al. 2017).

During the Ledoit-Wolf algorithm implementation, a bug was discovered in the `tawny` package, and reported: https://github.com/zatonovo/tawny/issues/1. The original MATLAB code was used as a reference to check results independently.

Data

The basic idea behind the data preference was to construct a set of instruments that provide exposure to different world economies. One straightforward way of doing so is to allocate between futures on leading indices. Quandl provides a dataset of continuous futures (CHRIS), where each series represents the daily price of the first futures contract that is being perpetually rolled over. Most of these series are commodities, but stock indices are also present, and their summary is presented in Table 1. The primary US index, S&P 500, is deliberately excluded and will serve as a benchmark.

For diversification purposes, two more futures from the same source were added to the list: VIX and precious metals (PMet). A total of 14 assets with an available range of historical values from 2014-01-07 till 2018-06-28 was considered, with an obvious skew towards European countries. Such imbalance will be quantified and corrected at the next step, before the optimization.

The risk-free rate was assumed to be time-dependent and was proxied by the 3M US Treasury rate (source: FRED). The historical GDP data (denominated in US dollars) was obtained from the World Bank.

Several irregularities were spotted in the raw futures dataset. A small number of entries in the Nikkei 225 and Nifty series are presumably incorrectly scaled. These were identified as outliers and rescaled correctly. No other data quality issues were observed.

Asset prices and the risk-free rate were used to calculate weekly excess returns. These constitute the primary dataset for optimization. There is a total of 223 observations for each time series.

All levels of returns, volatilities and Sharpe ratios, reported in this study, are annualized.

Quandl API code	Underlying	Country
CHRIS/LIFFE_Z	FTSE 100	United Kingdom
CHRIS/LIFFE_FCE	CAC 40	France
CHRIS/EUREX_FDAX	DAX	Germany
CHRIS/EUREX_FSMI	SMI	Switzerland
CHRIS/LIFFE_FTI	AEX	Netherlands
CHRIS/LIFFE_BXF	BEL20	Belgium
CHRIS/EUREX_FFOX	OMXH25	Finland
CHRIS/LIFFE_PSI	PSI 20	Portugal
CHRIS/MX_SXF	S&P/TSX 60	Canada
CHRIS/SGX_NK	Nikkei 225	Japan
CHRIS/CME_IBV	Ibovespa	Brazil
CHRIS/SGX_IN	Nifty	India
CHRIS/EUREX_FCPR	Precious metals	
CHRIS/CBOE_VX	VIX	

Table 1: Asset universe

Correlation analysis

High correlation between asset returns contradicts the idea of a well-diversified asset universe and is a potential problem for further optimization stability. To decorrelate, we investigate the correlation structure and remove several assets.

The correlation structure (using Spearman[2] correlation) is presented in Tables 2 and 3. As previously mentioned, there is a high correlation between European assets, so we leave FTSE 100 and remove others from our dataset, so we now have a dataset of 8 decorrelated assets. The significance is marked as follows: `p < .0001 ****; p < .001 ***; p < .01 **; p < .05 *.`

Note that two exogenous assets (precious metals and VIX) may indeed be considered as defensive due to zero-negative correlation with the rest of the assets.

	FTSE100	CAC40	DAX	SMI	AEX	BEL20
FTSE100						
CAC40	0.71****					
DAX	0.65****	0.89****				
SMI	0.72****	0.76****	0.73****			
AEX	0.77****	0.91****	0.87****	0.78****		
BEL20	0.66****	0.87****	0.81****	0.74****	0.85****	
PSI20	0.62****	0.73****	0.66****	0.61****	0.71****	0.69****

Table 2: Highly correlated assets

	FTSE100	OMXH25	TSX60	Nikkei225	Ibovespa	Nifty	PMet
FTSE100							
OMXH25	0.70****						
TSX60	0.62****	0.57****					
Nikkei225	0.49****	0.58****	0.49****				
Ibovespa	0.37****	0.32****	0.42****	0.19**			
Nifty	0.47****	0.43****	0.41****	0.40****	0.32****		
PMet	-0.05	-0.15*	-0.01	-0.29****	0.12	0.09	
VIX	-0.53****	-0.50****	-0.49****	-0.46****	-0.34****	-0.44****	0.04

Table 3: Narrowed asset universe

Examining structure with PCA

We got rid of the highly correlated assets, but there still may be a linear structure between the assets. If that's the case, then instead of asset bets we actually make bets on some factors, which constitute a low-dimensional space. This may again lead to allocation instability and contradicts the diversification idea.

We run the PCA (with data scaling) and present the results in Tables 4 and 5.

	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8
Standard deviation	1.9812	1.0841	0.8613	0.7834	0.7275	0.6324	0.5959	0.5096
Proportion of Variance	0.4906	0.1469	0.0927	0.0767	0.0662	0.0500	0.0444	0.0325
Cumulative Proportion	0.4906	0.6376	0.7303	0.8070	0.8732	0.9232	0.9675	1.0000

Table 4: PCA, importance of components

[2] The difference between Pearson and Spearman correlations is insignificant for all considered assets.

	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8
FTSE100	-0.43	0.02	-0.07	0.35	0.15	0.25	0.05	0.77
OMXH25	-0.41	-0.11	-0.13	0.25	0.48	0.26	0.38	-0.55
TSX60	-0.41	0.10	0.06	0.27	0.11	-0.38	-0.74	-0.19
Nikkei225	-0.38	-0.33	-0.08	-0.29	-0.03	-0.71	0.37	0.13
Ibovespa	-0.26	0.36	0.83	-0.27	0.07	0.06	0.15	-0.00
Nifty	-0.34	0.17	-0.40	-0.73	0.03	0.32	-0.24	0.00
PMet	0.07	0.84	-0.33	0.13	0.06	-0.31	0.26	0.00
VIX	0.38	-0.07	0.04	-0.20	0.85	-0.17	-0.14	0.20

Table 5: PCA, factor loadings

There are two important observations that follow from the PCA analysis. First, the leading components do not constitute the majority (85–90%) of the explained variance, so that the possible dimensionality reduction is not quite pronounced. That is a good sign — the asset universe is well diversified, and no asset is redundant. Second, some factors still capture meaningful structure within the data (see factor loadings, Table 5). The first factor obviously corresponds to the difference between the primary and the defensive assets ("All versus defensive"). Several factors bear geographical division, like the third ("Americas versus others"), the fourth ("Europe versus others"). The second and the fifth factors are dominated by the defensive assets. This interpretation is useful as a different way of characterizing asset allocation.

Key concepts and algorithms

Optimization objectives

The mean-variance optimization framework is well known. For our scenario with precomputed excess returns, the optimization problem for optimal allocation w^* between N assets may be formulated as

$$w^* = \underset{w}{\operatorname{argmax}} \left[w^T \mu - \frac{\lambda}{2} w^T \Sigma w \right], \tag{1}$$

where μ and Σ are the first and the second moments of the expected return distribution. Parameter λ is interpreted as an investor's risk aversion. Two important limiting cases are $\lambda = 0$ ("maximum return") and $\lambda = \infty$ ("minimum variance").

Several alternative objective functions are considered. The first is the Sharpe ratio maximization:

$$w^* = \underset{w}{\operatorname{argmax}} \left[\frac{w^T \mu}{\sqrt{w^T \Sigma w}} \right]. \tag{2}$$

The second is the tail risk minimization ("minimum VaR", "minimum ES"):

$$w^* = \underset{w}{\operatorname{argmax}} \left[w^T \mu - C_\alpha \sqrt{w^T \Sigma w} \right], \tag{3}$$

where C_α is a constant that is dependent on the confidence level α, namely $C_\alpha = -\Phi^{-1}(1-\alpha)$ for VaR and $C_\alpha = \frac{1}{1-\alpha} \int\limits_0^{1-\alpha} \Phi^{-1}(s) ds$.

The problem 1 is a quadratic programming problem, which degenerates to a linear problem if $\lambda = 0$. The analytical solution for it (given $\lambda \neq 0$) is

$$w^* = \frac{\Sigma^{-1} \mu}{\lambda}.$$

Both problems 2 and 3 are nonlinear, so they require an appropriate solver.

Unconstrained problems 1–3 are rarely solved in practice. It is most common to include the budget constraint, $\sum\limits_{i=1}^{N} w_i = 1$, or a set of boundary ("box", or "range") constraints, $l_i \leqslant w_i \leqslant u_i$ for some or all indices i. The addition of generic constraints turns the optimization problem into a task that is solved numerically, though for some specific cases the analytical solution is known. For instance, the "max Sharpe ratio" problem 2 is underdefined if no constraints are present — if the solution w^* exists, then any scaling kw^* for $k \neq 0$ is also a solution (leveraging the portfolio does not change the Sharpe ratio). We therefore will consider this problem with the budget constraint enabled, in which case

$$w^* = \frac{\Sigma^{-1} \mu}{\mathbb{1}^T \Sigma^{-1} \mu}$$

is the analytical solution. As a side note, setting $\lambda = 1$ for the mean-variance objective results in the optimal portfolio that also maximizes the logarithmic utility (Kelly portfolio).

Note that we do not include the risk-free rate in any of the optimization formulations, since we are operating on excess returns.

Checking Porthos: optimization

When implementing any numerical routine, one should convince others it works as intended. To do that, here we may use the known solutions listed above to verify that the code is returning correct results. We run the unconstrained problem 1 (with an arbitrary $\lambda = 5$) and the budget-constrained problem 3 on our data and compare results to the benchmark solutions (Table 6). The results match up to the order of 1E-16 (on par with the double numerical precision) and 1E-7, correspondingly. The second number is not perfect[3], but more than enough for our experiments.

