Interpolation – Part 1

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Today

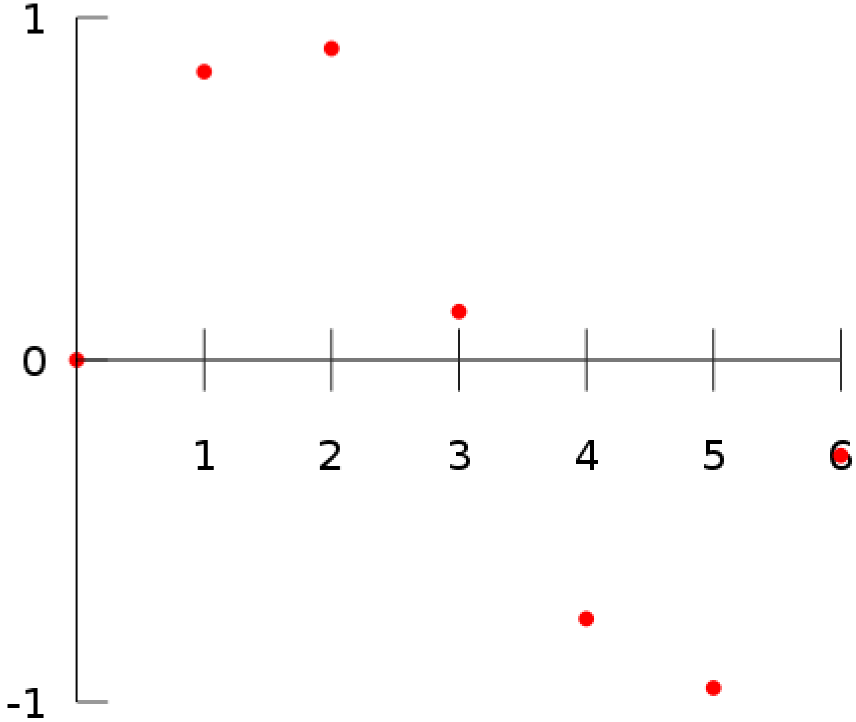
- What is interpolation?
- Lagrange interpolation

