Solving nonlinear equations in one variable

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Many slides are from Gerald W. Recktenwald (PSU) and Peter Arbenz (ETH)

Last week

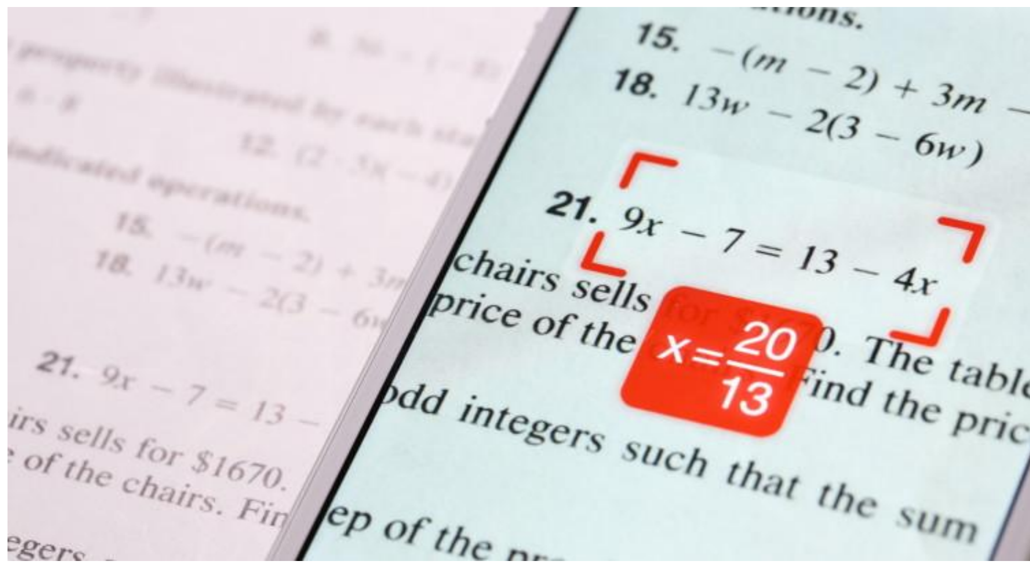
- Evaluating a polynomial
- Horner's method

$$f(x) = c_5x^4 + c_4x^3 + c_3x^2 + c_2x + c_1$$

$$= c_1+x\,(c_2+x\,(c_3+x\,(c_4+x\,(c_5))))$$

Today

- Solving an equation (one variable)



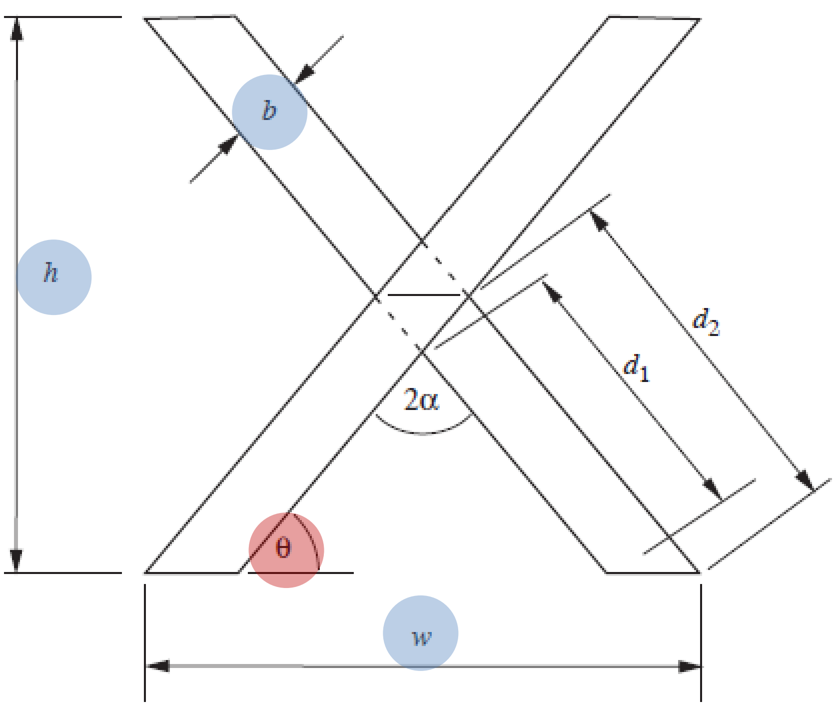
PhotoMath

- https://www.youtube.com/watch?v=XlbVB50mlh4

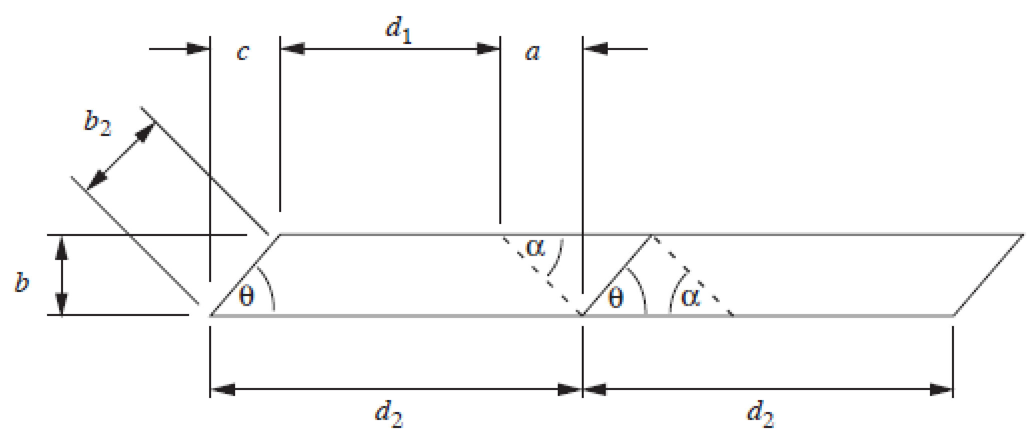
Example 1: Picnic Table Leg

- Computing the dimensions of a picnic table leg involves a root-finding problem.

