Aishwarye Omer
EE 456 # Homework - 07 September 16, 2020

Question: Design and train an MADALINE network to implement a 2-input XOR gate

Solution : The network will work on Bipolar Input, the function and the parameters are as follows:

- $f(0) = 1$ and $\alpha = 0.1$
- $Y_{in} = \sum_{1}^{n} V_i \times Z_i + b_3$
- Initially, $w_{11} = 0.3,\ w_{12} = -0.1,\ w_{21} = 0.2,\ w_{22} = 0.4$
- $b_1 = 0.1,\ b_2 = -0.2,\ b_3 = 0.5.$
- $v_1 = 0.4,\ v_2 = 0.4$
- The truth table for the 2-input XOR gate is as follows:

X_i	1	2	$(\oplus)T_i$
1	-1	-1	-1
2	-1	1	1
3	1	-1	1
4	1	1	-1

- Each X_i and T_i is a column vector.
- Error is calculated as: $E = 0.5\ *\ \sum_{i=1}^{n}(T_i - Y_i)^2$. Error should decrease after every cycle of input is applied.
- $W_{11_{new}} = W_{11_{old}}\ +\ \alpha \times X_i \times (T_i - Y_i)$
- $W_{12_{new}} = W_{12_{old}}\ +\ \alpha \times X_i \times (T_i - Y_i)$
- $W_{21_{new}} = W_{21_{old}}\ +\ \alpha \times X_i \times (T_i - Y_i)$
- $W_{22_{new}} = W_{22_{old}}\ +\ \alpha \times X_i \times (T_i - Y_i)$
- $b_{i\ new} = b_{i\ old}\ +\ \alpha \times (T_i - Y_i)$
- $z_1 = f(\ w_{11} * X_i + w_{21} * X_i + b_1\ )$
- $z_2 = f(\ w_{12} * X_i + w_{22} * X_i + b_2\ )$
- $y = f(\ v_1 * z_1 + v_2 * z_2 + b_3\ )$

Calculations: (next three iterations after the first training done in class i.e first time when weights and bias were updated)

- Apply $X_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $T_1 = -1$

  1. $z_1 \ = \ f(\ -0.1 + 0 - 0.1\ ) \ = \ f(-0.2) \ = \ -1$
  2. $z_2 \ = \ f(\ 0.3 - 0.2 - 0.4\ ) \ = \ f(-0.3) \ = \ -1$
  3. $y_1 \ = \ f(\ -0.4 - 0.4 + 0.3\ ) \ = \ f(-0.5) \ = \ -1 \rightarrow$ **ok**
  4. $E \ = \ 0$

- Apply $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $T_2 = 1$

  1. $z_1 \ = \ f(\ -0.1 + 0 - 0.1\ ) \ = \ f(-0.2) \ = \ -1$
  2. $z_2 \ = \ f(\ 0.3 + 0.2 - 0.4\ ) \ = \ f(0.1) \ = \ 1$
  3. $y_2 \ = \ f(\ -0.4 + 0.4 + 0.3\ ) \ = \ f(0.3) \ = \ 1 \rightarrow$ **ok**
  4. $E \ = \ 0$

- Apply $X_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T_3 = 1$

  1. $z_1 \ = \ f(\ 0.1 + 0 - 0.1\ ) \ = \ f(0) \ = \ 1$
  2. $z_2 \ = \ f(\ -0.3 - 0.2 - 0.4\ ) \ = \ f(-0.9) \ = \ -1$
  3. $y_3 \ = \ f(\ 0.4 - 0.4 + 0.3\ ) \ = \ f(0.3) \ = \ 1 \rightarrow$ **ok**
  4. $E \ = \ 0$

- Apply $X_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T_4 = -1$

  1. $z_1 \ = \ f(\ 0.1 + 0 - 0.1\ ) \ = \ f(0) \ = \ 1$
  2. $z_2 \ = \ f(\ -0.3 + 0.2 - 0.4\ ) \ = \ f(-0.5) \ = \ -1$
  3. $y_4 \ = \ f(\ 0.4 - 0.4 + 0.3\ ) \ = \ f(0.3) \ = \ 1 \rightarrow$ not ok
  4. Let's train $z_1,\ b_1,\ b_3$:

