Parallel Hierarchic Adaptive Stabilized Transient Analysis (PHASTA)

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I. Incompressible Navies-Stokes equation

Let us state incompressible Navies- Stokes equation (strong form)

$$u_{i,i} = 0 \tag{1}$$

$$\rho u_{i,t} + \rho u_j u_{i,j} + P_{,i} - \tau_{ij,j} - f_i = 0 \tag{2}$$

$$\overbrace{\rho(\underbrace{\frac{\partial v}{\partial t}}_{\text{Unsteady acceleration}} + \underbrace{v \cdot \nabla v}_{\text{Convective acceleration}})}^{\text{Inertia (per volume)}} = \overbrace{\underbrace{-\nabla p}_{\text{Pressure gradient}} + \underbrace{\mu \nabla^2 v}_{\text{Viscosity}} + \underbrace{f}_{\text{Other body forces}}}^{\text{Divergence of stress}}$$

(1) is mass conversation

(2) are 3 momentum conservation equation

Finite element approach uses weak form of equations.

Let us derive it:

Introducing new notation $\mathcal{L}_i \triangleq \mathrm{l.h.s}$ *(2)*

Let's write *(1)**q+$\sum_{i=1}^{3}$(2) $* \, w_i$; integrating it over our domain Ω (Γ≜ ∂ Ω)

$$0 = \int_{\Omega} \left[q u_{i,i} + w_i l_i\right] d\Omega = \int_{\Omega} \left[q u_{i,i} + w_i(\rho u_{i,t} + \rho u_j u_{i,j} + P_{,i} - \tau_{ij,j} - f_i)\right] d\Omega \tag{3}$$

In order to get weak form, we will integrate continuity and stress term of momentum equation by partly recall that

$$\int_{\Omega} aB_{,i} d\Omega = \int_{\Omega} [aB]_{,i} d\Omega - \int_{\Omega} a_{,i} B d\Omega$$

$$= \int_{\Gamma} aBn_i d\Gamma - \int_{\Omega} a_{,i} B d\Omega \tag{4}$$

Taking *(4)* into account we can rewrite *(3)* as:

$$0 = \int_{\Omega} \left[-q_{,i}u_i + w_i(\rho u_{i,t} + \rho u_j u_{i,j} - f_i)\right] + w_{i,j}\{-P\delta_{ij} + \tau_{ij}\}d\Omega$$
$$+ \int_{\Gamma} \left[q u_i n_i - w_i(\tau_{ij} - P\delta_{ij})n_j\right]d\Gamma \quad (5)$$

--Weak Galvevkin Form (this is the simplest form for I.N.S equation, which is unstable. On Page 9. We'll introduce a stabilization term, and on Page 9 we'll consider an additional mass conservation term (i.e. term, which makes the mass conservation equation satisfied).

Now let's see what we can do with eqn *(5)*, assuming that it doesn't need any extra terms

In order to solve *(5)*, we have to propose trial solution y(x,y,z,t) and weight function w(x,y,z)

w:

$$w_i = \sum_{A=1}^{n_{np}} N_i^A(x) w_i^A \quad (6)$$

$$\mathrm{q} = \sum_{A=1}^{n_{np}} N_P^A(x) q^A \quad (7)$$

y:

$$u_i = \sum_{B=1}^{n_{np}} N_i^B(x) u_i^B \quad (8)$$

$$u_{i,t} = \sum_{B=1}^{n_{np}} N_i^B(x) u_{i,t}^B \quad (9)$$

$$\mathrm{P} = \sum_{B=1}^{n_{np}} N_P^B(x) p^B \quad (10)$$

Here we used A for weight function and B for trial function

Let's plug *(6)* & *(7)* into *(5)*

$$\sum_{i=1}^{3}\sum_{A=1}^{n_{np}} w_i^A \left\{\int_\Omega \left[N_i^A[\rho\, u_{i,t} + \rho u_j u_{i,j} - f_i\right] + N_{i,j}^A[\tau_{ij} - P\sigma_{i,j}]]d\Omega \right.$$

$$+ \int_\Gamma -N_i^A\{\tau_{ij} - P\delta_{ij}\}d\Gamma\} \quad (11)$$

$$+ \sum_{A=1}^{n_{np}} q^A\{\int_\Omega -N_{p,i}^A u_i d\Omega + \int_\Gamma N_P^A u_i n_i d\Gamma\} = 0$$

Here we wrote momentum eq. contribution in the first term and continuity in the second term:

$$\sum_{i=1}^{3}\sum_{A=1}^{n_{np}} w_i^A \overset{m}{R}{}_i^A + \sum_{A=1}^{n_{np}} q^A \overset{c}{R}{}^A = 0 \quad (12)$$

Where $\overset{m}{R}{}_i^A$, i=1,2,3 are momentum residuals, $\overset{m}{R}{}_i^A$ is continuity residual

