Uniformisation theorem for flag bundles over Riemann surfaces

Anton Isopoussu

August 2, 2019

Abstract

We show that there is a simple extension of the Uniformisation Theorem to flag varieties of polystable vector bundles over Riemann surfaces.

1 Introduction

Throughout this chapter we let C be a curve and denote its fundamental group by Γ without reference to the choice of a base point. Let $\widehat{C}$ be the universal cover of C, which is one of the three model spaces given by the Uniformisation theorem. Let π be the canonical projection $\widehat{C} \to C$ and σ the covering action $\widehat{C} \times \Gamma \to \widehat{C}$.

Theorem 1. *Let E be a polystable vector bundle on C and let $\mathcal{F}l_r(E)$ be a flag bundle of E over C. All Kähler classes in $\mathcal{F}l_r(E)$ are cscK. In particular, $\mathcal{F}l_r(E)$ is K-semistable for all polarisations.*

We obtain a partial Yau-Tian-Donaldson correspondence for flag bundles on high genus curves using Theorem 1.

Theorem 2. *Let $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ be a polarised flag bundle on C.*

If E is polystable, the flag bundle $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ is K-semistable. If E is stable and $g \geq 2$, then the variety $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ is K-stable.

Finally, if E is simple and $g \geq 2$, the YTD correspondence holds for any line bundle $\mathcal{L}_\lambda(A)$ with $\lambda \in \mathcal{P}_\diamond(r)$ and A ample.

We prove the following Lemma in Section 3.

Lemma 3. *If the vector bundle E is simple and the genus satisfies $g \geq 2$, then the automorphism group of $\mathcal{F}l_r(E)$ is discrete.*

Proof of Theorem 2. The first statement follows directly from Theorem 1 and Proposition **??**.

For the second statement, we also need Lemma 3 and Proposition **??** which strengthens Proposition **??** in the case of a discrete automorphism group.

If E is polystable, the final statement follows from the second statement. If E is simple but not polystable, then we can construct a destabilising test configuration for $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ by Theorem **??**. $\square$

Remark 4. In order to prove a full YTD correspondence on flag bundles over curves one would need to analyse the delicate cases when $\mathcal{F}l_r(E)$ admits vector fields. By Equation (18) and the preceding discussion we see that this may happen when the base curve C is an elliptic curve and when E is properly polystable, that is, isomorphic to a direct sum of stable vector bundles of equal slopes. If the base curve C is isomorphic to $\mathbb{P}^1$, Grothendieck's theorem states that any holomorphic vector bundle E can be decomposed into a direct sum $\bigoplus_{i=1}^{r_E} \mathcal{O}_{\mathbb{P}^1}(m_i)$ for some $m_i \in \mathbb{Z}$ for $i = 1, \ldots, r_E$ [**?**].

2 Construction of flag bundles from representations of the fundamental group

Let G be an algebraic group and $\rho\colon \Gamma \to G$ be a representation. We define the *associated bundle with fibre* G [3]

$$\mathbb{E}_\rho = \widehat{C} \times G \big/ \Gamma \tag{1}$$

by the identification

$$(c, g) \sim (\sigma(\gamma, c), \rho(\gamma) g) \tag{2}$$

for $(c, g) \in \widehat{C} \times G$ and $\gamma \in \Gamma$. The quotient space $\mathbb{E}_\rho$ is an algebraic principal bundle over the curve C.

A representation $\rho\colon \Gamma \to \mathrm{GL}(e, \mathbb{C})$ determines a vector bundle E_ρ by setting

$$E_\rho = C \times \mathbb{C}^{r_E} \big/ \Gamma \tag{3}$$

by the identification in Equation (1) with $\mathrm{GL}(e, \mathbb{C})$ acting on $\mathbb{C}^{r_E}$ in the usual way. The vector bundle E_ρ and its associated frame bundle $\mathbb{E}_\rho$ have natural Zariski trivial algebraic structures since the fibre of $\mathbb{E}_\rho$ is $\mathrm{GL}(r_E, \mathbb{C})$ [4].

A *locally trivial holomorphic fibration* with fibre F is a holomorphic map $f\colon M \to M'$ of complex manifolds M and M' such that each point $x \in M'$ has an analytic neighborhood $U \subset M'$ such that the restriction of f to U is given by the first projection $U \times F \to U$.

