Universidad del País Vasco

Facultad de Ciencia y Tecnología

NSDE: Optional practises 1 and 2

Group 2

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Practise 1, optional part

2. The Lotka-Volterra model

Statement

The Lotka-Volterra equations, also known as the predator-prey equations, are a pair of first-order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change thorugh time according to the pair of equations:

$$\begin{cases} \frac{dx}{dt} = A\,x - B\,x\,y, \\ \frac{dy}{dt} = -C\,y + D\,x\,y, \end{cases}$$

where, x is the number of prey (for example, rabbits);

y is the number of some predator (for example, foxes);

$\frac{dy}{dt}$ and $\frac{dx}{dt}$ represent the growth rates of the two populations over time;

t represents time; and

A, B, C, D are positives parameters describing the interaction of the two species.

(c) Write a program in *Mathematica* that shows the evolution of the populations of foxes and rabbits, correspondent to $A = 0{,}5, B = 0{,}002, C = 0{,}1, D = 0{,}001$ and initial population of 250 rabbits and 100 foxes.

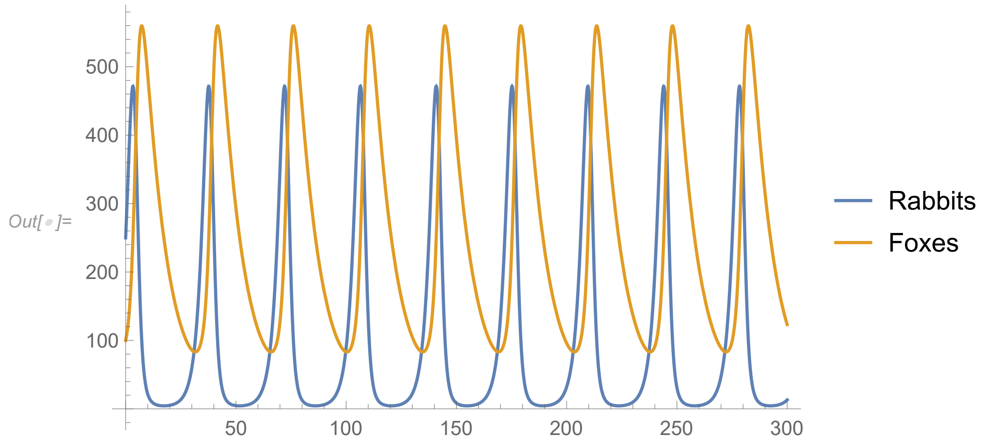
Solution

First of all, we need to write the system with the corresponding values of A, B, C and D.

$$\begin{cases} \frac{dx}{dt} = 0{,}5\,x - 0{,}002\,x\,y, \\ \frac{dy}{dt} = -0{,}1\,y + 0{,}001\,x\,y, \end{cases}$$

We also have to take into consider that, in order to solve this system, we need some initial conditions. This initial conditions are the initial populations of rabbits and foxes. We have x = rabbits and y = foxes. Making use of the information given in the statement we write x[0] = 250 and y[0] = 100.

All the information has been written in the *Mathematica* notebook and the result we get is the next one:



If we analyze the graph, we can see that the solution is periodical. Moreover, the population of rabbits is only bigger than the foxes' one at the initial part of each period, then the number of foxes increase so the rabbits population decreases quickly.

Practise 2, Optional Part

2.4 The heat equation on $\mathbb{R}$

Statement

$$\begin{cases} u_t = \nu\, u_{xx}, \quad \nu > 0, & \text{for} \quad (x,t) \in \times(0,\infty) \\ u(0,x) = \phi(x), & \text{for} \quad x \in . \end{cases} \tag{1}$$

the solution can be obtained explicitly as

$$u(t,x) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \exp[-(x-y)^2/4\nu t]\, \phi(y)\, dy. \tag{2}$$

(a) Write a program in *Mathematica* (be careful, it gives messages of error since division by zero appears and it is very slow) that

- Define the solution (2) for the particular problem with $\nu = 1$ and

$$\phi(x) = Exp(-x^2).$$

- Verify that the solution satisfies the heat equation (1).
- Plot the solution for $-6 \leq x \leq 6$ and $0 \leq t \leq 4$.
- Plot the contour plot of the solution, for example, for $-6 \leq x \leq 6$ and $0 \leq t \leq 4$.