	Mean-var, numerical	Mean-var, analytical	Max Sharpe, numerical	Max Sharpe, analytical
FTSE100	-0.7766	-0.7766	-0.5213	-0.5213
OMXH25	0.6143	0.6143	0.4124	0.4124
TSX60	0.4102	0.4102	0.2753	0.2753
Nikkei225	-0.0952	-0.0952	-0.0639	-0.0639
Ibovespa	0.1911	0.1911	0.1283	0.1283
Nifty	1.3273	1.3273	0.8910	0.8910
PMet	-0.3204	-0.3204	-0.2151	-0.2151
VIX	0.1390	0.1390	0.0933	0.0933

Table 6: Numerical results and analytical solutions

Shrinkage estimators

Many researchers have pointed out that the sample estimates of the vector of means and the covariance matrix have poor statistical properties and lead to numerically unstable allocation results, and therefore should not be used for optimization problems like 1–3. Several alternative techniques for the moments estimation were proposed, including EWMA and GARCH-based methods, as well as shrinkage estimators. We will use the Ledoit-Wolf algorithm, which may be outlined as follows. The covariance matrix is estimated as

$$\hat{\Sigma}_{Shrink} = \hat{\delta}^* F + (1 - \hat{\delta}^*) S,$$

where S is the sample covariance matrix, F is the shrinkage target. The coefficient $\hat{\delta}^* \in [0, 1]$ is the optimal shrinkage intensity: it delivers the minimum Frobenius norm of the difference between the shrinkage estimator and the "true" (in a strictly defined sense) covariance matrix. The details and the exact formulas for F and $\hat{\delta}^*$ are provided in the original paper (Ledoit & Wolf 2003).

Black-Litterman model

The innovation, proposed in (Black & Litterman 1992), is twofold. First, the vector of asset mean returns is itself a random variable, which is only known with some degree of uncertainty, $\mu \sim N(\pi, \tau\Sigma)$. To set π, an equilibrium argument is invoked, and $\pi = \lambda \Sigma w_m$, where w_m is the "market allocation", usually taken from a broad market index. The risk aversion parameter is calibrated to satisfy the equation $\lambda = \frac{w_m^T \pi}{w_m^T \Sigma w_m} = \frac{S_m}{\sqrt{w_m^T \Sigma w_m}}$, where S_m is the Sharpe ratio observed in the market. All these components form a prior distribution of the excess returns.

[3] There is room for improvement: the nonlinear solver may be calibrated by decreasing limits of the stopping criteria at the cost of sacrificing the computation speed.

The second innovation is that the model incorporates the expert views, expressed as a "pick" matrix P, where each entry is also subject to uncertainty, $P\mu \sim N(v, \Omega)$. This setup provides a way to derive the posterior distribution $N(\mu_{BL}, \Sigma_{BL})$, where

$$\mu_{BL} = \pi + \tau\Sigma P^T(\tau P\Sigma P^T + \Omega)^{-1}(v - P\pi)$$
$$\Sigma_{BL} = (1+\tau)\Sigma - \tau^2\Sigma P^T(\tau P\Sigma P^T + \Omega)^{-1}P\Sigma$$

The formula for the covariance matrix contains an extension proposed in (Meucci 2010), which is expected to provide additional computational stability.

The choice of the uncertainty parameter τ is being discussed in the literature and remains a controversial subject. Typical suggestions are $\tau \approx \frac{1}{T}$ (listed among other options in a comprehensive guide (Idzorek 2007)), or $\tau = 0.4$ according to (Meucci 2010). Our experiments indicate that changing τ does not have a significant impact on the allocation, so we choose the second option.

The question of picking the market Sharpe ratio is also non-trivial, but there exist long-term performance studies, and in our case a suitable answer is given in (Angelidis & Tessaromatis 2014): the global index-style portfolio has a Sharpe ratio around 0.5. We use this value later to obtain the market perspective on the risk aversion parameter and the equilibrium prior allocations.

Efficient frontier

The risk aversion parameter λ for the mean-variance problem 1 may be also interpreted as a parametrization for the efficient frontier. This fact is important for the allocation study and allows to examine not just a single portfolio, but the whole set of feasible portfolios. The efficient frontier may be therefore numerically calculated by shifting the risk aversion parameter and re-optimizing. But since the range $\lambda \in [0, \infty)$ is infinite, it is algorithmically more convenient to produce the range of target means $[\mu_{min}, \mu_{max}]$ (based on "minimum variance" and "maximum return" portfolios, correspondingly) and iterate over it instead. This is our approach, and the interval of possible target means is split into an arbitrary number of portfolios with defined target means (by default there are 20 portfolios in the efficient frontier).

We provide two key visualizations for the efficient frontier. The first is the standard risk-return profile, with individual assets and optimal portfolios. The second is also common for allocation robustness studies: it is an allocation profile along the efficient frontier. It is an insightful plot that shows the allocation dynamics when the desired level of return (and risk) changes.

Checking Porthos: efficient frontier

To verify the efficient frontier calculation, we use a traditional mean-variance setting with the budget and the long-only constraints. With these restrictions enabled simultaneously, we should obtain a typical curve that is bounded in the risk-return plane. In addition, non-negative allocations that sum up to one provide a straightforward allocation profile (no need to scale the allocation weights). These results are presented in Figures 1 and 2.

For additional control, we also consider two auxiliary efficient portfolios, a mean-variance portfolio with an arbitrary risk aversion of $\lambda = 5$ and a portfolio with the maximum Sharpe ratio.

Key points to verify these results are summarized below.

- Risk-return plane (Figure 1):
  - Both ends of the efficient frontier are correct (minimum variance, maximum return). The upper end is the portfolio with a 100% allocation in the asset having the highest return (Nifty);
  - Both auxiliary portfolios are efficient. The portfolio with low risk aversion is closer to the upper end of the curve;
  - All individual assets are within the feasible set.

- Allocation profile (Figure 2):
  - All allocations are positive and sum up to 1 (constraints are handled correctly);
  - There is a tendency towards corner solutions at the right end of the spectrum.

As a final check, we illustrate that the maximum Sharpe ratio is indeed achieved by the second auxiliary portfolio. To do that, we calculate the Sharpe ratio along the efficient frontier and list it along with the asset allocation in Table 7.

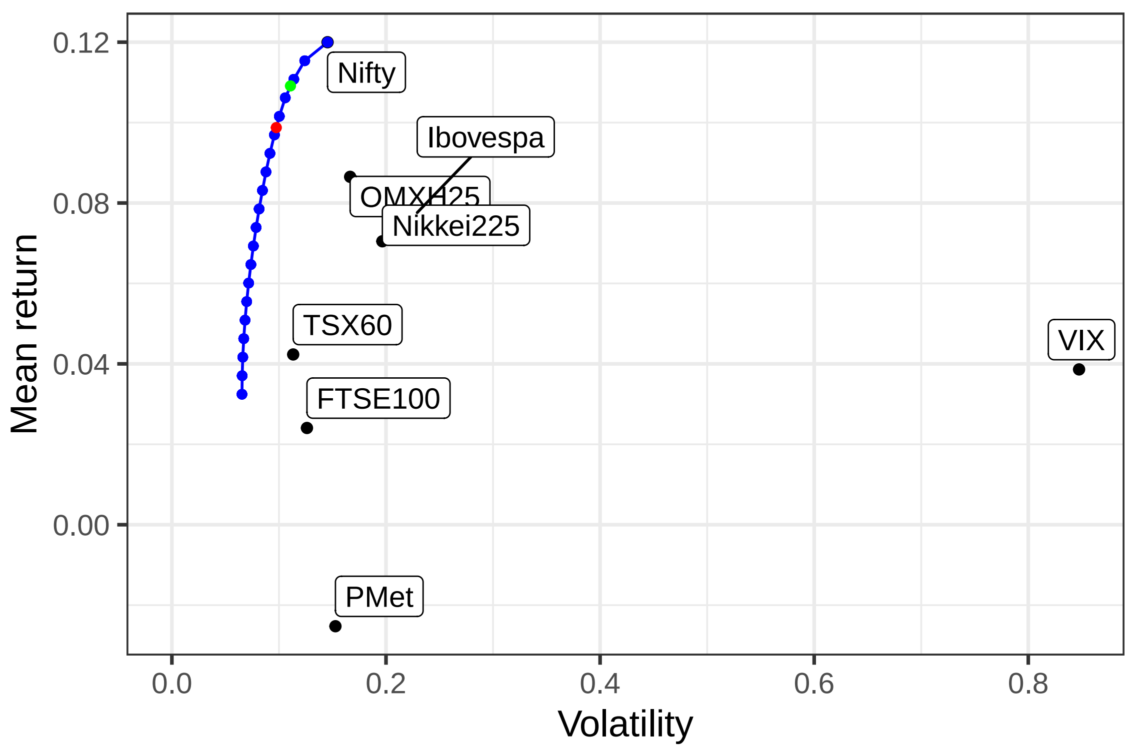

Figure 1: Efficient frontier, budget and long-only constraints. Green dot — mean-variance with $\lambda = 5$, red dot — max Sharpe.