Leg assembly



Detail of one leg



Slide credit: Gerald W. Recktenwald

- Dimensions of a the picnic table leg satisfy

$$\boxed{w\sin\theta = h\cos\theta + b}$$

- Given *w*, *h*, and *b*, what is the value of θ?
- An analytical solution for $\theta = f(w, h, b)$ exists, but is not obvious.
- Use a numerical root-finding procedure to find the value of θ that satisfies

$$f(\theta) = w\sin\theta - h\cos\theta - b = 0$$

$$\boxed{\rightarrow \text{ 方程式求根問題}}$$

Example 2: Kepler's equation

(計算行星的軌道)

$$\boxed{x - a\sin x = b}$$

- Given *a* and *b*, what is the value of *x*?
- $a = 0.2$, $b = \pi/3$, $x = ?$
- A numerical approach:

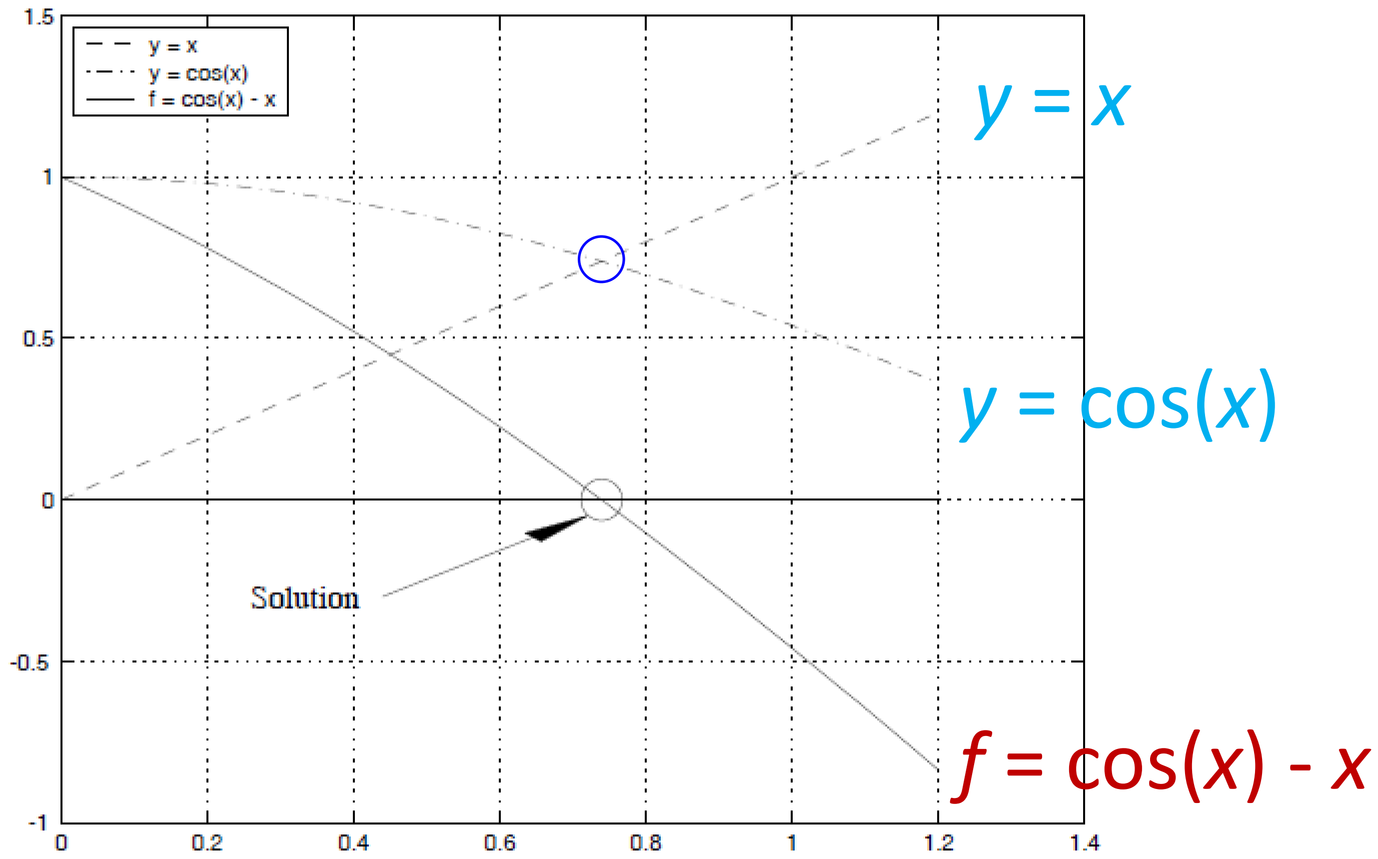
solve

$$f(x) = x - a\sin x - b = 0$$

Roots of $f(x) = 0$

- Any function of one variable x can be put in the form $f(x) = 0$? ***Yes!***
- Example:
  - To find x that satisfies $\cos(x) = x$,
  - Find the zero crossing of $f(x) = \cos(x) - x = 0$.

$\cos(x) = x, \quad x = ?$



y = x
y = cos(x)
f = cos(x) - x
Solution
1.5
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
y = x
y = cos(x)
f = cos(x) - x

Number of Roots

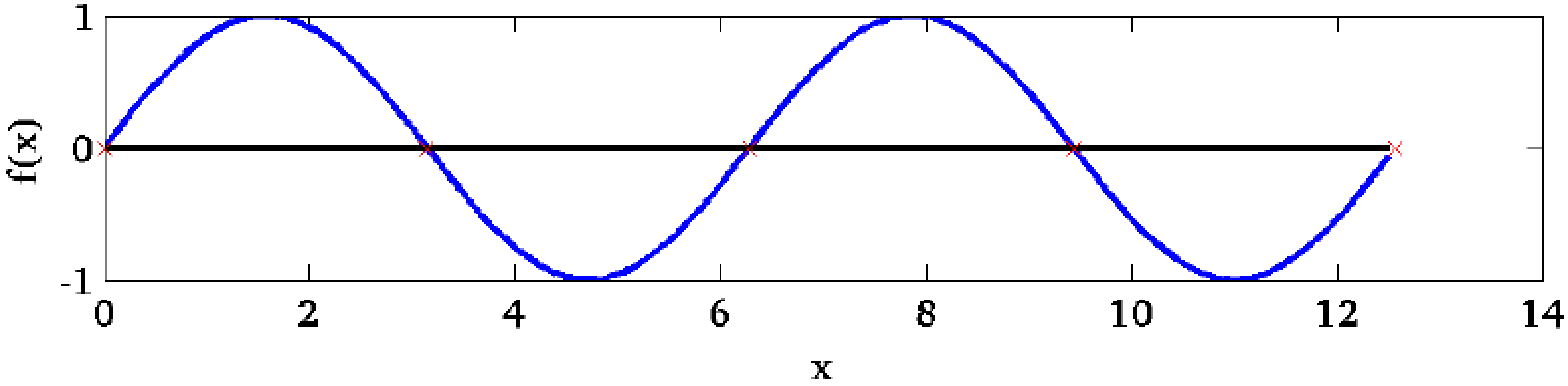
- In contrast to scalar linear equations

$$mx - n = 0 \Rightarrow x = \frac{n}{m},$$

nonlinear equations have an undetermined number of zeros.