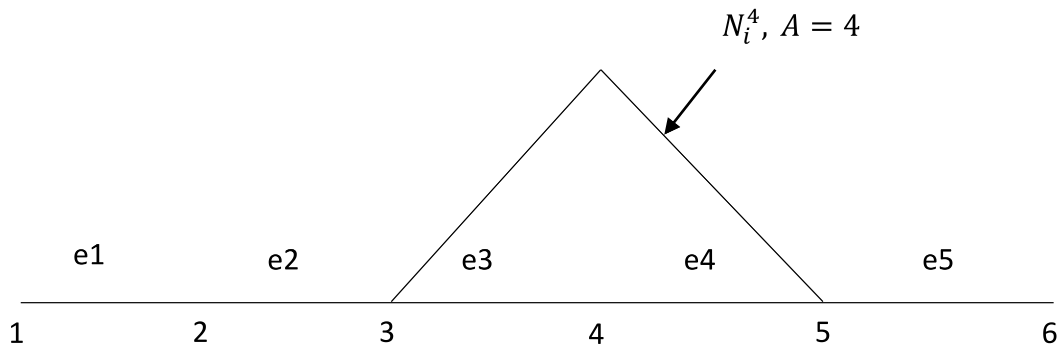
Provided that w_i^A & q^A are arbitrary we can conclude that

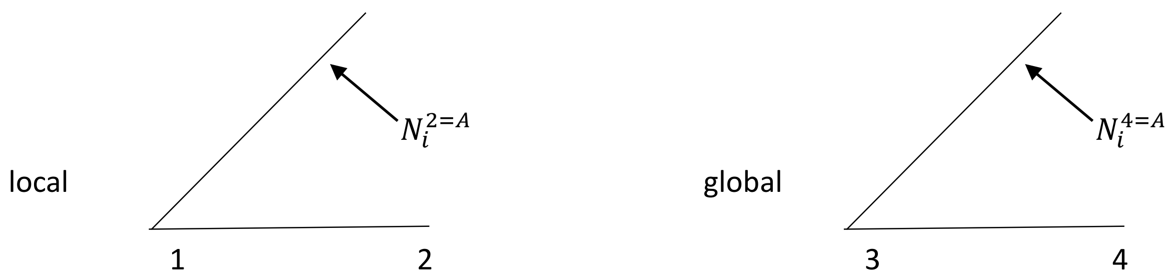
$$\widetilde{\overset{m}{R}} = \left\{ \overset{m}{R}{}_i^A \right\}_{3\times n_{np}} = 0 \quad (13)$$

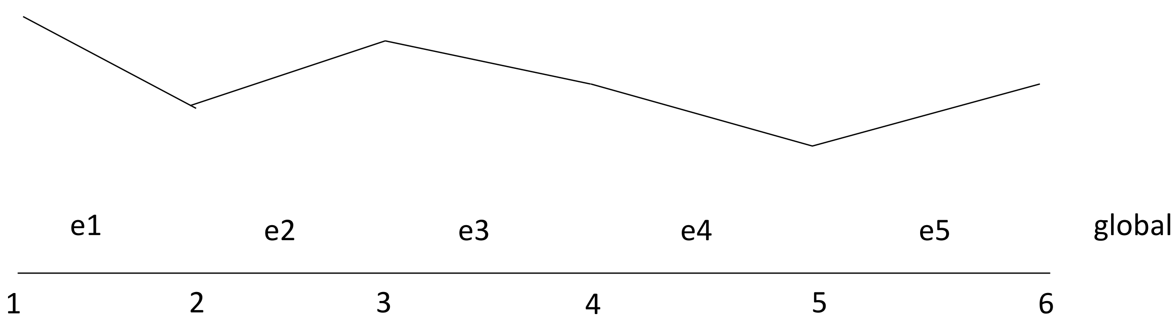
$$\widetilde{\overset{c}{R}} = \left\{ \overset{c}{R}{}^A \right\}_{1\times n_{np}} = 0 \quad (14)$$

We don’t evaluate integral *(11)* globally, we’ll do it from local point of view, which is more efficient. To emphasize the local point of view, we’ll use lower case superscripts a & B.

1d example


$N_i^4,\ A = 4$
e1
e2
e3
e4
e5
1
2
3
4
5
6


local
$N_i^{2=A}$
1
2
global
$N_i^{4=A}$
3
4


e1
e2
e3
e4
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global
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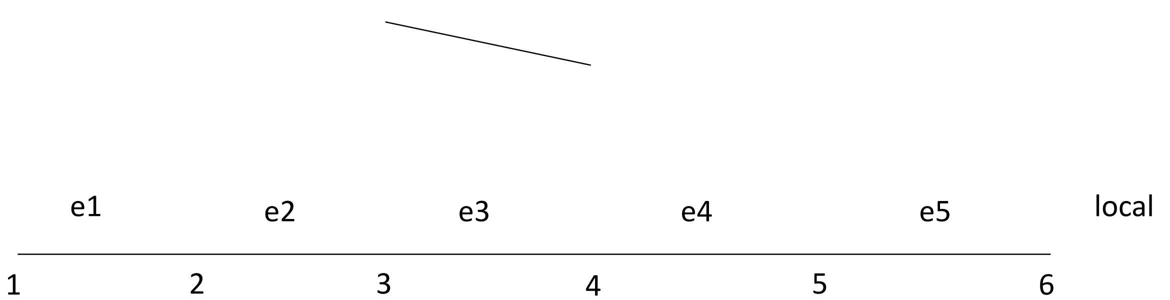

e1
e2
e3
e4
e5
local
1
2
3
4
5
6

$$f(x) = \sum_{A=1}^{n_{np}} N_i^A f^A$$

$$\overset{3}{f}\,(\mathrm{x}) = \sum_{a=1}^{n_a^1=2} N_i^a \, \overset{3}{f}{}^a$$

