Solving Nonlinear Systems of Equations

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Announcement

- No class next week (校運動會)

Today

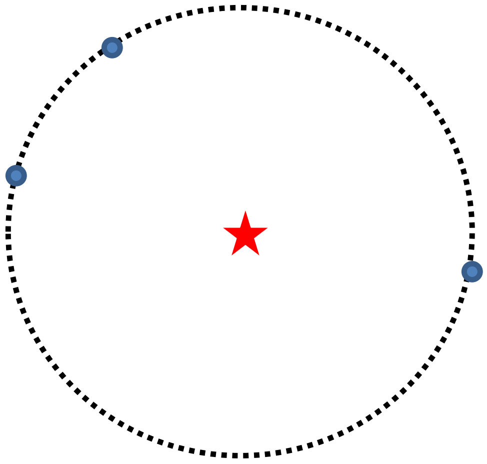
- Solving nonlinear systems of equations

$$x^2 - 2xy + y^2 = 3$$
$$x^2 + xy + y^2 = 12$$

$$2u^2 - 4u + v^2 + 3w^2 + 6w + 2 = 0$$
$$u^2 + v^2 - 2v + 2w^2 - 5 = 0$$
$$3u^2 - 12u + v^2 + 3w^2 + 8 = 0$$

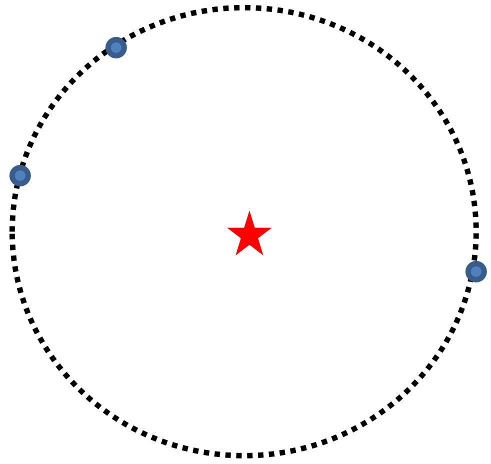
Today

- Solving nonlinear systems of equations
  - 代課老師小陳答應幫忙改100份數學考卷，題目為給定三個點，計算通過此三點的圓，求其圓心。但粗心的小陳忘記要參考答案！請你寫一個程式，輸入三個點(三點不會在同一直線上)，輸出通過此三點的圓之圓心位置。



Solving nonlinear systems

- 未知數?
  - x, y, R（圓心座標及半徑）
- 方程式?
  - 三個
  - 題目給的三個點到圓心的距離為R
- Multivariate Newton's method



Review: One-variable Newton's method

Given a scalar differentiable function $f(x)$,

1. Start from an initial guess x_0
2. For $i = 0, 1, 2, \ldots$ compute

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

until x_{i+1} satisfies some termination criterion

Extending to more variables

- Suppose we have 3 unknowns, 3 nonlinear equations:

$$\begin{aligned} f_1(u,v,w) &= 0 \\ f_2(u,v,w) &= 0 \\ f_3(u,v,w) &= 0 \end{aligned}$$

$$\boxed{\begin{gathered} 2u^2 - 4u + v^2 + 3w^2 + 6w + 2 = 0 \\ u^2 + v^2 - 2v + 2w^2 - 5 = 0 \\ 3u^2 - 12u + v^2 + 3w^2 + 8 = 0 \end{gathered}}$$

- Define the vector-valued function:

$$F(\underline{x}) = F(u,v,w) = (f_1, f_2, f_3)$$

where

$$\underline{x} = (u,v,w).$$

Jacobian matrix

- 3 variables: $\underline{x} = (u, v, w)$
- 3 functions: f_1, f_2, f_3

$$D_F(\underline{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{bmatrix}$$