Solution

The first thing we have done is finding the solution using the equation (2), for $\phi(x) = e^{-x^2}$ and $\nu = 1$. That way, the solution obtained using *Mathematica* is the following

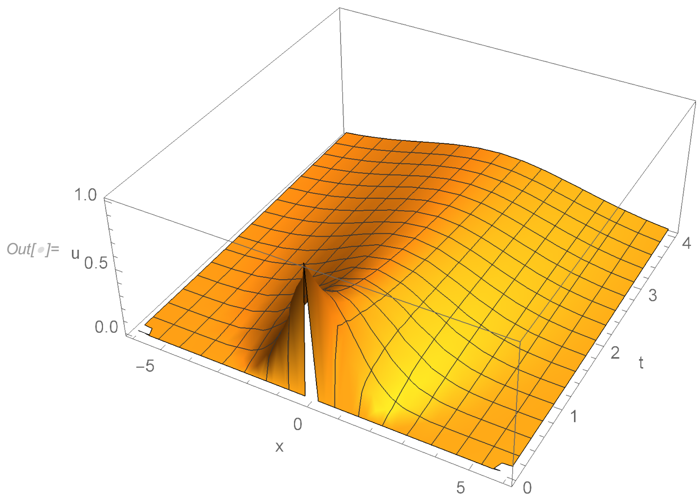
$$u(x,t) = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{4+\frac{1}{t}}\sqrt{t}}, \text{ if } \mathrm{Re}[\tfrac{1}{t}] \geq -4$$

Now, our goal is to verify that the solution we have obtained satisfies the heat equation. So we have to show that the following equations are satisfied:

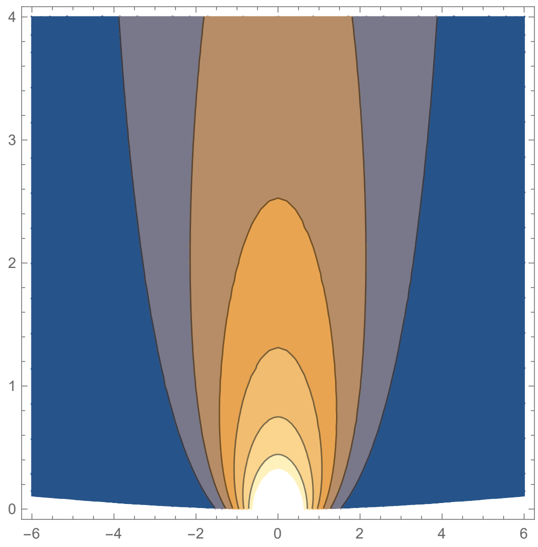
- $u_t = \nu u_{xx}$
- $u(x,0) = \phi(x)$

This is easy to prove using *Mathematica*.

The next thing we have done is to plot the solution for -6 $\leq x \leq$ 6 and 0 $\leq t \leq$ 4. This is the result:



The last thing we are asked to do is the contour plot of the solution for -6 $\leq$ x $\leq$ 6 and 0 $\leq$ t $\leq$ 4, this is the result:



2.5 The wave equation on $\mathbb{R}$

Enunciado

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0, & \text{for} \quad (x,t) \in \times(0,\infty) \\ u(0,x) = \phi(x), & \text{for} \quad x \in \\ u_t(0,x) = \psi(x), & \text{for} \quad x \in . \end{cases} \tag{3}$$

the solution can be obtained explicitly (d'Alembert's solution) as:

$$u(t,x) = \frac{1}{2}\,[\phi(x+c\,t)+\phi(x-c\,t)] + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t}\psi(s)\,ds. \tag{4}$$

