Celestial Mechanics Memorandum

S/C: spacecraft, **CoM**: Center of Mass, **EoM**: Equations of Motion, **g.c.**: generalized coordinates, **OE**: Orbital Elements, **OED**: OE Differences, **MOE**: Mean OE, **OOE**: Osculating OE,

Constants

$\mu = GM = 398600.8\ km^3/s^2$ *(Earth's gravitational constant)*
$r_\odot =$ *(Earth's radius)*
$J_2 = 1082.7 \cdot 10^{-6}$ *(Earth's second zonal harmonic)*

Basic Energy Notions

• Work

Work done by $\mathbf{f}$ and $\boldsymbol{\tau}$

$$dW = \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r} + \boldsymbol{\tau}(\boldsymbol{\theta}) \cdot d\boldsymbol{\theta} \tag{1}$$

• Conservative Forces

Conservative force $\mathbf{f}_{cons}$ if it is an **exact differential**.

$$\frac{df_x}{dy} = \frac{df_y}{dx}.$$

namely, it is path-independent, e.g.

$$W_{AB} = \int_A^B \mathbf{f}(\mathbf{r}_1) \cdot d\mathbf{r_1} = \int_A^B \mathbf{f}(\mathbf{r}_2) \cdot d\mathbf{r_2}.$$

• Mechanical Energy E

Total mechanical energy

$$E = \underbrace{K_{rot} + K_{trans}}_{\text{total kinetic}} + \underbrace{U_{grav} + U_{elas} + ..}_{\text{total potential}} + \underbrace{W_{nc}(\mathbf{f}_{ext}, \mathbf{f}_{fric}, ...)}_{\text{non-conservative}}.$$

Remark .1 *If $W_{nc} = 0$ the energy of the system is **conserved**.*

For a system with only $\mathbf{f}_{cons}$: *(Work-Energy Theorem)*

$$\begin{aligned} \Delta U_{AB} &= U_B - U_A = -W_{AB}, \\ \Delta K_{AB} &= -\Delta U_{AB} = W_{AB}. \end{aligned} \tag{2}$$

Remark .2 *Cons. work W_{AB} enables exchange between U and K.*

Force $\mathbf{f}_{cons}$ acting on conservative system via (1) and (2).

$$dU = -dW = -\mathbf{f} \cdot d\mathbf{r} \Rightarrow \mathbf{f} = -\nabla U = -\frac{\partial U}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{k}} \tag{3}$$

• Gravitational Potential Energy

Potential from Gauss's law of gravity: ***(Shell theorem)***

$$\textit{(field flux)}\ \Phi_g = \oint_{\partial V} \mathbf{g} \cdot d\mathbf{S} = -4\pi GM$$

with field $\mathbf{g} = g(r)\hat{\mathbf{n}}$, $d\mathbf{S} = dS\hat{\mathbf{n}}$ and $\partial V \equiv$ sphere, then

$$g(r) = -\frac{GM}{r^2} = \frac{|\mathbf{f}_g|}{m}, \tag{4}$$

and by integrating the gravity strength field

$$U_g(r) = -\int g(r)dr = -\frac{GM}{r}. \tag{5}$$

Newton's law of gravity from (3) and (4):

$$\mathbf{a}_g = \frac{\mathbf{F}_g}{m} = \frac{-\nabla U_g}{m} = -\frac{\mu}{|\mathbf{r}|^3}\mathbf{r}. \tag{6}$$

Potential from spherical harmonic expansion in ECEF: ***(Oblateness)***

$$\begin{aligned} U_g(r, \Phi, \Lambda) = \frac{\mu}{r}\Big\{1 + \textstyle\sum_{n=2}^{\infty}\Big[\left(\frac{r_\odot}{r}\right)^n J_n P_{n0}(\cos\Phi) + \\ + \textstyle\sum_{m=1}^{n}\Big[\left(\frac{r_\odot}{r}\right)^n (C_{nm}\cos m\Lambda + S_{nm}\sin m\Lambda) P_{nm}(\cos\Phi)\Big]. \end{aligned} \tag{7}$$