Number of Roots

- $f(x) = \sin(x)$
- On $[a, b] = [0, 4\pi]$ there are **???** roots.



Finding roots

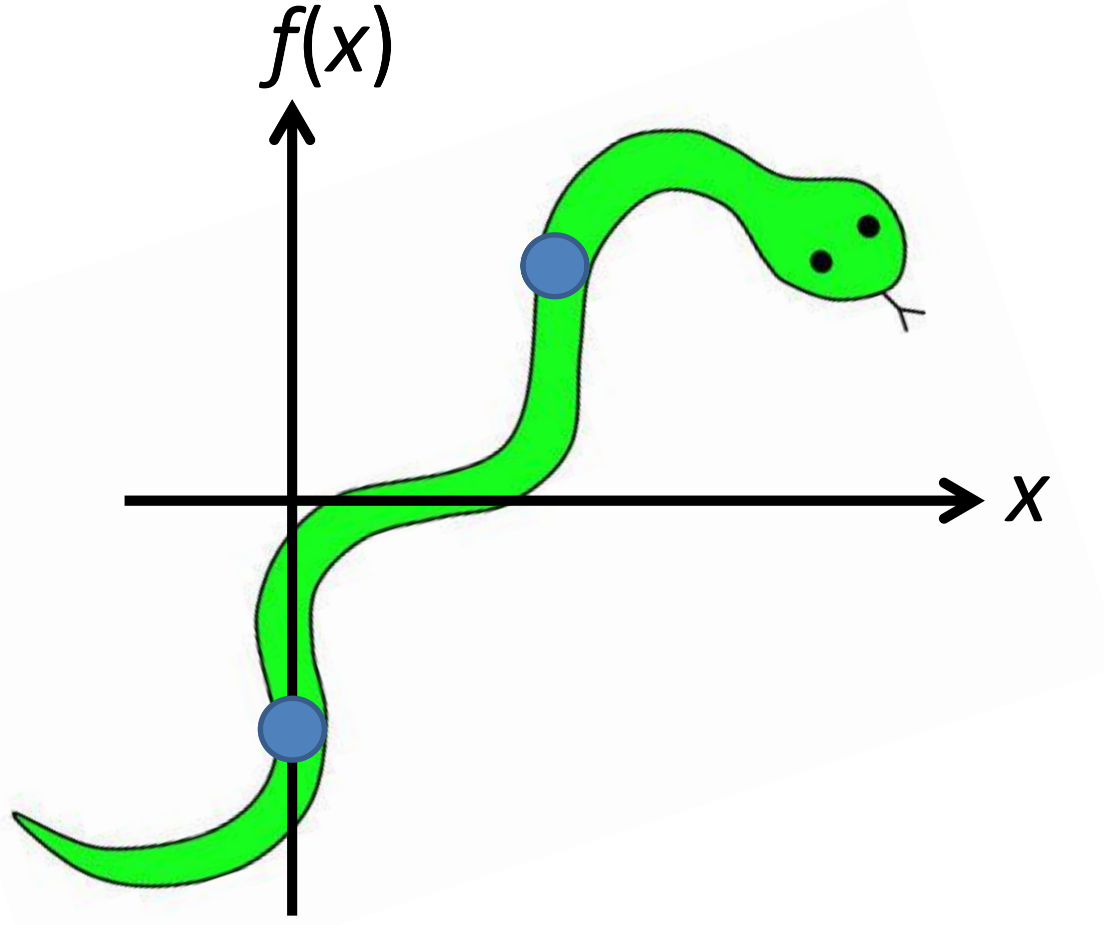
$$f(x) = x^3 + x - 1 = 0$$

$$x = ?$$

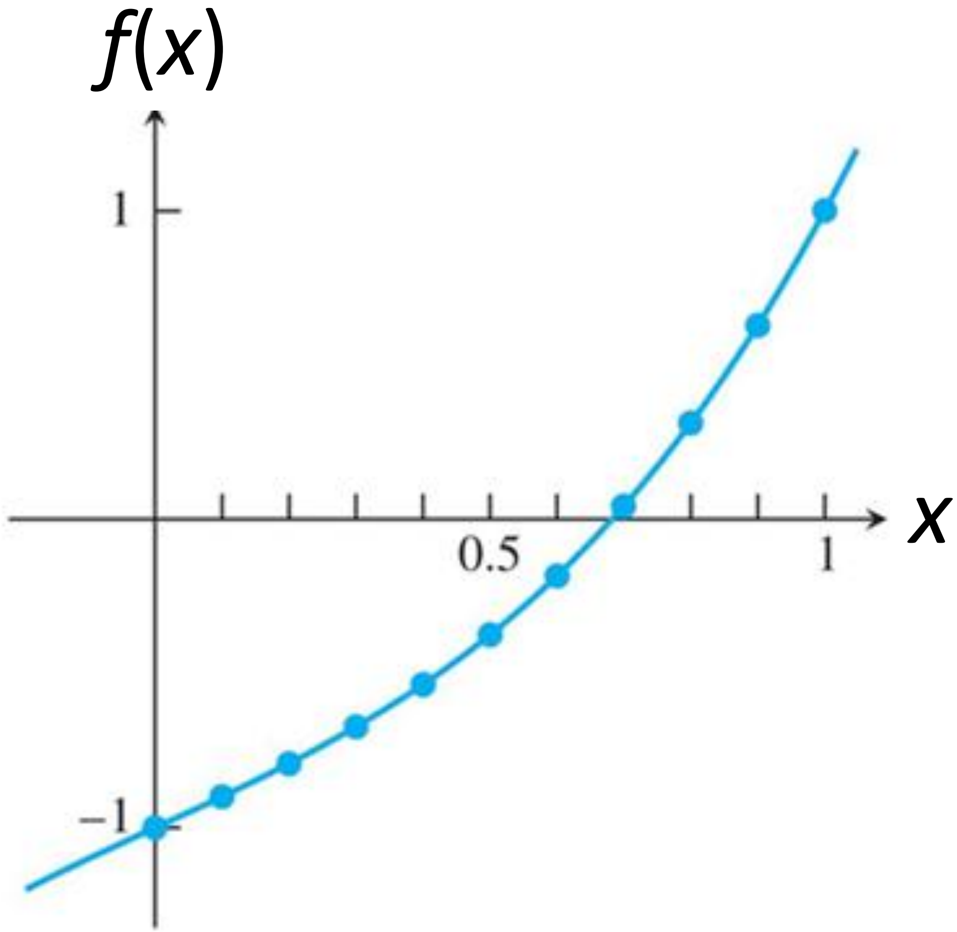
幾個解?
範圍為?

