Tautological projections on the moduli space of abelian varieties

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Introduction

Fix a number $g \geq 1$. We are interested in the intersection theory of $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties (X, θ).

The vector bundle associating to an abelian variety X the space $\mathrm{H}^0(X, \Omega_X)$ is the *Hodge bundle*, denoted by $\mathbb{E}_g$. It can be extended to any toroidal compactification of $\mathcal{A}_g$, and it is the source of the first collection of interesting classes, the *lambda classes*:

$$\lambda_i = c_i(\mathbb{E}_g) \in \mathsf{CH}^i(\mathcal{A}_g) \text{ or } \mathsf{CH}^i(\overline{\mathcal{A}}_g)$$

The *tautological ring* is the subring $\mathsf{R}^*(\mathcal{A}_g)$ generated by the lambda classes. In [7] it was shown that

$$\mathsf{R}^*(\mathcal{A}_g) = \frac{\mathbb{Q}[\lambda_1, \dots, \lambda_g]}{\langle c(\mathbb{E}_g \oplus \mathbb{E}_g^\vee) - 1, \lambda_g \rangle}$$

The tautological projection

Definition (Canning-Molcho-Oprea-Pandharipande, [1]):
Let α be a cycle class on $\mathcal{A}_g$; it's tautological projection is the unique class $\mathsf{taut}(\alpha) \in \mathsf{R}^*(\mathcal{A}_g)$ such that

$$\int_{\overline{\mathcal{A}}_g} p(\lambda_i)\lambda_g \cap \overline{\alpha} = \int_{\overline{\mathcal{A}}_g} p(\lambda_i)\lambda_g \cap \mathsf{taut}(\alpha)$$

for any polynomial p in the lambda classes, for any toroidal compactification $\overline{\mathcal{A}}_g$.

Two key results ensure that taut is well-defined:

- $\mathsf{R}^*(\mathcal{A}_g)$ is a Gorenstein ring, by the result of [7].
- $\lambda_g = 0$ when restricted to $\overline{\mathcal{A}}_g \smallsetminus \mathcal{A}_g$, as shown in [1].

We are interested in calculating the tautological projection of a cycle class α coming from an interesting locus inside $\mathcal{A}_g$, and studying its *shadow*

$$\alpha - \mathsf{taut}(\alpha).$$

Noether-Lefschetz cycles

The **Noether-Lefschetz locus** inside $\mathcal{A}_g$ parametrizes abelian varieties with Picard rank at least 2. It is a countable union of closed subvarieties of $\mathcal{A}_g$ parametrizing either:

- Abelian varieties having an abelian subvariety.
- Abelian varieties with real multiplication.

These give rise to a very rich collection of cycles. The simplest ones are the product cycles:

$$[\mathcal{A}_u \times \mathcal{A}_{g-u}] \in \mathsf{CH}^{u(g-u)}(\mathcal{A}_g).$$

We know how to calculate $\mathsf{taut}([\mathcal{A}_u \times \mathcal{A}_{g-u}])$, see [1]. For example,

$$\mathsf{taut}([\mathcal{A}_1 \times \mathcal{A}_{g-1}]) = \frac{1}{24}\frac{4g}{|B_{2g}|}\lambda_{g-1},$$

where B_{2i} is the Bernouilli number. Moreover, in [2] it is shown that the shadow of $[\mathcal{A}_1 \times \mathcal{A}_5]$ is not zero (!).

The rest of the irreducible components of NL parametrizing non-simple abelian varieties are of the form

$$\mathsf{NL}_{g,\delta} = \left\{ \begin{array}{c} (X,\theta) \in \mathcal{A}_g \text{ having an abelian subvariety } Y \\ \text{of dimension } u \text{ and such that } \theta_{|Y} \text{ is of type } \delta \end{array} \right\},$$

where $u \leq g/2$ and $\delta = (d_1, \dots, d_u)$ is a list of numbers such that $d_i \mid d_{i+1}$ for all i. When $u = 1$ we denote them by $\mathsf{NL}_{g,d}$. These are the cycles that we will focus on.