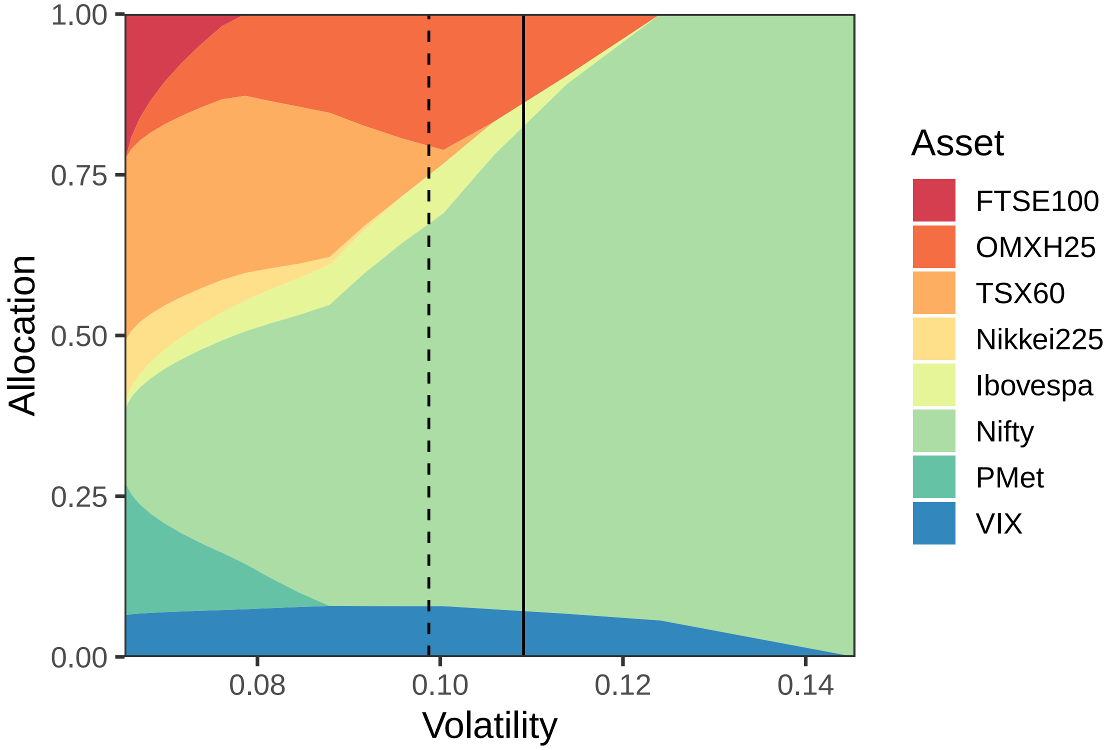

Figure 2: Efficient frontier allocations, budget and long-only constraints. Solid line — mean-variance with $\lambda = 5$, dashed line — max Sharpe.

FTSE100	OMXH25	TSX60	Nikkei225	Ibovespa	Nifty	PMet	VIX	Sharpe ratio
0.000	0.122	0.000	0.000	0.027	0.782	0.000	0.070	0.985
0.000	0.201	0.064	0.000	0.074	0.583	0.000	0.079	1.013
0.243	0.000	0.280	0.097	0.007	0.087	0.222	0.064	0.496
0.000	0.193	0.091	0.001	0.072	0.564	0.000	0.079	1.012
0.000	0.211	0.021	0.000	0.077	0.611	0.000	0.079	1.012
0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.825

Table 7: Efficient frontier portfolios. From top to bottom: mean-variance with $\lambda = 5$, max Sharpe, min variance, two portfolios with high (but not maximum) Sharpe ratio, max return.

Allocation study

The allocation study is conducted using several scenarios with varying inputs. The baseline scenario is the mean-variance with sample estimates. We are interested in the following alternatives:

- Varying optimization objectives:
  - different risk aversion parameters for the mean-variance problem;
  - Sharpe ratio maximization;
  - VaR/ES minimization.
- Varying moments estimation:
  - Ledoit-Wolf shrinkage estimators;
  - Black-Litterman model: without views and with simple discretionary views[4].

The main question of this section is to find out what scenarios provide the most robust allocation. This is not a trivial task, because we are operating on real data, and there is no known correct answer. We therefore treat the allocation robustness in a two-fold fashion, as an allocation stability a) along the efficient frontier, and b) with regard to the market data input. Either way, we build not just one, but a set of portfolios that have similar inputs, and analyze the difference in allocation. For a single setup there may be dozens of portfolios, so it is essential to come up with a quantitative measure of robustness. One such metric of variability is proposed in (Idzorek & Bertsch 2004), called SDS (style drift score):

$$SDS = \sqrt{\sum_{i=1}^{N} \mathrm{Var}(w_{ik})},$$

where k is the parametrization of the portfolio set K. We propose another metric, AAP (average allocation perturbation), defined as a Frobenius norm of the column-demeaned matrix of weights $W = \{w_{ik}\}_{i=1,\ldots,N;k\in K}$. Both SDS and AAP have a reference point of zero (no variability in weights). The third metric, BPC (boundary percentage) indicates what is the proportion of asset allocations that ended up limited by boundary constraints. A similar metric is used in the robustness study (Idzorek 2006). This metric tracks corner solutions.

The construction of the efficient frontier was described earlier. The stability test with regards to the market data input is designed similar to the well known k-fold validation routine. In our study, we drop a single period of a fixed length (10% of the historic data by default), optimize the objective, and repeat the procedure by shifting the dropped period. Naturally, the allocation should be changing to reflect the difference in the market, but we expect the allocation to be relatively stable.

In this section, we do not use the budget and the long-only constraints. Instead, we only limit the exposure in individual assets by applying box constraints $-1 \leqslant w_i \leqslant 1$. This reflects the leverage that can be obtained by trading futures contracts.

Mean-variance optimization, risk aversion and shrinkage estimators

Our first experiment studies the effect of the risk aversion parameter and of the Ledoit-Wolf shrinkage estimators. There are three options considered for the risk aversion: $\lambda = 0.1$ (low risk

[4]Views generated by machine learning methods are considered in the backtesting section.

aversion), $\lambda = 1$ (neutral risk aversion) and $\lambda = 4.699$ ("market" risk aversion). The last value was obtained by applying the Black-Litterman equilibrium argument, where the "market allocation" is proxied using weights, proportional to the respective country GDP. For precious metals and VIX, a conservative value of $\frac{1}{2N}$ was chosen to reflect the defensive role of these assets.

The efficient frontier for the baseline scenario and the scenario with Ledoit-Wolf shrinkage is outlined in Table 8, using a subset of efficient portfolios, which are indexed from 0 (minimum variance) to 20 (maximum return). The Ledoit-Wolf curve is also provided in Figure 3, along with the optimal portfolios. Note how the change of constraints is reflected in the shape of the frontier: the feasible set is now larger, and the maximum achievable Sharpe ratio is also higher.

Table 8 is also accompanied by Figure 4, which illustrates the dynamics of the allocation. Note that for the y-axis we illustrate the absolute size of the position, which explains the noticeable dip in FTSE 100 (see portfolios 18–19 in Table 8).

	FTSE100	OMXH25	TSX60	Nikkei225	Ibovespa	Nifty	PMet	VIX	Sharpe ratio
BS0 (LW0)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
BS2	-0.085	0.067	0.045	-0.010	0.021	0.146	-0.035	0.015	1.079
LW2	-0.083	0.066	0.043	-0.010	0.021	0.146	-0.036	0.015	1.067
BS6	-0.426	0.337	0.225	-0.052	0.105	0.728	-0.176	0.076	1.079
LW6	-0.414	0.330	0.215	-0.049	0.105	0.732	-0.181	0.074	1.067
BS10	-0.885	0.839	0.555	0.017	0.263	1.000	-0.287	0.158	1.048
LW10	-0.859	0.829	0.541	0.022	0.265	1.000	-0.301	0.155	1.036
BS14	-0.981	1.000	1.000	0.482	0.642	1.000	-0.441	0.300	0.945
LW14	-0.973	1.000	1.000	0.472	0.642	1.000	-0.475	0.291	0.935
BS18	-0.126	1.000	1.000	1.000	1.000	1.000	-0.822	0.496	0.838
LW18	-0.157	1.000	1.000	1.000	1.000	1.000	-0.876	0.480	0.830
BS19	0.615	1.000	1.000	1.000	1.000	1.000	-1.000	0.579	0.803
LW19	0.636	1.000	1.000	1.000	1.000	1.000	-1.000	0.566	0.794
BS 20 (LW20)	1.000	1.000	1.000	1.000	1.000	1.000	-1.000	1.000	0.683

Table 8: Efficient frontier allocations. BS — baseline, LW — Ledoit-Wolf scenarios.

Finally, the allocation stability is summarized in Table 9. The first two rows account for the efficient frontier stability, so these are independent from the risk aversion. The rest of the table corresponds to the k-fold stability.

	AAP	SDS	BPC
Frontier: sample moments	9.083	1.096	0.050
Frontier: Ledoit-Wolf	9.110	1.097	0.050
Sample moments, $\lambda = 0.1$	7.729	0.680	0.950
Ledoit-Wolf, $\lambda = 0.1$	7.749	0.684	0.950
Sample moments, $\lambda = 1$	6.818	1.123	0.637
Ledoit-Wolf, $\lambda = 1$	6.917	1.122	0.625
Sample moments, $\lambda = 4.699$	6.579	0.652	0.125
Ledoit-Wolf, $\lambda = 4.699$	6.429	0.637	0.112

Table 9: Stability metrics for different risk aversion. Top — frontier stability, bottom — k-fold stability.

Key insights of this stage are:

- Moderate and low levels of risk aversion tend to lead to corner solutions (BPC metric, Table 9) and are undercompensated for the risk taken (Sharpe ratio, Table 8). The GDP-induced risk aversion that is calculated using the Black-Litterman equilibrium argument provides a well-diversified and stable allocation (AAP and SDS metrics, Table 9).

