

Celestial Mechanics Memorandum

S/C: spacecraft, **CoM**: Center of Mass, **EoM**: Equations of Motion, **g.c.:** generalized coordinates, **OE**: Orbital Elements, **OED**: OE Differences, **MOE**: Mean OE, **OOE**: Osculating OE,

Constants

$\mu = GM = 398600.8 \text{ km}^3/\text{s}^2$ (*Earth's gravitational constant*)

$r_{\odot} =$ (*Earth's radius*)

$J_2 = 1082.7 \cdot 10^{-6}$ (*Earth's second zonal harmonic*)

Basic Energy Notions

• Work

Work done by **f** and **τ**

$$dW = \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r} + \boldsymbol{\tau}(\boldsymbol{\theta}) \cdot d\boldsymbol{\theta} \quad (1)$$

• Conservative Forces

Conservative force **f**_{cons} if it is an **exact differential**.

$$\frac{df_x}{dy} = \frac{df_y}{dx}.$$

namely, it is path-independent, e.g.

$$W_{AB} = \int_A^B \mathbf{f}(\mathbf{r}_1) \cdot d\mathbf{r}_1 = \int_A^B \mathbf{f}(\mathbf{r}_2) \cdot d\mathbf{r}_2.$$

• Mechanical Energy *E*

Total mechanical energy

$$E = \underbrace{K_{rot} + K_{trans}}_{\text{total kinetic}} + \underbrace{U_{grav} + U_{elas} + \dots}_{\text{total potential}} + \underbrace{W_{nc}(\mathbf{f}_{ext}, \mathbf{f}_{fric}, \dots)}_{\text{non-conservative}}.$$

Remark .1 If $W_{nc} = 0$ the energy of the system is **conserved**.

For a system with only **f**_{cons}: (*Work-Energy Theorem*)

$$\begin{aligned} \Delta U_{AB} &= U_B - U_A = -W_{AB}, \\ \Delta K_{AB} &= -\Delta U_{AB} = W_{AB}. \end{aligned} \quad (2)$$

Remark .2 Cons. work W_{AB} enables exchange between U and K . Force **f**_{cons} acting on conservative system via (1) and (2).

$$dU = -dW = -\mathbf{f} \cdot d\mathbf{r} \Rightarrow \mathbf{f} = -\nabla U = -\frac{\partial U}{\partial x} \hat{\mathbf{i}} - \frac{\partial U}{\partial y} \hat{\mathbf{j}} - \frac{\partial U}{\partial z} \hat{\mathbf{k}} \quad (3)$$

• Gravitational Potential Energy

Potential from Gauss's law of gravity: (*Shell theorem*)

$$(\text{field flux}) \quad \Phi_g = \oint_{\partial V} \mathbf{g} \cdot d\mathbf{S} = -4\pi GM$$

with field $\mathbf{g} = g(r)\hat{\mathbf{n}}$ and $d\mathbf{S} = dS\hat{\mathbf{n}}$, then

$$g(r)4\pi r^2 = 4\pi GM \Rightarrow g(r) = -\frac{GM}{r^2}. \quad (4)$$

and by integrating the gravity strength field

$$U_g(r) = \int g(r)dr = -\frac{GM}{r}. \quad (5)$$

Newton's law of gravity from (3) and (4):

$$\mathbf{a}_g = \frac{\mathbf{F}_g}{m} = \frac{-\nabla U_g}{m} = -\frac{\mu}{|\mathbf{r}|^3} \mathbf{r}. \quad (6)$$

Potential from spherical harmonic expansion in ECEF: (*Oblateness*)

$$\begin{aligned} U_g(r, \Phi, \Lambda) &= \frac{\mu}{r} \left\{ 1 + \sum_{n=2}^{\infty} \left[\left(\frac{r_{\odot}}{r} \right)^n J_n P_{n0}(\cos \Phi) + \right. \right. \\ &\quad \left. \left. + \sum_{m=1}^n \left[\left(\frac{r_{\odot}}{r} \right)^n (C_{nm} \cos m\Lambda + S_{nm} \sin m\Lambda) P_{nm}(\cos \Phi) \right] \right\}. \end{aligned} \quad (7)$$