The tautological projection of $[\mathsf{NL}_{g,\delta}]$

Theorem (I.L., [4]):
For a fixed u, the tautological projections of all the $[\mathsf{NL}_{g,\delta}]$ are all proportional. In the $u = 1$ case, we can compute the proportionality factor:

$$\mathsf{taut}([\mathsf{NL}_{g,d}]) = \frac{1}{24} \cdot \frac{4d^{2g-1}}{g|B_{2g}|}\prod_{p|d}(1 - p^{2-2g})\lambda_{g-1}. \tag{1}$$

and in general, the proportionality factor can be reduced to a linear algebra problem over finite rings.

Idea of the proof: The spaces $\mathsf{NL}_{g,\delta}$ are parametrized by a finite morphism

$$\beta_{g,\delta} : \mathcal{A}_{u,\delta}^{\mathsf{lev}} \times \mathcal{A}_{g-u,\delta}^{\mathsf{lev}} \to \mathcal{A}_g,$$

where $\mathcal{A}_{u,\delta}^{\mathsf{lev}}$ is the moduli space of polarized abelian varieties of type δ with a level structure given by a symplectic basis of the kernel of the polarization. By showing that the map $\beta_{g,\delta}$ extends to the boundary while still preserving the splitting of the pullback of the Hodge bundle, we prove the theorem.

The homomorphism property

We say that two cycles α, β in $\mathsf{CH}^*(\mathcal{A}_g)$ have the *homomorphism property* if

$$\mathsf{taut}(\alpha) \cdot \mathsf{taut}(\beta) = \mathsf{taut}(\alpha \cdot \beta).$$

In principle, there is no reason to believe that this holds, but it does in a lot of instances. In [4], we show that any pair of NL-cycles parametrizing non-simple abelian varieties have the homomorphism property.

A modular form

Following an idea of Greer, consider the slightly different locus

$$\widetilde{\mathsf{NL}}_{g,d} = \left\{ \begin{array}{c} (X,\theta) \text{ such that there is an homomorphism } f; E \to X \\ \text{from an elliptic curve } E \text{ such that } \deg(f^*\theta) = d \end{array} \right\}.$$

It can also be seen as $\pi(\gamma^{-1}(C_d))$, where $\pi : \mathcal{A}_1 \times \mathcal{A}_g \to \mathcal{A}_g$ is the projection and

$$\gamma : \mathcal{A}_1 \times \mathcal{A}_g \to \mathrm{KM}(2g, g)$$

sends two Hodge structures V and W to $\mathrm{Hom}(V, W)$. $\mathrm{KM}(2g, g)$ is a Kudla-Millson space of weight 2 Hodge structures of type $(g, 2g, g)$ and C_d are the special cycles considered in the Kudla-Millson program. This suggests that they form a modular form. Indeed, since

$$[\widetilde{\mathsf{NL}}_{g,d}] = \sum_{\hat{d}|d} \sigma_1\left(\frac{d}{\hat{d}}\right)[\mathsf{NL}_{g,\hat{d}}],$$

and using the result (1) before, we see that

$$\mathsf{taut}\left(\frac{(-1)^g}{24}\lambda_{g-1} + \sum_{d\geq 1}[\widetilde{\mathsf{NL}}_{g,d}]q^d\right)$$

is an Einsenstein series. A similar construction can be done for NL-cycles with $u > 1$.

Modularity conjecture I (Greer-I.L.-Lian)

$$\frac{(-1)^g}{24}\lambda_{g-1} + \sum_{d\geq 1}[\widetilde{\mathsf{NL}}_{g,d}]q^d$$

is the Fourier expansion of a modular form of weight $2g$ with values in $\mathsf{CH}^{g-1}(\mathcal{A}_g)$.

The subspaces of $\mathrm{CH}^{g-1}(\mathcal{A}_g)$ generated by the $[\mathsf{NL}_{g,d}]$ are invariant under the action of the Hecke algebra, they are one dimensional for $g \leq 5$ and at least two dimensional for $g = 6$. In analogy with the situation for the moduli space of $K3$ surfaces, it makes sense to conjecture the following:

Modularity conjecture II (Pixton)

The subape of $\mathsf{CH}^{g-1}(\mathcal{A}_g)$ spanned by Noether-Lefschetz cycles resembles the space of modular forms of weight $2g$.