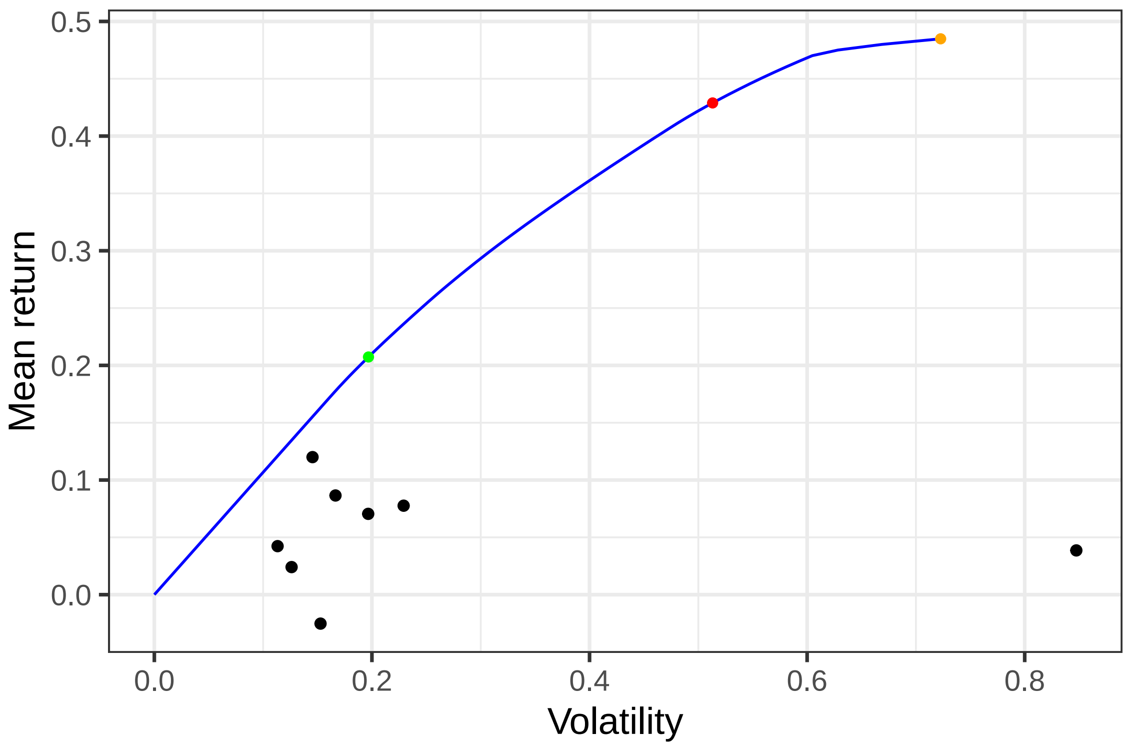

Figure 3: Efficient frontier, box constraints, Ledoit-Wolf estimators. Green dot — GDP-induced risk aversion, red dot — $\lambda = 1$, orange dot — $\lambda = 0.1$.

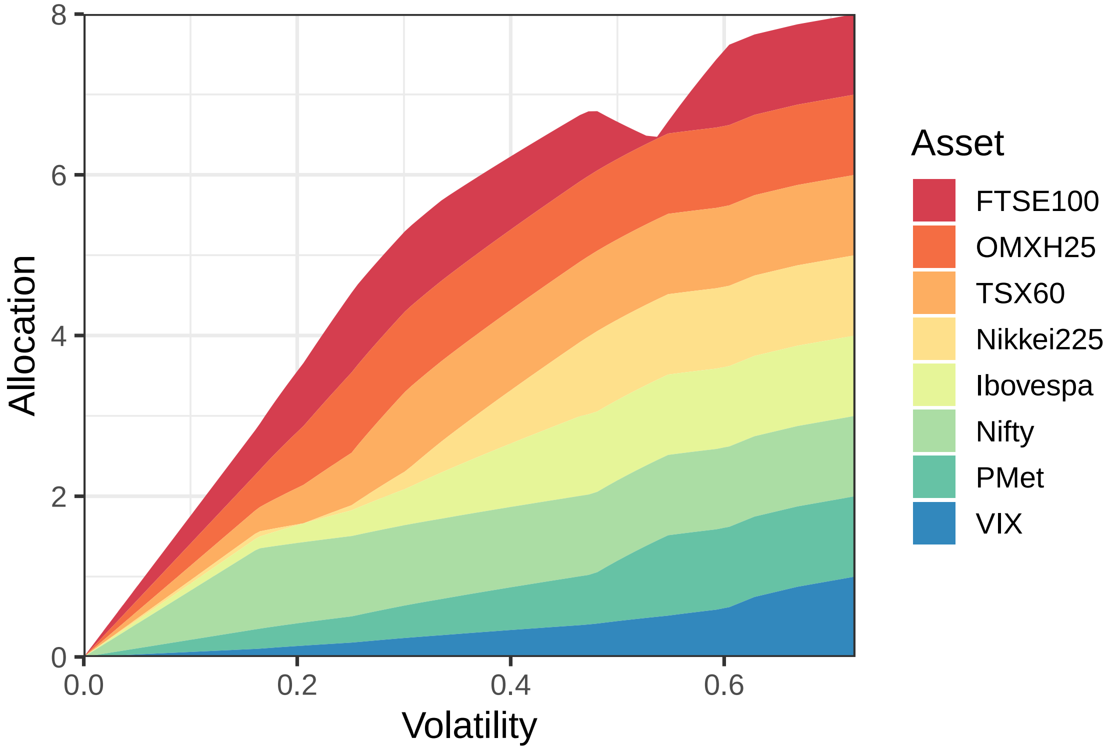

Figure 4: Efficient frontier absolute allocations, box constraints, Ledoit-Wolf estimators

- The effect of Ledoit-Wolf shrinkage is not very strong, but noticeable. The stability tends to worsen for moderate and low risk aversions, but improves for all tracked metrics when allocating according to the relatively high risk aversion. This comes at a cost of decreased Sharpe ratio, though the average decrease is only of order of 0.01 (annualized).
- There is a pronounced effect of having a very volatile asset (VIX) in the portfolio on the shape of the efficient frontier (Figure 3): the right end of the curve extends far into the high risk, high reward region. This results in a shift of the λ spectrum: even $\lambda = 1$ results in an aggressive allocation, so the well-diversified region without too many corner solutions is captured by relatively high risk aversion parameter values (i.e., more penalization for the taken risk).
- The aggressiveness of the allocation grows towards the right end of the efficient frontier (see Figure 4), and the growth is uniform across assets, with one noticeable exception. The position in FTSE 100 changes direction from short to long quite sharply between 50% and 60% target yearly volatility, sacrificing the diversification effect for more desired risk. There is a similar behavior in Nikkei 225 (portfolios 6–10, Table 8), but less pronounced.

Alternative optimization objectives

Our second experiment covers the influence of various optimization objectives. Apart from the previously used mean-variance optimization and the Sharpe ratio maximization, we now add minimization of value at risk (VaR) and expected shortfall (ES). Both new objectives have a confidence parameter α, which is chosen to be $\alpha = 0.99$, $\alpha = 0.95$ and $\alpha = 0.8$.

In this section, our baseline solution is the best allocation from the previous section: the mean-variance objective and the GDP-induced market risk aversion, plus the Ledoit-Wolf shrinkage.

One immediate difficulty for minimum VaR/ES objectives is the lack of constraints. Indeed, with $[-1, 1]$ box constraints only the allocations will tend to zero, avoiding any returns at all! In other words, an additional scaling is required to produce any nontrivial allocations. We attempt to introduce such scaling into the optimization problem (rather than applying an arbitrary scaling afterwards), by adding a budget constraint, so that all allocations sum up to the same value as for our known baseline: 1.43.

The efficient frontier for these constraints and three optimal min-ES portfolios are plotted in Figure 5. All solutions are closely allocated, so for clarity the whole set of solutions is provided in Table 10.

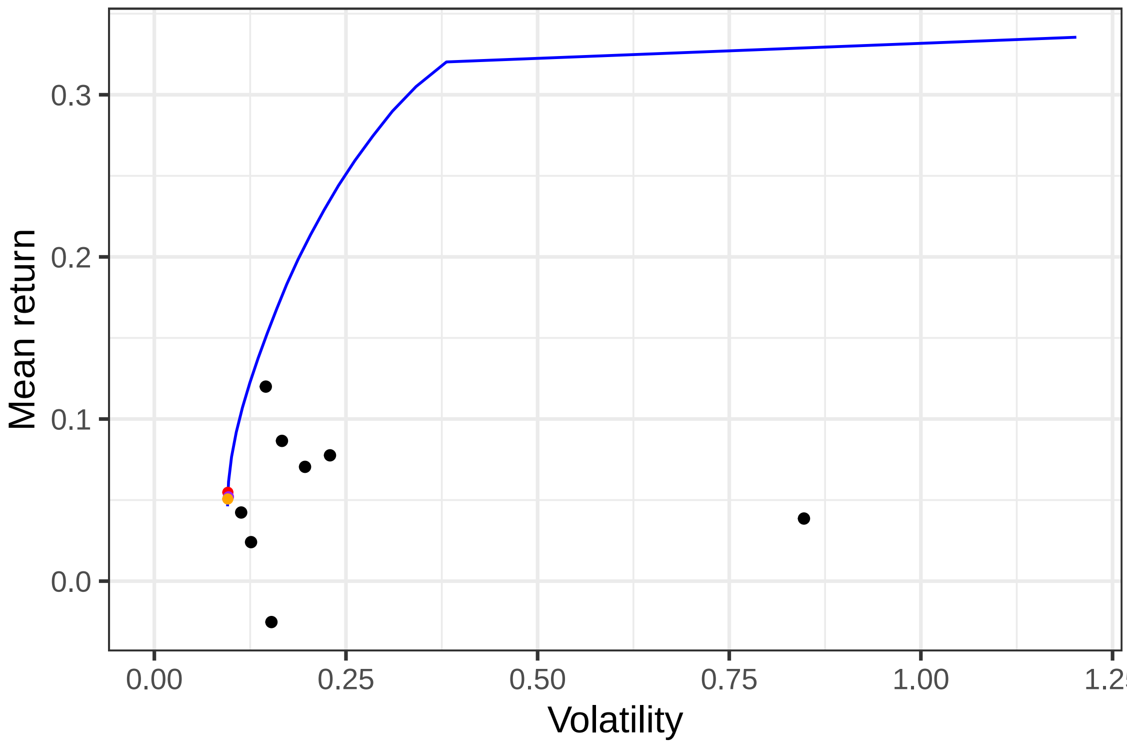

Figure 5: Efficient frontier, min VaR/ES objective. Orange dot — $\alpha = 0.99$, purple dot — $\alpha = 0.95$, red dot — $\alpha = 0.8$.

