Assignment 3

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Question 1

Suppose we have a number of teachers and a number of courses and we want to assign the courses to the teachers. Let $T = \{T_1, \ldots, T_n\}$ be a set of teachers and let $C = \{C_1, \ldots, C_m\}$ be a set of courses. Each course $c \in C$ is taught by a set $T(c)$ of the teachers ($T(c) \subset T$).

Let k (such that $k \leq n$) be a given natural number. We need to know if we can hire at most k teachers to cover all the m courses. We define the following boolean variables:

- Teachers t_i for $i = 1, \ldots, n$, where t_i is true if and only if the teacher is hired.
- Courses c_j for $j = 1, \ldots, m$ where c_j is true if and only if there is a teacher that teaches c_j.

Consider the following points in answering Questions 1.1 to 1.4:

- You can only use the variables t_i for $i = 1, \ldots, n$ and c_j for $j = 1, \ldots, m$ in your answers to questions Questions 1.1 to 1.4.
- The answer to Questions 1.1 to 1.4 are supposed to be written in terms of SAT and SMT formulas. No English words are required to explain the answers. This part of the assignment is marked based on the correctness of the SAT/SMT formulas you have written and English explanations will be ignored.
- You can use the following function in your answer to Questions 1.1 to 1.4. Let $l \leq n$ be a given natural number. We define:

$$subsets(T, l) = \{T' \subseteq T \mid |T'| = l\}$$

where $|T|$ denotes the cardinality of a set. So the function $subsets(T, l)$ denotes all subsets of T that include l elements.

Question 1.1

Provide a constraint (formula) that is true if and only if all the courses in C are taught by at least one teacher.

ANS:

$$\bigwedge_{j=1\ldots m} c_j$$

for each $j \in 1$ to m

$(c_1 \ldots \vee \ldots c_m)$

Question 1.2

Provide a constraint (formula) that is true if and only if at most k teachers are hired. Make sure that your formula is in CNF.

ANS:

The formula is true if and only if at most 'k' teachers are hired. Constraint is $k(t_1, t_2, \ldots t_n)$ where $1 \leq k \leq n$

$$\boxed{Subset\ (C, t_1)}$$

$$Subset\ (T, l) = \{T' \subseteq T \mid |T'| = l\}.$$

Here, each disjunction represents constraints $\sim (\bigwedge_{i \in T} t_i)$ for all subsets T of $\{1, \ldots n\}$ of size $k+1$.

$$\boxed{t_k \wedge (\sim t_k \vee 1)}$$

$$\bigwedge_{C \in Subsets(\sim,\, k+1)} \ \bigvee_{t \in C} \sim t$$

$$t_k \wedge (\sim t_k \vee \bot)$$

Question 1.3

Write a constraint (formula) that captures the following statement: A course is covered only if at least one teacher who teaches it is hired. Make sure your formula is in CNF.

ANS:

A course is covered only if at least one teacher who teacher is hired

C_j is true only if there is teacher that teaches C_j.

For courses at least teachers hired

$T' = (t_1 \vee t_2 \dots\dots \vee t_n)$

Course is covered by at least one teacher.

P_{ij} = true if. only C_j is assigned with i^{th} teacher.

$$\boxed{(\sim P_{j1} \wedge \sim P_{j2} \dots \wedge \sim P_{jk}) = \bigwedge_{i \in \{1, 2, \dots k\}} \sim P_{ji}}$$

$$\bigwedge_{j=1,\dots m} \sim C_j \vee \Big(\bigvee_{t \in T(C_j)} t\Big)$$

$$(C_j \rightarrow t) \wedge t$$

$$\boxed{(\sim C_j \vee t) \wedge t}$$

Question 1.4

We have so far solved the problem of hiring at most k teachers. Now assume that we want exactly k teachers to be hired, such that all the courses are taught. What other constraints are required (in addition to the ones you listed in 1.1, 1.2 and 1.3)? Write a CNF formula for the additional constraints.

ANS:

We want exactly k teachers to be hired.

Exactly k teachers hired.

Exactly $k(T)$ = atleast $k(T) \wedge$ atmost $k(T)$.