The Gorenstein kernel for $\mathcal{M}_g^{\mathrm{ct}}$

Consider the moduli space of curves of compact type, the open subspace of $\overline{\mathcal{M}}_g$ where the Torelli map

$$\mathrm{Tor} : \mathcal{M}_g^{ct} \to \mathcal{A}_g$$

is still well defined. The tautological ring $\mathsf{R}^*(\mathcal{M}_g^{ct})$ was also conjectured to be Gorenstein by Faber, but we now know that this is false except in small genera. However, there is a λ_g-evaluation map

$$\mathsf{CH}^{2g-3}(\mathcal{M}_g^{ct}) \longrightarrow \mathbb{Q}$$

that gives rise to the λ_g-pairing

$$\mathsf{CH}^k(\mathcal{M}_g^{ct}) \longrightarrow \mathsf{R}^{2g-3-k}(\mathcal{M}_g^{ct})^*. \tag{2}$$

The kernel of (2) is the Gorenstein kernel. In [2] it is shown that the pullback via Torelli of the shadow of $[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ is a non-zero, tautological element of the Gorenstein kernel. For higher d we cannot expect the shadow of $[\mathsf{NL}_{g,d}]$ to be tautological on $\mathcal{M}_g^{ct}$, but it lands on the kernel of the pairing (2).

Gromov-Witten Theory of a moving elliptic curve

The key lemma to relating the $\mathsf{NL}_{g,d}$ cycles and Gromov-Witten theory is a result of Greer-Lian that $\mathrm{Tor}^*([\widetilde{\mathsf{NL}}_{g,d}])$ is the virtual class of the moduli space of stable maps to a moving elliptic curve, which are connected to the genus 1 invariants of the Hilbert scheme of points in $\mathbb{C}^2$. These invariants are mostly unkown, except for one conjecture made in by Pandharipande and Tseng about the simplest non-trivial invariant:

$$\langle (2) \rangle_1^{\mathrm{Hilb}(\mathbb{C}^2,d)} = \frac{(t_1+t_2)^2}{24t_1t_2}\left(\mathrm{Tr}_d + \sum_{k=2}^{d-1}\frac{\sigma_1(d-k)}{d-k}\mathrm{Tr}_k\right) \tag{3}$$

where Tr_k is the q-series obtained from trace of the operator of quantum multiplication by the divisor class of the Hilbert scheme of k points in $\mathbb{C}^2$ and t_1, t_2 are the torus weights (see [6] for the precise definitions).

After a long sequence of transformations, and using that NL-cycles have the homorphism property, we arrive at the following result ([4] and [5]):

Shadows of NL-cycles, the λ_g-pairing and $\mathrm{Hilb}(\mathbb{C}^2)$

Theorem (I.L.-Pandharipande, [4, 5]):
The following are equivalent:

- $[\mathsf{NL}_{g,d}]$ and $[\mathrm{Tor}(\mathcal{M}_g^{ct})]$ have the homomorphism property.
- The pullback via Torelli of the shadow of $[\mathsf{NL}_{g,d}]$ is in the kernel of (2).
- Equation (3) holds.
- If s_0 is the zero-section to the universal elliptic curve $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$,

$$\langle \tau_1(s_0)\lambda_g\lambda_{g-2} \rangle_{g,d}^{\pi} = \frac{|B_{2g-2}|}{24(2g-2)!}\sigma_{2g-1}(d) \tag{4}$$

Finally, an evaluation of the LHS of (4) using the geometry of ellipticaly fibered $K3$ surfaces shows that all the statements above hold, see [5].

Further directions

- Tautological projections of any special cycle.
- Looking for a failure of the homomorphism property, related to the search for non-tautological classes.
- Gromov-Witten theory of a moving abelian variety.
- Extend the picture presented here to the compactifications of $\mathcal{A}_g$.

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