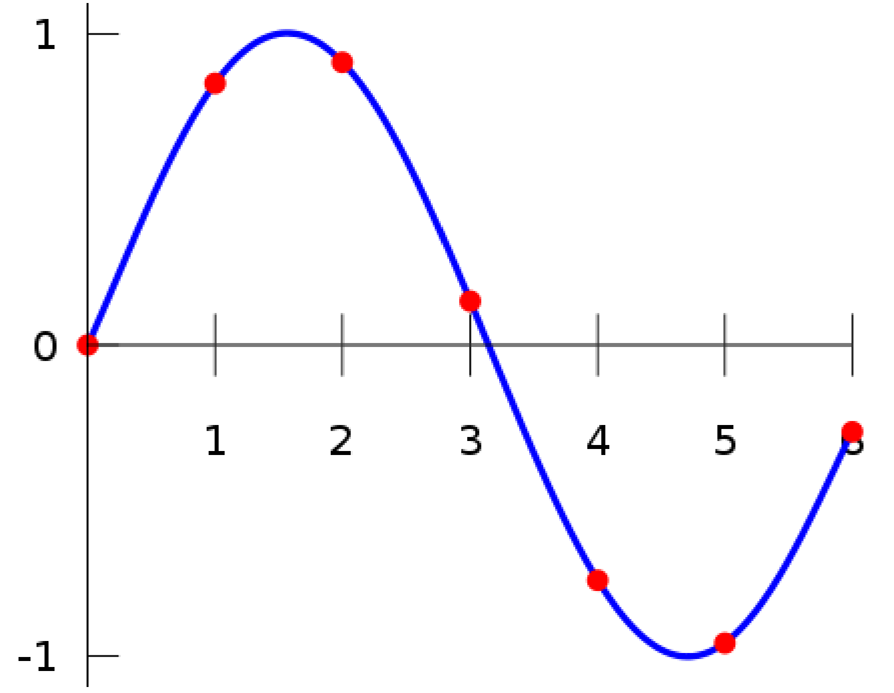
Interpolating data

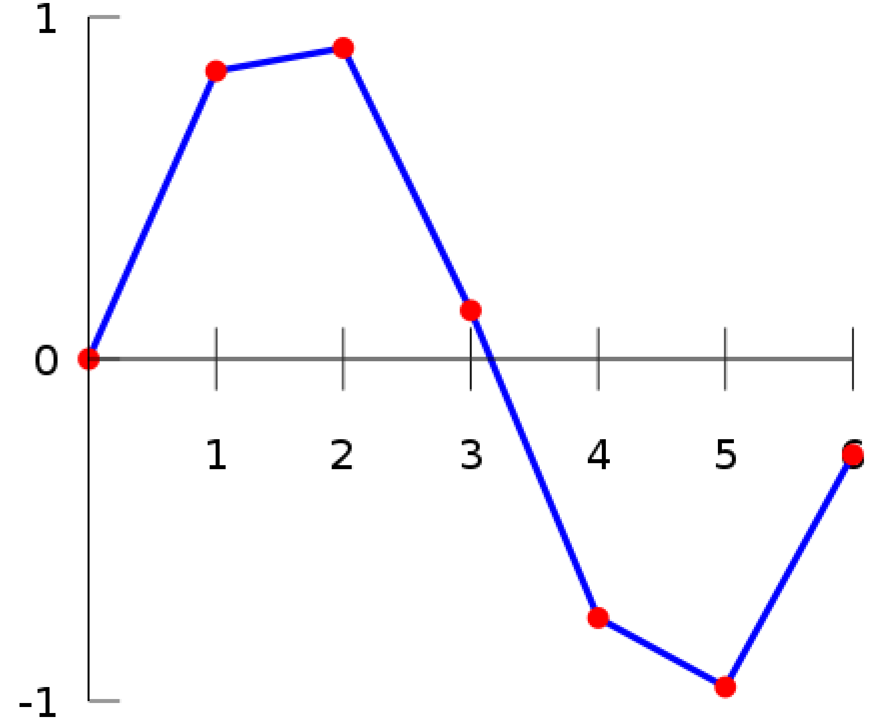
- We are given a collection of data samples $\{(x_i, y_i)\}_{i=0}^{n}$.
  - x_i: the abscissas
  - y_i: the data values
- Want to find a function $v(x)$ which can be used to estimate sampled function for $x \neq x_i$.

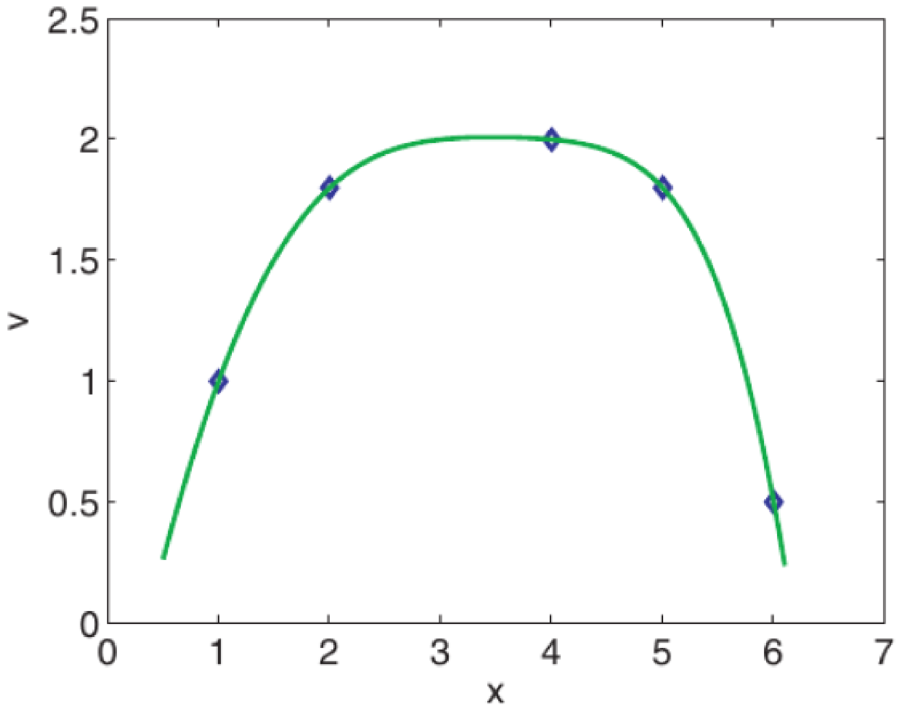
$$\boxed{\text{Interpolation:} \qquad v(x_i) = y_i, \quad i = 0, 1, \ldots, n.}$$