Theorem 5. *Suppose that E is polystable vector bundle over a (complex, smooth, projective) curve C. Let $\bar{P}_r$ denote the image of the parabolic subgroup $P_r \subset \mathrm{GL}(r_E, \mathbb{C})$ in $\mathrm{PGL}(r_E, \mathbb{C})$. Then there exists representation $\rho\colon \Gamma \to \mathrm{PGL}(r_E, \mathbb{C})$ such that the holomorphic quotient map*

$$\widehat{C} \times \mathrm{PGL}(r, E) \big/ \bar{P}_r \to \mathcal{F}l_r(E) \tag{4}$$

is a holomorphic locally trivial fibration with fibre Γ.

Proof of Theorem 5. Let $\mathbb{E}$ be the frame bundle of E and define the *projectivised frame bundle*

$$\bar{\mathbb{E}} := \mathbb{E}/\mathbb{G}_m, \tag{5}$$

where $\mathbb{G}_m$ acts via the inclusion

$$\lambda \mapsto \lambda I \in \mathrm{GL}(r_E, \mathbb{C}) \tag{6}$$

for $\lambda \in \mathbb{G}_m$. By the Narasimhan-Seshadri Theorem **??** there exists a representation $\rho : \Gamma \to \mathrm{PGL}(r_E, \mathbb{C})$ such that $\bar{\mathbb{E}}$ is the associated bundle

$$\bar{\mathbb{E}} = \left(\widehat{C} \times \mathrm{PGL}(r_E, \mathbb{C})\right)/\Gamma. \tag{7}$$

of the representation ρ Since multiples of the identity matrix are contained in P_r we can write

$$\mathcal{F}l_r(E) = \bar{\mathbb{E}}/\bar{P}_r. \tag{8}$$

Hence the representation ρ induces an action of Γ on $\mathcal{F}l_r(E)$. The double quotient

$$\widehat{C} \times \mathrm{PGL}(r_E, \mathbb{C}) \longrightarrow \mathbb{E} \longrightarrow \mathcal{F}l_r(E) \tag{9}$$

can be factorised in two ways. We define the map

$$\hat{\pi} : \widehat{C} \times \mathrm{PGL}(r_E, \mathbb{C})/\bar{P}_r \longrightarrow \mathcal{F}l_r(E) \tag{10}$$

by

$$(x, g\bar{P}_r) \mapsto \left(\sigma(\Gamma, x), \rho(\Gamma)g\bar{P}_r\right) \in \mathcal{F}l_r(E). \tag{11}$$

The map $\hat{\pi}$ fits into the diagram

$$\begin{tikzcd}
\widehat{C} \times \mathrm{PGL}(r_E, \mathbb{C}) \arrow[r] \arrow[d] & \mathbb{E} \arrow[d] \\
\widehat{C} \times \mathrm{PGL}(r_E, \mathbb{C})/\bar{P}_r \arrow[r, "\hat{\pi}"] & \mathcal{F}l_r(E)
\end{tikzcd}$$

and is a locally trivial holomorphic fibration with fibre Γ, since π is. □

3 Constant scalar curvature Kähler metrics on flag bundles and K-polystability

We begin with a proof of Theorem 1, then turn to the proof of Lemma 3.

Proof of Theorem 1. Let G denote the group $\mathrm{PGL}(r_E, \mathbb{C})$. The Picard group of $\mathcal{F}l_r(E)$ is generated by line bundles of the form $\mathcal{L}_\lambda(A)$ where λ is in $\mathcal{P}(r)$ and A is a line bundle on C by Lemma **??**.

Fix a line bundle $M = \mathcal{L}_\lambda \otimes A$ with $A \in \operatorname{Pic} C$ and $\lambda \in \mathcal{P}(\lambda)$. Let

$$\pi \colon \widehat{C} \times G/P_r \to \mathcal{F}l_r(E) \tag{12}$$

be the projection constructed in Theorem 5.