Potential with only J_2-perturbation to the first order:

$$U_g^{J_2} = -\frac{\mu}{|\mathbf{r}|} - \frac{J_2 \mu r_\odot^2}{2|\mathbf{r}|^3}\left(3\left[\frac{z}{|\mathbf{r}|}\right]^2 - 1\right) + O[J_2^2]. \tag{8}$$

Rigid Body Notions

Defined inertial $i := \{o, \hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\}$ and body $b := \{o', \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ frames

$$\mathbf{r}_{op} \equiv \mathbf{r}_{op/i}, \quad \mathbf{s}_{o'p} \equiv \mathbf{r}_{o'p/b} \tag{9}$$

Properties of *body* frame centered in $c \equiv CoM$. For every $p \in B$

$$\int_B \mathbf{r}_{cp}dm \approx \textstyle\sum_p \mathbf{r}_{cp}m_p = 0, \quad \int_B \dot{\mathbf{r}}_{cp}dm \approx \sum_p \dot{\mathbf{r}}_{cp}m_p = 0. \tag{10}$$

Vector differentiation in moving frame b: ***(Transport theorem)***

$$\dot{\mathbf{u}}_{/b}]_i = \dot{\mathbf{u}}_{/b}]_b + \{\boldsymbol{\omega}_{ib} \times \mathbf{u}\}_{/b}. \tag{11}$$

Momentum of a point p w.r.t. o with vector $\mathbf{r}_{op} = \mathbf{r}_{oo'} + \mathbf{r}_{o'p}$.

$$\begin{aligned} d\mathbf{p}_{p/i} &= \dot{\mathbf{r}}_{op}dm = (\dot{\mathbf{r}}_{oo'} + \dot{\mathbf{r}}_{o'p} + \{\boldsymbol{\omega}_{ib} \times \mathbf{r}_{o'p}\}_{/i})dm, \\ d\mathbf{h}_{op/i} &= \{\mathbf{r}_{op} \times d\mathbf{p}_p\}_{/i} = \{\mathbf{r}_{op} \times (\dot{\mathbf{r}}_{oo'} + \dot{\mathbf{r}}_{o'p} + \{\boldsymbol{\omega}_{ib} \times \mathbf{r}_{o'p}\}_{/i})\}dm. \end{aligned}$$

Remark .3 *A **rigid-body** has no rel. velocities, i.e. $\dot{\mathbf{s}}_{o'p}]_b = 0$.*

• Angular momentum

AM of a body B with respect to point o' seen from body frame b

$$\mathbf{h}_{o'B/b} = \int_{p \in B} d\mathbf{h}_{op/b} = \int_B \mathbf{s}_{op}dm \times \{\boldsymbol{\omega}_{ib} \times \mathbf{s}_{o'p}\}_{/b} = \mathbf{J}_{B/b}\boldsymbol{\omega}_{ib/b} \tag{12}$$

If $o' = c$ (CoM), then $\int_B \mathbf{r}_{op}dm \times \mathbf{r}_{oc} = \int_B \mathbf{s}_{cp}dm \times \mathbf{s}_{cc} = \mathbf{0}$ [1].

• Torque

According to Newton-Euler EoM, $\boldsymbol{\tau}$ is the cause of $\dot{\mathbf{h}}]_i$

$$\boldsymbol{\tau} = \dot{\mathbf{h}}]_i = \dot{\mathbf{h}}]_b + \boldsymbol{\omega} \times \mathbf{h} = \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}. \tag{13}$$

• Kinetic energy

Translational and rotational Kinetic energies from momentum

$$K_t = \int \mathbf{p}d\mathbf{v} = \tfrac{1}{2}m|\mathbf{v}|^2, \quad K_r = \int \mathbf{h}d\boldsymbol{\omega} = \tfrac{1}{2}|\boldsymbol{\omega}|^2, \tag{14}$$

• Lagrangian $L(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}})$

The *Lagrangian* is defined for a set of g.c $\boldsymbol{\chi} \in \mathbb{R}^N$

$$L := K(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}) - U(\boldsymbol{\chi}). \tag{15}$$

Euler-Lagrange from the Lagrangian, with constraints $\mathbf{g}_1(\boldsymbol{\chi}), \mathbf{g}_2(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, t)$, and Lagrange multipliers $\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_n$.