Examples

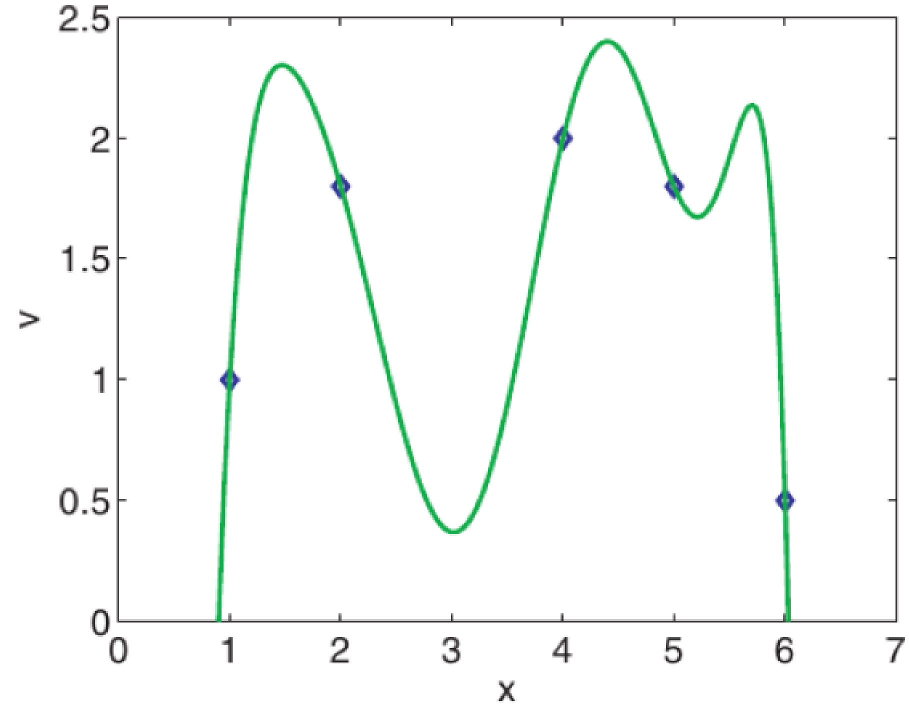








Reasonable



Unreasonable

Interpolating data

- Interpolation is the reverse of evaluation.
  - Evaluation (e.g., nested multiplication): given a polynomial, evaluate a y-value for a given x-value.
  - Interpolation: given these points, compute a polynomial that can generate them.

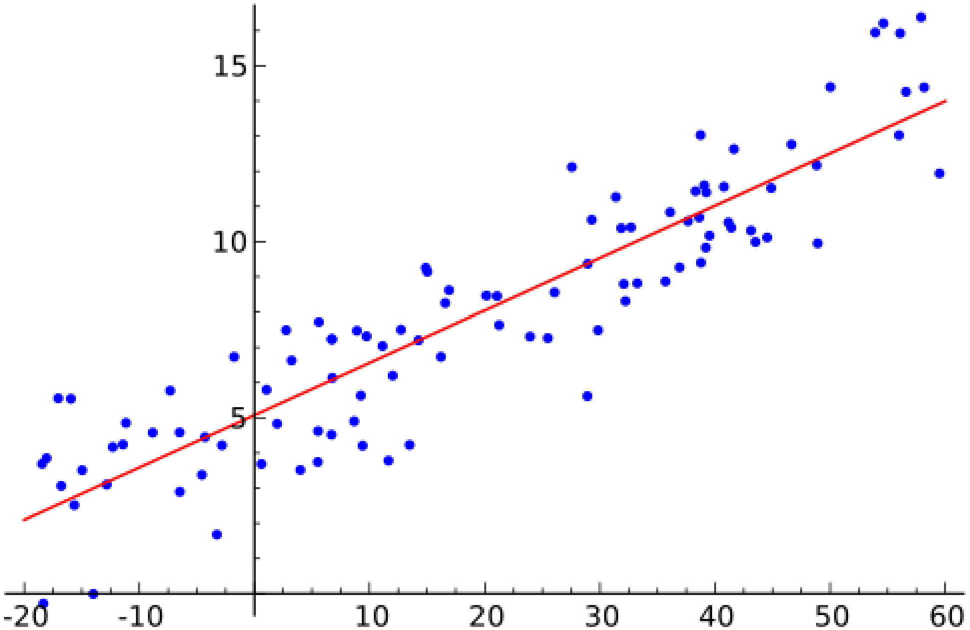
Interpolating data

- Why?
  - We often get discrete data from sensors or computations, but we want information as if the data were not discretely sampled.
- Want a reasonable looking interpolation. If possible, $v(x)$ should be inexpensive to evaluate for a given x.