There is a Kähler-Einstein (hence cscK) metric ω_0 in $c_1(\mathcal{L}_\lambda)$, unique up to the action of G, by results of Koszul and Matsushima [2]. Let s_0 be the (constant) scalar curvature of ω_0. Let ω_A be a constant scalar curvature metric such that $2\pi[\omega_A] = c_1(A)$ with scalar curvature s_1 and let ω_1 be the pullback to $\widehat{C}$. Since $\omega_0 + \omega_1$ is Γ-invariant, it descends to a form ω on $\mathcal{F}l_r(E)$ with constant scalar curvature $s_0 + s_1$. $\square$

Let V be a complex vector space of dimension r_E. In order to apply a classical result of Demazure, we regard $\mathcal{F}l_r(V)$ as a quotient of $\mathrm{PGL}(r, V)$. Let Q_r be the image of a stabiliser of an r-flag of subspaces in $\mathrm{PSL}(r_E, \mathbb{C})$ and let $\mathfrak{q}_r$ be its Lie algebra. Also let $\mathfrak{psl}(r_E, \mathbb{C})$ denote the Lie algebra of $\mathrm{PSL}(r_E, \mathbb{C})$. We have a well known exact sequence

$$0 \longrightarrow (\mathrm{PSL}(r_E, \mathbb{C}) \times \mathfrak{q}_r)/Q_r \longrightarrow \mathrm{PSL}(r_E, \mathbb{C})/Q_r \times \mathfrak{psl}(r_E, \mathbb{C}) \longrightarrow \mathcal{T}_{\mathcal{F}l_r(V)} \longrightarrow 0. \tag{13}$$

where Q_r acts on $\mathfrak{q}_r$ by the adjoint action and $\mathcal{T}_{\mathcal{F}l_r(V)}$ is the tangent bundle.

It follows from results of Demazure and Bott [1, Section 4.8] that we have

$$H^i\left(\mathcal{F}l_r(V), \mathcal{T}_{\mathcal{F}l_r(V)}\right) = \begin{cases} \mathfrak{psl}(r_E, \mathbb{C}), \text{ if } i = 0 \\ 0, \text{ otherwise.} \end{cases} \tag{14}$$

Let $p : \mathcal{F}l_r(E) \to C$ be the projection. Since $\mathcal{F}l_r(E)$ is Zariski locally trivial on C, this generalises in a straightforward manner. Let h be a hermitian metric on E and let $\mathrm{End}^0(E)$ denote the sheaf of trace-free endomorphisms on E. Let U be a Zariski open set in C such that

$$\mathcal{F}l_r(E) \cong U \times \mathcal{F}l_r(V). \tag{15}$$

We have a natural identification

$$\left(\mathcal{E}nd^0(E)/\mathbb{C}\right)\big|_U \cong \mathcal{O}_B\big|_U \otimes \mathfrak{psl}(r_E, \mathbb{C}), \tag{16}$$

where the $\mathbb{C}$ denotes the constant sheaf included in $\mathcal{E}nd^0(E)$ as multiples of the identity. Let $\mathcal{V}_{\mathcal{F}l_r(E)}$ denote the relative tangent bundle of $\mathcal{F}l_r(E)$ with respect to the projection p. We obtain from Equation (14)

$$R^i p_* \mathcal{V}_{\mathcal{F}l_r(E)} = \begin{cases} \mathcal{E}nd^0(E)/\mathbb{C} \text{ if } i = 0 \text{ and} \\ 0 \text{ otherwise,} \end{cases} \tag{17}$$

Proof of Lemma 3. We must show that the vector space $H^0(\mathcal{F}l_r(E), \mathcal{T}_{\mathcal{F}l_r(E)})$ is trivial. We have the exact sequence

$$0 \longrightarrow \mathcal{V}_{\mathcal{F}l_r(E)} \longrightarrow \mathcal{T}_{\mathcal{F}l_r(E)} \longrightarrow p^*\mathcal{T}_C \longrightarrow 0 \tag{18}$$

where $\mathcal{T}_C$ is the tangent bundle of the curve C. It suffices to show that $H^0(\mathcal{F}l_r(E), \mathcal{V}_{\mathcal{F}l_r(E)}) = 0$ since $H^0(C, \mathcal{T}_C) = 0$ as the genus $g(C)$ satisfies $g(C) > 1$. The vector bundle E is simple, therefore we have $H^0(C, \mathcal{E}nd(E)) = \mathbb{C} \cdot \mathrm{Id}_E$. The claim follows by identifying $H^0(C, \mathcal{E}nd^0(E))$ as a subspace of $H^0(C, \mathcal{E}nd(E))$. □

References

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