Potential with only J_2 -perturbation to the first order:

$$U_g^{J_2} = -\frac{\mu}{|\mathbf{r}|} - \frac{J_2 \mu r_{\odot}^2}{2|\mathbf{r}|^3} \left(3 \left[\frac{z}{|\mathbf{r}|} \right]^2 - 1 \right) + O[J_2^2]. \quad (8)$$

Rigid Body Notions

Defined inertial $i := \{o, \hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\}$ and body $b := \{o', \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ frames

$$\mathbf{r}_{op} \equiv \mathbf{r}_{op/i}, \quad \mathbf{s}_{o'p} \equiv \mathbf{r}_{o'p/b} \quad (9)$$

Properties of *body* frame centered in $c \equiv CoM$. For every $p \in B$

$$\int_B \mathbf{r}_{cp} dm \approx \sum_p \mathbf{r}_{cp} m_p = 0, \quad \int_B \dot{\mathbf{r}}_{cp} dm \approx \sum_p \dot{\mathbf{r}}_{cp} m_p = 0. \quad (10)$$

Vector differentiation in moving frame b : (**Transport theorem**)

$$\dot{\mathbf{u}}_{/b}]_i = \dot{\mathbf{u}}_{/b}]_b + \{\boldsymbol{\omega}_{ib} \times \mathbf{u}\}_{/b}. \quad (11)$$

Momentum of a point p w.r.t. o with vector $\mathbf{r}_{op} = \mathbf{r}_{oo'} + \mathbf{r}_{o'p}$.

$$\begin{aligned} d\mathbf{p}_{p/i} &= \dot{\mathbf{r}}_{op} dm = (\dot{\mathbf{r}}_{oo'} + \dot{\mathbf{r}}_{o'p} + \{\boldsymbol{\omega}_{ib} \times \mathbf{r}_{o'p}\}_{/i}) dm, \\ d\mathbf{h}_{op/i} &= \{\mathbf{r}_{op} \times d\mathbf{p}_p\}_{/i} = \{\mathbf{r}_{op} \times (\dot{\mathbf{r}}_{oo'} + \dot{\mathbf{r}}_{o'p} + \{\boldsymbol{\omega}_{ib} \times \mathbf{r}_{o'p}\}_{/i})\}_{/i} dm. \end{aligned}$$

Remark .3 A **rigid-body** has no rel. velocities, i.e. $\dot{\mathbf{s}}_{o'p}]_b = 0$.

• Angular momentum

AM of a body B with respect to point o' seen from body frame b

$$\mathbf{h}_{o'B/b} = \int_{p \in B} d\mathbf{h}_{op/b} = \int_B \mathbf{s}_{op} dm \times \{\boldsymbol{\omega}_{ib} \times \mathbf{s}_{o'p}\}_{/b} = \mathbf{J}_{B/b} \boldsymbol{\omega}_{ib/b} \quad (12)$$

If $o' = c$ (CoM), then $\int_B \mathbf{r}_{op} dm \times \mathbf{r}_{oc} = \int_B \mathbf{s}_{cp} dm \times \mathbf{s}_{cc} = \mathbf{0}$ [1].

• Torque

According to Newton-Euler EoM, **τ** is the cause of $\dot{\mathbf{h}}]_i$

$$\boldsymbol{\tau} = \dot{\mathbf{h}}]_i = \dot{\mathbf{h}}]_b + \boldsymbol{\omega} \times \mathbf{h} = \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}. \quad (13)$$

• Kinetic energy

Translational and rotational Kinetic energies from momentum

$$K_t = \int \mathbf{p} d\mathbf{v} = \frac{1}{2} m |\mathbf{v}|^2, \quad K_r = \int \mathbf{h} d\boldsymbol{\omega} = \frac{1}{2} |\boldsymbol{\omega}|^2, \quad (14)$$

• Lagrangian $L(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}})$

The *Lagrangian* is defined for a set of g.c $\boldsymbol{\chi} \in \mathbb{R}^N$

$$L := K(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}) - U(\boldsymbol{\chi}). \quad (15)$$

Euler-Lagrange from the Lagrangian, with constraints $\mathbf{g}_1(\boldsymbol{\chi}), \mathbf{g}_2(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, t)$, and Lagrange multipliers $\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_n$.