At least k teachers hired

$$\text{atleast } k(T) = \text{atmost}_{n-k} \left(\{ \sim t_i \mid t_i \in T \} \right)$$

Atmost k teachers hired

$$\text{atmost } k(T) = \bigwedge_{\substack{T' \subseteq \{1, \ldots n\} \\ |T'| = k+1}} \bigvee_{i \in T} \neg t_i$$

Now; exactly $k(T)$ is given by

$$\bigwedge_{c \in \text{subsets}(T,\, n-k+1)} \bigvee_{t \in C} t$$

$$t_k \leftrightarrow c_j$$

$$(t_k \rightarrow c_j) \wedge (c_j \rightarrow t_k)$$

$$\boxed{(\sim t_k \vee c_j) \wedge (\sim c_j \vee t_k)}$$

Question 4.

Construct a resolution refutation graph for the following unsatisfiable formula:

$$p_1 \wedge p_2 \wedge p_3 \wedge (\neg p_1 \vee q) \wedge (\neg p_2 \vee \neg q \vee r) \wedge (\neg p_3 \vee \neg r)$$

ANS:

Step 1: Convert the formula into Conjunctive Normal Form (CNF) set of Clauses S.

$S = \{\{p_1\}, \{p_2\}, \{p_3\}, \{\neg p_1, q\}, \{\neg p_2, \neg q, r\}, \{\neg p_3, \neg r\}\}$

Step 2: The Resolution Rule is applied to all possible pairs of clauses that contains complementary literals.

Complimentary literals are:

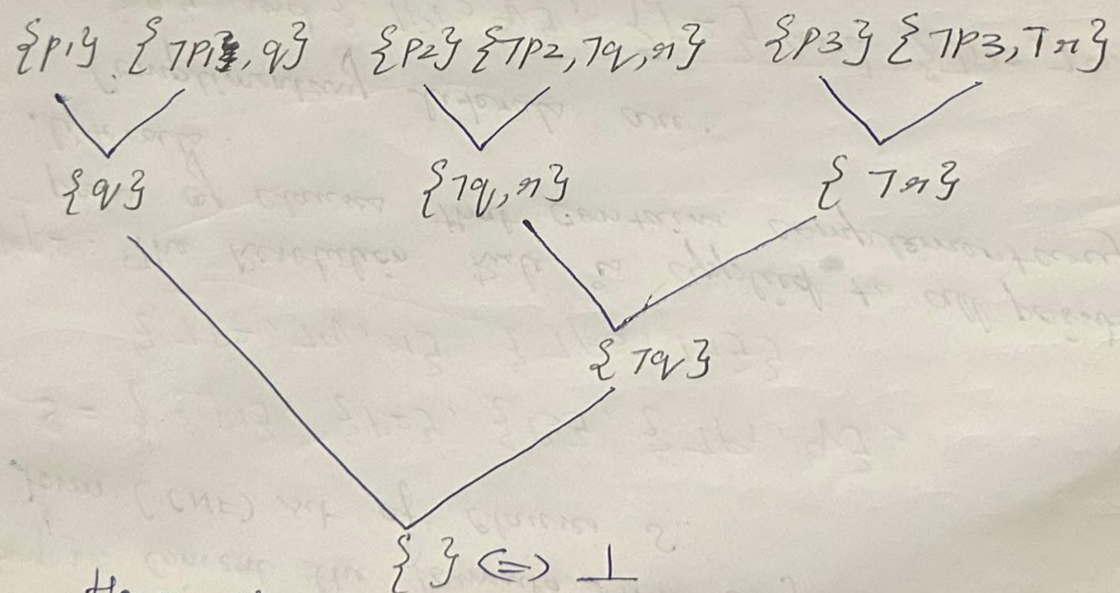
$\{p_1\}$ and $\{\neg p_1, q\}$, $\{p_2\}$ and $\{\neg p_2, \neg q, r\}$

$\{p_3\}$ and $\{\neg p_3, r\}$

We can remove the literals that are repeated. The complementary literals in the sentence are eliminated (as a validity).

If not, and if it is not already a part of the clause set S, it is added to S and taken into account for subsequent resolution inferences.

Step3: If after applying the resolution rule, an empty clause is derived, then it is Unsatisfiable.



Hence UNSAT.

Question 5.

Apply the congruence closure algorithm to decide the satisfiability of the following formula:

$$f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$$

Provide the level of detail as in Lecture 10. In particular, show the intermediate partitions (sets of congruence classes) after each merge or propagation step, together with a brief explanation of how the algorithm arrived at that partition (e.g., "by literal f(x) = y, merge f(x) with y").