(a) Write a program in *Mathematica* that

- Define the solution (4) for the particular problems with $c = 1$ and

$$\phi(x) = Exp(-x^2), \qquad \psi(x) = 0.$$

  and

$$\phi(x) = 0, \qquad \psi(x) = Exp(-x^2).$$

- Verify that the solutions satisfy the wave equation (3).
- Plot the solutions at $t = 0, 1, 2, 3$, for $-6 \le x \le 6$ (use command `Plot`).
- Plot the 3D graphic of the solutions, for example, for $-6 \le x \le 6$ and $0 \le t \le 6$ (use command `Plot3D`).
- Plot the contour plot of the solutions, for example, for $-6 \le x \le 6$ and $0 \le t \le 6$ (use command `ContourPlot`).
- What is the function that is part of the solution of problem (5)? How is it defined?

Solution

We are going to split the solution of this (3) problem in two. On the one hand we will solve the problem when $\phi(x) = Exp(-x^2)$ and $\psi(x) = 0$, and on the other hand we will solve the problem when $\phi(x) = 0$ and $\psi(x) = Exp(-x^2)$.

Case 1

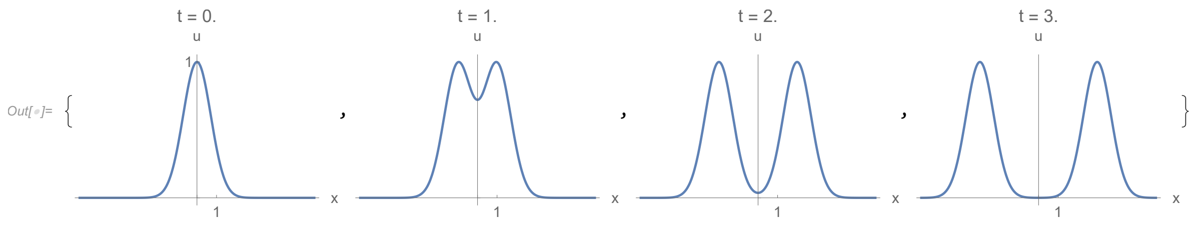
In this case, we will study the solution for $\phi(x) = Exp(-x^2)$ and $\psi(x) = 0$. First we have found the solution for the problem with c = 1, easily done using *Mathematica*. This is the solution:

Out[ ]= $\frac{1}{2}\left(e^{-(-t+x)^2} + e^{-(t+x)^2}\right)$

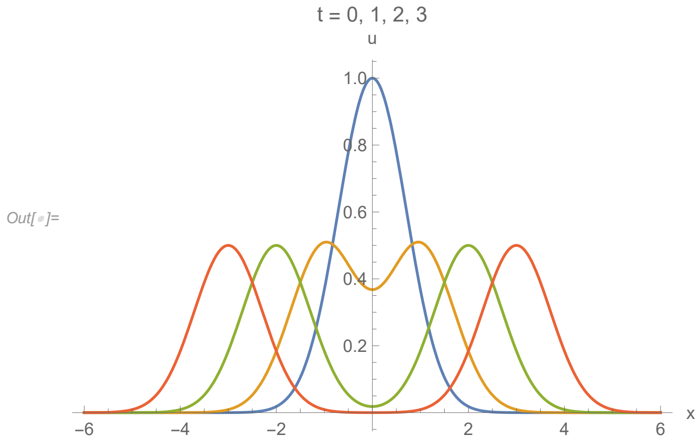
In order to verify that the solution we have found satisfies the wave equation, we have to prove that the next equations are satisfied:

