T1

$\xi \sim R(0, \theta)$ $\quad \theta > 0 \quad$ $\vec{x}_n$ - выборка

$\tilde{\theta}_1 = 2\bar{x} = 2\frac{1}{n}\sum_{i=1}^{n} x_i \qquad \tilde{\theta}_2 = \min x_i \qquad \tilde{\theta}_3 = \max x_i$

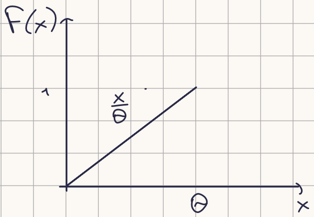
$\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$

$p(x) - \frac{1}{\theta}\{(0, \theta)\}$

$M[\xi] = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_0^{\theta} \frac{x}{\theta}\, dx = \frac{\theta}{2}$

$M[\xi^2] = \int_{-\infty}^{\infty} x^2\, p(x)\, dx = \int_0^{\theta} \frac{x^2}{\theta}\, dx = \frac{\theta^2}{3}$

$D[\xi] = \frac{\theta^2}{4} - \frac{\theta^2}{3} = \frac{\theta^2}{12}$



1 $\quad \tilde{\theta}_1 = 2\bar{x} \qquad \forall \theta > 0 \quad M[\tilde{\theta}_1] = \theta$

$M\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{n}\sum_{i=1}^{n} M[x_i] = \frac{2}{n} n M[\xi] = M[\xi] = \theta$

$\tilde{\theta}_1$ - оценка несмещенная

$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum x_i\right] = \frac{4}{n}\sum D[x_i] = \frac{4}{n} D[\xi] = \frac{\theta^2}{3n} \to 0, \quad n \to \infty$

оценка состоятельная по дост условию

2 $\quad \tilde{\theta}_2 = \min x_i$

$M[\tilde{\theta}_2] = \int_{-\infty}^{\infty} y\, \varphi(y)\, dy$

$\Phi(y) = 1 - (1 - F(y))^n$

$\varphi(y) = \Phi'(y) = n(1 - F(y))^{n-1}\, p(y)$

$\frac{y}{\theta}$ " на $(0, \theta)$ $\qquad$ " $\frac{1}{\theta}\{(0, \theta)\}$

$= n\left(1 - \frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}$, на $(0, \theta)$

$$M[\tilde{\theta}_2] = \int_0^{\theta} n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{y}{\theta}\,dy = \left\{t = 1-\frac{y}{\theta}\right\}$$

$$= -\int_1^0 n t^{n-1}\,\theta(1-t)\,dt = \int_0^1 n\theta t^{n-1}dt - \int_0^1 n\theta t^{n}dt =$$

$$= \theta\left[1-\frac{n}{n+1}\right] = \frac{\theta}{n+1}$$ — оценка смещенная

$$\theta_2' = (n+1)\underbrace{x_{min}}_{\tilde{\theta}_2}$$ — несмещенная

$$M[\hat{\theta}_2'] = \theta$$

$$M[\tilde{\theta}_2^2] = \int_0^{\theta} n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}y^2\,dy = -\int_1^0 n t^{n-1}\theta^2(1-t^2)\,dt =$$

$$n\theta^2\left[\frac{1}{n} - 2\frac{1}{n+1} + \frac{1}{n+2}\right] = \frac{\cancel{n}\theta(\cancel{n^2}+\cancel{3n}+2-\cancel{2n^2}-\cancel{4n}+\cancel{n^2}+\cancel{n})}{\cancel{n}(n+1)(n+2)}$$

$$= \frac{2\theta^2}{(n+1)(n+2)}$$

$$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\,\frac{2n+2-n-2}{(n+1)^2(n+2)}$$

$$D[\tilde{\theta}_2'] = (n+1)^2 D[\tilde{\theta}_2] = \frac{\theta^2 n}{n+2} \underset{n\to\infty}{\not\to} 0$$

по определению $\tilde{\theta}_2'$

$$\forall\theta>0\ \ \forall\varepsilon>0 \hookrightarrow P(|\tilde{\theta}_2' - \tilde{\theta}| \ge \varepsilon) \longrightarrow 0$$

$$P(|\tilde{\theta}_2' - \theta| \ge \varepsilon) \ge P(\tilde{\theta}_2' \ge \theta+\varepsilon) = P((n+1)x_{min} \ge \theta+\varepsilon)$$

$$= P\left(x_{min} \ge \frac{\theta+\varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\theta+\varepsilon}{n+1}\right) = 1-\left(1-\left(1-F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right)$$

$$= \left(1-F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n = \left(1-\left(\frac{\theta+\varepsilon}{\theta(n+1)}\right)\right)^n \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$$