Example: 2 unknowns, 2 equations

$$\boxed{\begin{aligned} v - u^3 &= 0 \\ u^2 + v^2 - 1 &= 0 \end{aligned}}$$

$\underline{x} = (u, v)$

- $f_1 = v - u^3$
- $f_2 = u^2 + v^2 - 1$
- $\partial f_1 / \partial u = -3u^2$
- $\partial f_1 / \partial v = 1$
- $\partial f_2 / \partial u = 2u$
- $\partial f_2 / \partial v = 2v$

$$D_F(\underline{x}) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$$

Multivariate Newton's method

- The Taylor expansion for $F(\underline{x})$ around $\underline{x}_0$ is

$$F(\underline{x}) = F(\underline{x}_0) + D_F(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0) + O(\underline{x} - \underline{x}_0)^2.$$

- Let $\underline{r}$ be the root, $\underline{x}_0$ be the current guess,

$$\underline{0} = F(\underline{r}) \approx F(\underline{x}_0) + D_F(\underline{x}_0) \cdot (\underline{r} - \underline{x}_0)$$

$$-F(\underline{x}_0) \approx D_F(\underline{x}_0) \cdot (\underline{r} - \underline{x}_0)$$

$$-D_F(\underline{x}_0)^{-1} F(\underline{x}_0) \approx (\underline{r} - \underline{x}_0)$$

destination current guess

Multivariate Newton's method: Algorithm

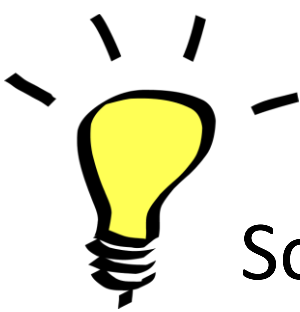
$\underline{x}_0$ = initial vector

$\underline{x}_{k+1} = \underline{x}_k \underbrace{- \left(D_F(\underline{x}_k)\right)^{-1} F(\underline{x}_k)}_{\underline{s}}$ for $k = 0,1,2,\ldots$

$$\underline{s} = -\left(D_F(\underline{x}_k)\right)^{-1} F(\underline{x}_k)$$

$$D_F(\underline{x}_k)\underline{s} = -F(\underline{x}_k)$$

A $\underline{x}$ $\underline{b}$



Solving A$\underline{x}$ = $\underline{b}$

$\underline{s}$ is the solution of $D_F(\underline{x}_k)\underline{s} = -F(\underline{x}_k)$

Multivariate Newton's method: Algorithm

$\underline{x}_0$ = initial vector

$\underline{x}_{k+1} = \underline{x}_k - \left(D_F(\underline{x}_k)\right)^{-1} F(\underline{x}_k)$ for $k = 0,1,2,\ldots$

$\underline{x}_0$ = initial vector

solve $D_F(\underline{x}_k)\underline{s} = -F(\underline{x}_k)$

$\underline{x}_{k+1} = \underline{x}_k + \underline{s}$ for $k = 0,1,2,\ldots$

Example

- Find a solution of the following system with starting guess (1, 2)

$$v - u^3 = 0$$
$$u^2 + v^2 - 1 = 0.$$

- $f_1 = \; v - u^3$
- $f_2 = \; u^2 + v^2 - 1$
- $\partial f_1/\partial u = -3u^2$
- $\partial f_1/\partial v = 1$
- $\partial f_2/\partial u = 2u$
- $\partial f_2/\partial v = 2v$

$$D_F(u, v) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$$

$\underline{x}_0$ = initial vector

solve $D_F(\underline{x}_k)\underline{s} = -F(\underline{x}_k)$

$\underline{x}_{k+1} = \underline{x}_k + \underline{s}$ for $k = 0, 1, 2, \ldots$

$F = \begin{bmatrix} v - u^3 \\ u^2 + v^2 - 1 \end{bmatrix}$

$D_F(u, v) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$

- Start point $\underline{x}_0$ = (1, 2), solve

  u v

$$\begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- Get $\underline{s}$ = (0, –1)
- Update $\underline{x}$

  $\underline{x}_1 = \underline{x}_0 + \underline{s}$ = (1, 2) + (0, –1) = (1, 1)

