Representing Functions as Power Series

Consider

$$\boxed{\frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots = \sum_{n=0}^{\infty} u^n \quad |u| < 1} \tag{1}$$

EXAMPLE 1: Express $\dfrac{1}{1+x}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

Putting $u = -x$ in (1), we get

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n \quad \Longrightarrow \quad \boxed{\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n}$$

with the interval of convergence

$$|-x| < 1 \quad \Longrightarrow \quad |x| < 1 \quad \Longrightarrow \quad \boxed{(-1,1)}$$

EXAMPLE 2: Express $\dfrac{1}{5+x}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{1}{5+x} = \frac{\frac{1}{5}}{1+\frac{x}{5}} = \frac{1}{5}\cdot\frac{1}{1+\frac{x}{5}} = \frac{1}{5}\cdot\frac{1}{1-\left(-\frac{x}{5}\right)}$$

Putting $u = -x/5$ in (1), we get

$$\frac{1}{1-\left(-\frac{x}{5}\right)} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^n} \quad \Longrightarrow \quad \frac{1}{5}\cdot\frac{1}{1-\left(-\frac{x}{5}\right)} = \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^{n+1}}$$

Therefore

$$\boxed{\frac{1}{5+x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^{n+1}}}$$

with the interval of convergence

$$\left|-\frac{x}{5}\right| < 1 \quad \Longrightarrow \quad \left|\frac{x}{5}\right| < 1 \quad \Longrightarrow \quad |x| < 5 \quad \Longrightarrow \quad \boxed{(-5,5)}$$

EXAMPLE 3: Express $\dfrac{1}{1+x^2}$ as a power series and find the interval of convergence.

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Solution: We have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Putting $u = -x^2$ in (1), we get

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n \quad \Longrightarrow \quad \boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}}$$

with the interval of convergence

$$|-x^2| < 1 \quad \Longrightarrow \quad |x^2| < 1 \quad \Longrightarrow \quad x^2 < 1 \quad \Longrightarrow \quad \boxed{(-1,1)}$$

EXAMPLE 4: Express $\dfrac{1}{1+x^5}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)}$$

Putting $u = -x^5$ in (1), we get

$$\frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty}(-x^5)^n \quad \Longrightarrow \quad \boxed{\frac{1}{1+x^5} = \sum_{n=0}^{\infty}(-1)^n x^{5n}}$$

with the interval of convergence

$$|-x^5| < 1 \quad \Longrightarrow \quad |x^5| < 1 \quad \Longrightarrow \quad |x| < 1 \quad \Longrightarrow \quad \boxed{(-1,1)}$$

EXAMPLE 5: Express $\dfrac{x^5}{7-9x^3}$ as a power series and find the interval of convergence.

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Solution: We have

$$\frac{x^5}{7-9x^3}=\frac{\frac{x^5}{7}}{1-\frac{9x^3}{7}}=\frac{x^5}{7}\cdot\frac{1}{1-\frac{9x^3}{7}}$$

Putting $u=\dfrac{9x^3}{7}$ in (1), we get

$$\frac{1}{1-\frac{9x^3}{7}}=\sum_{n=0}^{\infty}\left(\frac{9x^3}{7}\right)^n=\sum_{n=0}^{\infty}\frac{9^n x^{3n}}{7^n} \quad\Longrightarrow\quad \frac{x^5}{7}\cdot\frac{1}{1-\frac{9x^3}{7}}=\frac{x^5}{7}\cdot\sum_{n=0}^{\infty}\frac{9^n x^{3n}}{7^n}=\sum_{n=0}^{\infty}\frac{9^n x^{3n+5}}{7^{n+1}}$$

thus

$$\boxed{\frac{x^5}{7-9x^3}=\sum_{n=0}^{\infty}\frac{9^n x^{3n+5}}{7^{n+1}}}$$

with the interval of convergence

$$\left|\frac{9x^3}{7}\right|<1 \quad\Longrightarrow\quad |x^3|<\frac{7}{9} \quad\Longrightarrow\quad \boxed{\left(-\sqrt[3]{\frac{7}{9}},\sqrt[3]{\frac{7}{9}}\right)}$$