ANS:

Congruence closure algorithm Steps:

1. Confirm if it is first order logic (FOL).
2. Place the sub terms in its congruent class.

⇒ {f(x)}, {g(x)}, {f(y)}, {f(g(x))}, {g(f(x))},
{x}, {f(g(f(y)))}, {g(y)}.

3. Merge the classes for t1 and t2 for each positive literal t1 = t2 in F.
4. Since, f(g(x)) = g(f(x)), merge f(g(x)) and g(f(x)). Consider the next equality f(g(f(y))) = x, merge them.

$$\{f(g(x)),\ g(f(x))\},$$
$$\{f(x)\},\ \{g(x)\},\ \{f(y)\},\ \{g(y)\}$$
$$\{f(g(f(y))),\ x\}$$

5. Consider the next equality f(y) = x, so merge them.

$$\Rightarrow \{f(g(x)),\ g(f(x))\},\ \{f(x)\},$$
$$\{g(x)\},\ \{g(y)\},\ \{f(g(f(y))),\ x,\ f(y)\}$$

6. f(g(f(y))) = f(g(x)) is the same because f(y) = x.

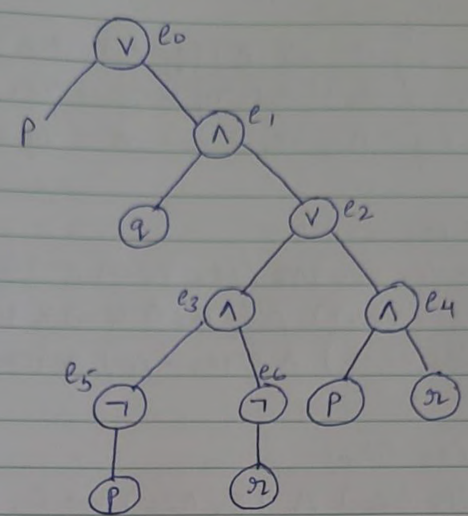
$$\therefore \{f(g(x)),\ g(f(x)),\ x,\ f(y),\ f(g(f(y)))\},$$
$$\{f(x)\},\ \{g(x)\},\ \{g(y)\}$$

The closure of the subset doesn't introduce any additional congruences. Consequently, it's unsatisfiable. Since both g(f(x)) and x were part of the same set, the assertion g(f(x)) ≠ x is incorrect. Hence, it's unsatisfiable.

Question 2.

Use Tseitin's transformation to convert $p \vee (q \wedge ((\neg p \wedge \neg r) \vee (p \wedge r)))$ into CNF.

ANS:



$e_0 \leftrightarrow (p \vee e_1)$ — ①

$e_1 \leftrightarrow (q \wedge e_2)$ — ②

$e_2 \leftrightarrow (e_3 \vee e_4)$ — ③

$e_3 \leftrightarrow (e_5 \wedge e_6)$ — ④

$e_5 \leftrightarrow \neg p$ — ⑤

$e_6 \leftrightarrow \neg r$ — ⑥

$e_4 \leftrightarrow (p \wedge r)$ — ⑦

1. $e_0 \leftrightarrow (p \vee e_1)$

$(e_0 \rightarrow (p \vee e_1)) \wedge ((p \vee e_1) \rightarrow e_0)$

$(\neg e_0 \vee (p \vee e_1)) \wedge ((\neg p \wedge \neg e_1) \vee e_0)$

$(\neg e_0 \vee p \vee e_1) \wedge ((\neg p \vee e_0) \wedge (\neg e_1 \vee e_0))$

2. $e_1 \leftrightarrow (q \wedge e_2)$

$(e_1 \rightarrow (q \wedge e_2)) \wedge ((q \wedge e_2) \rightarrow e_1)$

$(\neg e_1 \vee (q \wedge e_2)) \wedge ((\neg q \vee \neg e_2) \vee e_1)$

$(\neg e_1 \vee q) \wedge (\neg e_1 \vee e_2) \wedge (\neg q \vee \neg e_2 \vee e_1)$

3. $e_2 \leftrightarrow (e_3 \vee e_4)$

$(e_2 \rightarrow (e_3 \vee e_4) \wedge ((e_3 \vee e_4) \rightarrow e_2)$

$(\neg e_2 \vee (e_3 \vee e_4)) \wedge ((\neg e_3 \wedge \neg e_4) \vee \cancel{\neg} e_2)$

$(\neg e_2 \vee (e_3 \vee e_4)) \wedge (\neg e_3 \vee e_2) \wedge (\neg e_4 \vee e_2)$