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\chi}}}\right) - \frac{\partial L}{\partial \boldsymbol{\chi}} + \underbrace{\nabla_{\boldsymbol{\chi}}\mathbf{g}_1^\top \boldsymbol{\lambda}_h}_{\text{holonomic}} - \underbrace{\nabla_{\dot{\boldsymbol{\chi}}}\mathbf{g}_2^\top \boldsymbol{\lambda}_n}_{\text{non-holonomic}} = \underbrace{\mathbf{Q} + \mathbf{D}}_{\text{ext. forces}} \tag{16}$$

Remark .4 *For mechanical systems, $\frac{\partial L}{\partial \dot{\boldsymbol{\chi}}} = 0$.*

• Hamiltonian $H(\boldsymbol{\chi}, \mathbf{p}, t)$

The *Hamiltonian* is defined with $\mathbf{p} = \nabla_{\dot{\boldsymbol{\chi}}} L(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}})$

$$H := \mathbf{p}^\top \dot{\boldsymbol{\chi}} - L(\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}). \tag{17}$$

For g.c. $\boldsymbol{\chi} \in \mathbb{R}^N$ and momentum $\mathbf{p} \in \mathbb{R}^N$.

$$\dot{\mathbf{p}} = -\nabla_{\boldsymbol{\chi}} H, \quad \dot{\boldsymbol{\chi}} = \nabla_{\mathbf{p}} H. \tag{18}$$

for conserververvative systems $H(\boldsymbol{\chi}, \mathbf{p}, t) = H(\boldsymbol{\chi}, \mathbf{p}) = E_{tot}$.

Remark .5 *EoM can be derived from (16) or (18).*

Quaternions

A unitary quaternion defining a rotation from a to b

$$\mathbf{q}_a^b := [\eta\ \boldsymbol{\varepsilon}^\top]^\top \in \mathbb{H} : \eta^2 + |\boldsymbol{\varepsilon}|^2 = 1. \tag{19}$$

with alternative representations as follows ($\boldsymbol{\vartheta} \in \mathbb{R}^3$)

$$\mathbf{q}_a^b = \mathbf{q}\{\boldsymbol{\vartheta}\} = e^{\boldsymbol{\vartheta}} = [\cos(|\boldsymbol{\vartheta}|/2)\ \tfrac{\boldsymbol{\vartheta}^\top}{|\boldsymbol{\vartheta}|}\sin(|\boldsymbol{\vartheta}|/2)]^\top. \tag{20}$$

Conversion from $\mathbf{q}_a^b$ to $\mathbf{R}_a^b$ is achieved by

$$\mathbf{R}_a^b\{\mathbf{q}_a^b\} = \mathbf{I}_3 + 2\eta[\boldsymbol{\varepsilon}]_\times + 2[\boldsymbol{\varepsilon}]_\times^2. \tag{21}$$

and useful approximations of small local rotations

$$\Delta\mathbf{q}_\ell \approx [1\ \Delta\boldsymbol{\vartheta}_\ell^\top/2]^\top, \quad \Delta\mathbf{R}_\ell \approx \mathbf{I} + [\Delta\boldsymbol{\vartheta}_\ell]_\times. \tag{22}$$