THEOREM (**Differentiation and Integration of Power Series**): If the power series

$$\sum c_n(x-a)^n$$

has radius of convergence $R>0$, then the function f defined by

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+\ldots=\sum_{n=0}^{\infty}c_n(x-a)^n$$

is differentiable (and therefore continuous) on an interval $(a-R,a+R)$ and

(i) $$\boxed{f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+\ldots=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}}$$

(ii) $$\boxed{\int f(x)dx=C+c_0(x-a)+c_1\frac{(x-a)^2}{2}+c_2\frac{(x-a)^3}{3}+\ldots=C+\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

REMARK: Although the Theorem above says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

EXAMPLE 6: Express $\dfrac{1}{(1-x)^2}$ as a power series and find its radius of convergence.

Solution: Differentiating both sides of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

we get

$$\boxed{\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}}$$

By the Theorem above the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, $R = 1$.

EXAMPLE 7: Express $\ln(1-x)$ as a power series and find its radius of convergence.

Solution: Integrating both sides of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

we get

$$-\ln(1-x) = \int \frac{1}{1-x}dx \quad = \int (1 + x + x^2 + \ldots)dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C$$

To determine the value of C we put $x = 0$ in this equation and obtain

$$-\ln(1-0) = C \quad \Longrightarrow \quad C = 0$$

Therefore

$$\boxed{\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots = -\sum_{n=1}^{\infty} \frac{x^n}{n}} \tag{2}$$

The radius of convergence is the same as for the original series: $R = 1$.

REMARK: One can show that (2) is also true if $x = -1$. Substituting $x = -1$ into the formula, we get

$$\boxed{\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}}$$

EXAMPLE 8: Express $\tan^{-1} x$ as a power series and find its radius of convergence.

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Solution: By Example 3 we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Integrating both sides of this equation, we get

$$\tan^{-1} x = \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + \ldots) dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

To determine the value of C we put $x = 0$ in this equation and obtain

$$\tan^{-1} 0 = C \quad \Longrightarrow \quad C = 0$$

Therefore

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} \tag{3}$$

The radius of convergence is the same as for the original series: $R = 1$.

REMARK: One can show that (3) is also true if $x = \pm 1$. Substituting $x = 1$ into the formula, we get

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}}$$

EXAMPLE 9: Express $\displaystyle\int \frac{1}{1+x^5} dx$ as a power series.

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Solution: We first notice that integrating of $\frac{1}{1+x^5}$ by hand is incredibly difficult. In fact,

$$\int \frac{1}{1+x^5}dx = -\frac{1}{20}\ln\left(2x^2 - x - \sqrt{5}x + 2\right)\sqrt{5} - \frac{1}{20}\ln\left(2x^2 - x - \sqrt{5}x + 2\right)$$

$$+\arctan\left(\frac{4x - 1 - \sqrt{5}}{\sqrt{10 - 2\sqrt{5}}}\right)\frac{1}{\sqrt{10 - 2\sqrt{5}}} - \frac{1}{5}\arctan\left(\frac{4x - 1 - \sqrt{5}}{\sqrt{10 - 2\sqrt{5}}}\right)\sqrt{5}\frac{1}{\sqrt{10 - 2\sqrt{5}}}$$

$$-\frac{1}{20}\ln\left(2x^2 - x + \sqrt{5}x + 2\right) + \frac{1}{20}\ln\left(2x^2 - x + \sqrt{5}x + 2\right)\sqrt{5}$$

$$+\arctan\left(\frac{4x - 1 + \sqrt{5}}{\sqrt{10 + 2\sqrt{5}}}\right)\frac{1}{\sqrt{10 + 2\sqrt{5}}} + \frac{1}{5}\arctan\left(\frac{4x - 1 + \sqrt{5}}{\sqrt{10 + 2\sqrt{5}}}\right)\sqrt{5}\frac{1}{\sqrt{10 + 2\sqrt{5}}}$$

$$+\frac{1}{5}\ln(x+1) + C$$

By Example 4 we have

$$\frac{1}{1+x^5} = \sum_{n=0}^{\infty}(-1)^n x^{5n}$$

Integrating both sides of this equation, we get

$$\boxed{\int \frac{1}{1+x^5}dx = \int \sum_{n=0}^{\infty}(-1)^n x^{5n}dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{5n+1}}{5n+1} = C + x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \dots}$$