Where

$$\overset{3}{f}{}^1 \triangleq \overset{3}{f}$$

$$\overset{3}{f}{}^2 \triangleq \overset{4}{f}$$

in our example

Now we have to break up the integral

$$\int_{\Omega} (.)d\Omega = \sum_{e=1}^{n_{ee}} \int_{\Omega^e} (.)d\Omega^e \tag{15}$$

Let’s consider in detail one of elements, and write out the continuity residual for it:

$$\overset{c}{R}{}^{ae} = \int_{\Omega^e} -N_{p,i}^a u_i d\Omega + \int_{\Gamma^e} N_p^a u_i n_i d\Gamma \tag{16}$$

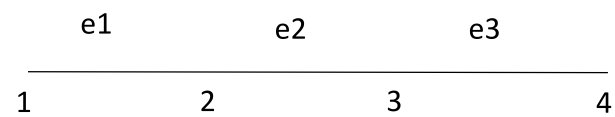
Note: Γ^e is only the portion of the boundary of Ω^e that is part of real Γ

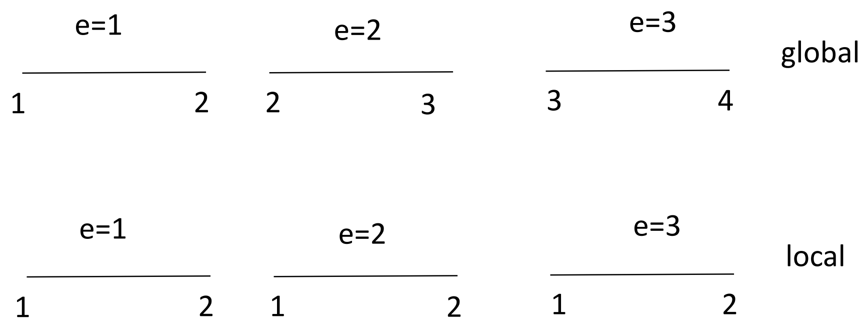
How to collect integral back according to *(15)*? (Put together all element integrals)

We can generate a connectivity table:

el # \ local node #	1	2
1	1	2
2	2	3
3	3	4

↳ this is 1d example.





In the code we have ien (1:npro, 1:hshl), where 1:npro is el # and 1:hshl is local node #

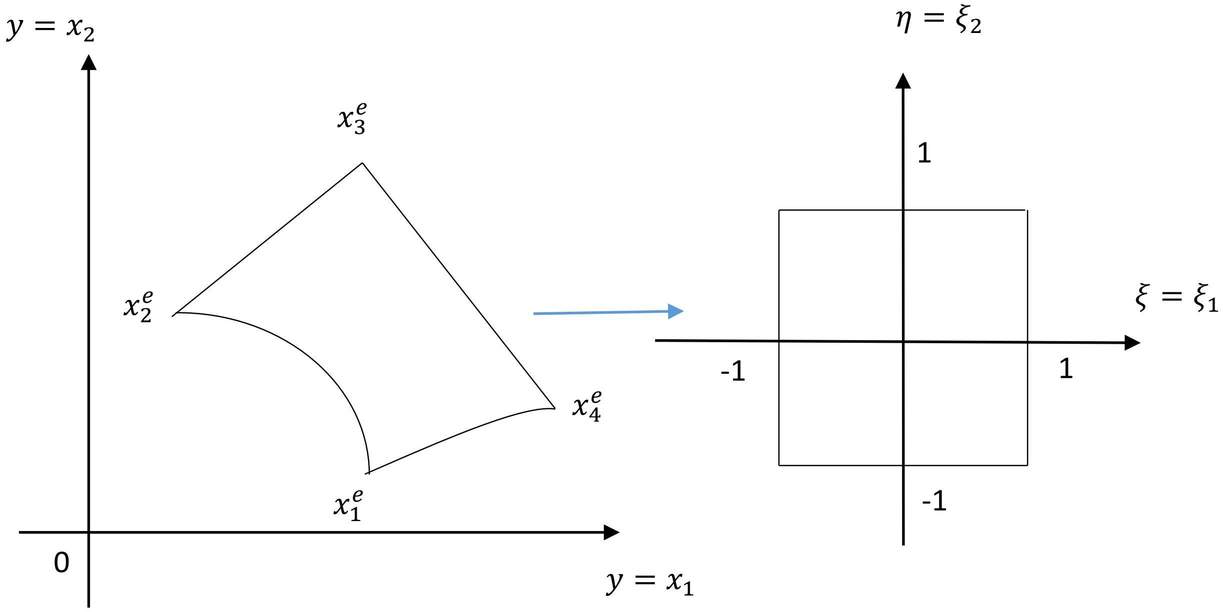
Using this table we can do assemble operation $\overset{c}{R}^{A} = \mathbb{A}_{e=1}^{n_{ee}} \overset{c}{R}^{ae}$

