Discrete and Algorithmic Geometry
Sheet 4

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Problem G*. *Enumerate, up to combinatorial equivalence, all balanced configurations $\mathcal{V}$ of n vectors in $\mathbb{Z}^e$ whose coordinates are all at most m in absolute value, such that*

(1) *the maximum m is achieved by some $v \in \mathcal{V}$,*

(2) *and such that no hyperplane spanned by $e-1$ of the vectors strictly separates exactly one vector from the others.*

For this, recall that a vector configuration $\mathcal{V} = (v_1, \ldots, v_n)$ is balanced if $\sum_i v_i = 0$; that no hyperplane defined by $e-1$ elements of $\mathcal{V}$ separates exactly one vector from the others iff the Gale dual of $\mathcal{V}$ is in convex position; and that two vector configurations are combinatorially equivalent if they define the same oriented matroid.

This problem can be divided in two parts

1. Find all “diferent” vector configurations
2. Identify those configurations that correspond to the same polytope (combinatorial equivalence).

Pseudocode
Trivial algorithm: check all the possibilities and after that check if they are combinatorially equivalent. $\mathcal{O}(m^{e(n-1)})$.
This is really inefficient!
Note that up to combinatorial equivalence we can reduce the number of possibilities to $\mathcal{O}(m^{e(n-1)}/(|BC_e|n!))$ and $|BC_e| = 2^e e!$, so, it can be done much more efficiently than the algorithm above.

1. Dynamic programming? Calculate the $\mathcal{V}(n, e, m)$ using all the other configurations $\mathcal{V}(n', e', m')$ where $n' < n$, $e' < e$ and $m' < m$.

   Basic cases: For $m = 0$, the only configuration we can choose is n zero vectors. For $n = 1$, we can take every possible vector. (Estic molt espès i no se m'acudeixen altres casos base, a banda $e = 0$, que és una parida i no semblen rellevants. Si $e = 1$, triar n vectors en dimensió 1 ja és prou merda.)

   Induction: (We add a vector to the configuration) $\mathcal{V}(n+1, e, m)$ Take a configuration $v = \{v_1, \ldots, v_n\} \in \mathcal{V}(n, e, m)$ for each vector v_i in this configuration consider all the configurations that keep constant this v_i and at all the other v_j ($j \neq i$), we add the vectors of all the configurations of $\mathcal{V}(n, e, m)$ and the spare vector take as the $n+1$.

   $\mathcal{V}(n, e+1, m)$

   (We incrementally consider larger boxes) $\mathcal{V}(n, e, m+1)$ Assume we have generated all configurations in $\mathcal{V}(n, e, m)$, then the only new configurations are the ones with at least one vector of length $m+1$. So, for every $i \in [1, n]$, choose i vectors in the boundary and $n - i$ as in $\mathcal{V}(n - i, e, m)$. It remains to be checked which vectors of the boundary can be avoided

   Similar Idea but...
   Since there must be a vector that achieved m, we consider $v_1 = (m, 0, \ldots, 0)$. Then, we need to compute all the vectors such that $\sum_{i=2}^{n} = -v_1$. This can be done inductively on how many zero vectors are there.

Furthermore, we should check the hyperplane condition.
This need to be done for

$$S = \{(m,0,\dots,0),(m,1,0,\dots,0),(m,2,0,\dots,0),\dots,(m,m,0,\dots,0),(m,m,1,0,\dots,0),\dots,(m,\dots,m)\}.$$

If is this is done, taking in account the simetries of the cube and the rotation of the vectors, we have all the possible configurations (com ho veieu? Crec que aixi estaria... no? No se si m'estic deixant algun vector).
Example: if $e=2$, $n=3$ and $m=3$

$$\begin{pmatrix}3\\0\end{pmatrix} = -\begin{pmatrix}-1\\0\end{pmatrix} - \begin{pmatrix}-2\\0\end{pmatrix} = -\begin{pmatrix}-1\\\pm 1\end{pmatrix} - \begin{pmatrix}-2\\\mp 1\end{pmatrix} = -\begin{pmatrix}-1\\\pm 2\end{pmatrix} - \begin{pmatrix}-2\\\mp 2\end{pmatrix} = -\begin{pmatrix}-1\\\pm 3\end{pmatrix} - \begin{pmatrix}-2\\\mp 3\end{pmatrix}$$

$$\begin{pmatrix}3\\1\end{pmatrix} = \dots$$

$$\begin{pmatrix}3\\2\end{pmatrix} = \dots$$

$$\begin{pmatrix}3\\3\end{pmatrix} = \dots$$

2. In order to distinguish between different polytopes, we want to know the facets.