Columbia Unversity
COMSW4771
Dr. James McInerney
Homework 0

Submitted By: Alexander Stein, as5281

September 10th, 2017

Problem 1.1

(i) What is the marginal distribution of X?

Table 1: Marginal Probabilities

	Y1	Y2	Y3	P_x(i)
X1	0.1	0.2	0.3	0.6
X2	0.2	0.1	0.1	0.4
P_y(j)	0.3	0.3	0.4	1.00

(ii) What is Pr[Y=1 | X=2]?

P(Y=1|X=2) = P(Y=1, X=2) / P(X=2)
P(Y=1|X=2) = (0.2)/(0.4)
P(Y=1|X=2) = 1/2

(iii) Let f : x $\rightarrow$ x^2. What is E[f(x)|Y=1]?

$E(x|Y = y)$
$= \sum_{x \in X} xP(X = x|Y = y)$

$= \sum_{x \in X} xP(x = x, Y = y)/P(Y = y)$

$E(f(x)|Y = 1)$
$= \sum_{x \in X} x^2 P(X = x, Y = 1)/P(Y = 1)$
$= (1^2) * (0.1)/(0.3) + (2^2) * (0.2)/(0.3)$
$= 1/3 + 8/3$
$= 3$

Problem 1.2

(i) Verify that $\frac{1}{\theta} e^{-x/\theta}$ is a probability distribution.

To verify, we will show that the integral from 0 to ∞ is 1

$$\int_0^\infty \frac{1}{\theta} e^{x/\theta} dx$$

$$u = -\frac{x}{\theta} \rightarrow dx = -\theta du$$

$$= -\frac{1}{\theta} \int_0^\infty \theta e^u du$$

$$= -e^u = -e^{-x/\theta} \mid_0^\infty$$

$$= (-e^{-\infty/\theta}) - (-e^{-0/\theta})$$

$$= 1$$

(ii) What is E[X] ?

$$\mu = E[X] = \int_{-\infty}^\infty x f(x) dx = \frac{1}{\theta} \int_0^\infty e^{-x/\theta}$$

$$u = x/\theta dx = \theta du$$

$$= \theta \int u e^{-u} du$$

$$f = u \text{ and } g' = e^{-u}$$

$$f' = 1 \text{ and } g = -e^{-u}$$

$$= fg - \int f'g \rightarrow -u\theta e^{-u} + \theta \int e^{-u} du$$
$$= -ue^{-u} - e^{-u} = -xe^{-x/\theta} - \theta e^{-x/\theta} \mid_0^\infty$$
$$= (0) - (-\theta) = \theta$$

(iii) What is Var(X) ?

$$Var(X) = E((X - \mu)^2) = E(X^2) - E(X)^2 = E(X^2) - \mu$$
$$\text{so all we need to find is } E(X^2)$$
$$E(X^2) = \int_0^\infty \frac{x^2}{\theta} e^{-x/\theta} dx$$
$$u = x/\theta \text{ and } dx = \theta du$$
$$= \theta^2 \int u^2 e^{-u} du$$
$$f = u^2 \text{ and } g' = e^{-u}$$
$$f' = 2u \text{ and } g = -e^{-u}$$
$$= fg - \int f'g \rightarrow \theta^2[-u^2 e^{-u} - \int_0^\infty -2ue^{-u} du]$$
$$f_1 = u \text{ and } g_1' = e^{-u}$$
$$f_1' = 1 \text{ and } g_1 = -e^{-u}$$
$$= \theta^2[-u^2 e^{-u} + 2[-ue^{-u} + \int_0^\infty -e^{-u} du]]$$
$$= \theta^2[-u^2 e^{-u} - 2ue^{-u} - 2e^{-u}]$$
$$= -(x^2 e^{-x/\theta} + 2\theta x e^{-x/\theta} + 2\theta^2 e^{-x/\theta}) \mid_0^\infty$$
$$= (0) - (-2\theta^2) = 2\theta^2$$

$$Var(X) = E(X^2) - E(X)^2 = 2\theta^2 - \theta^2 = \theta^2$$

Problem 1.3

Table 2: Burglary Probabilities

	Robbery(5%)	No Robbery(95%)
Alarm	99%	10%
No Alarm	1%	90%

We are effectively using Bayes Theroem to solve this problem:

$$P(\text{"}true\_alarm\text{"}) = P(Robbery) * P(Alarm \mid Robbery)$$

$$= (0.05) * (0.99) = 0.0495$$

$$P(\text{"}false\_alarm\text{"}) = P(No\_Robbery) * P(Alarm|No\_Robbery)$$

$$= (0.95) * (0.10) = 0.095$$

$$P(\text{"}any\_alarm\text{"}) = P(\text{"}true\_alarm\text{"}) + P(\text{"}false\_alarm\text{"})$$

$$= 0.1445$$

$$P(Robbery \mid \text{"}any\_alarm\text{"}) = \frac{P(\text{"}true\_alarm\text{"})}{P(\text{"}any\_alarm\text{"})} = \frac{0.0495}{0.1445} = 0.34256$$

Problem 2.1

Subspace S is spanned by $a_1 = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 4 \end{vmatrix}$, $a_2 = \begin{vmatrix} 2 \\ 8 \\ 3 \\ 2 \end{vmatrix}$, $a_3 = \begin{vmatrix} 3 \\ 10 \\ 6 \\ 6 \end{vmatrix}$.

(i) what isthe dimension of the subspace S?

we can see that $a_1 + a_2 - a_3 = 0$, But there is no scalar multiple k such that $ka_1 = a_2$. Thus, a_1 and a_2 are linearly independent, and the dimension of S is 2.

(ii) Project the vector $\begin{vmatrix} 6 \\ 5 \\ 9 \\ 2 \end{vmatrix}$ onto the subspace S

First we need an orthogonal basis, we will use Graham-Schmitt Process

$$v_1 = a_1$$

$$v_2 = a_2 - \frac{(a_2, v_1)}{||v_1||^2} v_1$$

$$v_2 = \begin{vmatrix} 5/6 \\ 17/3 \\ -1/2 \\ -8/3 \end{vmatrix}$$

Now we can project...

$$proj_s u = \frac{(u, v_1)}{||v_1||^2} v_1 + \frac{(u, v_2)}{||v_2||^2} v_2 = 1.7v_1 + 0.58v_2$$

Problem 2.2 - SKIPPED, don't know how

Problem 3.1 - SKIPPED, don't know how

Problem 4.1 (see attached images for plots)

CODE:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.io import loadmat
from scipy import linalg as LA

mat_dict = {}
mat_dict.update(loadmat('hw0data.mat'))

M = mat_dict['M']

print "The dimensions of matrix M: ", M.shape
print "The 4th row of M: ", M[3,:]
print "The 5th column of M: ", M[:,4]
print "Mean value of the 5th column of M: ", np.mean(M[:,4])

print "histogramming the 4th row..."
plt.hist(M[3,:], rwidth=0.5)
plt.title("Histogram of M[3,:]")
plt.xlabel("Value")
plt.ylabel("Frequency")
plt.show()

M_square = np.dot(np.transpose(M), M)
e_vals, e_vecs = LA.eig(M_square)