Which can be in the code as:

$$
\begin{aligned}
&\overset{c}{R}^{A} = 0 \\
&do\ i = 1, n_{el} \\
&do\ a = 1, n_{en} \\
&A = ien(e,a) \\
&\overset{c}{R}^{A} = \overset{c}{R}^{A} + \overset{c}{R}^{ca} \\
&end\ do \\
&end\ do
\end{aligned}
$$

II. Mappings

In order to make the integrals easier to evaluate, we need to map elements to a canonical element, e.g. in 2d



Mapping will do for us following transformations:

$$x(\xi) \leftrightarrow \xi(x)$$

$$\Omega^e \leftrightarrow \blacksquare$$

$$\Gamma^e \leftrightarrow \blacksquare_\Gamma$$

$$d\Omega^e \leftrightarrow d\blacksquare$$

$$d\Gamma^e \leftrightarrow d\blacksquare_\Gamma$$

Derivation

$$\frac{\partial N^a}{\partial x_i} = N^a_{,i}(\xi) = \frac{\partial N^a}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = N^a_{,\xi_j}\xi_{j,i} \quad (17)$$

Where $\xi_{j,i}$ will be different for each element

Integrals:

$$\int_{\Omega^e} f(x)\, d\Omega = \int_{\blacksquare} f\big(x(\xi)\big) D(\xi) d\blacksquare \quad (18)$$

$$D(\xi) = \det(x, \xi) = \det(x_i, \xi_j) \quad (19)$$

--Jacobian of the mapping x(ξ)

The advantage of using mappings:

Each element looks the same after applying the mapping→ all of the elements will be treated by the same routine (if they are the same type, like triangular, quadrilateral, tetrahedron, wedge, brick and pyramid, of course).

Let's rewrite *(16)* using mapping

$$\overset{c}{R}^{a} = \int_{\blacksquare} -N^{a}_{P,i}\, u_i D d\blacksquare \quad + \int_{\blacksquare_\Gamma} N^{a}_{P} u_i n_i D_\Gamma \, d\blacksquare_\Gamma \qquad (20)$$

Q: How do we evaluate integrals?

A: Using Gauss's quadrature

$$\overset{c}{R}^{ae} = \int_{\blacksquare} \overset{c}{r}^{ae}(\xi) d\blacksquare = \sum_{k=1}^{n_{int}} \overset{c}{r}^{ae}(\xi_k) w_R \qquad (21)$$

Where $\overset{c}{r}^{ae}(\xi_k)$- function value at point ξ_k, w_k- weight for the k^{th} quadrature point

We mentioned on Page 2 that Galerkin form *(5)* is unstable and we need an extra term to make it stable

$$\sum_{e=1}^{n_{ee}} \int_{\Omega^e} \left[\underbrace{\tau_M \left\{ u_j w_{i,j} + \frac{q_{,i}}{\rho} \right\} \mathcal{L}_i}_{\text{Momentum conservation}} + \underbrace{\tau_c w_{j,j} u_{i,i}}_{\text{continuity}} \right] d\Omega^e \qquad (22)$$

The first term represents the momentum conservation and the second term represents the continuity.

To make continuity equation satisfied, we also have to add a mass conservation term to the Galerkin form *(5)*.

$$\sum_{e=1}^{n_{ee}} \int_{\Omega^e} \left[w_i \rho \overset{\Delta}{u}_j\, u_{i,j} + \tau \left\{ \frac{\rho \overset{\Delta}{u}_j}{\tau_M} \right\} w_{i,j} \left\{ \frac{\rho \overset{\Delta}{u}_k}{\tau_M} \right\} u_{i,k} \right] d\Omega^e \qquad (23)$$

Adding together eqns *(5)*, *(22)* & *(23)*, we can write

$$0 = \int_{\Omega} \left\{ -\overset{1}{q_{,i}u_i} + w_i\{ \overset{2}{\rho u_{i,t}} + \overset{3}{\rho u_j u_{i,j}} - \overset{4}{f_i} \right\} + w_{i,j} \left\{ \overset{5}{\tau_{ij}} - \overset{6}{p\delta_{ij}} \right\}\} d\Omega$$

$$+ \int_{\Gamma} \left\{ \overset{7}{q u_i n_i} - w_i\{ \overset{8}{\tau_{ij}} - \overset{9}{p\delta_{ij}} \} n_j \right\} d\Gamma + \sum_{e=1}^{n_{el}} \int_{\Omega^e} [\tau_M\{ \overset{10}{u_j w_{i,j}}$$

$$+ \overset{11}{\frac{q_{,i}}{\rho}} \}\mathcal{L}_i + \overset{12}{\tau_c w_{j,j} u_{i,i}}] \mathrm{d}\Omega^e \tag{24}$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left[\overset{13}{w_i \rho \overset{\Delta}{u}_j u_{i,j}} + \overset{14}{\tau \left\{ \frac{\rho \overset{\Delta}{u}_j}{\tau_M} \right\} w_{i,j} \left\{ \frac{\rho \overset{\Delta}{u}_k}{\tau_M} \right\} u_{i,k}} \right] \mathrm{d}\Omega^e$$