• Time-derivative

In the fixed i and moving b reference frame

$$\begin{aligned} \dot{\mathbf{q}}_{/i} &:= \tfrac{1}{2}\begin{bmatrix}0\\ \boldsymbol{\omega}_\ell\end{bmatrix} \otimes \mathbf{q}(t) = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_\ell)\mathbf{q}(t), & \boldsymbol{\omega}_\ell &\equiv \boldsymbol{\omega}_{/i}, \\ \dot{\mathbf{q}}_{/b} &:= \tfrac{1}{2}\mathbf{q}(t) \otimes \begin{bmatrix}0\\ \boldsymbol{\omega}_\ell\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\boldsymbol{\varepsilon}^\top\\ \eta\mathbf{I} - [\boldsymbol{\varepsilon}]_\times\end{bmatrix}\boldsymbol{\omega}_\ell, & \boldsymbol{\omega}_\ell &\equiv \boldsymbol{\omega}_{/b}. \end{aligned} \tag{23}$$

where $\boldsymbol{\omega}_\ell$ is local angular perturbation in the frame $\mathbf{q}_a^b$.

$$\boldsymbol{\omega}_\ell = \frac{d\vartheta_\ell}{dt} = \lim_{\Delta t \to 0}\frac{\Delta\vartheta_\ell}{\Delta t}. \tag{24}$$

and $\boldsymbol{\Omega}(\boldsymbol{\omega})$ and $[\boldsymbol{\omega}]_\times$ are skew symmetric matrices

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) := \begin{bmatrix}0 & -\boldsymbol{\omega}^\top\\ \boldsymbol{\omega} & -[\boldsymbol{\omega}]_\times\end{bmatrix}, \quad [\boldsymbol{\omega}]_\times := \begin{bmatrix}0 & \omega_z & -\omega_y\\ -\omega_z & 0 & \omega_x\\ \omega_y & -\omega_x & 0\end{bmatrix}. \tag{25}$$

Analogously, the rotation matrix kinematics are:

$$\dot{\mathbf{R}}_a^b = [\boldsymbol{\omega}_b^{ab}]_\times\mathbf{R}_a^b = \mathbf{R}_a^b[\boldsymbol{\omega}_a^{ab}]_\times. \tag{26}$$

• First-order integration: ($\bar{\omega} = \frac{\omega_{n+1} + \omega_n}{2}$)

$$\mathbf{q}_{n+1} \approx \mathbf{q}_n \otimes \left(\mathbf{q}\{\overline{\omega}_n\Delta t\} + \frac{\Delta t^2}{24}\begin{bmatrix}0\\ \boldsymbol{\omega}_n \times \boldsymbol{\omega}_{n+1}\end{bmatrix}\right), \tag{27}$$

Keplerian Orbits

An orbit $\mathcal{O}(\boldsymbol{\kappa})$ defined by constant elements $\boldsymbol{\kappa} = \{a, e, i, \omega, \Omega\}$.

• *Vis-viva*

Since $E = \frac{v^2}{2} - \frac{\mu}{r}$ mantained along orbit $E_{apo} = E_{peri}$, leading to

$$v^2 = \mu\left(\tfrac{2}{r} - \tfrac{1}{a}\right). \tag{28}$$

• Orbital radius and period

In terms of the zero-order (Keplerian) $f_{(0)}$, $E_{(0)}$ and $a_{(0)}$.

$$r_t = \frac{a(1-e^2)}{1+e\cos f} = a(1 - e\cos E), \quad T = 2\pi\sqrt{\frac{a^3}{\mu}}. \tag{29}$$

- **Anomaly angles M, E, f and time t**

Note that n referes to mean motion.

$$\begin{aligned} M =& 2\tan^{-1}\left[\sqrt{\tfrac{1-e}{1+e}}\tan\tfrac{f}{2}\right] - \tfrac{e\sqrt{1-e^2}\sin f}{1+e\cos f} \\ =& E - e\sin E = \sqrt{\mu/a^3}(t-\tau) = n(t-\tau). \end{aligned} \tag{30}$$

- **Semi-latus rectum**

Relations between the orbit parameter p, a, b and e

$$p = \tfrac{h^2}{\mu} = \tfrac{b^2}{a} = a(1-e^2). \tag{31}$$

- **Important relations**

$$\begin{aligned} \textbf{Lagrange's formula}: \mathbf{r}\cdot\mathbf{v} = \tfrac{\mu}{h} re\sin f. \\ \textbf{Hodograph relation}: \mathbf{v} = \tfrac{\mu}{h^2}\mathbf{h}\times\left(\mathbf{e}+\tfrac{\mathbf{r}}{r}\right). \end{aligned} \tag{32}$$

Transformations

PQW→ECI: $R = R_z(\Omega)R_x(i)R_z(\omega)$.