- $u_{tt} - c^2 u_{xx} = 0$
- $u(x,0) = \phi(x)$
- $u_t(x,0) = \psi(x)$

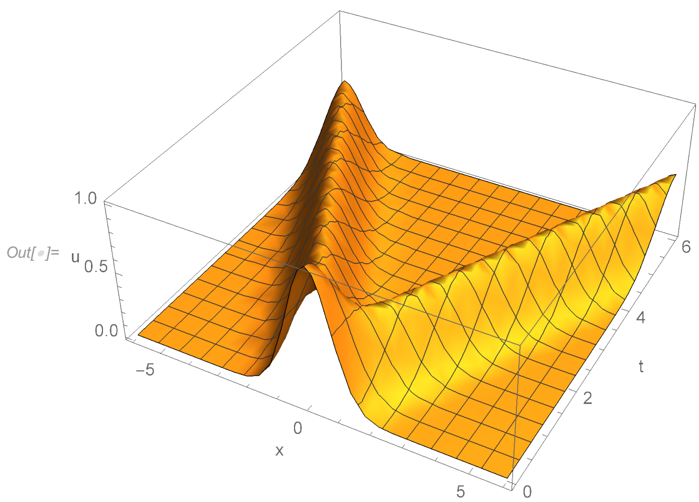
Now we have to plot the solutions at t = 0, 1, 2, 3, for -6 $\leq x \leq$ 6.



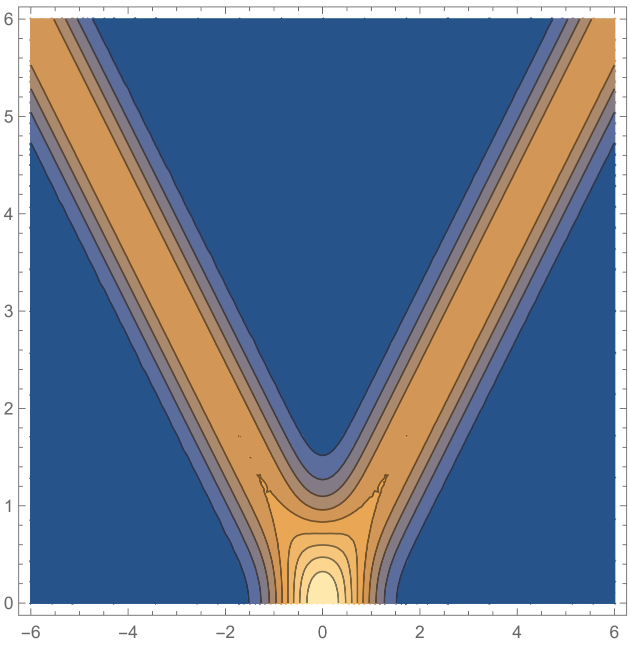
If we plot all the graphs we have just obtained, in a unique graph, we get:



The 3D plot in this case is the next one:



Out[∘]=



Case 2

Now, we have that $\phi(x) = 0$ and $\psi(x) = Exp(-x^2)$. As in the previous example, using *Mathematica* we reach to:

Out[∘]= $\frac{1}{4}\sqrt{\pi}$ (Erf[t - x] + Erf[t + x])

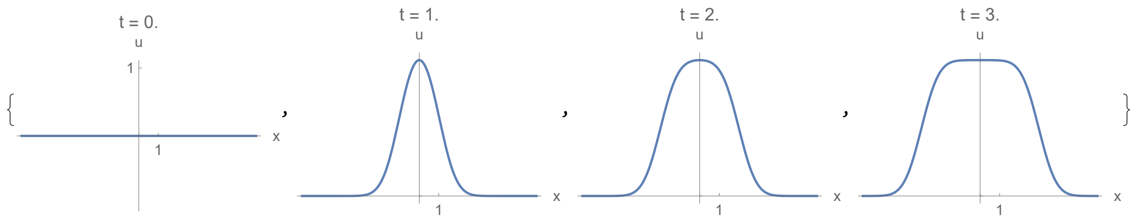
To be sure that the solution is correct, we have to check that it satisfies:

- $u_{tt} - c^2 u_{xx} = 0$
- $u(x, 0) = \phi(x)$
- $u_t(x, 0) = \psi(x)$

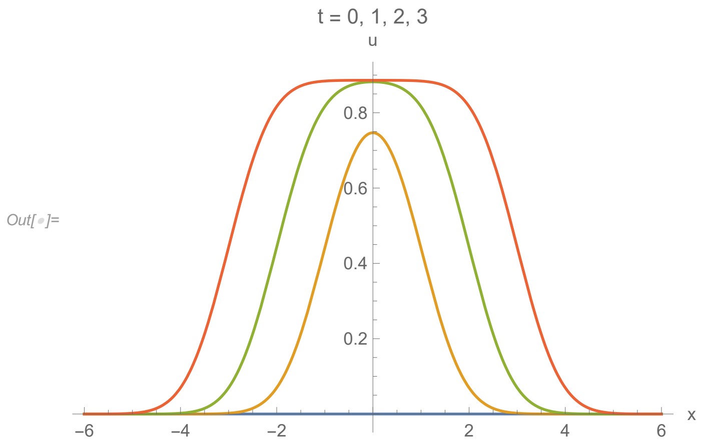
Using *Mathematica* one more time, we obtain that it is the correct solution.

Plotting the solution,

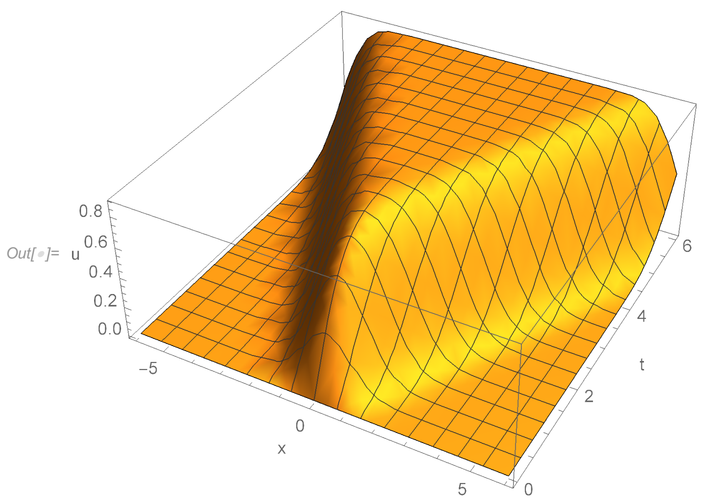
Out[∘]=



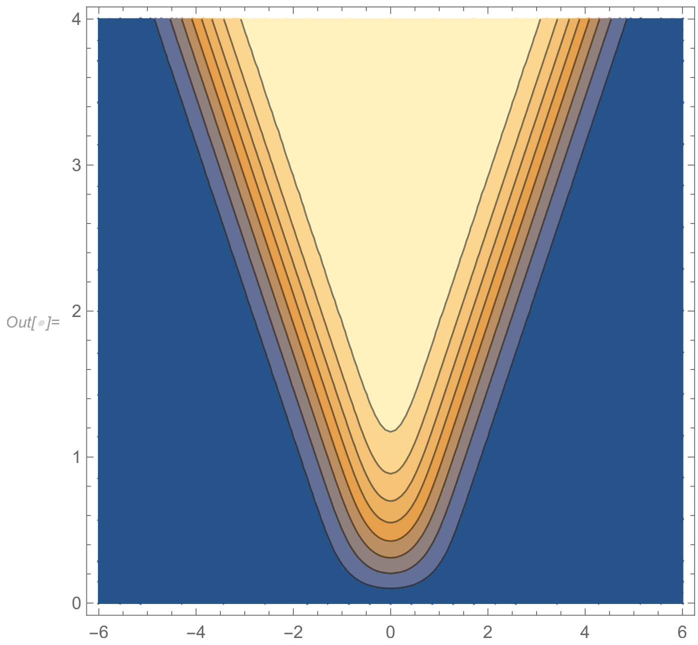
If we plot all the graphs in just one, we get the next graph:



Considering the 3D graphic for $-6 \leq x \leq 6$ and $0 \leq t \leq 6$,



And finally, we plot the contour plot of solutions,



2.6 The Laplace's equation on $\mathbb{R}^2$

Statement

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{for} \quad (x,y) \in \mathbb{R}^2 \\ u(x,0) = \phi(x), & \text{for} \quad x \in \mathbb{R} \\ u_y(x,0) = \psi(x), & \text{for} \quad x \in \mathbb{R} \end{cases} \tag{5}$$

the solution can be obtained explicitly (d'Alembert's solution) as:

$$u(x,y) = \frac{1}{2}\left[\phi(x+i\,y) + \phi(x-i\,y)\right] - \frac{i}{2}\int_{x-i\,y}^{x+i\,y} \psi(s)\,ds. \tag{6}$$

(a) Write a program in *Mathematica* that

- Define the solution (6) for the particular problem with

$$\phi(x) = \frac{1}{1+x^2}, \qquad \psi(x) = 0.$$

- Verify that the solution satisfies the Laplace's equation (5).
- Plot the 3D graphic of the solution for $-2 \leq x, y \leq 2$.
- Plot the contour plot of the solution, for example, for $-2 \leq x, y \leq 2$.

Solution

In order to find the solution, we use (6) formula. Then, the next functions are needed.

$$\phi(x+iy) = \tfrac{1}{1+(x+iy)^2} \qquad\qquad \phi(x-iy) = \tfrac{1}{1+(x-iy)^2}$$

Since $\psi(x) = 0$, we eliminate the integral that is written in the solution formula due to

$$\int_{x-i\,y}^{x+i\,y} 0 \;\; \mathrm{ds.} = 0$$

The solution we obtain making use of *Mathematica* is the next one:

$$u(x,y) = \frac{1}{2}\left(\frac{1}{1+(x-iy)^2} + \frac{1}{1+(x+iy)^2}\right)$$

In order to verify the solution we have to do three calculations, which are the next ones:

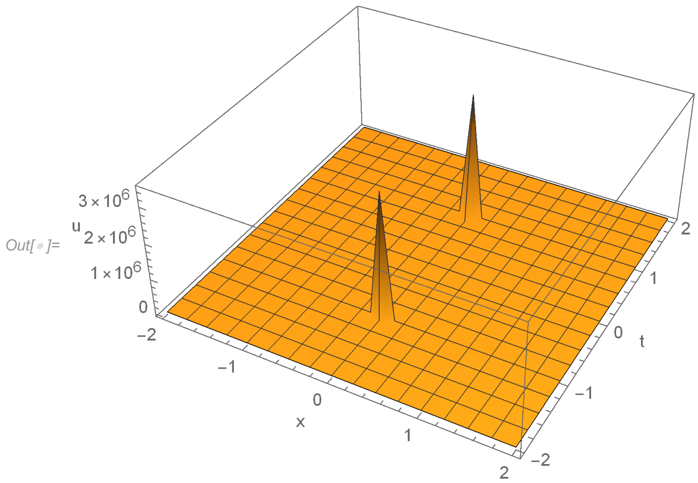
- $u_{xx} + u_{yy} = 0$
- $u(x,0) = \phi(x)$
- $u_y(x,0) = \psi(x)$

Firstly, we calculate $u_{xx} + u_{yy}$. Making use of *Mathematica* we get that the solution of the calculation we have just mentioned is 0.

Secondly, we calculate $u(x,0)$. Easily, with *Mathematica*, we check that $u(x,0) = \phi(x)$.

And, finally, we calculate $u_y(x,0)$. *Mathematica* gives us the solution $u_y(x,0) = 0$ which coincides with $\psi(x)$. This way we have verified that the solution we get is correct.

Now, we plot the solution in 3D. The result is the next one:



Finally, we have drawn the contour plot of the solution for -2 ≤ x and y ≤ 2. The contour plot is the next one:

Out[ ]=