4. $e_3 \leftrightarrow (e_5 \wedge e_6)$

$(e_3 \rightarrow (e_5 \wedge e_6) \wedge ((e_5 \wedge e_6) \rightarrow e_3)$

$(\neg e_3 \vee (e_5 \wedge e_6)) \wedge ((\neg e_5 \vee \neg e_6) \vee e_3)$

5. $e_5 \leftrightarrow \neg p$

$(e_5 \rightarrow \neg p) \wedge (\neg p \rightarrow e_5)$

$(\neg e_5 \vee \neg p) \wedge (p \vee e_5)$

6. $e_6 \leftrightarrow \neg r$

$(e_6 \rightarrow \neg r) \wedge (\neg r \rightarrow e_6)$

$(\neg e_6 \vee \neg r) \wedge (r \vee e_6)$

7. $e_4 \leftrightarrow (p \wedge r)$

$(e_4 \rightarrow (p \wedge r)) \wedge ((p \wedge r) \rightarrow e_4)$

$(\neg e_4 \vee (p \wedge r)) \wedge (\neg (p \wedge r) \vee e_4)$

$(\neg e_4 \wedge (p \wedge r)) \wedge ((\neg p \vee \neg r) \vee e_4)$

$(\neg e_4 \vee p) \wedge (\neg e_4 \vee r) \wedge (\neg p \vee \neg r \vee e_4)$

So,

$e_0 \wedge (e_0 \leftrightarrow (p \vee e_1)) \wedge (e_1 \leftrightarrow (q \wedge e_2)) \wedge$
$(e_2 \leftrightarrow (e_3 \vee e_4)) \wedge (e_3 \leftrightarrow (e_5 \wedge e_6)) \wedge$
$(e_5 \leftrightarrow \neg p) \wedge (e_6 \leftrightarrow \neg r) \wedge (e_4 \leftrightarrow (p \wedge r))]$

is equivalent to

$e_0 \wedge (\neg e_0 \vee p \vee e_1) \wedge (\neg p \vee e_0) \wedge (\neg e_1 \vee e_0)$
$\wedge (\neg e_1 \vee q) \wedge (\neg e_1 \vee e_2) \wedge (\neg q \vee \neg e_2 \vee e_1) \wedge$
$(\neg e_2 \vee e_3 \vee e_4) \wedge (\neg e_3 \vee e_2) \wedge (\neg e_4 \vee e_2) \wedge$
$(\neg e_3 \vee e_5) \wedge (e_3 \vee e_6) \wedge (\neg e_5 \vee \neg e_6 \wedge e_3) \wedge$
$(\neg e_5 \vee \neg p) \wedge (p \vee e_5) \wedge (\neg e_5 \vee \neg r) \wedge (r \vee e_6$
$\wedge (\neg e_4 \vee p) \wedge (e_4 \vee r) \wedge (\neg p \vee \neg r \vee e_4)$

Question 3.

Which of the Boolean formulae below are satisfiable, and which ones are unsatisfiable?

- $p \vee (q \wedge r)$
- $\neg((p \wedge (p \rightarrow r)) \rightarrow r)$
- $\neg(p \wedge ((p \rightarrow q) \rightarrow q))$

ANS:

If we get a true value for the expression for any possible combination of the literals, then the formula becomes SAT.

∧∨ ¬ →

1. p ∨ (q ∧ r)

This equation only contains pure literals, hence this is SAT.

It is true when p = q = r = true.

2. ¬ ((p ∧ (p → r)) → r))

¬(¬(p ∧ (p → r)) ∨ r))

(p ∧ (p → r)) ∧ ¬ r)

(p ∧ (¬p ∨ r) ∧¬r)

Now, we have CNF form, we can apply DPLL

Taking unit propagation on p, we get (r ∧¬r) which cannot be true.

Similarly, if we take unit propagation on r, we get (p ∧ ¬p) which cannot be true again.

Hence, the expression is UNSAT.

3. ¬(p ∧ ((p → q) → q))

¬(p ∧ ((¬p ∨ q) → q))

¬(p ∧ (¬(¬p ∨ q) ∨ q))

¬(p ∧ (p ∧ ¬q) ∨ q))

¬((p ∧ ¬q) ∨ q))

(¬p ∨ q) ∧ q)

Removing pure literal ¬p

q ∧ q

Hence, this is SAT.