print "Top three eigenvalues of M_transpose * M: ", e_vals[0:3]
```

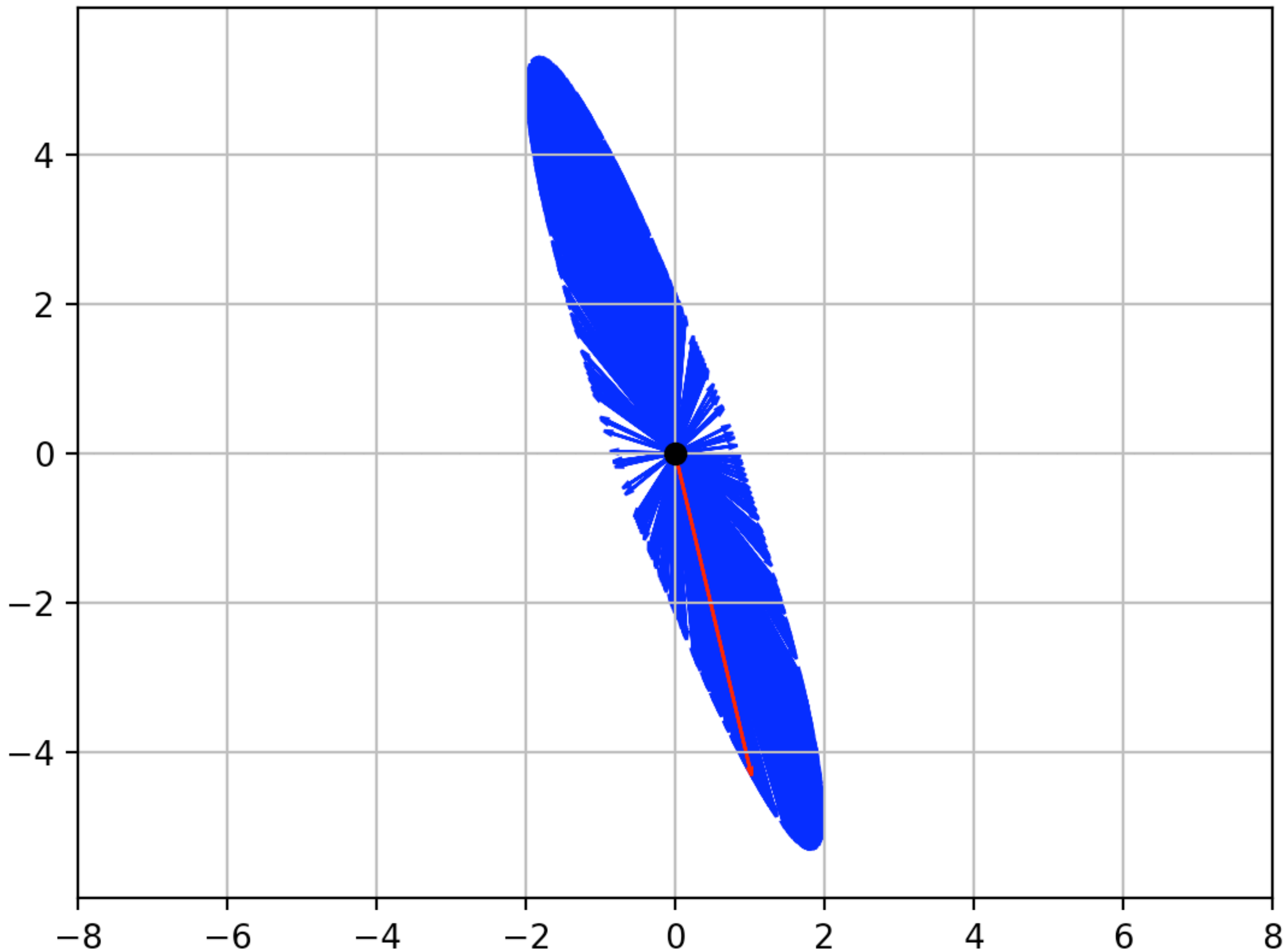
Problem 4.2 (see attached images for plots)

CODE:

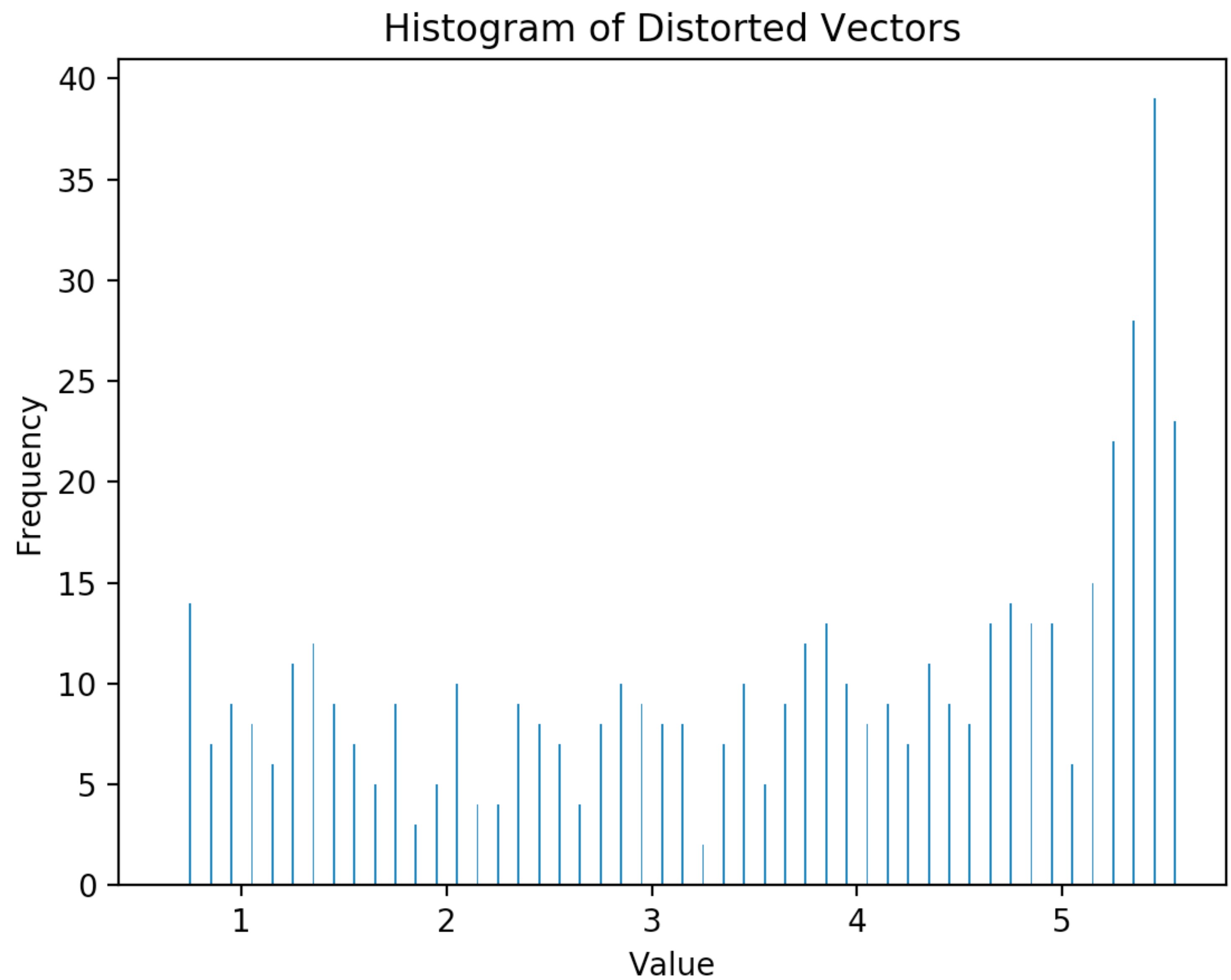
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.io import loadmat
from scipy import linalg as LA
```

```
L = np.matrix('1.25 -1.5; -1.5 5')

print L

# 500 2-vectors from Gaussian distribution, normalized
R = np.random.normal(0,1,(500,2))
for i, row in enumerate(R):
    R[i,:] = row / np.linalg.norm(row)

# Distorted Vectors
R_hat = np.dot(R,L)

# Eigenvalues of L
e_vals, e_vecs = LA.eig(L)
lambda_max = e_vals.max()
lambda_min = e_vals.min()
v_max = e_vecs[1];
print "Lambda Max: ", lambda_max
print "V_max: ", v_max
print "Lambda Min: ", lambda_min

# Magnitudes of R_hat
R_hat_mag = np.array([np.linalg.norm(row) for row in R_hat])

# Histogram of Distorted Vectors' Magnitudes
print "histogramming Distorted Vectors Magnitudes"
plt.hist(R_hat_mag, rwidth=0.1, bins=np.arange(0.6, 5.7, 0.1))
plt.title("Histogram of Distorted Vectors")
plt.xlabel("Value")
plt.ylabel("Frequency")
plt.show()

# Plot of All the Distorted Vectors and compare to lambda_max
print "plotting Distorted Vectors vs. V_max"
rows,cols = R_hat.shape

for i,l in enumerate(range(0,rows)):
```

```
    plt.axes().arrow(0,0,R_hat[i,0], \
      R_hat[i,1],head_width=0.05,head_length=0.1,color = 'b')

l_v_max = np.dot(L,v_max)
plt.axes().arrow(0,0,l_v_max[0,0], \
  l_v_max[0,1],head_width=0.05,head_length=0.1,color = 'r')

plt.plot(0,0,'ok') #<-- plot a black point at the origin
plt.axis('equal')  #<-- set the axes to the same scale
plt.xlim([-8,8]) #<-- set the x axis limits
plt.ylim([-8,8]) #<-- set the y axis limits
plt.grid(b=True, which='major') #<-- plot grid lines
plt.show()
```