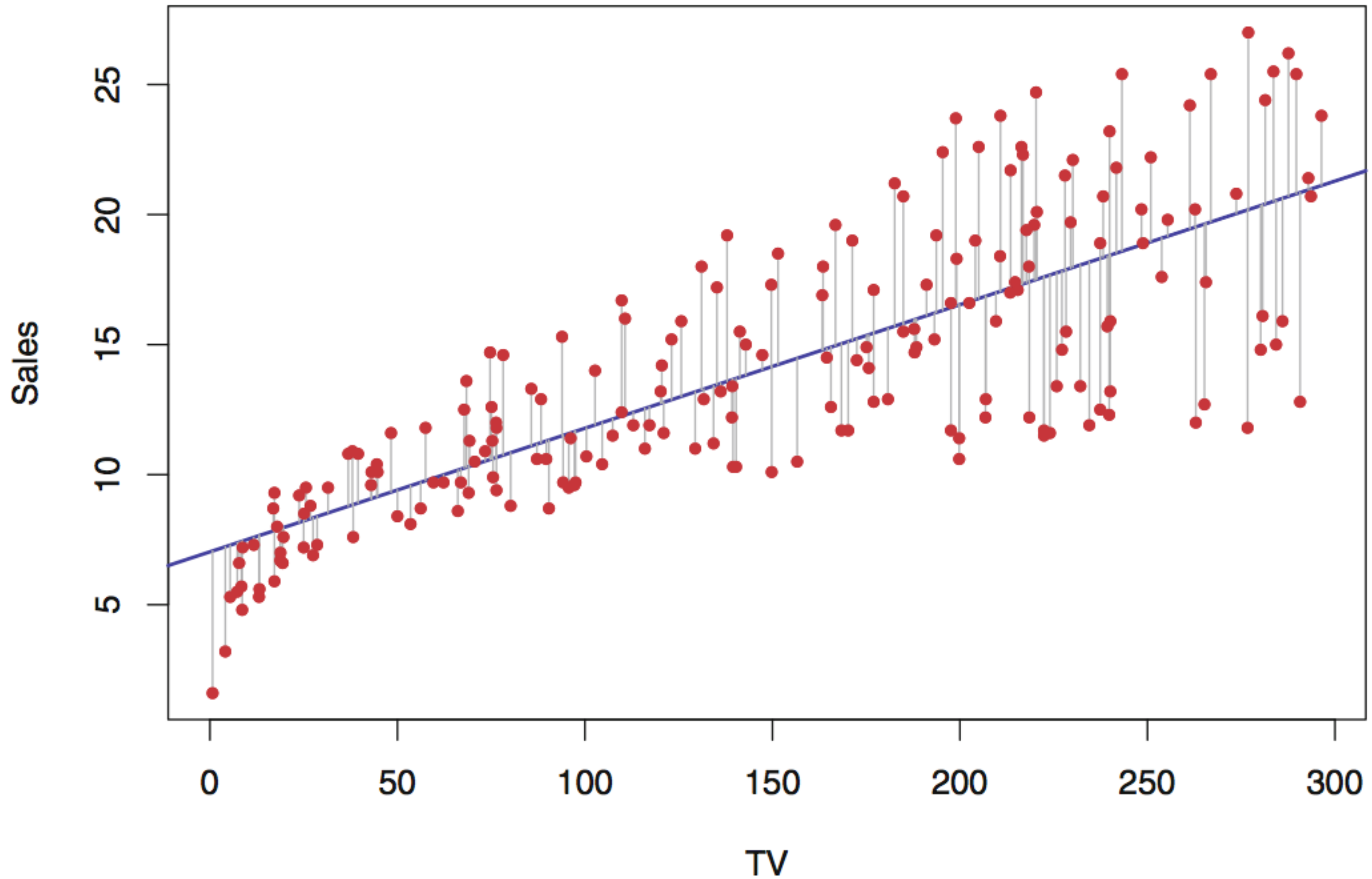
Interpolation, extrapolation, approximation

- Interpolation
  - The new value $x \neq x_i$ is inside the range of the interpolation points $x_0, x_1, \ldots, x_n$.
- Extrapolation
  - The new value z is outside this range.
- Approximation
  - Some norm $\|v - y\|$ of the difference of the vectors $\underline{v} = [v(x_0), \ldots, v(x_n)]$ and $\underline{y} = [y_0, \ldots, y_n]$ is minimized.