$$f(x) = x^3 + x - 1 = 0$$

- Must have a root between 0 and 1
  - $f(0) = -1$
  - $f(1) = 1$
  - $f(0)f(1) < 0$



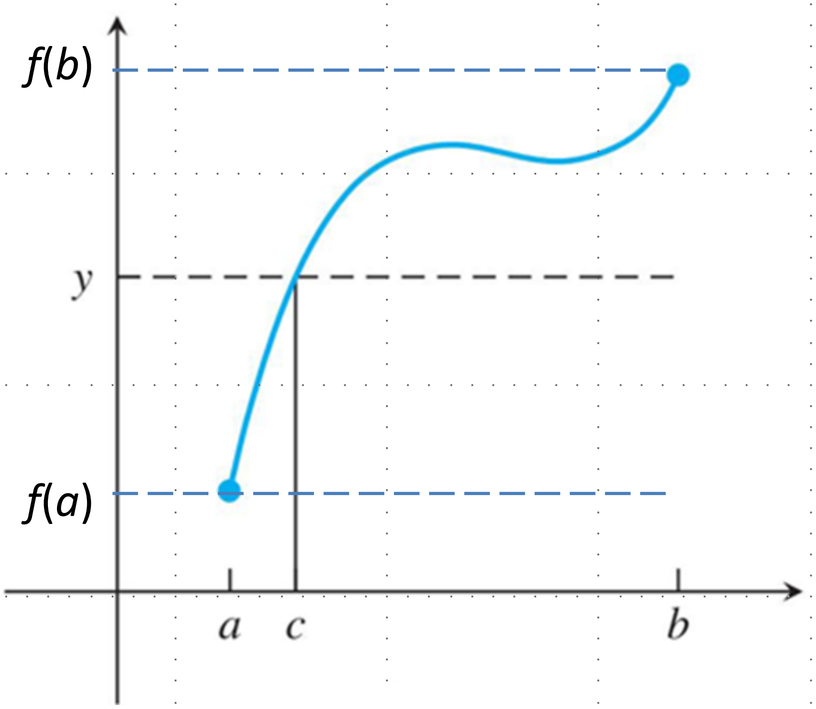
$$f(x) = x^3 + x - 1 = 0$$



Iterative methods for finding roots

- Starting with an initial guess/iterate x_0 we generate a sequence of iterates x_1, x_2, … that (hopefully) converges to a root of the function.
- A rough knowledge of the root's location is required.
- Could probe the function and try to find two arguments *a*, *b* s.t. $f(a)\,f(b) < 0$.
  – Intermediate Value Theorem: $\exists\, x^*$ in the interval (a,b)

Intermediate value theorem



- Let f be a continuous function on the interval $[a, b]$. If y is a number between $f(a)$ and $f(b)$, then there exists a number c with $a \leqq c \leqq b$ such that $f(c) = y$.

Stopping an iterative procedure

- Various criteria are used to check (almost) convergence: We terminate iterating after *n* iterations if:

$$|x_n - x_{n-1}| < \mathtt{atol},$$

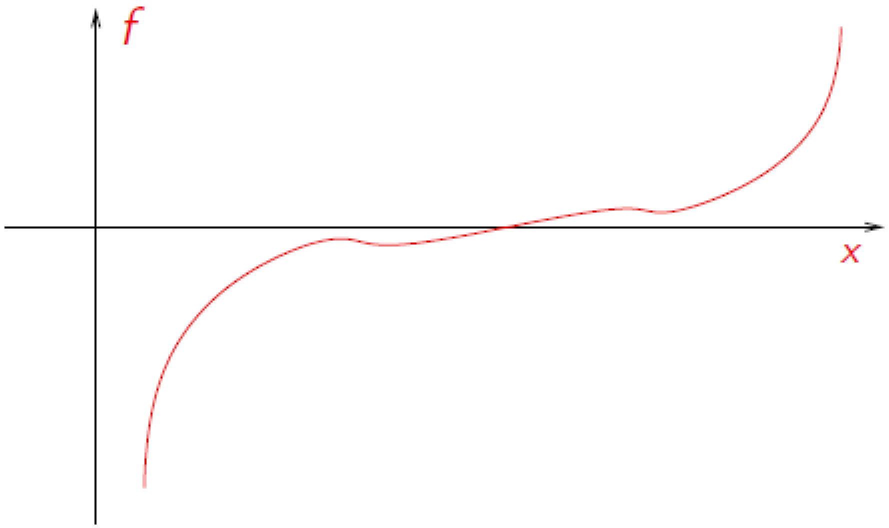
$$|x_n - x_{n-1}| < \mathtt{rtol}\,|x_n|,$$

