Behavioral System Theory in Optimization-Based Control

Formal Methods for AI-Enabled Cyber-Physical Systems

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Motivation

- Traditional control methods often rely on explicit modeling - not always practical
- Limits the usability of learning-based control systems
- Behavioral System Theory offers a solution

Behavioral System Theory

Key Assumptions

1. **Persistency of Excitation:** A signal $u \in \mathbb{R}^{mT}$ is PE of order L if $H_L(u)$ has full row rank.
2. **LTI'ness:** The system $\Sigma = (T, W, \mathscr{B})$ satisfies:
   (a) **Linearity:** $\mathscr{B}$ is a vector space: $\mathscr{B}(\alpha_1 \cdot w_1 + \alpha_2 \cdot w_2) = \alpha_1 \cdot \mathscr{B}(w_1) + \alpha_2 \cdot \mathscr{B}(w_2)$
   (b) **Time Invariance:** $B \subseteq \sigma B$, where σ is the shift operator: $(\sigma w)(t) = w(t+1)$
   (c) **Completeness:** $w|_{[t_0,t_1]} \in \mathscr{B}|_{[t_0,t_1]} \implies w \in \mathscr{B}$
3. **Sufficient trajectory length T:** $T \geq (m+1)(T_{\text{ini}} + N + n(B)) - 1$, where:
   - N: Prediction horizon.
   - $n(B)$: System state dimension.
   - m: Input dimension.

Traditional control relies on state-space models described by:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t),$$

where (A, B, C, D) are system matrices. Behavioral System Theory (BST) offers a data-driven alternative using Hankel matrices:

$$H_L(u) = \begin{bmatrix} u_1 & u_2 & \cdots & u_{T-L+1} \\ u_2 & u_3 & \cdots & u_{T-L+2} \\ \vdots & \vdots & \ddots & \vdots \\ u_L & u_{L+1} & \cdots & u_T \end{bmatrix},$$

constructed from input trajectories over a window of length L. According to the Fundamental Lemma [3], if u^d is persistently exciting of order $L + n(\mathscr{B})$, then:

$$(u, y) \in \mathscr{B} \iff \exists \alpha \in \mathbb{R}^{T-L+1}$$

$$s.t. \quad \begin{bmatrix} H_L(u^d) \\ H_L(y^d) \end{bmatrix} \alpha = \begin{bmatrix} u \\ y \end{bmatrix}.$$

Data-Enabled Predictive Control (DeePC) [2]

Classical Optimal Tracking Objective

$$\min_{u,x,y} \sum_{k=0}^{N-1} (\|y_k - r_{t+k}\|_Q^2 + \dots \|u_k\|_R^2)$$

$$s.t. \quad x_0 = \hat{x}(t),$$
$$x_{k+1} = Ax_k + Bu_k,$$
$$y_k = Cx_k + Du_k,$$
$$u_k \in \mathcal{U}, y_k \in \mathcal{Y} \quad \forall k \in [0, N-1]$$

System dimensions:

- $n = 2$
- $p = 1$
- $m = 1$

Input Constraints:

- Control inputs $[N]$: $u_{min} = 0$, $u_{max} = 100$

Output Constraints:

- System outputs $[m]$: $y_{min} = -10$, $y_{max} = 10$

Initial and Target States:

- Initial State: $x_{\text{ini}} = [9 \quad 2]^\top$.
- Target State y_{target}: 0.

Filter Parameters:

- T_{ini}: 1 (Initial trajectory length).
- N: 10 (Prediction horizon).
- Q: $100 \cdot \mathbf{I}$ (Output weight matrix).
- R: $0.01 \cdot \mathbf{I}$ (Input weight matrix).

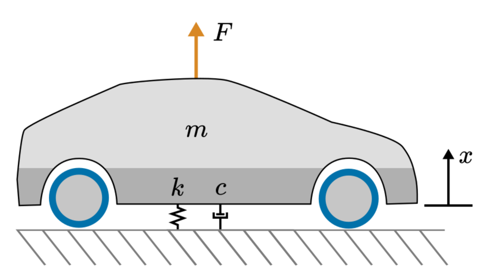
▶ **DeePC Control Objective**

$$\min_{g,u,y} \sum_{k=0}^{N-1} (\|y_k - r_{t+k}\|_Q^2 + \|u_k\|_R^2)$$

$$s.t. \quad \begin{pmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{pmatrix} g = \begin{pmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \\ y \end{pmatrix},$$