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{\boldsymbol{\chi}}} \right) - \frac{\partial L}{\partial \boldsymbol{\chi}} + \underbrace{\nabla_{\boldsymbol{\chi}} \mathbf{g}_1^{\top} \boldsymbol{\lambda}_h}_{\text{holonomic}} - \underbrace{\nabla_{\dot{\boldsymbol{\chi}}} \mathbf{g}_2^{\top} \boldsymbol{\lambda}_n}_{\text{non-holonomic}} = \underbrace{\mathbf{Q} + \mathbf{D}}_{\text{ext. forces}} \quad (16)$$

Remark .4 For mechanical systems, $\frac{\partial L}{\partial \dot{\boldsymbol{\chi}}} = 0$.

• Hamiltonian $H(\boldsymbol{\chi}, \mathbf{p}, t)$

The *Hamiltonian* is defined with $\mathbf{p} = \nabla_{\dot{\boldsymbol{\chi}}} L(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}})$

$$H := \mathbf{p}^{\top} \dot{\boldsymbol{\chi}} - L(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}). \quad (17)$$

For g.c. $\boldsymbol{\chi} \in \mathbb{R}^N$ and momentum $\mathbf{p} \in \mathbb{R}^N$.

$$\dot{\mathbf{p}} = -\nabla_{\boldsymbol{\chi}} H, \quad \dot{\boldsymbol{\chi}} = \nabla_{\mathbf{p}} H. \quad (18)$$

for conservervative systems $H(\boldsymbol{\chi}, \mathbf{p}, t) = H(\boldsymbol{\chi}, \mathbf{p}) = E_{tot}$.

Remark .5 EoM can be derived from (16) or (18).

Quaternions

A unitary quaternion defining a rotation from a to b

$$\mathbf{q}_a^b := [\boldsymbol{\eta} \boldsymbol{\varepsilon}^{\top}]^{\top} \in \mathbb{H} : \boldsymbol{\eta}^2 + |\boldsymbol{\varepsilon}|^2 = 1. \quad (19)$$

with alternative representations as follows ($\boldsymbol{\vartheta} \in \mathbb{R}^3$)

$$\mathbf{q}_a^b = \mathbf{q}\{\boldsymbol{\vartheta}\} = e^{\boldsymbol{\vartheta}} = [\cos(|\boldsymbol{\vartheta}|/2) \quad \frac{\boldsymbol{\vartheta}^{\top}}{|\boldsymbol{\vartheta}|} \sin(|\boldsymbol{\vartheta}|/2)]^{\top}. \quad (20)$$

Conversion from \mathbf{q}_a^b to \mathbf{R}_a^b is achieved by

$$\mathbf{R}_a^b\{\mathbf{q}_a^b\} = \mathbf{I}_3 + 2\boldsymbol{\eta}[\boldsymbol{\varepsilon}]_{\times} + 2[\boldsymbol{\varepsilon}]_{\times}^2. \quad (21)$$

and useful approximations of small local rotations

$$\Delta \mathbf{q}_{\ell} \approx [1 \quad \Delta \boldsymbol{\vartheta}_{\ell}^{\top}/2]^{\top}, \quad \Delta \mathbf{R}_{\ell} \approx \mathbf{I} + [\Delta \boldsymbol{\vartheta}_{\ell}]_{\times}. \quad (22)$$