Where

$$\overset{\Delta}{u}_j = -\tau_M \frac{\mathcal{L}_j}{\rho},\ \bar{u}_j = u_j + \overset{\Delta}{u}_j = u_j - \tau_M \frac{\mathcal{L}_j}{\rho} \tag{25}$$

Now we can put all terms of eq. *(24)* into residual vectors *(13)* & *(14)*.

Note that

$-\overset{m}{R}{}^{A}$ goes to res(1:n_{np}; 1:3)

$-\overset{c}{R}{}^{A}$ goes to res(1:n_{np}; 4)

- We assemble these global residuals from element level integrations

Let’s state that local form of the residuals (which are stored in -rl(:,:,:) in the code:

$$\overset{m}{R}{}_i^a = \int_{■} [N_i^a \left\{ \overset{2}{\rho u_{i,t}} + \overset{3+13}{\rho u_k u_{i,k}} - \overset{4}{f_i} \right\} + N_{i,k}^a \left\{ \overset{5}{\tau_{ik}} - \overset{6}{p\delta_{ik}} \right\} + \overset{10}{N_{i,k}^a \tau_M u_k}\ \mathcal{L}_i$$

$$+ \overset{12}{N_{i,i}\tau_c u_{k,k}} + \overset{14}{N_{i,l}\tau u_{i,k}}\ \mathcal{L}_l \mathcal{L}_k] \mathrm{D}d■ - \int_{■_\Gamma} N_i^a (\overset{8}{\tau_{ik}} - \overset{9}{p\delta_{ik}}) n_k D_r d■_\Gamma \tag{26}$$

$$\overset{c}{R}{}^a = \int_{■} -\overset{1+11}{N_{p,k}^a u_k}\ \mathrm{D}d■ + \int_{■_\Gamma} N_p^a u_k n_k D_r d■_\Gamma \tag{27}$$

Note: $\tau_{ik} = \mu(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i})$

In terms *(15)*, *(18)* & *(20)* of eq *(26)*, we have the following undefined so far parameters: (e2stab.f)

$$\tau_M = [\left(\frac{2C_1}{\Delta t}\right)^2 + u_e u_{in} g_{em} + C_2 D^2 g_{em} g_{em}]^{-1/2} \tag{28}$$

$$\tau_c = \frac{\rho}{8\tau_M g_u} \tag{29}$$

$$\tau = \frac{\tau_M}{\sqrt{\mathcal{L}_l \mathcal{L}_m g_m}} \tag{30}$$

Where g_{ij} is the metric tensor:

$$g_{ij} = \xi_{k,i}\xi_{k,j} = \frac{d\xi_k}{dx_i}\frac{d\xi_k}{dx_j} \tag{31}$$

III. Implicit time integration

Let’s describe the i^{th} iteration, moving from time step n to time step n+1, according to Generalized Alpha Method

Let’s first write it out for velocity vector

$$\overset{n+1}{u}{}^{B(i)} = \overset{n}{u}{}^{B} + \Delta t\, \overset{n}{u}{}^{B}_{,t} + \Delta t \cdot \gamma(\overset{n+1}{u}{}^{B(i)}_{,t} - \overset{n}{u}{}^{B}_{,t}) \tag{OD 1}$$

$$\overset{m}{R}{}^{A}\,(\overset{n+\alpha_m}{u}{}^{B(i)}_{,t}\,;\ \overset{n+\alpha_f}{u}{}^{B(i)}\,;\ \overset{n+\alpha_f}{p}{}^{B(i)}) \cong 0 \tag{OD 2}$$

$$\overset{n+\alpha_m}{u}{}^{B(i)}_{,t} = \overset{n}{u}{}^{B}_{,t} + \alpha_m(\overset{n+1}{u}{}^{B(i)}_{,t} - \overset{n}{u}{}^{B}_{,t}) \tag{OD 3}$$

$$\overset{n+\alpha_f}{u}{}^{B(i)} = \overset{n}{u}{}^{B} + \alpha_f(\overset{n+1}{u}{}^{B(i)} - \overset{n}{u}{}^{B}) \tag{OD 4}$$