не состоятельная

$$\tilde{\theta}_2 \quad \forall\theta>0\ \ \forall\varepsilon>0 \hookrightarrow P(|\hat{\theta}_2 - \theta| \ge \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$P(|\tilde{\theta}_2 - \theta| \ge \varepsilon) = P(\tilde{\theta}_2 < \theta-\varepsilon) + \overset{=0}{P(\tilde{\theta}_2 > \theta+\varepsilon)} =$$

$$= P(x_{min} < \theta-\varepsilon) = \underset{\varepsilon<\theta}{\Phi(\theta-\varepsilon)} = 1-\left(1-\frac{\theta\varepsilon}{\theta}\right)^n = 1-\left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1$$

не состоятельная оценка

• $\tilde{\Theta}_3 = X_{max}$

$\Phi(z) = F(z)^n$

$\varphi(z) = nF(z)^{n-1}\, p(z) = n\left(\frac{z}{\Theta}\right)^{n-1}\frac{1}{\Theta}$ $\qquad \{[0,\Theta]\}$

$M[\tilde{\Theta}_3] = \int_0^{\Theta} n\frac{z^n}{\Theta^n}\,dz = \frac{n}{n+1}\Theta$ — смещенная

$\tilde{\Theta}_3' = \frac{n+1}{n}\tilde{\Theta}_3$ — несмещенная

$M[\tilde{\Theta}_3^2] = \int_{-\infty}^{\infty} z^2\varphi(z)\,dz = \int_0^{\Theta} n\frac{z^{n+1}}{\Theta^n}\,dz = \frac{n}{n+2}\Theta^2$

$D[\tilde{\Theta}_3] = M[\tilde{\Theta}_3^2] - M^2[\tilde{\Theta}_3] = \Theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right] = \Theta^2\frac{n}{(n+1)^2(n+2)} \to 0$

$D[\tilde{\Theta}_3'] = \left(\frac{n+1}{n}\right)^2 D[\tilde{\Theta}_3] = \Theta^2\frac{1}{n(n+2)} \xrightarrow[n\to\infty]{} 0$ состоятельная

по определению

$\forall\Theta>0\ \forall\varepsilon>0 \hookrightarrow P(|\tilde{\Theta}_3'-\Theta|\ge\varepsilon) \xrightarrow[n\to\infty]{} 0$

$P(|\max x_i - \Theta| \ge \varepsilon) = P(\max x_i \ge \Theta+\varepsilon) + P(\max x_i \le \Theta-\varepsilon) =$

$= 1 - \underbrace{P(\max x_i < \Theta+\varepsilon)}_{(F(\Theta+\varepsilon))^n = 1} + \underbrace{P(\max x_i < \Theta-\varepsilon)}_{(F(\Theta-\varepsilon))^n = \left(1-\frac{\varepsilon}{\Theta}\right)^n \xrightarrow[n\to\infty]{} 0} + \underbrace{P(\max x_i = \Theta-\varepsilon)}_{0}$

$\Rightarrow \tilde{\Theta}_3' \xrightarrow{P} 0$ состоятельна

$$\hat{\Theta}_4 = x_1 + \frac{1}{(n-1)} \sum_{i=2}^{n} x_i \qquad \xi \sim R(0,\Theta),\ \Theta > 0 \qquad M\xi = \frac{\Theta}{2}$$

$$M[\hat{\Theta}_4] = M[x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i] = M[x_1] + \frac{1}{n-1}\sum_{i=2}^{n} M[x_i] = \frac{\Theta}{2} + \frac{\Theta}{2} = \Theta$$

не смещенная

- $D[\hat{\Theta}_4] = D[x_1 + \frac{1}{n-1}\sum x_i] = D[\xi] + \frac{1}{(n-1)^2}(n-1)D[\xi] =$