• Time-derivative

In the fixed i and moving b reference frame

$$\begin{aligned} \dot{\mathbf{q}}_{/i} &:= \frac{1}{2} \begin{bmatrix} 0 \\ \boldsymbol{\omega}_{\ell} \end{bmatrix} \otimes \mathbf{q}(t) = \frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}_{\ell}) \mathbf{q}(t), & \boldsymbol{\omega}_{\ell} &\equiv \boldsymbol{\omega}_{/i}, \\ \dot{\mathbf{q}}_{/b} &:= \frac{1}{2} \mathbf{q}(t) \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega}_{\ell} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\boldsymbol{\varepsilon}^{\top} \\ \boldsymbol{\eta} \mathbf{I} - [\boldsymbol{\varepsilon}]_{\times} \end{bmatrix} \boldsymbol{\omega}_{\ell}, & \boldsymbol{\omega}_{\ell} &\equiv \boldsymbol{\omega}_{/b}. \end{aligned} \quad (23)$$

where $\boldsymbol{\omega}_{\ell}$ is local angular perturbation in the frame \mathbf{q}_a^b .

$$\boldsymbol{\omega}_{\ell} = \frac{d\boldsymbol{\vartheta}_{\ell}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \boldsymbol{\vartheta}_{\ell}}{\Delta t}. \quad (24)$$

and $\boldsymbol{\Omega}(\boldsymbol{\omega})$ and $[\boldsymbol{\omega}]_{\times}$ are skew symmetric matrices

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) := \begin{bmatrix} 0 & -\boldsymbol{\omega}^{\top} \\ \boldsymbol{\omega} & -[\boldsymbol{\omega}]_{\times} \end{bmatrix}, \quad [\boldsymbol{\omega}]_{\times} := \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}. \quad (25)$$

Analogously, the rotation matrix kinematics are:

$$\dot{\mathbf{R}}_a^b = [\boldsymbol{\omega}_b^{ab}]_{\times} \mathbf{R}_a^b = \mathbf{R}_a^b [\boldsymbol{\omega}_a^{ab}]_{\times}. \quad (26)$$

• First-order integration: ($\bar{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}_{n+1} + \boldsymbol{\omega}_n}{2}$)

$$\mathbf{q}_{n+1} \approx \mathbf{q}_n \otimes \left(\mathbf{q}\{\bar{\boldsymbol{\omega}}_n \Delta t\} + \frac{\Delta t^2}{24} \begin{bmatrix} 0 \\ \boldsymbol{\omega}_n \times \boldsymbol{\omega}_{n+1} \end{bmatrix} \right), \quad (27)$$

Keplerian Orbits

An orbit $\mathcal{O}(\boldsymbol{\kappa})$ defined by constant elements $\boldsymbol{\kappa} = \{a, e, i, \omega, \Omega\}$.

• Vis-viva

Since $E = \frac{v^2}{2} - \frac{\mu}{r}$ maintained along orbit $E_{apo} = E_{peri}$, leading to

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right). \quad (28)$$

• Orbital radius and period

In terms of the zero-order (Keplerian) $f_{(0)}$, $E_{(0)}$ and $a_{(0)}$.

$$r_t = \frac{a(1-e^2)}{1+e \cos f} = a(1-e \cos E), \quad T = 2\pi \sqrt{\frac{a^3}{\mu}}. \quad (29)$$

• **Anomaly angles M , E , f and time t**

Note that n refers to mean motion.

$$\begin{aligned} M &= 2 \tan^{-1} \left[\sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} \right] - \frac{e \sqrt{1-e^2} \sin f}{1+e \cos f} \\ &= E - e \sin E = \sqrt{\mu/a^3} (t - \tau) = n(t - \tau). \end{aligned} \quad (30)$$

• **Semi-latus rectum**

Relations between the orbit parameter p , a , b and e

$$p = \frac{h^2}{\mu} = \frac{b^2}{a} = a(1 - e^2). \quad (31)$$

• **Important relations**

$$\begin{aligned} \text{Lagrange's formula : } \mathbf{r} \cdot \mathbf{v} &= \frac{\mu}{h} r e \sin f. \\ \text{Hodograph relation : } \mathbf{v} &= \frac{\mu}{h^2} \mathbf{h} \times \left(\mathbf{e} + \frac{\mathbf{r}}{r} \right). \end{aligned} \quad (32)$$

Transformations

PQW \rightarrow **ECI**: $R = R_z(\Omega) R_x(i) R_z(\omega)$.