$$|f(x_n)| < \mathtt{ftol},$$

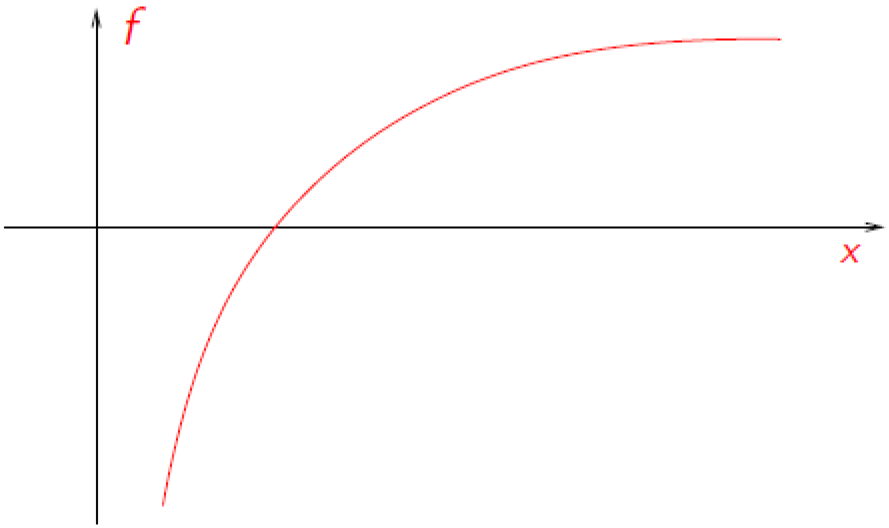
  where atol, rtol, ftol are user-specified constants.
- A combination of the first two is

$$|x_n - x_{n-1}| < \mathtt{tol}\,(1 + |x_n|)$$

Termination criteria



$|f(\mathbf{x}_k)|$ small $\not\Rightarrow$ $|\mathbf{x} - \mathbf{x}^*|$ small



$|f(\mathbf{x}_k)|$ small $\Rightarrow$ $|\mathbf{x} - \mathbf{x}^*|$ small

Today: two iterative approaches

- Bisection
- Fixed point iteration

猜數字 (1~100)

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Bisection

- 猜數字
- Method for finding a root of a scalar equation $f(x) = 0$ in an interval $[a, b]$
- Assumption: $f(a)\, f(b) < 0$
- Since f is continuous there must be a zero $x^* \in [a, b]$

1. Compute midpoint m of the interval and check $f(m)$
2. Depending on the sign of $f(m)$, we can decide if $x^* \in [a, m]$ or $x^* \in [m, b]$
   - Of course, if $f(m) = 0$ then we are done.

Bisection

- Given $f(\cdot)$ $\quad$ $f(x) = x^3 + x - 1$
- Given a range $[a, b]$ $\quad$ $[0, 1]$
- Determine a stopping condition

$(b - a) < 10^{-6}$ or $f((a+b)/2) \approx 0$

Compute the roots of $f(x) = 0$

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750

Bisection: Example

$f(x) = x^3 + x - 1$

a	*b*	*mid*	*f*(*mid*)
0	1	0.5	-0.3750
0.5	1	0.75	0.1719

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309
0.625	0.75	0.6875	0.0125

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309
0.625	0.75	0.6875	0.0125
0.625	0.6875	0.6563	-0.0611

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309
0.625	0.75	0.6875	0.0125
0.625	0.6875	0.6563	-0.0611
0.6563	0.6875	0.6719	-0.0248

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0.6563	0.6875	0.6719	-0.0248
0.6719	0.6875	0.6797	-0.0063
0.6797	0.6875	0.6836	0.0031
0.6797	0.6836	0.6816	-0.0016
0.6816	0.6836	0.6826	0.0006
⋮	⋮	⋮	⋮

Today: two iterative approaches

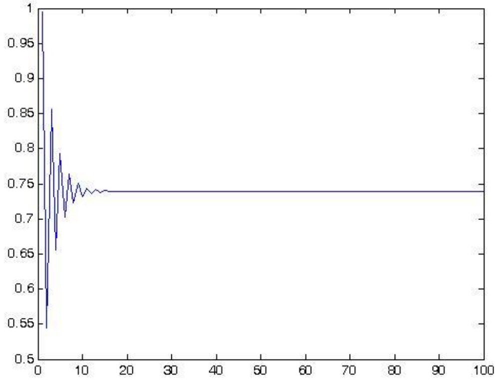
- Bisection
- Fixed point iteration

Fixed point iteration

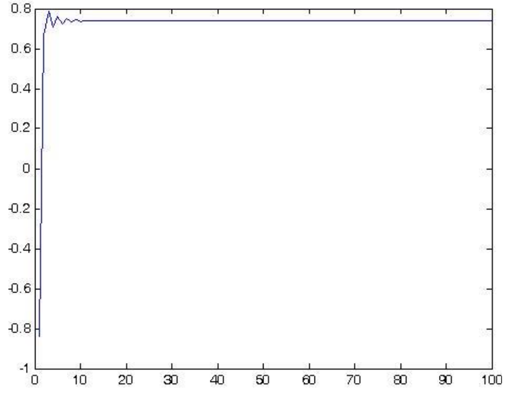
What happens in the following example?

- Let *x* be an arbitrary number.
- Repeat computing *x* = cos(*x*)

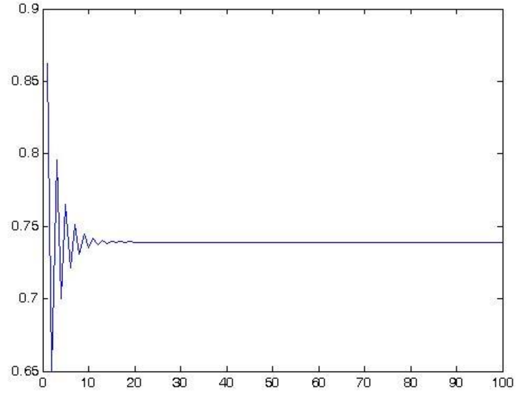
The number converges to 0.7390851332.



x = 0.1



x = 10



x = 100

Fixed point iteration

- Problem $f(x) = 0$ can be rewritten as

$$x = g(x).$$

(There are many ways to do this.)

We are looking for a **fixed point**.

Fixed point iteration

- Definition: The real number x is a **fixed point** of a function g if $g(x) = x$.
- Example
  - The fixed point of $\cos(x)$ is 0.7390851332.
- What is (or are) the fixed point(s) of $g(x) = x^3$?