	FTSE100	OMXH25	TSX60	Nikkei225	Ibovespa	Nifty	PMet	VIX	Sharpe ratio
MVO, $\lambda = 4.699$	-0.741	0.676	0.441	-0.014	0.216	1.000	-0.280	0.133	1.053
Max Sharpe	-0.354	0.283	0.184	-0.042	0.090	0.626	-0.155	0.064	1.067
Min VaR, $\alpha = 0.99$	0.320	0.004	0.410	0.129	0.014	0.165	0.300	0.090	0.537
Min ES, $\alpha = 0.99$	0.324	0.002	0.411	0.130	0.014	0.160	0.302	0.090	0.530
Min VaR, $\alpha = 0.95$	0.307	0.011	0.410	0.126	0.017	0.179	0.292	0.090	0.558
Min ES, $\alpha = 0.95$	0.316	0.007	0.410	0.128	0.015	0.169	0.297	0.090	0.543
Min VaR, $\alpha = 0.8$	0.264	0.035	0.409	0.118	0.024	0.225	0.267	0.092	0.627
Min ES, $\alpha = 0.8$	0.299	0.016	0.410	0.125	0.018	0.187	0.287	0.091	0.571

Table 10: Allocations for different optimization objectives

Allocation stability metrics are provided in Table 11.

	AAP	SDS	BPC
Frontier: MVO	9.110	1.097	0.050
Frontier: min VaR/ES	8.386	0.985	0.000
MVO, $\lambda = 4.699$	6.429	0.637	0.112
Max Sharpe ratio	3.369	0.309	0.000
Min VaR, $\alpha = 0.99$	1.832	0.101	0.000
Min ES, $\alpha = 0.99$	1.849	0.100	0.000
Min VaR, $\alpha = 0.95$	1.779	0.104	0.000
Min ES, $\alpha = 0.95$	1.815	0.102	0.000
Min VaR, $\alpha = 0.8$	1.638	0.117	0.000
Min ES, $\alpha = 0.8$	1.749	0.106	0.000

Table 11: Stability metrics for different optimization objectives. Top — frontier stability, bottom — k-fold stability.

Here are the key insights of this stage.

- The necessity to scale the min-VaR/ES solution artificially essentially adds another scaling parameter, so now instead of calibrating the risk aversion we need to calibrate two parameters — the confidence level and the scaling parameter. It is unclear how to that properly, since the allocation sensitivity towards the confidence level α is weak (see Figure 5).
- The previous point is further aggravated by the fact that the α spectrum does not cover the efficient frontier, so the interpretability of the investor's risk preference is obscured.
- There is little difference in the optimal allocation (Table 10). There are no short positions at all. When the confidence level is fixed, the min-ES portfolio has lower Sharpe ratios than the min-VaR, and both of them have Sharpe ratios significantly lower compared to the well-diversified MVO.
- When comparing the min-VaR and the min-ES objectives (Table 11), the former is more stable according to the AAP metric, but less stable in terms of the SDS metric. The dependence on the confidence level is also counterintuitive: with increasing α, the SDS stability grows, but the AAP stability falls. Allocations are very risk averse, so there are no corner solutions at all.
- The effect of VIX on the efficient frontier observed in the previous section is even more pronounced.

Applying Black-Litterman

We have already discussed how we use the equilibrium argument to obtain the market risk aversion. This gives us the prior, π, which we call "market prior". We also consider the so-called "uninformed prior", where $\pi = \left(\frac{1}{N}, \frac{1}{N}, ..., \frac{1}{N}\right)$, which is a popular choice for discrete distributions in many Bayesian applications.

As an aside, there is an opportunity for a sanity check, when there are no views, a uniform prior and no budget constraint. In that case, the posterior allocation should always stay uniform, stretching the leverage along the efficient frontier. The top left panel in Figure 6 indicates that this property holds.

Apart from our baseline (albeit trivial) scenario — no views, uninformed prior — we consider three more: by adding a simple view, by using the market prior, and by doing both of these. Our simple (e.g. not derived from anywhere) view is as follows: we believe that emerging markets

(Ibovespa and Nifty) will outperform all other assets (excluding the defensive ones) by 5% p.a. The pick matrix is then $P = \left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, 0, 0\right)$ and the view is 0.05 with uncertainty constant equal to 1 (the same level of uncertainty as observed in the market).

The results of four scenarios are presented in Figures 6–7 and Tables 12–13.

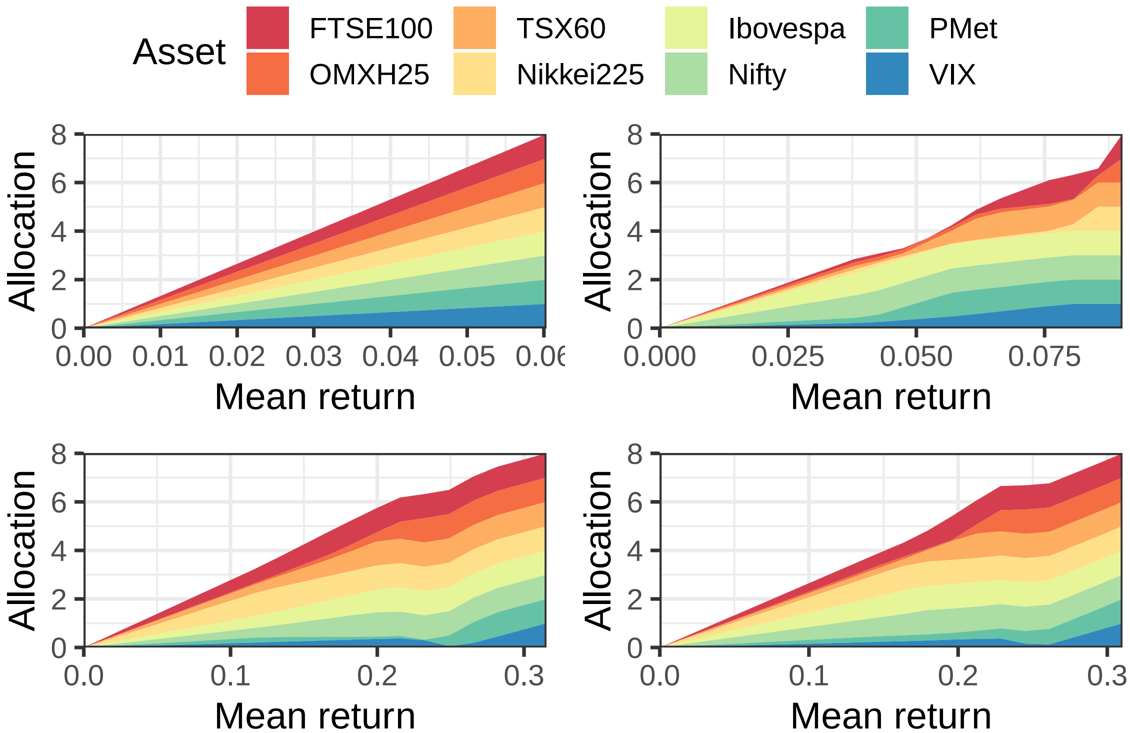

Figure 6: Absolute allocation profiles for various BL inputs. Top left — no views, uninformed prior. Top right — simple view, uninformed prior. Bottom left — no views, market prior. Bottom right — simple view, market prior.

	FTSE100	OMXH25	TSX60	Nikkei225	Ibovespa	Nifty	PMet	VIX	Sharpe ratio
Top left	0.089	0.089	0.089	0.089	0.089	0.089	0.089	0.089	0.073
Top right	-0.061	-0.061	-0.061	-0.061	0.390	0.390	0.089	0.089	0.127
Bottom left	0.131	0.012	0.078	0.210	0.106	0.088	0.045	0.045	0.348
Bottom right	0.100	-0.019	0.047	0.179	0.168	0.150	0.045	0.045	0.365

Table 12: Optimal allocations for various BL inputs. Top left —– no views, uninformed prior. Top right —- simple view, uninformed prior. Bottom left —– no views, market prior. Bottom right —– simple view, market prior.

Key conclusions of this stage are summarized below.