RTN \rightarrow **ECI**: $R = R_z(\Omega) R_x(i) R_z(\omega + f)$.

Non-Keplerian Orbits

They osculate and drift, i.e. $\dot{\boldsymbol{\kappa}}_t \neq 0$, depending on the type of perturbations considered. Generally, we use LPEs or GPEs. Section based on [2].

• **J_2 Effects**

The Earth oblateness produces **short-**, **long-**period oscillations and **secular drift**.

Remark .6 *The secular drift is better studied using the MOE, instead of OOE, since short- and long-period oscillations are removed.*

$$\begin{aligned} (\text{Nodal drift}) \quad \dot{\Omega} &= -\frac{3}{2} J_2 \left(\frac{r_{\odot}}{p} \right)^2 n \cos i, \\ (\text{Apsidal drift}) \quad \dot{\omega} &= \frac{3}{4} J_2 \left(\frac{r_{\odot}}{p} \right)^2 n (5 \cos^2 i - 1), \\ \dot{M} &= n + \frac{3}{4} J_2 \left(\frac{r_{\odot}}{p} \right)^2 n \sqrt{1 - e^2} (3 \cos^2 i - 1). \end{aligned} \quad (33)$$

• **Lagrange Planetary Equations LPE (disturbance \tilde{R} is conservative)**

$$\begin{aligned} \dot{a} &= -\frac{2a^2}{k} \frac{\partial \tilde{R}}{\partial \tau}, \quad \dot{e} = -\frac{a(1-e^2)}{ke} \frac{\partial \tilde{R}}{\partial \tau} - \frac{1}{e} \sqrt{\frac{1-e^2}{ka}} \frac{\partial \tilde{R}}{\partial \omega} \\ \frac{di}{dt} &= \frac{1}{\sqrt{ka(1-e^2)} \sin i} \left[\cos i \frac{\partial \tilde{R}}{\partial \omega} - \frac{\partial \tilde{R}}{\partial \Omega} \right], \\ \dot{\Omega} &= \frac{1}{\sqrt{ka(1-e^2)} \sin i} \frac{\partial \tilde{R}}{\partial i}, \quad \dot{\omega} = \frac{1}{e} \sqrt{\frac{1-e^2}{ka}} \left[\frac{\partial \tilde{R}}{\partial e} - \frac{e \cot i}{1-e^2} \frac{\partial \tilde{R}}{\partial i} \right]. \end{aligned}$$

• **Gauss Planetary Equations GPE (disturbance S, T, N non-conservative)**

Remark .7 \tilde{R} and STN usually involve an anomaly angle directly. For convenience, we use one of them as **independent variable**. Then,

$$\dot{\boldsymbol{\kappa}}_i = \frac{d\boldsymbol{\kappa}_i}{df} \frac{df}{dt}. \quad (34)$$

Remark .8 When inserting zero-order $a_{(0)}, \dots, \Omega_{(0)}$ (Keplerian orbit) into LPE or GPE and integrate we obtain first-order solutions.