Fixed point iteration: Approach

- Given a function $f(x)$, select a function $g(x)$ such that

$$f(x) = 0 \longrightarrow g(x) = x.$$

- Then
  - x_0 = initial guess
  - $x_{i+1} = g(x_i)$ for $i = 0, 1, 2, \ldots$
- Until x_{i+1} satisfies some termination criterion

注!意

- There are many ways to transform $f(x) = 0$ into fixed point form! Not all of them are “good” in terms of convergence.
- Example: $x^3 + x - 1 = 0$ → $g(x) = x$

$$x = 1 - x^3$$

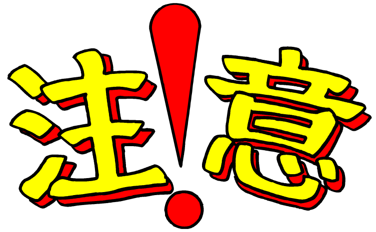
$$x = \sqrt[3]{1 - x}$$

$$x = \frac{1 + 2x^3}{1 + 3x^2}$$

(+$2x^3$ on both sides)

$3x^3 + x - 1 = 2x^3$

$(3x^2 + 1)x = 2x^3 + 1$



- There are many ways to transform $f(x) = 0$ into fixed point form! Not all of them are "good" in terms of convergence.
- Example: $x^3 + x - 1 = 0$ → $g(x) = x$

$$x = 1 - x^3 \quad \Rightarrow g(x) = 1 - x^3$$

$$x = \sqrt[3]{1 - x} \quad \Rightarrow g(x) = \sqrt[3]{1 - x}$$

$$x = \frac{1 + 2x^3}{1 + 3x^2} \Rightarrow g(x) = \frac{1 + 2x^3}{1 + 3x^2}$$

Which one will work?

- There are many ways to transform $f(x) = 0$ into fixed point form! Not all of them are “good” in terms of convergence.
- Example: $x^3 + x - 1 = 0$ $\rightarrow g(x) = x$

$$x = 1 - x^3 \quad \Rightarrow g(x) = 1 - x^3$$

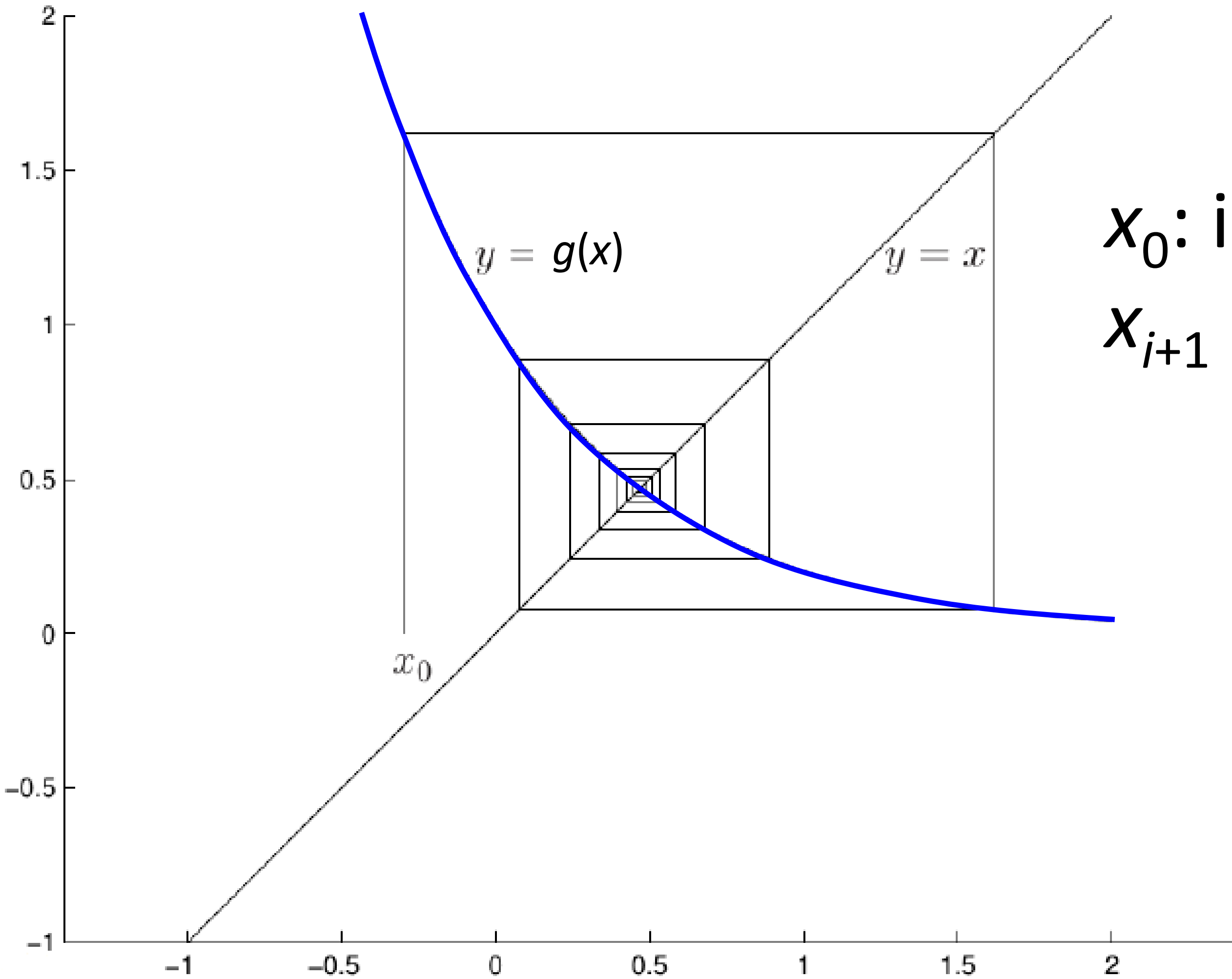
$$x = \sqrt[3]{1 - x} \quad \Rightarrow g(x) = \sqrt[3]{1 - x}$$