     $\rightarrow\ w_{11_{new}} \ = \ 0.1 + 0.1 \times 1 \times (-1 - 1) \ = \ -0.1$

     $\rightarrow\ w_{21_{new}} \ = \ 0 + 0.1 \times 1 \times (-1 - 1) \ = \ -0.2$

     $\rightarrow\ b_{1_{new}} \ = \ -0.1 + 0.1 \times (-1 - 1) \ = \ -0.3$

     $\rightarrow\ b_{3_{new}} \ = \ 0.3 + 0.1 \times (-1 - 1) \ = \ 0.1$
  5. Rest of the weights and bias remain same:
  6. $w_{12} = -0.3,\ w_{22} = 0.2$
  7. $b_2 = -0.4$ and $v_1 = 0.4,\ v_2 = 0.4$

- Next iteration starts from here.
- Again, Apply $X_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \ T_1 = -1$

1. $z_1 \ = \ f(\ 0.1 + 0.2 - 0.3\ ) \ = \ f(0) \ = \ 1$
2. $z_2 \ = \ f(\ 0.3 - 0.2 - 0.4\ ) \ = \ f(-0.3) \ = \ -1$
3. $y_1 \ = \ f(\ 0.4 - 0.4 + 0.1\ ) \ = \ f(0.1) \ = \ 1 \rightarrow$ not ok
4. Let's train $z_1, \ b_1, \ b_3$:

$$\begin{aligned}
&\rightarrow w_{11_{new}} \ = \ -0.1 + 0.1 \times -1 \times (-1-1) \ = \ 0.1 \\
&\rightarrow w_{21_{new}} \ = \ -0.2 + 0.1 \times -1 \times (-1-1) \ = \ 0 \\
&\rightarrow b_{1_{new}} \ = \ -0.3 + 0.1 \times (-1-1) \ = \ -0.5 \\
&\rightarrow b_{3_{new}} \ = \ 0.1 + 0.1 \times (-1-1) \ = \ -0.1
\end{aligned}$$

5. Rest of the weights and bias remain same:
6. $w_{12} = -0.3, \ w_{22} = 0.2$
7. $b_2 = -0.4.$
8. $v_1 = 0.4, \ v_2 = 0.4$

- Apply $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ T_2 = 1$

1. $z_1 \ = \ f(\ -0.1 + 0 - 0.5\ ) \ = \ f(-0.6) \ = \ -1$
2. $z_2 \ = \ f(\ 0.3 + 0.2 - 0.4\ ) \ = \ f(0.1) \ = \ 1$
3. $y_2 \ = \ f(\ -0.4 + 0.4 - 0.1\ ) \ = \ f(-0.1) \ = \ -1 \rightarrow$ not ok
4. Let's train $z_2, \ b_2, \ b_3$:

$$\begin{aligned}
&\rightarrow w_{12_{new}} \ = \ -0.3 + 0.1 \times -1 \times (1-(-1)) \ = \ -0.5 \\
&\rightarrow w_{22_{new}} \ = \ 0.2 + 0.1 \times 1 \times (1-(-1)) \ = \ 0.4 \\
&\rightarrow b_{2_{new}} \ = \ -0.4 + 0.1 \times (1-(-1)) \ = \ -0.2 \\
&\rightarrow b_{3_{new}} \ = \ -0.1 + 0.1 \times (1-(-1)) \ = \ 0.1
\end{aligned}$$

5. Rest of the weights and bias remain same:
6. $w_{11} = 0.1, \ w_{21} = 0$
7. $b_1 = -0.5.$
8. $v_1 = 0.4, \ v_2 = 0.4$

- Apply $X_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ T_3 = 1$

1. $z_1 \ = \ f(\ 0.1 + 0 - 0.5\ ) \ = \ f(-0.4) \ = \ -1$
2. $z_2 \ = \ f(\ -0.5 - 0.4 - 0.2\ ) \ = \ f(-1.1) \ = \ -1$