Spacecraft Dynamics

• **Newton Planetary Equations under J_2 -perturbation**

$$\begin{aligned} \ddot{x} &= -\frac{\mu x}{|\mathbf{r}|^3} - 3x \frac{J_2 \mu r_{\odot}^2}{2|\mathbf{r}|^5} \left[1 - 5 \left(\frac{z}{|\mathbf{r}|} \right)^2 \right] + \frac{f_x}{m}, \\ \ddot{y} &= -\frac{\mu y}{|\mathbf{r}|^3} - 3y \frac{J_2 \mu r_{\odot}^2}{2|\mathbf{r}|^5} \left[1 - 5 \left(\frac{z}{|\mathbf{r}|} \right)^2 \right] + \frac{f_y}{m}, \\ \ddot{z} &= -\frac{\mu z}{|\mathbf{r}|^3} - 3z \frac{J_2 \mu r_{\odot}^2}{2|\mathbf{r}|^5} \left[3 - 5 \left(\frac{z}{|\mathbf{r}|} \right)^2 \right] + \frac{f_z}{m}. \end{aligned} \quad (35)$$

• **Quaternion-Based Attitude Dynamics with RWs**

\mathbf{J}_s : S/C inertia, \mathbf{h}_w : RWs angular momentum, $\boldsymbol{\tau}_w$: RWs torque, and $\boldsymbol{\tau}_d$: disturbance torques. All quantities expressed in b frame.

$$\begin{aligned} \dot{\mathbf{q}}_i^b &= \frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}_{ib/b}) \mathbf{q}_i^b, \\ \dot{\boldsymbol{\omega}}_{ib/b} &= \mathbf{J}_s^{-1} (-\boldsymbol{\omega}_{ib/b} \times \mathbf{J}_s \boldsymbol{\omega}_{ib/b} - \boldsymbol{\omega}_{ib/b} \times \mathbf{h}_{w/b} + \boldsymbol{\tau}_{d/b} - \boldsymbol{\tau}_{w/b}), \\ \dot{\mathbf{h}}_{w/b} &= \boldsymbol{\tau}_{w/b} \end{aligned} \quad (36)$$

Remark .9 Note that it is convenient to align the RWs axis of rotation with the b frame, which usually also coincides with the principal axis of inertia of the S/C.

Relative Spacecraft Dynamics

• **Clohessey-Wiltshire Equations (CW)**

• **Hill-Clohessey-Wiltshire Equations (HCW)**

• **Relative state through OED**

The relative states in the *Hill* frame (i.e. RTN), $\delta \mathbf{r}_{/RTN}$, as a function of f and OED, $\delta \boldsymbol{\kappa}$.

$$[\delta x \ \delta y \ \delta z]^{\top}_{/RTN}(f) = \Phi_{\delta \boldsymbol{\kappa}}^{\delta r} [\delta a \ \delta e \ \delta i \ \delta \omega \ \delta \Omega \ \delta M]^{\top} \quad (37)$$

with radius r is at position f , we have

$$\Phi_{\delta \boldsymbol{\kappa}}^{\delta r} = \begin{bmatrix} \frac{r}{a} & 0 & 0 \\ -a \cos f & \frac{r \sin f}{\eta^2} (2 + e \cos f) & 0 \\ 0 & 0 & r \sin \theta \\ 0 & r & 0 \\ 0 & r \cos i & -r \cos \theta \sin i \\ \frac{ae \sin f}{\eta} & \frac{r \sin f}{\eta^3} (1 + e \cos f)^2 \delta M & 0 \end{bmatrix}. \quad (38)$$

The change in f can also be expressed in terms of $\delta \boldsymbol{\kappa}$.

$$\delta f = \frac{(1 + e \cos f)^2}{\eta^3} \delta M + \frac{\sin f}{\eta^2} (2 + e \cos f) \delta e, \quad (39)$$

where $\eta = \sqrt{1 - e^2}$ in both (38) and (39).

Remark .10 Eq. (37) is a linearization, thus loses accuracy if $\delta r \sim r$. If $\delta a \neq 0$, then the orbital periods between $\mathcal{O}(\boldsymbol{\kappa})$ and $\mathcal{O}(\boldsymbol{\kappa} + \delta \boldsymbol{\kappa})$ change, yielding secular growth in $\delta M, \delta E$ and δf .

References

- [1] J. R. Wertz. Spacecraft attitude determination and control. 1994.
- [2] Schaub H., and Junkins, J. L., Analytical Mechanics of Space Systems, AIAA, 2009.