$$x = \frac{1 + 2x^3}{1 + 3x^2} \Rightarrow g(x) = \frac{1 + 2x^3}{1 + 3x^2}$$

$x_0 = 0.5$

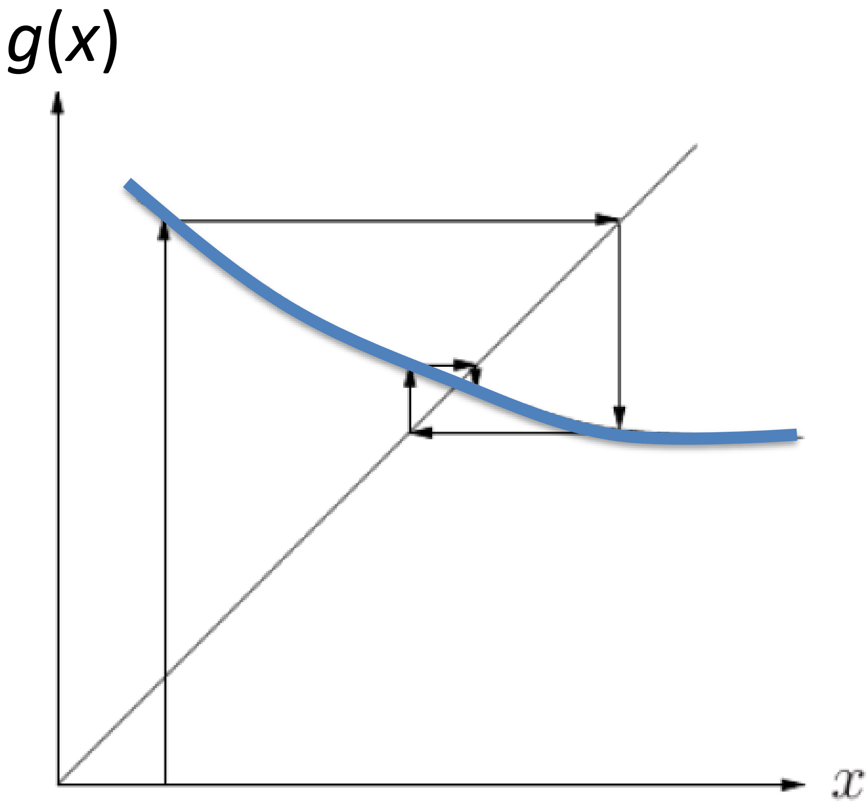
$g(x) = 1 - x^3$	$g(x) = \sqrt[3]{1-x}$		$g(x) = \dfrac{1+2x^3}{1+3x^2}$
0.5000	0.5000		0.5000
0.8750	0.7937		0.7143
0.3301	0.5909		0.6832
0.9640	0.7424		0.6823
0.1041	0.6363	0.6807	0.6823
0.9989	0.7138	0.6835	0.6823
0.0034	0.6590	0.6815	0.6823
1.0000	0.6986	0.6829	0.6823
0.0000	0.6704	0.6819	0.6823
1.0000	0.6907	0.6826	0.6823
0.0000	0.6763	0.6821	0.6823
1.0000	0.6866	0.6825	0.6823
0.0000	0.6792	0.6822	0.6823
	0.6845	0.6824	
		0.6823	
		0.6824	

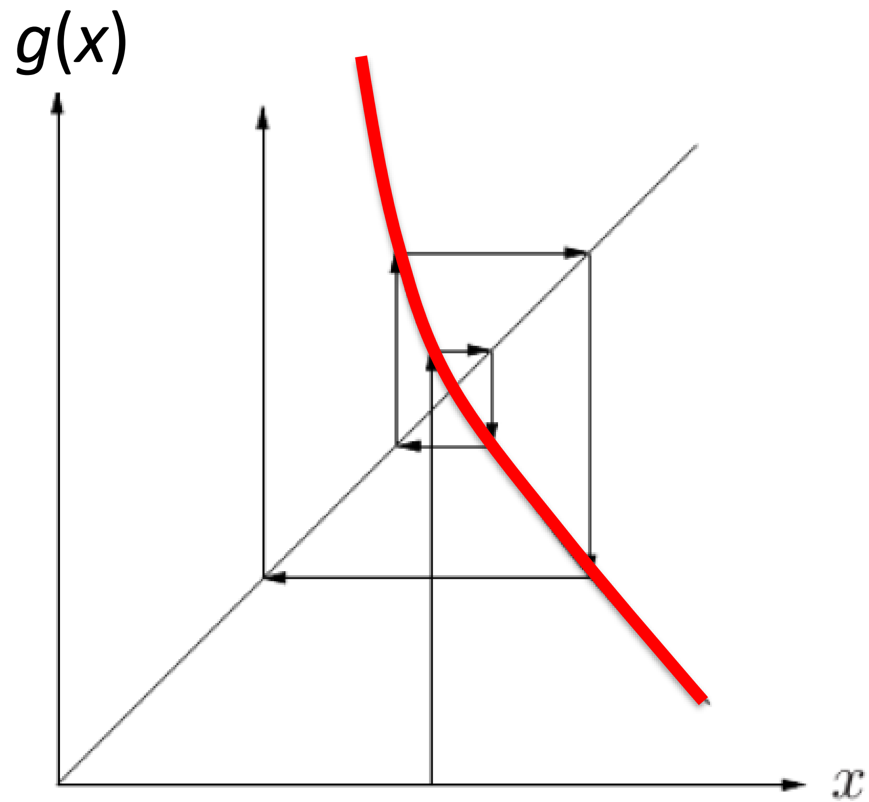
start with x_0 on the x-axis
go parallel to the y-axis to the graph of $F \equiv g$
move parallel to the x-axis to the graph $y = x$
go parallel to the y-axis to the graph of F