Same equation for pressure:

$$\overset{n+1}{P}{}^{B(i)} = \overset{n}{P}{}^{B} + \Delta \mathrm{t}\, \overset{n}{P}{}^{B}_{,t} + \Delta \mathrm{t}\gamma(\overset{n+1}{P}{}^{B(i)}_{,t} - \overset{n}{P}{}^{B}_{,t}) \tag{P 1}$$

$$\overset{c}{R}{}^{a}\,(\overset{n+\alpha_m}{u}{}^{B(i)}_{,t}\,,\ \overset{n+\alpha_f}{u}{}^{B(i)}\,,\ \overset{n_f}{p}{}^{B}_{,t}) \cong 0 \tag{P 2}$$

$$\overset{n+\alpha_m}{p}{}^{B(i)}_{,t} = \overset{n}{p}{}^{B}_{,t} + \alpha_m(\overset{n+1}{p}{}^{B(i)}_{,t} - \overset{n}{p}{}^{B}_{,t} \tag{P 3}$$

$$\overset{n+\alpha_f}{p}{}^{B(i)} = \overset{n}{p}{}^{B} + \alpha_f(\overset{n+1}{p}{}^{B(i)}_{,t} - \overset{n}{p}{}^{B}_{,t}) \tag{P 4}$$

How predictor-multicorrector works?

1) predict $\overset{n+1}{u}{}^{(i)} = \overset{n}{u}$, $\overset{n+1}{p}{}^{(i)} = \overset{n}{p}$

2) invert *(OD 1)*:

$$\overset{n+1}{u}{}^{(i)}_{,t} = \frac{1}{\gamma\Delta t}\left[\overset{n+1}{u}{}^{(i)} - \overset{n}{u}\right] + (1-\frac{1}{\gamma})\overset{n}{u}_{,t}$$

Invert *(P 1)*

$$\overset{n+1}{p}{}^{(i)}_{,t} = \frac{1}{\gamma\Delta t}\left[\overset{n+1}{p}{}^{(i)} - \overset{n}{p}\right] + (1-\frac{1}{\gamma})\overset{n}{p}_{,t}$$

3) evaluate: $\overset{n+\alpha_m}{u}{}^{(i)}_{,t}$; $\overset{n+\alpha_f}{u}{}^{(i)}$; $\overset{n+\alpha_f}{p}{}^{(i)}_{,t}$; $\overset{n+\alpha_f}{p}{}^{(i)}$

4) evaluate: $\overset{m}{R}{}^{A} \neq 0$’ $\overset{c}{R}{}^{A} \neq 0$

Reture to 1) increasing the iteration number i until residuals will be close enough to zero

Q: How we will choose $\overset{n+\alpha_f}{u}{}^{B}$, $\overset{n+\alpha_m}{u}{}^{B}_{,t}$, $\overset{n+\alpha_f}{p}{}^{B}$ in order to get zero residuals?

A: we will use an analog of Newton’s method ($x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{\left.\frac{df}{dx}\right|_{x^{(i)}}}$ for $f(x) = 0$)

Let’s linearize eqn *(OD 2)* and *(P 2)* about $\overset{n+1}{u}{}^{B}_{,t}$ and $\overset{n+1}{p}{}^{B}_{,t}$:

$$\sum_{B=1}^{n_{np}}\{\frac{\partial \overset{m}{R}{}^{A}}{\partial \overset{n+1}{u}{}^{B}_{,t}}\Delta \overset{n+1}{u}{}^{B}_{,t} + \frac{\partial \overset{m}{R}{}^{A}}{\partial \overset{n+1}{p}{}^{B}_{,t}}\Delta \overset{n+1}{p}{}^{B}_{,t}\} = -\overset{m}{R}{}^{A} \quad (32)$$

$$\sum_{B=1}^{n_{np}}\{\frac{\partial \overset{c}{R}{}^{A}}{\partial \overset{n+1}{u}{}^{B}_{,t}}\Delta \overset{n+1}{u}{}^{B}_{,t} + \frac{\partial \overset{c}{R}{}^{A}}{\partial \overset{n+1}{p}{}^{B}_{,t}}\Delta \overset{n+1}{p}{}^{B}_{,t}\} = -\overset{c}{R}{}^{A} \quad (33)$$

Introducing new notation we can rewrite these equations as:

$$\sum_{B=1}^{n_{np}}\{K^{AB}\Delta \overset{n+1}{u}{}^{B}_{,t} + G^{AB}\Delta \overset{n+1}{p}{}^{B}_{,t}\} = -\overset{m}{R}{}^{A} \quad (34)$$

$$\sum_{B=1}^{n_{np}}\{D^{AB}\Delta \overset{n+1}{u}{}^{B}_{,t} + C^{AB}\Delta \overset{n+1}{p}{}^{B}_{,t}\} = -\overset{c}{R}{}^{A} \tag{35}$$

Or, in matrix form

$$\begin{bmatrix} K & G \\ D & C \end{bmatrix}\begin{Bmatrix} \Delta u_{,t} \\ \Delta p_{,t} \end{Bmatrix} = \begin{Bmatrix} -\overset{m}{R} \\ \\ -\overset{c}{R} \end{Bmatrix} = -\{R\} \tag{36}$$