- The Black-Litterman model results indicate a strong dependence on the prior. The chosen priors — the uninformed and the market ones — are not that far from each other, but the optimal allocations (Table 12) are quite different, and the same applies for the whole frontier (Figure 6).
- Black-Litterman estimates show a significant sensitivity towards inputs (see how the asset location changes on the risk-return plane, Figure 7), distorting the whole efficient frontier. It is therefore essential to do a proper calibration of BL inputs.

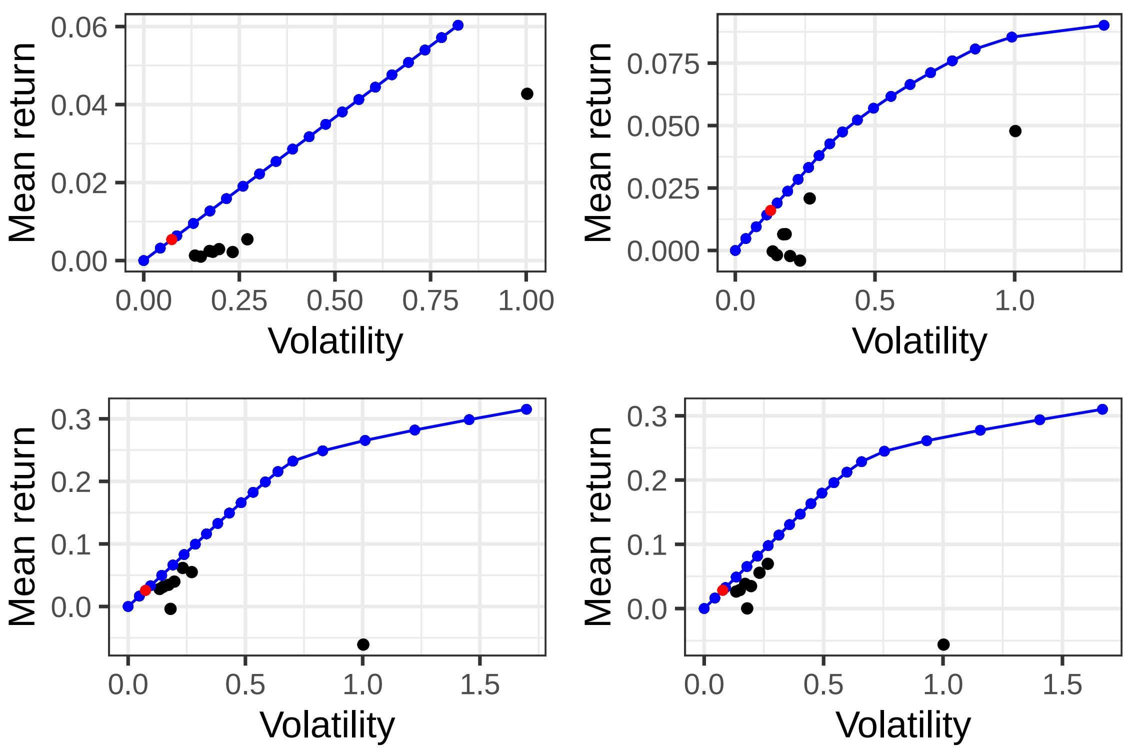

Figure 7: Efficient frontiers for various BL inputs. Top left — no views, uninformed prior. Top right — simple view, uninformed prior. Bottom left — no views, market prior. Bottom right — simple view, market prior. Red dot — optimal allocations.

	AAP	SDS	BPC	AAP	SDS	BPC
Top left	3.839	0.881	0.050	0.000	0.000	0.000
Top right	7.832	1.088	0.050	2.413	0.022	0.000
Bottom left	7.361	1.094	0.050	0.755	0.000	0.000
Bottom right	6.265	1.065	0.050	0.876	0.005	0.000

Table 13: Allocation stability for various BL inputs. Top left —– no views, uninformed prior. Top right —- simple view, uninformed prior. Bottom left —– no views, market prior. Bottom right —– simple view, market prior. Left part — frontier stability, right — k-fold stability.

- There is a clear allocation "inheritance effect" of the prior: for the uninformed prior, optimal allocations are equal for emerging/non-emerging markets, and the same picture for the defensive asset for both priors. This effect may be undesirable, since it rather reflects the lack of knowledge about assets.
- Adding a simple view has a well-pronounced effect on the allocation (see Ibovespa and Nifty allocations, Table 12), and enhances the portfolio performance. For reference, an additional comparison is provided in Figure 8 that demonstrates this effect for a different set of constraints (same as previously considered, long-only and budget).
- Stability metrics (Table 13) imply that
  - as expected, the baseline scenario with the uninformed prior and no views is "absolutely stable" regardless of the market inputs. Substituting with the market prior changes the allocation composition (AAP metric is non-zero), but leaves the portfolio insensitive to the market data (SDS metric is zero);
  - the scenario with the simple view and the market prior is the most stable along the efficient frontier;
  - the BPC metric suggests that all portfolios are free of corner solutions, which suggests that the positions are underleveraged.
- Overall, the performance of these allocations is far from being acceptable. There are several reasons for that. First, the aforementioned sensitivity towards BL inputs (i.e., a finer tuning may be required). Second, the set of constraints may have been chosen inappropriately for such setting, and poor performance may be a result of a missing leverage scaling. Finally, the provided Sharpe ratios are just parametric estimates, not produced by running actual backtests. The proper comparison is conducted in the next section.

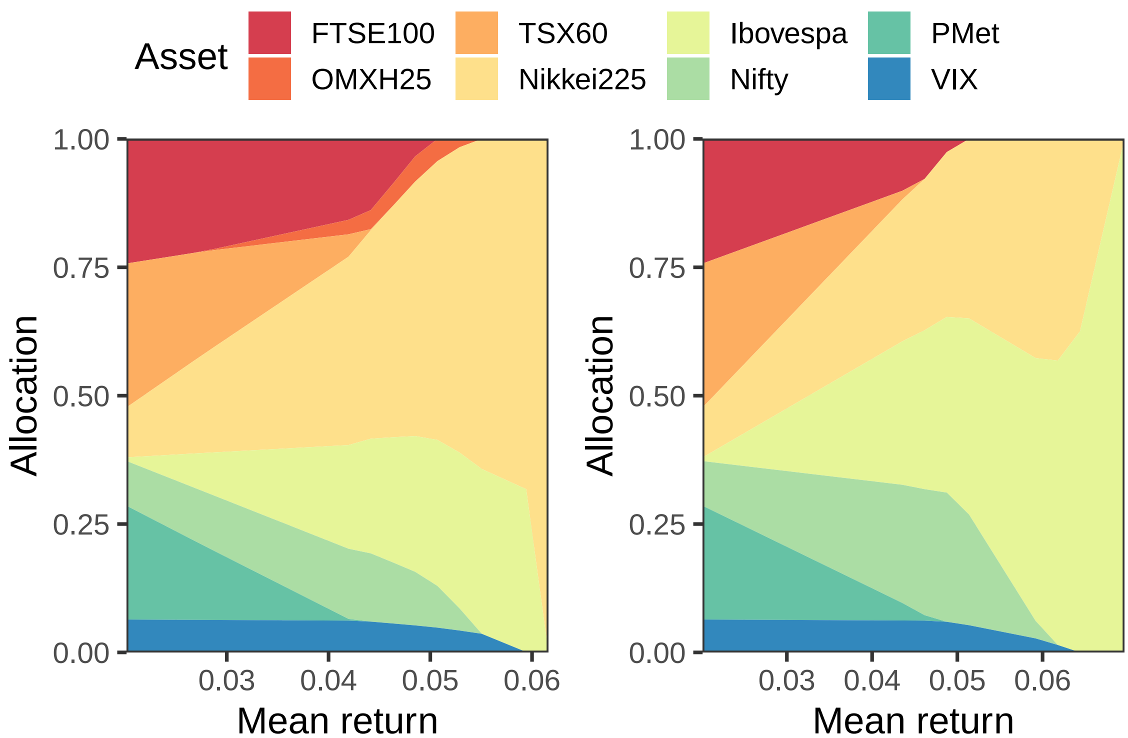

Figure 8: Efficient frontier, market prior, long-only and budget constraints. Left — no views, right — simple view in favor of emerging markets.

Backtesting

Methodology

Studying the single allocation and its' properties is an important topic, but the strategy cannot be considered tradable unless it shows good performance within the backtesting range. To get a proper understanding of the allocation strategy we employ a rolling rebalancing scheme, when the optimization is applied on a window of historical data and the optimal allocation is held for some time horizon. We therefore have two new parameters[5] — N_{train} and N_{test}. Thus, the allocation at each time t_i is always based on the available historical data of fixed length $[t_{i-N_{train}}, t_i]$, and the actual performance is aggregated by keeping this allocation for $[t_{i+1}, t_{i+1+N_{test}}]$. Then the window shifts and the procedure repeats.

The previous section indicates that using $[-1, 1]$ box constraints tends to pull asset allocations towards zero. This turns out to be an important issue when constructing a portfolio with dynamic rebalancing. None of the considered backtesting scenarios yielded acceptable performance, so we do not include these results and consider only the long-only and the budget constraints when attempting to search for the best algorithm in terms of its' historical performance.

View generation via machine learning

Machine learning methods and techniques are very popular nowadays, but one should exercise caution when applying them to financial applications, including portfolio optimization. It is a general consensus that an attempt to predict asset returns based on their previous history is a futile effort regardless of the complexity of the underlying algorithm. The idea of applying machine learning to generate views for the Black-Litterman model, however, seems like an interesting opportunity. It is also a more "fair" approach for backtesting, when we do not have a history of recorded views and try to avoid inventing our own views, which are inevitably biased, since we possess the knowledge of what happened in the market during and after the backtesting period.