Approximation: Example



Approximation: Example



Today

- What is interpolation?
- Lagrange interpolation

Lagrange interpolation

- Given n data points (x_1, y_1), ..., (x_n, y_n), the polynomial that interpolates the points is:

$$\boxed{P(x) = y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x)}$$

$$L_k(x) = \frac{(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

- Degree of $P(x)$?

Lagrange interpolation (n = 3)

- Given n data points $(x_1, y_1), \ldots, (x_n, y_n)$, the polynomial of degree $d = n-1$ that interpolates the points is:

$$P_2(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

$P(x_1) = ?$

$P(x_2) = ?$

$P(x_3) = ?$

Lagrange interpolation: Example 1

- Find an interpolating polynomial for the data points (0, 1), (2, 2) and (3, 4).

$$P(x) = 1\frac{(x-2)(x-3)}{(0-2)(0-3)} + 2\frac{(x-0)(x-3)}{(2-0)(2-3)} + 4\frac{(x-0)(x-2)}{(3-0)(3-2)}$$

$$= \frac{1}{6}(x^2 - 5x + 6) + 2\left(-\frac{1}{2}\right)(x^2 - 3x) + 4\left(\frac{1}{3}\right)(x^2 - 2x)$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x + 1.$$

Check $P(0) = ?$, $P(2) = ?$, $P(3) = ?$

Lagrange interpolation: Example 1

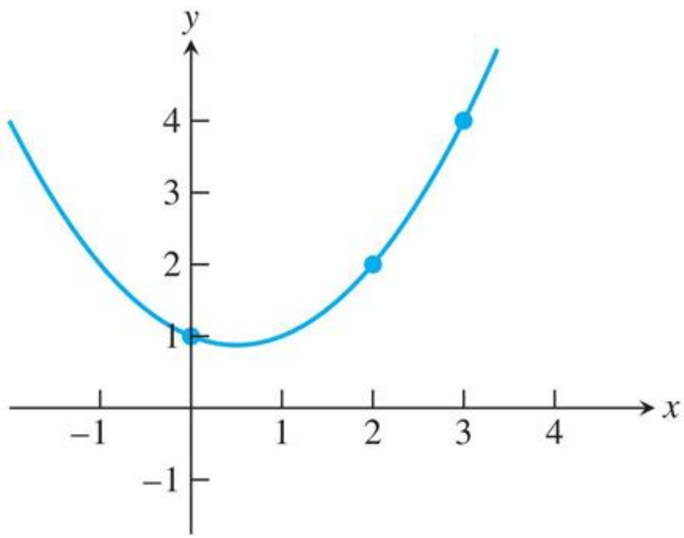
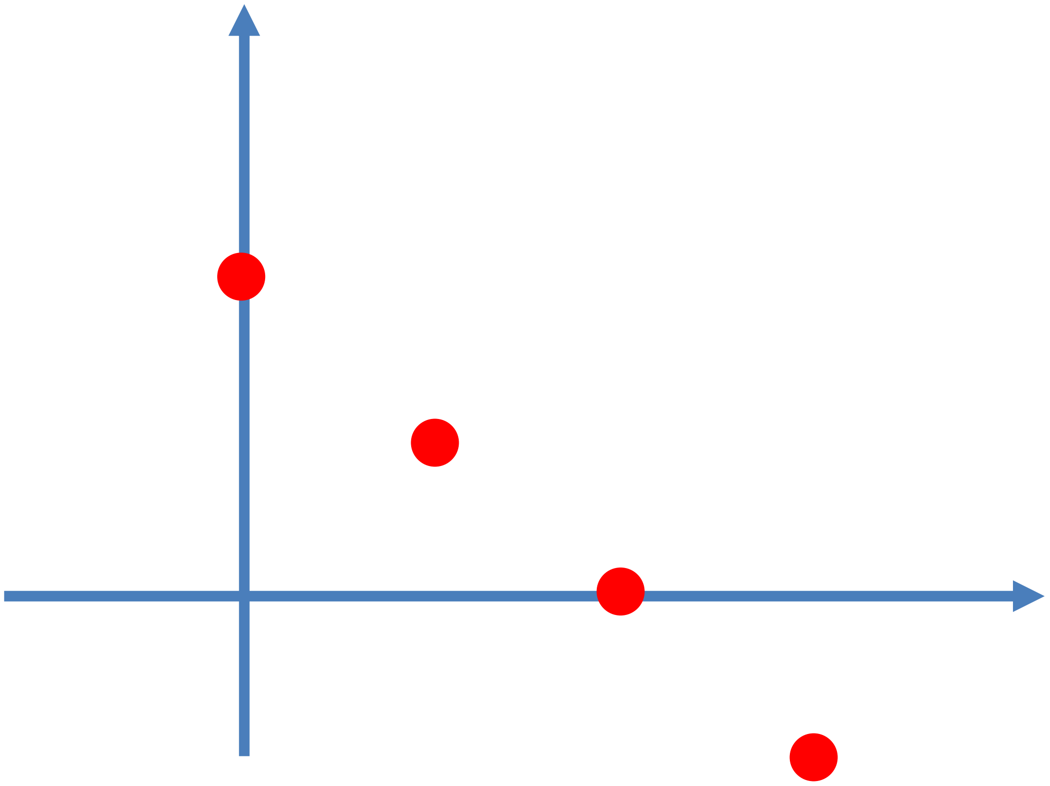