Let’s write the alternate form of eq. *(27)* for L.N.S tangent of continuity

$$\overset{C4LNS}{R^{a}} = \int_{■} N_p^a u_{k,k} Dd■ + \int_{■} N_{p,k}^a \tau_M \frac{\mathcal{L}_k}{\rho} Dd■ \tag{37}$$

Now we write matrices K, G D & C at the element level:

$$R^{aB} = \frac{\partial \overset{m}{R}{}^{a}}{\partial \overset{n+1}{u}{}^{B}_{,t}} = R_{ij}^{aB} = \frac{\partial \overset{m}{R}{}^{a}_{i}}{\partial \overset{n+1}{u}{}^{B}_{j,t}} = \sum_{C=1}^{n_{np}}\{\frac{\partial \overset{m}{R}{}^{a}_{i}}{\partial \overset{n+\alpha_f}{u}{}^{C}_{k}}\frac{\partial \overset{n+\alpha_f}{u}{}^{C}_{k}}{\partial \overset{n+1}{u}{}^{B}_{j,t}} + \frac{\partial \overset{m}{R}{}^{a}_{i}}{\partial \overset{n+\alpha_m}{u}{}^{C}_{k,t}}\frac{\partial \overset{n+\alpha_m}{u}{}^{C}_{k,t}}{\partial \overset{n+\alpha_m}{u}{}^{B}_{j,t}}\}$$

$$= \frac{\partial \overset{m}{R}{}^{a}_{i}}{\partial \overset{n+\alpha_f}{u}{}^{B}_{j}}\alpha_f\gamma\Delta t + \frac{\partial \overset{m}{R}{}^{a}_{i}}{\partial \overset{n+\alpha_m}{u}{}^{B}_{j,t}}\alpha_m \tag{38}$$

$$G_i^{aB} = \frac{\partial \overset{m}{R}{}^{a}_{i}}{\partial \overset{n+1}{p}{}^{B}_{,t}} = \alpha_f\gamma\Delta t\frac{\partial \overset{m}{R}{}^{a}_{i}}{\partial \overset{n+\alpha_f}{p}{}^{B}} \tag{39}$$

$$D_j^{aB} = \frac{\partial \overset{C4LNS}{R^{a}}}{\partial \overset{n+1}{u}{}^{B}_{j,t}} = \frac{\partial \overset{C4LNS}{R^{a}}}{\partial \overset{n+\alpha_f}{u}{}^{B}_{j}}\alpha_f\gamma\Delta t + \frac{\partial \overset{C4LNS}{R^{a}}}{\partial \overset{n+\alpha_m}{u}{}^{B}_{j,t}}\alpha_m \tag{40}$$

$$C^{aB} = \frac{\partial \overset{C4LNS}{R^a}}{\partial \overset{n+1}{p_{,t}}} = \frac{\partial \overset{C4LNS}{R^a}}{\partial \overset{n+\alpha_f}{p^B}} \alpha_f \gamma \Delta t \tag{41}$$

Let us write out some derivations in terms of shape function

$$\frac{\partial \overset{n+\alpha_m}{u^B_{i,t}}}{\partial \overset{n+\alpha_m}{u^B_{j,t}}} = \frac{\partial(\sum_{C=1}^{n_{en}} N_i^C \overset{n+\alpha_m}{u^C_{i,t}})}{\partial \overset{n+\alpha_m}{u^B_{j,t}}} = \sum_{C=1}^{n_{en}} N_i^C \delta_{ij}\delta_{eB} = N_i^B \delta_{ij} \tag{42}$$

$$\frac{\partial \overset{n+\alpha_f}{u_{i,k}}}{\partial \overset{n+\alpha_f}{u^B_j}} = \frac{\partial(\sum_{C=1}^{n_{en}} N_{i,k}^C \overset{n+\alpha_f}{u_i})}{\partial \overset{n+\alpha_f}{u_{j,t}}} = \sum_{C=1}^{n_{en}} N_{i,k}^C \delta_{ij}\delta_{eB} = N_{i,k}^B \delta_{i,j} \tag{43}$$

$$\frac{\partial P}{\partial \overset{n+\alpha_f}{p^B}} = N_p^B \tag{44}$$

$$\frac{\partial \overset{n+\alpha_f}{u_{k,i}}}{\partial \overset{n+\alpha_f}{u^B_j}} = N_{k,i}\delta_{kj} \tag{45}$$

$$\frac{\partial \overset{n+\alpha_f}{u_{k,k}}}{\partial \overset{n+\alpha_f}{u^B_j}} = N_{j,j}^B \tag{46}$$

Consider matrices *(38)*~ *(42)* in detail:

$$\begin{aligned} K_{ij}^{aB} = \int_{\blacksquare} & N_i^a \rho \alpha_{m_u} N_i^B \delta_{ij} + \alpha_{f_u}\gamma_u \Delta t N_i^a \rho u_k N_{i,k}^B \delta_{ij} + \alpha_{f_u}\gamma_u \Delta t N_{i,k}^a \{\mu N_{j,k}^B \\ & + \tau_M u_k \rho u_e N_{j,e}^B + \tau \mathcal{L}_k \mathcal{L}_e N_{j,e}^B\} \\ & + \alpha_{f_u}\gamma_u \Delta t \{\mu N_{i,j}^a N_{j,i}^B + N_{i,j}^a \tau_C N_{j,j}^B\} Dd\blacksquare \end{aligned} \tag{47}$$

$$G_i^{aB} \approx -\alpha_{f_p}\gamma_p \Delta t \int_{\blacksquare} N_{i,i}^a N_p^B Dd\blacksquare \tag{48}$$