3. $y_3 \;=\; f(\;-0.4 - 0.4 + 0.1\;) \;=\; f(-0.7) \;=\; -1 \rightarrow$ not ok
4. Let's train $z_1,\; b_1,\; b_3$:

   $\rightarrow\; w_{11_{new}} \;=\; 0.1 + 0.1 \times 1 \times (1 - (-1)) \;=\; 0.3$
   $\rightarrow\; w_{21_{new}} \;=\; 0 + 0.1 \times -1 \times (1 - (-1)) \;=\; -0.2$
   $\rightarrow\; b_{1_{new}} \;=\; -0.5 + 0.1 \times (1 - (-1)) \;=\; -0.3$
   $\rightarrow\; b_{3_{new}} \;=\; 0.1 + 0.1 \times (1 - (-1)) \;=\; 0.3$
5. Rest of the weights and bias remain same:
6. $w_{12} = -0.5,\; w_{22} = 0.4$
7. $b_2 = -0.2.$
8. $v_1 = 0.4,\; v_2 = 0.4$

- Apply $X_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\; T_4 = -1$

1. $z_1 \;=\; f(\;0.3 - 0.2 - 0.3\;) \;=\; f(-0.2) \;=\; -1$
2. $z_2 \;=\; f(\;-0.5 + 0.4 - 0.2\;) \;=\; f(-0.3) \;=\; -1$
3. $y_4 \;=\; f(\;-0.4 - 0.4 + 0.3\;) \;=\; f(-0.5) \;=\; -1 \rightarrow$ **ok**

- Next iteration starts here:
- $w_{11} = 0.3,\; w_{21} = -0.2,\; b_1 = -0.3$
- $w_{12} = -0.5,\; w_{22} = 0.4,\; b_2 = -0.2,\; b_3 = 0.3$
- Again, Apply $X_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\; T_1 = -1$

1. $z_1 \;=\; f(\;-0.3 + 0.2 - 0.3\;) \;=\; f(-0.4) \;=\; -1$
2. $z_2 \;=\; f(\;0.5 - 0.4 - 0.2\;) \;=\; f(-0.1) \;=\; -1$
3. $y_1 \;=\; f(\;-0.4 - 0.4 + 0.3\;) \;=\; f(-0.5) \;=\; -1 \rightarrow$ **ok**

- Apply $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\; T_2 = 1$

1. $z_1 \;=\; f(\;-0.3 - 0.2 - 0.3\;) \;=\; f(-0.8) \;=\; -1$
2. $z_2 \;=\; f(\;0.5 + 0.4 - 0.2\;) \;=\; f(0.7) \;=\; 1$
3. $y_2 \;=\; f(\;-0.4 + 0.4 + 0.3\;) \;=\; f(0.3) \;=\; 1 \rightarrow$ **ok**

- Apply $X_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\; T_3 = 1$

1. $z_1 \;=\; f(\;0.3 + 0.2 - 0.3\;) \;=\; f(0.2) \;=\; 1$

2. $z_2 \; = \; f(\; -0.5 - 0.4 - 0.2 \;) \; = \; f(-1.1) \; = \; -1$
3. $y_3 \; = \; f(\; 0.4 - 0.4 + 0.3 \;) \; = \; f(0.3) \; = \; 1 \rightarrow$ **ok**

- Apply $X_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; T_4 = -1$

1. $z_1 \; = \; f(\; 0.3 - 0.2 - 0.3 \;) \; = \; f(-0.2) \; = \; -1$
2. $z_2 \; = \; f(\; -0.5 + 0.4 - 0.2 \;) \; = \; f(-0.3) \; = \; -1$
3. $y_4 \; = \; f(\; -0.4 - 0.4 + 0.3 \;) \; = \; f(-0.5) \; = \; -1 \rightarrow$ **ok**

Therefore the network is set for 2-input XOR gate