Figure 3.1 Interpolation by parabola. The points (0,1), (2,2), and (3,4) are interpolated by the function $P(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 1$.

Lagrange interpolation: Example 2

- Find an interpolating polynomial for the data points (0, 2), (1, 1), (2, 0) and (3, -1).



Lagrange interpolation: Example 2

- Find an interpolating polynomial for the data points (0, 2), (1, 1), (2, 0) and (3, -1).

$$P(x) = 2\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + 1\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$

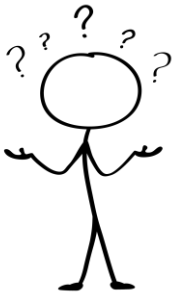
$$+ 0\frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} - 1\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$$

$$= -\frac{1}{3}(x^3 - 6x^2 + 11x - 6) + \frac{1}{2}(x^3 - 5x^2 + 6x) - \frac{1}{6}(x^3 - 3x^2 + 2x)$$

$$= -x + 2.$$

A linear function!

Question



- Given *n* data points
  - How many polynomials that interpolate the points?
  - How many polynomials of degree $n-1$ or less that interpolate the points?
- Lagrange interpolation: Is $P(x)$ ***unique***?

$$\boxed{\begin{aligned} P(x) &= y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x) \\ &= a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \end{aligned}}$$

Yes

Theorem. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n points in a plane with distinct x_i. Then there exists **one and only one** polynomial P of degree $n-1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \ldots, n$.

Proof sketch (by contradiction)

- Suppose $P(x)$ and $Q(x)$ have degree at most $n-1$ and both interpolate all n points.
- Now, define $H(x) = P(x) - Q(x)$.
  - Degree of $H(x)$?
  - $H(x_i) = P(x_i) - Q(x_i) = y_i - y_i = 0$
  - $0 = H(x_1) = H(x_2) = \ldots = H(x_n)$
  - $H(x)$ has a degree of at most n-1 and it has n roots.

Proof sketch (by contradiction)

- Suppose $P(x)$ and $Q(x)$ have degree at most $n-1$ and both interpolate all n points.
- Now, define $H(x) = P(x) - Q(x)$.
  - Degree of $H(x)$?
  - $H(x_i) = P(x_i) - Q(x_i) = y_i - y_i = 0$
  - $0 = H(x_1) = H(x_2) = \ldots = H(x_n)$
- H is the identically zero polynomial, and $P(x) \equiv Q(x)$.

Side information

A degree d polynomial can have at most d roots, unless it is the identically zero polynomial.

程式練習

And, please upload your program on moodle.

- Please estimate the 1980 population:

Year	population
1960	3039585530
1970	3707475887
1980	**?**
1990	5281653820
2000	6079603571