We consider four candidate algorithms:

- Logistic regression;
- Support Vector Machines (SVM);
- Naive Bayes classifier;
- Random forests.

All of these belong to the supervised learning class and are suitable for non-binary classification.

The setting for every algorithm is exactly the same, as well as the training input and the test data for predictions. The same idea as for the backtesting procedure applies, when N_{train} records are used to train the classifier and generate the next view. N_{test} here has a meaning of the rolling window length. These parameters are not necessarily the same as for the backtesting procedure.

We assign bearish/bullish labels (the range of labels is $\{-2, -1, 0, 1, 2\}$) based on the training data, that is transformed from the series of returns as follows. If the value of return exceeds twice the value of the rolling standard deviation, we assign +2. If it is in range of plus one and plus two standard deviations, we assign +1, if within plus and minus one standard deviation — 0, and analogously for the negative return.

It is possible to have the generated prediction that only consists of zeroes. That is considered normal, and in such case no views are passed on to the Black-Litterman model.

[5] The naming is borrowed from calibration techniques of machine learning algorithms.

Results

For this section, the baseline scenario is the infamous $\frac{1}{N}$ strategy, which keeps a static equally weighted allocation in each asset. Despite its' simplicity, it is actually not that simple to beat it, if the asset universe is chosen appropriately. For our data, the Sharpe ratio of such strategy is surprisingly high[6]: 0.925.

Before we consider the best results, we note that the resulting optimization and backtesting framework is a convenient tool for extensive studies of the subject. There are numerous possibilities in the search for the optimal strategy, and not every investigated scenario made it to the final list. Some findings are worth being mentioned without too much detail.

There is little to no difference in backtesting performance between the baseline scenario and the MVO with varied risk aversion. Adding shrinkage estimators also has little effect. Varying N_{train} and N_{test} for the backtesting routine and/or for the ML classifiers yields either a similar or worse performance, compared to the best strategies discussed below. Minimum VaR/ES objectives for various confidence levels show a very limited success, with the ES objective having smaller Sharpe ratios compared to the VaR objective.

Table 14 outlines strategies that achieved the best performance. S0 is the baseline, $\frac{1}{N}$ strategy. All other strategies use the Black-Litterman model: S1–S5 — without views or with simple views, S6–S9 — with ML-generated views. S3–S4 have the same "simple" (which is static over the whole backtesing period) view towards emerging markets as in the previous section. In all cases, the market prior is used (the uninformed prior underperforms in the majority of scenarios).

For S5 (minimum VaR), high confidence levels result in poor performance, so the empirical range with acceptable results is $\alpha \in [0.5, 0.6]$ (penalizing the left half of the return distribution). For S5, $\alpha = 0.52$. ML-generated views achieve better results when used in conjunction with the maximum Sharpe objective.

All dynamic strategies S1–S9 have $N_{train} = 100$ and $N_{test} = 10$ for the backtesing routine, $N_{train} = 50$ and $N_{test} = 10$ for the ML classifiers.

Strategy	Opt. objective	Views
S0	Static allocation	
S1	MVO	No
S2	Max Sharpe	No
S3	MVO	Simple
S4	Max Sharpe	Simple
S5	Min VaR	No
S6	Max Sharpe	Log regression
S7	Max Sharpe	SVM
S8	Max Sharpe	Naive Bayes
S9	Max Sharpe	Random forest

Table 14: Description of optimal strategies

Experiment results are presented in Tables 15–16 and Figures 9–10. Key observations of the backtesting study are aggregated below.

- It is quite difficult to beat the static $\frac{1}{N}$ allocation, and many strategies that provided robust single-period allocations were discarded or found to be of little use. However, at least 9

[6] This is another evidence that the assets were chosen well.