x_0: initial guess
$x_{i+1} = g(x_i)$

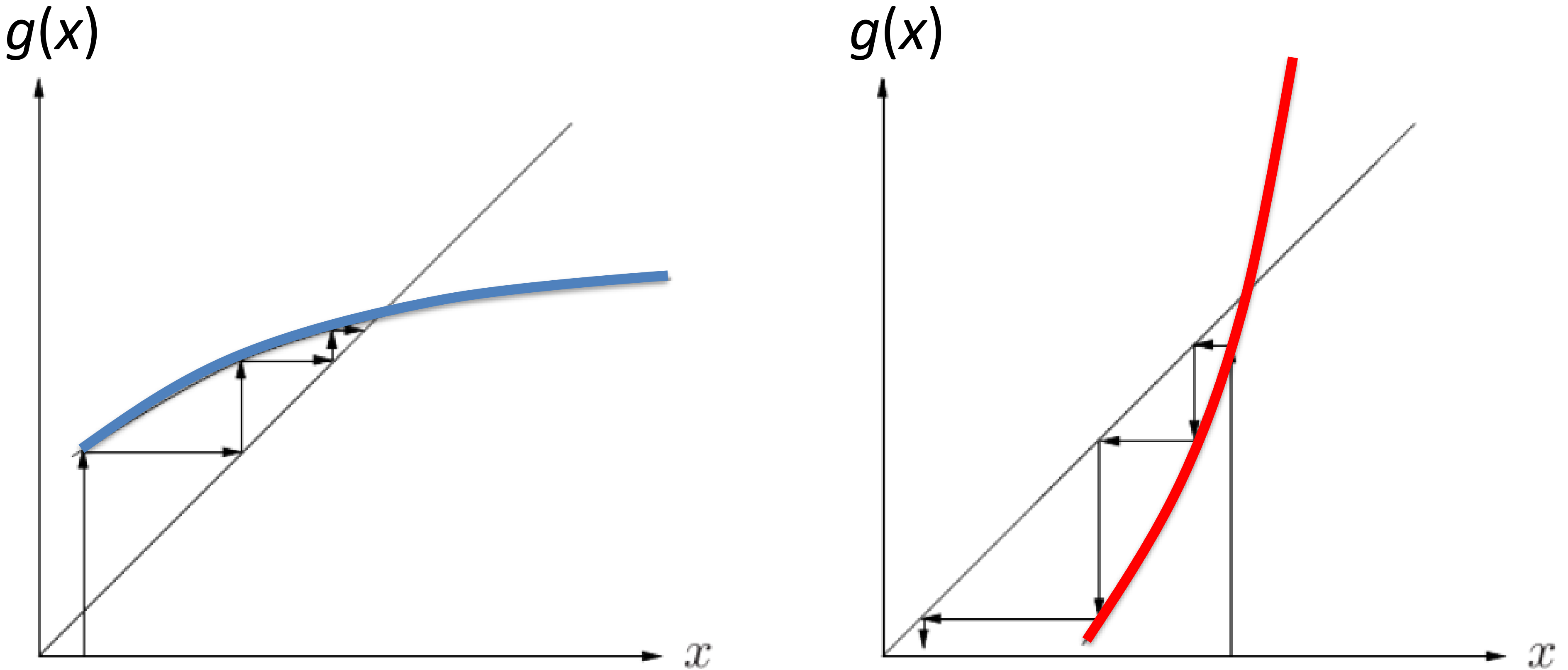
Geometric interpretation





x_0 is close to the root

Geometric interpretation



$$\boxed{g(x) = 1 - x^3}$$

$$\boxed{g(x) = \sqrt[3]{1 - x}}$$

$$\boxed{g(x) = \frac{1 + 2x^3}{1 + 3x^2}}$$

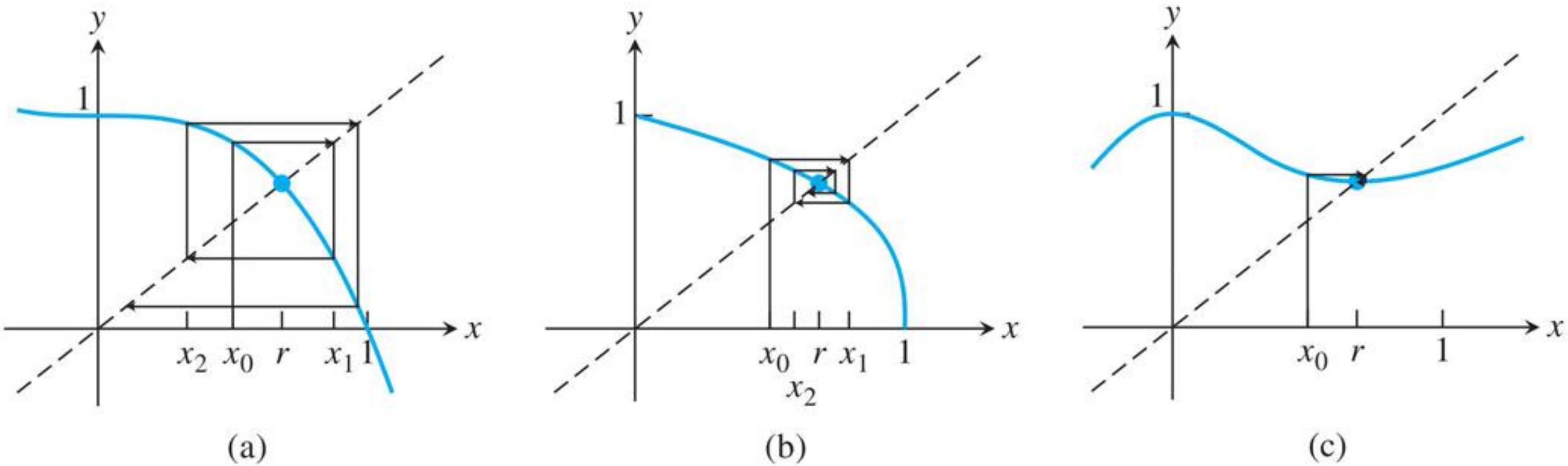


Figure 1.3 Geometric view of FPI. The fixed point is the intersection of $g(x)$ and the diagonal line. Three examples of $g(x)$ are shown together with the first few steps of FPI. (a) $g(x) = 1 - x^3$ (b) $g(x) = (1 - x)^{1/3}$ (c) $g(x) = (1 + 2x^3)/(1 + 3x^2)$

Convergence

- $S = |g'(r)| < 1$

Will explain this next time. Stay tuned. ☺

程式練習(二選一)

And, please upload your program on moodle.

請寫一個程式計算方程式的根, 輸出至小數點以下四位

Bisection

1. $\boxed{x^3 = 9}$

2. $\boxed{x - x^{1/3} - 2 = 0}$ $3 < x < 4$

Fixed-point iteration

1. $\boxed{\sin x = 6x + 5}$

2. $\boxed{x^3 = 2x + 2}$