4
2
0
−2
−4
−8
−6
−4
−2
0
2
4
6
8


Histogram of Distorted Vectors
Frequency
0
5
10
15
20
25
30
35
40
1
2
3
4
5
Value

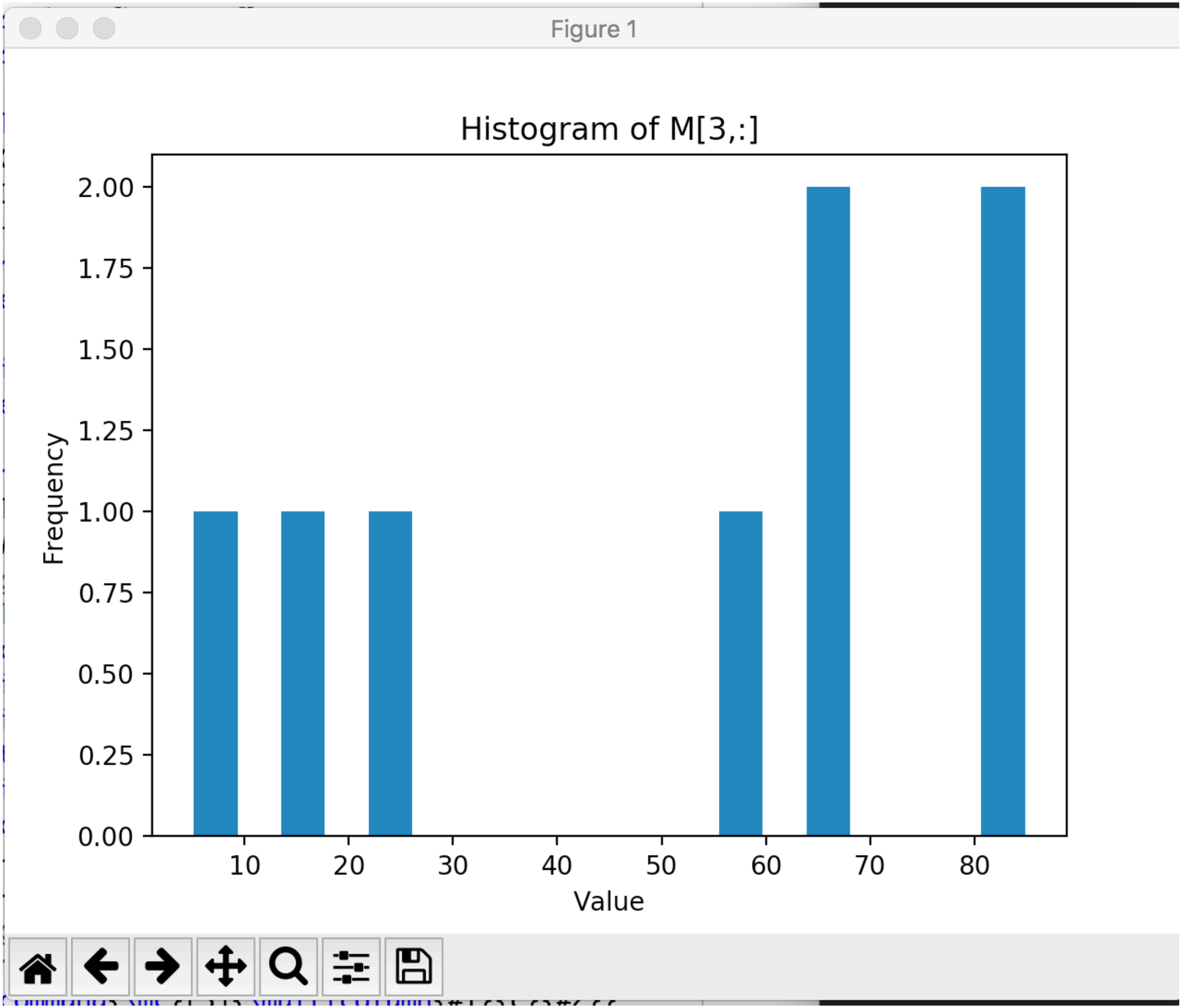


Figure 1
Histogram of M[3,:]
Frequency
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
10
20
30
40
50
60
70
80
Value