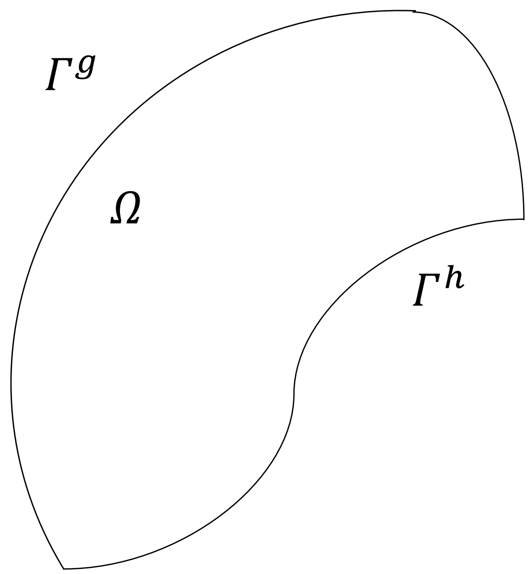
$$D_j^{aB} \approx \alpha_{f_u}\gamma_u\Delta t \int_{■} N_p^a N_{j,j}^B Dd■ = -\{G^{Ba}\}^T \tag{49}$$

$$C^{aB} \approx \int_{■} N_{p,k}^a \tau_m N_{p,k}^B \alpha_{f_p}\gamma_p \Delta t Dd■ \tag{50}$$

IV. Boundary condition

Generally, we can have 2 types of boundary

Conditions:



1) $\varphi = \varphi^g$ on Γ^g

3 names of the same thing $\begin{cases} Dirichlet \\ essential \\ nodal \end{cases}$

2) $\frac{\partial \varphi}{\partial n} = \varphi^h$ on Γ^h

$$\begin{cases} new \\ natural \\ face \end{cases}$$

Plus, we already described “periodical” boundary condition

Let’s consider essential B.C’s

$$y = \begin{Bmatrix} p \\ u \\ v \\ w \\ T \end{Bmatrix}$$

$$q = \begin{Bmatrix} u_r^g \\ u_s^g \\ u_t^g \end{Bmatrix}$$

--vector of quantities, which can be prescribed

(we are not considering pressure because this type of boundary condition is almost not applicable for it)

$$u = (u_r, u_s, u_t) \leftarrow local\ coordinate\ system\ (may\ not\ be\ aligned\ with\ Cartesian)$$

In general, we can express one in terms of another using direction cosines:

$$u_r^g = C_1^r u_1 + C_2^r u_2 + C_3^r u_3$$

$$u_s^g = C_1^s u_1 + C_2^s u_2 + C_3^s u_3$$

$$u_t^g = C_1^t u_1 + C_2^t u_2 + C_3^t u_3$$

In order to have diagonal dominance in matrix C_i^j, we can recorder r, s, t in such way that

$$|C_1^r| \geq |C_{i\neq 1}^r|$$

$$|C_2^s| \geq |C_{i\neq 2}^s|$$

$$|C_3^t| \geq |C_{i\neq 3}^t|$$

Let's express q in terms of given ? and solution variables

$$g(u) = \begin{cases} C_i^r u_i \\ C_i^s u_i \\ C_i^t u_i \end{cases}$$

Inverting this: (for u_1 only)

$$\hat{g} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{u_1^g}{C_1^r} - \dfrac{C_2^r}{C_1^r} u_2 - \dfrac{C_3^r}{C_1^r} u_3 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\Lsh \; this\ is\ done\ in\ itr6e.f$$

Now we have correct the weight function to be sure that they satisfy homogeneous counterpart of the Dirichlet B.C.:

$$w_i \Leftarrow \frac{\partial q_i}{\partial u_j} w_j$$

So substituting $\frac{\partial q_i}{\partial u_j}$ we can write

$$w_1 \Leftarrow -\frac{C_2^r}{C_1^r} w_2 - \frac{C_3^r}{C_1^r} w_3,$$

$$w_2 \Leftarrow -w_2$$

$$w_3 \Leftarrow w_3$$

So each possible B.C. can be thought of as a transformation applied to w:

$$w^{new} = Sw$$

$$S = \begin{matrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}, \ \ where\ \alpha \triangleq \frac{C_2^r}{C_1^r}, \beta \triangleq \frac{C_3^r}{C_1^r}$$

$$S^T = \begin{matrix} 0 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{matrix}$$

$$\overset{m}{R} \Leftarrow S^T \overset{m}{R}\ , \qquad LHS \Leftarrow S^T LHS\ S^T$$