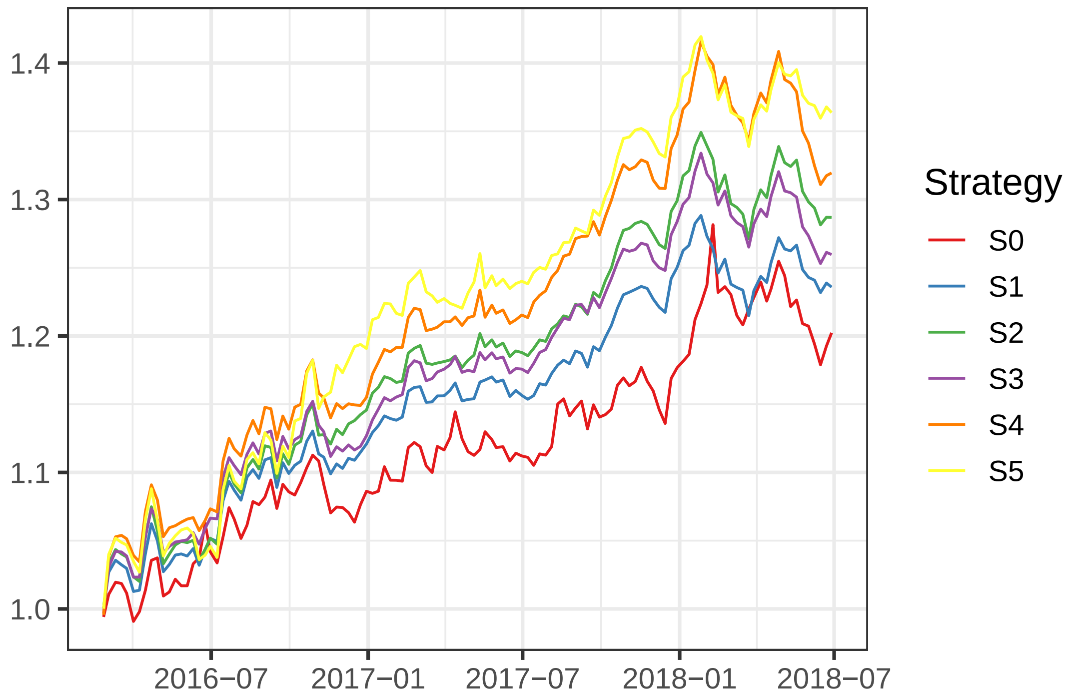

Figure 9: P&L for optimal strategies, strategies without ML views

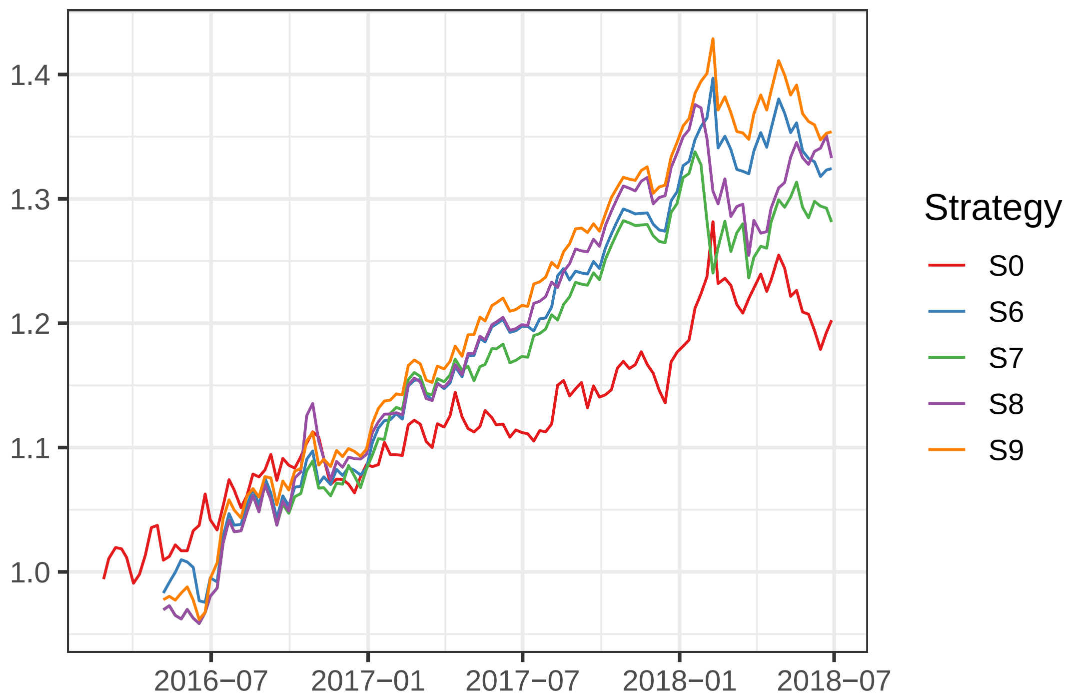

Figure 10: P&L for optimal strategies, strategies with ML views

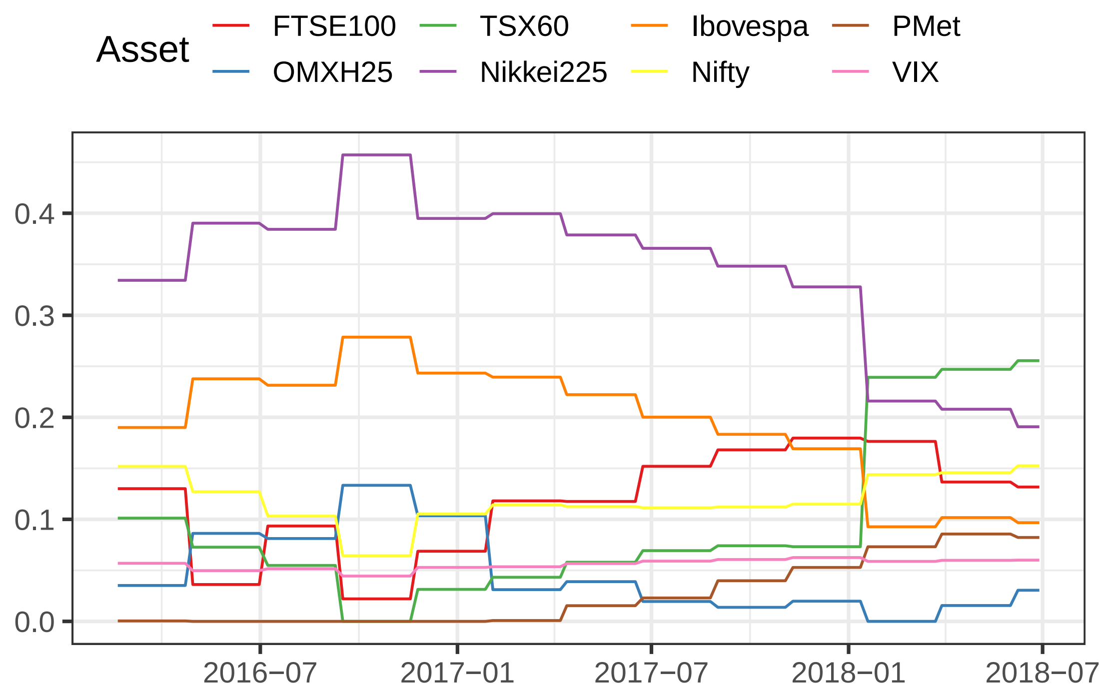

Figure 11: Asset allocations for S5

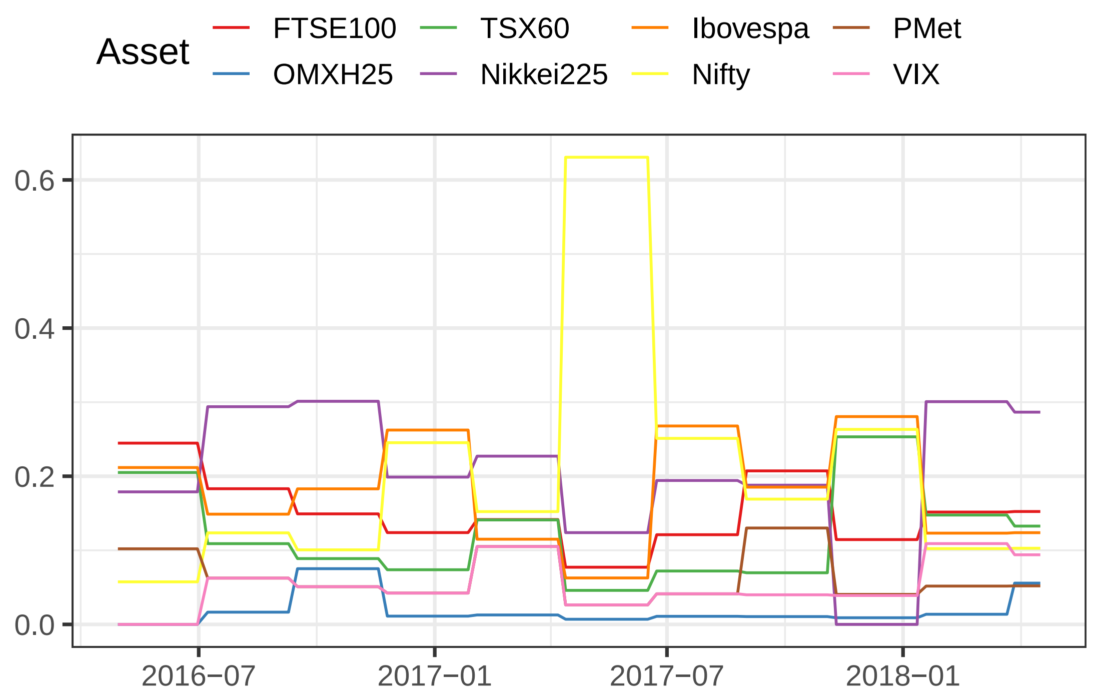

Figure 12: Asset allocations for S9

	Ann. return	Ann. volatility	Ann. Sharpe ratio	Max drawdown	VaR (95%)	ES (95%)
S0	0.081	0.088	0.925	0.080	-0.018	-0.023
S1	0.094	0.066	1.413	0.057	-0.014	-0.017
S2	0.113	0.074	1.519	0.058	-0.016	-0.019
S3	0.103	0.069	1.485	0.061	-0.014	-0.017
S4	0.124	0.080	1.561	0.074	-0.015	-0.019
S5	0.140	0.087	1.611	0.057	-0.015	-0.021
S6	0.138	0.081	1.697	0.057	-0.014	-0.024
S7	0.121	0.086	1.407	0.076	-0.019	-0.028
S8	0.141	0.089	1.592	0.088	-0.019	-0.027
S9	0.150	0.079	1.902	0.057	-0.016	-0.023

Table 15: Performance summary for optimal strategies. VaR and ES are reported as weekly. For S9, since random forest is an RNG-dependent method, this result is unstable. An average of 10 replications yields an average Sharpe ratio 1.713 with a standard deviation of 0.243.

	FTSE100	OMXH25	TSX60	Nikkei225	Ibovespa	Nifty	PMet	VIX
S0	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
S1	0.143	0.027	0.223	0.214	0.108	0.146	0.078	0.061
S2	0.183	0.017	0.109	0.294	0.149	0.124	0.063	0.063
S3	0.098	0.000	0.179	0.172	0.188	0.226	0.078	0.061
S4	0.110	0.000	0.044	0.230	0.257	0.234	0.063	0.063
S5	0.132	0.031	0.256	0.191	0.097	0.153	0.082	0.060
S6	0.152	0.056	0.133	0.287	0.124	0.103	0.052	0.094
S7	0.082	0.007	0.325	0.408	0.067	0.055	0.028	0.028
S8	0.036	0.003	0.824	0.058	0.029	0.024	0.012	0.012
S9	0.152	0.056	0.133	0.287	0.124	0.103	0.052	0.094

Table 16: Final allocations for optimal strategies. The last day of the backtest is June 28, 2018.

classes of strategies overperform the equally weighted strategy by a significant margin in terms of the Sharpe ratio while having similar risk metrics (drawdowns, VaR/ES, see Table 15).

- The Black-Litterman model with the market prior achieves the best performance, especially when used with the maximum Sharpe optimization objective.
- ML views generation is a viable option, which tends to increase Sharpe ratios. There are several drawbacks though — it adds the need to calibrate the input and the parameters of each classifier. It also has an effect on the allocation structure, making it less stable (compare Figures 11–12). The question of prediction quality also requires attention.
- Overall, the number of parameters for calibration (rolling window lengths, optimization objectives and constraints, approaches to BL priors) grows as we choose more and more complicated procedures, so that the problem of picking all these settings is now an optimization problem of its' own. As a result, it is easy to get a suboptimal strategy, or, on the other hand, overfit it.

One important practical question is how the optimal strategy is exposed to the known market benchmarks. To address that question, in Table 17 we provide CAPM betas for three strategies, S0, S5 and S9, against S&P 500 and MSCI world index, both of which are tradable. Both S5 and S9 have betas close to zero, which is a good sign for an investor who looks for diversification.

	S0	S&P 500	MSCI World
S0	1.000	-0.384	-0.427
S5	0.456	0.083	0.225
S9	0.508	0.017	-0.017

Table 17: CAPM betas versus benchmarks

As a final note on the backtesting performance, if we recall our PCA interpretation of factor loadings (Table 5), we may provide another perspective on the allocation decisions: as uncorrelated bets on risk factors (for a detailed discussion, see e.g. (Meucci et al. 2015)). Average bets for strategies S0, S5 and S9 are given in Table 18. Note how the strongest bet for all three strategies is on the "All versus defensive" factor. Geographical factors are treated differently, e.g. both S5 and S9 bet against European indices, but disagree on the direction of betting on American indices.

	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8
S0	-12.511	6.211	-2.216	-0.322	6.106	-2.869	-1.255	1.836
S5	-14.215	0.580	0.977	-5.032	3.311	-7.231	1.201	4.654
S9	-14.527	4.772	-1.176	-5.473	3.575	-3.275	-3.018	5.872

Table 18: PCA bets of optimal strategies

Conclusion

The presented study covers essential steps of asset allocation strategy research, starting from the asset universe specification and decorrelation, advancing to the problem of picking the appropriate optimizer along with appropriate estimators of the covariance matrix, and running proper backtests on a list of top performers.

The initial asset universe of index futures is examined for linear structure using Spearman correlations and PCA and narrowed down accordingly. We then introduce key approaches towards portfolio optimization along with the R package `porthos`, which was developed from scratch to assist the study goals and provide full reproducibility.

The allocation study on the chosen set of assets is conducted by varying algorithm inputs (optimization objectives, constraints, estimators) *ceteris paribus* and examining the allocation stability. The latter goal is achieved by building efficient frontiers and metrics that track corner solutions.

We conclude the study by choosing the best candidate strategies and conducting a comprehensive backtesting on a historical period of four and a half years long. We introduce a novel approach to view generation for the Black-Litterman model by performing a one-step prediction of an up-down move using a set of machine learning classifiers. Such strategy is shown to be viable when the history of discretionary views is not available for backtesting purposes.

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