$\underline{x}_0$ = initial vector
solve $D_F(\underline{x}_k)\underline{s} = -F(\underline{x}_k)$
$\underline{x}_{k+1} = \underline{x}_k + \underline{s}$ for $k = 0, 1, 2, \ldots$

$F = \begin{bmatrix} v - u^3 \\ u^2 + v^2 - 1 \end{bmatrix}$

$D_F(u, v) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$

- $\underline{x}_1 = (1, 1)$, solve

  u v

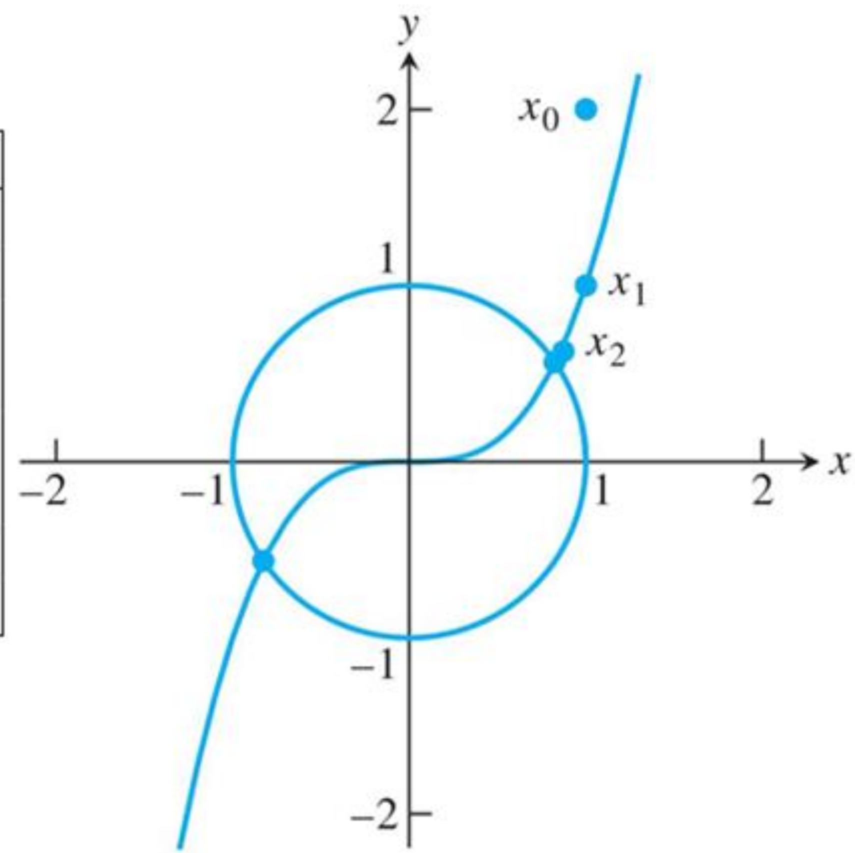
$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Get $\underline{s} = (-1/8, -3/8)$
- Update $\underline{x}$

  $\underline{x}_2 = \underline{x}_1 + \underline{s} = (1, 1) + (-1/8, -3/8) = (7/8, 5/8)$

$$v - u^3 = 0$$
$$u^2 + v^2 - 1 = 0$$

step	u	v
0	1.00000000000000	2.00000000000000
1	1.00000000000000	1.00000000000000
2	0.87500000000000	0.62500000000000
3	0.82903634826712	0.56434911242604
4	0.82604010817065	0.56361977350284
5	0.82603135773241	0.56362416213163
6	0.82603135765419	0.56362416216126
7	0.82603135765419	0.56362416216126



程式練習

And, please upload your program on moodle.

- Please return *x*, *y* and *z* of the following nonlinear system.

$$\begin{cases} x + y + z = 3 \\ x^2 + y^2 + z^2 = 5 \\ e^x + xy - xz = 1 \end{cases}$$