  $D[\xi] = \frac{\Theta^2}{12}$

  $= \frac{\Theta^2}{12}\frac{n}{n-1} \not\to 0 \qquad n \to \infty$

  достаточное условие не работает

- $\hat{\Theta}_4 \xrightarrow{P} \Theta$

  - $\xi_n \xrightarrow{P} \xi \qquad \eta_n \xrightarrow{P} \eta \quad \Rightarrow \quad \xi_n + \eta_n \xrightarrow{P} \xi + \eta$

  - $x_1 \xrightarrow{P} x_1$

    $\frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} M[\xi] = \frac{\Theta}{2}$

  - ЗБЧ Хинчина

    $\xi_1, \ldots \xi_n$ незав, один распред, тогда

    $\frac{1}{n}\sum_{i=1}^{n}\xi_i \xrightarrow{P} M[\xi]$

$$\hat{\Theta}_4 = x_1 + \frac{1}{(n-1)}\sum_{i=2}^{n} x_i \xrightarrow{P} x_1 + \frac{\Theta}{2}$$

не является состоятельной ×

$\tilde{\Theta}_1 = 2\overline{x}$

$\tilde{\Theta}_3' = \frac{n+1}{n} x_{max}$ сравним эфф

$D[\tilde{\Theta}_1] = \frac{\Theta^2}{3n}$

$D[\tilde{\Theta}_3'] = \frac{\Theta^2}{n(n+2)}$ эффективнее

$\frac{\Theta^2}{n(n+2)} < \frac{\Theta^2}{3n} \quad \forall \Theta > 0$

$3n < n^2 + 2n \quad n^2 > n \quad \forall n > 1$

$\tilde{\Theta}_3'$ эффективнее

T3

$$p(x) = \begin{cases} \dfrac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \theta > 0 \qquad n = 3$$

$$n = 3 \qquad \tilde{\theta}_1 = \overline{x} \qquad \hat{\theta}_2 = x_{(2)}$$

$$M[\xi] = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_0^{\infty} x\, e^{-\frac{x}{\theta}} \frac{1}{\theta} dx = \frac{1}{\theta}\int_0^{\infty} x e^{-\frac{x}{\theta}} dx$$

$$= \frac{1}{\theta}\left(-\theta x e^{-\frac{x}{\theta}}\Big|_0^{+\infty} + \theta \int_0^{+\infty} e^{-\frac{x}{\theta}} dx\right) = \int_0^{+\infty} e^{-\frac{x}{\theta}} dx = \theta e^{-\frac{x}{\theta}}\Big|_0^{+\infty} = \theta$$

$$M[\xi^2] = \int_0^{+\infty} x^2 e^{-\frac{x}{\theta}} \frac{1}{\theta} dx = \frac{1}{\theta}\int_0^{+\infty} x^2 e^{-\frac{x}{\theta}} dx = \left\{t = \frac{x}{\theta}\right\} = \theta^2 \int_0^{\infty} t^2 e^{-t} dt$$

$$= \underbrace{-\theta^2 t^2 e^{-t}\Big|_0^{+\infty}}_{0} + 2\theta^2 \int_0^{\infty} t e^{-t} dt = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = 2\theta^2 - \theta^2 = \theta^2$$

1 $\quad \tilde{\theta}_1 = \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$M\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} M[x_i] = M[x_i] = \theta$ - оценка несмещена

$$D[\hat{\theta}_1] = D\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} D[x_i] = \frac{\theta^2}{n} = \frac{\theta^2}{3}$$

$$\hat{\theta}_2 = x_{(2)}$$

$$F(x) = \int_0^x p(t)\, dt = 1 - e^{-\frac{x}{\theta}}$$

$$\varphi = n p(y)\, C_{n-1}^{k-1} (1 - F(y))^{n-k} (F(y))^{k-1} = \left\{\begin{matrix} k = 2 \\ n = 3 \end{matrix}\right\} =$$

$$= \frac{6}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)$$

$$M[\hat{\theta}_2] = 6\int_0^{\infty} \underbrace{\underbrace{\frac{x}{\theta}\left(e^{-\frac{2x}{\theta}}\right.}_{1} - e^{-\frac{3x}{\theta}}\Big)}_{2} dx$$

① $= -\frac{1}{2}\int_0^{\infty} x\, d e^{-\frac{2x}{\theta}} = -\frac{1}{2}\left[x e^{-\frac{2x}{\theta}} - \int_0^{\infty} e^{-\frac{2x}{\theta}} dx\right] = \frac{\theta}{4}$