RTN→ECI: $R = R_z(\Omega)R_x(i)R_z(\omega+f)$.

Non-Keplerian Orbits

They osculate and drift, i.e. $\dot{\boldsymbol{\kappa}}_t \neq 0$, depending on the type of perturbations considered. Generally, we use LPEs or GPEs. Section based on [2].

- **J_2 Effects**

The Earth oblateness produces **short-**, **long-**period oscillations and **secular drift**.

Remark .6 *The secular drift is better studied using the MOE, instead of OOE, since short- and long-period oscillations are removed.*

$$\begin{aligned} \textit{(\textbf{Nodal drift})}\ \dot{\bar{\Omega}} &= -\tfrac{3}{2}J_2\left(\tfrac{r_\odot}{p}\right)^2 n\cos i, \\ \textit{(\textbf{Apsidal drift})}\ \dot{\bar{\omega}} &= \tfrac{3}{4}J_2\left(\tfrac{r_\odot}{p}\right)^2 n\left(5\cos^2 i - 1\right), \\ \dot{\bar{M}} &= n + \tfrac{3}{4}J_2\left(\tfrac{r_\odot}{p}\right)^2 n\sqrt{1-e^2}(3\cos^2 i - 1). \end{aligned} \tag{33}$$

- **Lagrange Planetary Equations LPE (disturbance $\tilde{R}$ is conservative)**

$$\begin{aligned} \dot{a} &= -\tfrac{2a^2}{k}\tfrac{\partial\tilde{R}}{\partial\tau}, \quad \dot{e} = -\tfrac{a(1-e^2)}{ke}\tfrac{\partial\tilde{R}}{\partial\tau} - \tfrac{1}{e}\sqrt{\tfrac{1-e^2}{ka}}\tfrac{\partial\tilde{R}}{\partial\omega} \\ \tfrac{di}{dt} &= \tfrac{1}{\sqrt{ka(1-e^2)}\sin i}\left[\cos i\tfrac{\partial\tilde{R}}{\partial\omega} - \tfrac{\partial\tilde{R}}{\partial\Omega}\right], \\ \dot{\Omega} &= \tfrac{1}{\sqrt{ka(1-e^2)}\sin i}\tfrac{\partial\tilde{R}}{\partial i}, \quad \dot{\omega} = \tfrac{1}{e}\sqrt{\tfrac{1-e^2}{ka}}\left[\tfrac{\partial\tilde{R}}{\partial e} - \tfrac{e\cot i}{1-e^2}\tfrac{\partial\tilde{R}}{\partial i}\right]. \end{aligned}$$

- **Gauss Planetary Equations GPE (disturbance S, T, N non-conservative)**

Remark .7 *$\tilde{R}$ and STN usually involve an anomaly angle directly. For convenience, we use one of them as **independent variable**. Then,*

$$\dot{\boldsymbol{\kappa}}_i = \tfrac{d\boldsymbol{\kappa}_i}{df}\tfrac{df}{dt}. \tag{34}$$

Remark .8 *When inserting zero-order $a_{(0)}, ..., \Omega_{(0)}$ (Keplerian orbit) into LPE or GPE and integrate we obtain first-order solutions.*