② $= -\frac{1}{3}\left[x e^{-\frac{3x}{\theta}}\Big|_0^{\infty} - \int_0^{\infty} e^{-\frac{3x}{\theta}} dx\right] = \frac{\theta}{9}$

$M[\tilde{\theta}_2] = 6\left[\frac{\theta}{4} - \frac{\theta}{9}\right] = \frac{5\theta}{6}$

$\hat{\theta}_2$ – смещенная

$\hat{\theta}_2' = 6/5\ \tilde{\theta}_2 = 6/5\ x_{(2)}$ несмещ

2

$D[\tilde{\theta}_1] = D[\bar{x}] = \frac{1}{n^2} n D[y] = \frac{\theta^2}{n} = \frac{\theta^2}{3}$

$M(\theta_2^2) = n(n-1)\frac{1}{\theta}\int_0^{+\infty} y^2\left(e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta}n}\right)dy =$

$= \frac{n(n-1)}{\theta}\int_0^{\infty} y^2 e^{-\frac{y}{\theta}(n-1)} dy - \frac{n(n-1)}{\theta}\int_0^{\infty} y^2 e^{-\frac{y}{\theta}n} dy =$

$= \frac{n\theta^2}{(n-1)^2}\int_0^{\infty} t^2 e^{-t} dt - \frac{(n-1)\theta^2}{n^2}\int_0^{+\infty} t^2 e^{-t} dt =$

$= n(n-1)\frac{1}{\theta}\left(\frac{2\theta^3}{(n-1)^3} - \frac{2\theta^3}{n^3}\right) = 2\left(\frac{n\theta}{(n-1)^2} - \frac{(n-1)\theta^2}{n^2}\right) =$

$= 2\theta^2\frac{3n^2-3n+1}{n^2(n-1)^2} = \frac{19\theta^2}{6}$

$D[\tilde{\theta}_2] = \frac{19\theta^2}{18} - \frac{25}{36}\theta^2 = \frac{13}{36}\theta^2$

$D[\tilde{\theta}_2'] = \frac{36}{25} D[\tilde{\theta}_2] = \frac{13}{25}\theta^2$

Сравнение

$D[\tilde{\theta}_1] = \frac{\theta^2}{3} < D[\tilde{\theta}_2'] = \frac{13}{25}\theta^2 \quad \forall n$

$\hat{\theta}_1$ – эффективнее

3 регулярность

$p(x)$ непрер дифф по $\Theta > 0$

$$\frac{d}{d\theta}\int_{-\infty}^{\infty} p(x)\,dx = \frac{d}{d\theta}\left(\frac{1}{\theta}\int_0^{\infty} e^{-\frac{x}{\theta}}dx\right) = \frac{d}{d\theta}\left(-\frac{1}{\theta}\,\theta e^{-\frac{x}{\theta}}\right)\Big|_0^{\infty} =$$

$$= \frac{d}{d\theta}1 = 0$$

$I(\theta)$ - полож огр на Θ и $I_y(\theta)$ непр на Θ

$$I(\theta) = M\left[\frac{d\ln p(x)}{d\theta}\right]^2 = M\left[\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2\right] = \int_0^{\infty}\frac{(x-\theta)^2}{\theta^4}\frac{e^{-\frac{x}{\theta}}}{\theta}\,dx$$

$$= \frac{1}{\theta^5}\int_0^{\infty} x^2 e^{-\frac{x}{\theta}}dx + \frac{1}{\theta^3}\int_0^{\infty} e^{-\frac{x}{\theta}}dx - \frac{2}{\theta^4}\int_0^{\infty} x e^{-\frac{x}{\theta}}dx = \frac{1}{\theta^2}$$

регулярна

$\hat{\Theta}_1$ - несмещ, $D[\hat{\Theta}_1]$ - огр $\overset{\text{на } \forall \text{ комп}}{\Rightarrow}$ регулярна по дост

$\hat{\Theta}_2$ - несмещ, $D[\hat{\Theta}_2]$ - огр $\overset{\text{на } \forall \text{ комп}}{\Rightarrow}$ регулярна по дост

$\Rightarrow$ можно Крамера - Рао

$$D[\Theta_2] \geq \frac{1}{3\frac{1}{\theta^2}} = \frac{\theta^2}{3}$$

$\Rightarrow \Theta_1$ эффективнее