Spacecraft Dynamics

- **Newton Planetary Equations under J_2-perturbation**

$$\begin{aligned} \ddot{x} &= -\tfrac{\mu x}{|\mathbf{r}|^3} - 3x\tfrac{J_2\mu r_\odot^2}{2|\mathbf{r}|^5}\left[1-5\left(\tfrac{z}{|\mathbf{r}|}\right)^2\right] + \tfrac{f_x}{m}, \\ \ddot{y} &= -\tfrac{\mu y}{|\mathbf{r}|^3} - 3y\tfrac{J_2\mu r_\odot^2}{2|\mathbf{r}|^5}\left[1-5\left(\tfrac{z}{|\mathbf{r}|}\right)^2\right] + \tfrac{f_y}{m}, \\ \ddot{z} &= -\tfrac{\mu z}{|\mathbf{r}|^3} - 3z\tfrac{J_2\mu r_\odot^2}{2|\mathbf{r}|^5}\left[3-5\left(\tfrac{z}{|\mathbf{r}|}\right)^2\right] + \tfrac{f_z}{m}. \end{aligned} \tag{35}$$

- **Quaternion-Based Attitude Dynamics with RWs**

$\mathbf{J}_s$: S/C inertia, $\mathbf{h}_w$: RWs angular momentum, $\boldsymbol{\tau}_w$: RWs torque, and $\boldsymbol{\tau}_d$: disturbance torques. All quantities expressed in b frame.

$$\begin{aligned} \dot{\mathbf{q}}_i^b &= \tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_{ib/b})\mathbf{q}_i^b, \\ \dot{\boldsymbol{\omega}}_{ib/b} &= \mathbf{J}_s^{-1}(-\boldsymbol{\omega}_{ib/b}\times\mathbf{J}_s\boldsymbol{\omega}_{ib/b} - \boldsymbol{\omega}_{ib/b}\times\mathbf{h}_{w/b} + \boldsymbol{\tau}_{d/b} - \boldsymbol{\tau}_{w/b}), \\ \dot{\mathbf{h}}_{w/b} &= \boldsymbol{\tau}_{w/b} \end{aligned} \tag{36}$$

Remark .9 *Note that it is combinient to align the RWs axis of rotation with the b frame, which usually also coincides with the principal axis of inertia of the S/C.*

Relative Spacecraft Dynamics

- **Clohessy-Wiltshire Equations (CW)**

- **Hill-Clohessy-Wiltshire Equations (HCW)**

- **Relative state through OED**

The relative states in the *Hill* frame (i.e. RTN), $\delta\mathbf{r}_{/RTN}$, as a function of f and OED, $\delta\boldsymbol{\kappa}$.

$$[\delta x\ \delta y\ \delta z]^\top_{/RTN}(f) = \Phi^{\delta r}_{\delta\kappa}\,[\delta a\ \delta e\ \delta i\ \delta\omega\ \delta\Omega\ \delta M]^\top \tag{37}$$

with radius r is at position f, we have

$$\Phi^{\delta r}_{\delta\kappa} = \begin{bmatrix} \tfrac{r}{a} & 0 & 0 \\ -a\cos f & \tfrac{r\sin f}{\eta^2}(2+e\cos f) & 0 \\ 0 & 0 & r\sin\theta \\ 0 & r & 0 \\ 0 & r\cos i & -r\cos\theta\sin i \\ \tfrac{ae\sin f}{\eta} & \tfrac{r\sin f}{\eta^3}(1+e\cos f)^2\delta M & 0 \end{bmatrix}. \tag{38}$$

The change in f can also be expressed in terms of $\delta\boldsymbol{\kappa}$.

$$\delta f = \tfrac{(1+e\cos f)^2}{\eta^3}\delta M + \tfrac{\sin f}{\eta^2}(2+e\cos f)\delta e, \tag{39}$$

where $\eta = \sqrt{1-e^2}$ in both (38) and (39).

Remark .10 *Eq. (37) is a linearization, thus loses accuracy if $\delta r \sim r$. If $\delta a \neq 0$, then the orbital periods between $\mathcal{O}(\boldsymbol{\kappa})$ and $\mathcal{O}(\boldsymbol{\kappa}+\delta\boldsymbol{\kappa})$ change, yielding secular growth in $\delta M, \delta E$ and δf.*

References

[1] J. R. Wertz. Spacecraft attitude determination and control. 1994.

[2] Schaub H., and Junkins, J. L., Analytical Mechanics of Space Systems, AIAA, 2009.

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