$$u_k \in \mathcal{U}, y_k \in \mathcal{Y} \quad \forall k \in [0, N-1]$$

▶ **System: Mass-Spring Damper**

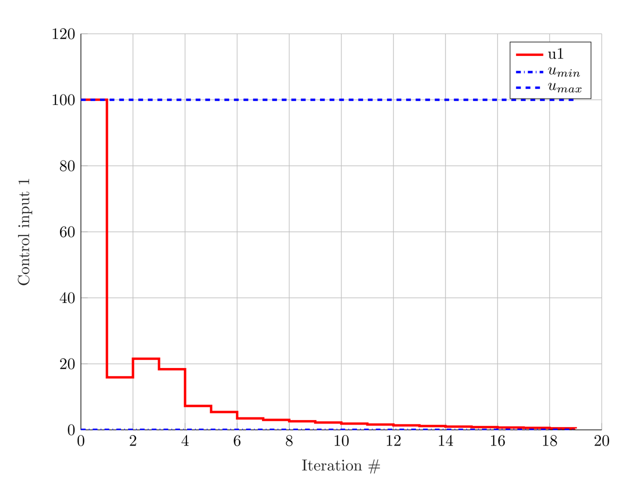


$$A = \begin{bmatrix} 1 & dt \\ -\frac{k \cdot dt}{m} & 1 - \frac{b \cdot dt}{m} \end{bmatrix},$$

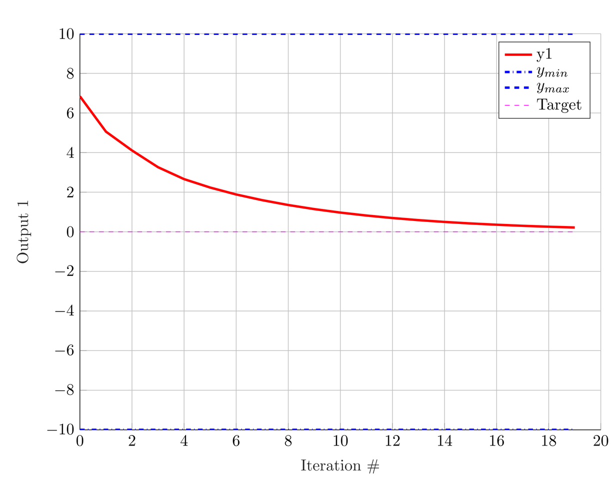
$$B = \begin{bmatrix} 0 \\ \frac{dt}{m} \end{bmatrix},$$

$$\mathbf{C} = [1 \quad 0], \quad \mathbf{D} = 0.$$

▶ **Simulation Results**



(a) Control Inputs



(b) System Outputs

▶ The tracking problem is solvable in the BST framework

Data-Driven Safety Filter (DDSF) [1]

Classical Safety Filter

$$\min_{u_{[0,N-1]}} \|u_0(t) - u_l(t)\|_R^2$$

$$\text{s.t.} \quad x_{k+1} = Ax_k + Bu_k,$$
$$x_0(t) = x_0,$$
$$x_k(t) \in \mathcal{X} \quad \forall k \in [0, N]$$
$$u_k(t) \in \mathcal{U} \quad \forall k \in [0, N-1].$$

Behavioral Safety Filter

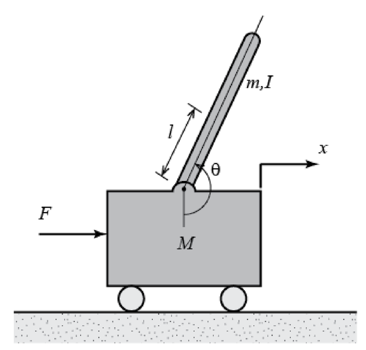
$$\min_{\alpha(t), \bar{u}(t), \bar{y}(t)} \|\bar{u}_0(t) - u_l(t)\|_R^2 \quad \text{s.t.}$$

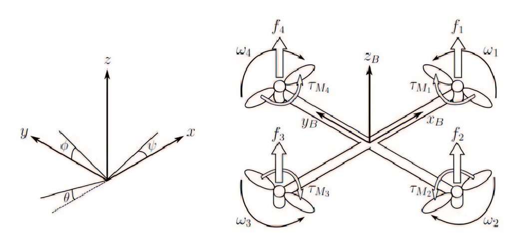
$$\begin{bmatrix} \bar{u}_{[-T_{\text{ini}}, N+T_{\text{ini}}-1]}(t) \\ \bar{y}_{[-T_{\text{ini}}, N+T_{\text{ini}}-1]}(t) \end{bmatrix} = \begin{bmatrix} H_L(u^d) \\ H_L(y^d) \end{bmatrix} \alpha(t),$$

$$\begin{bmatrix} \bar{u}_{[-T_{\text{ini}}, -1]}(t) \\ \bar{y}_{[-T_{\text{ini}}, -1]}(t) \end{bmatrix} = \begin{bmatrix} u_{[t-T_{\text{ini}}, t-1]} \\ y_{[t-T_{\text{ini}}, t-1]} \end{bmatrix},$$

$$\bar{u}_k(t) \in \mathcal{U}, \bar{y}_k(t) \in \mathcal{Y} \quad \forall k \in [0, N-1],$$
$$\bar{u}_k(t), \bar{y}_k(t) \in S_f, \quad \forall k \in [N, N+T_{\text{ini}}-1].$$

▶ **Systems: Inverted Pendulum, Quadrotor**



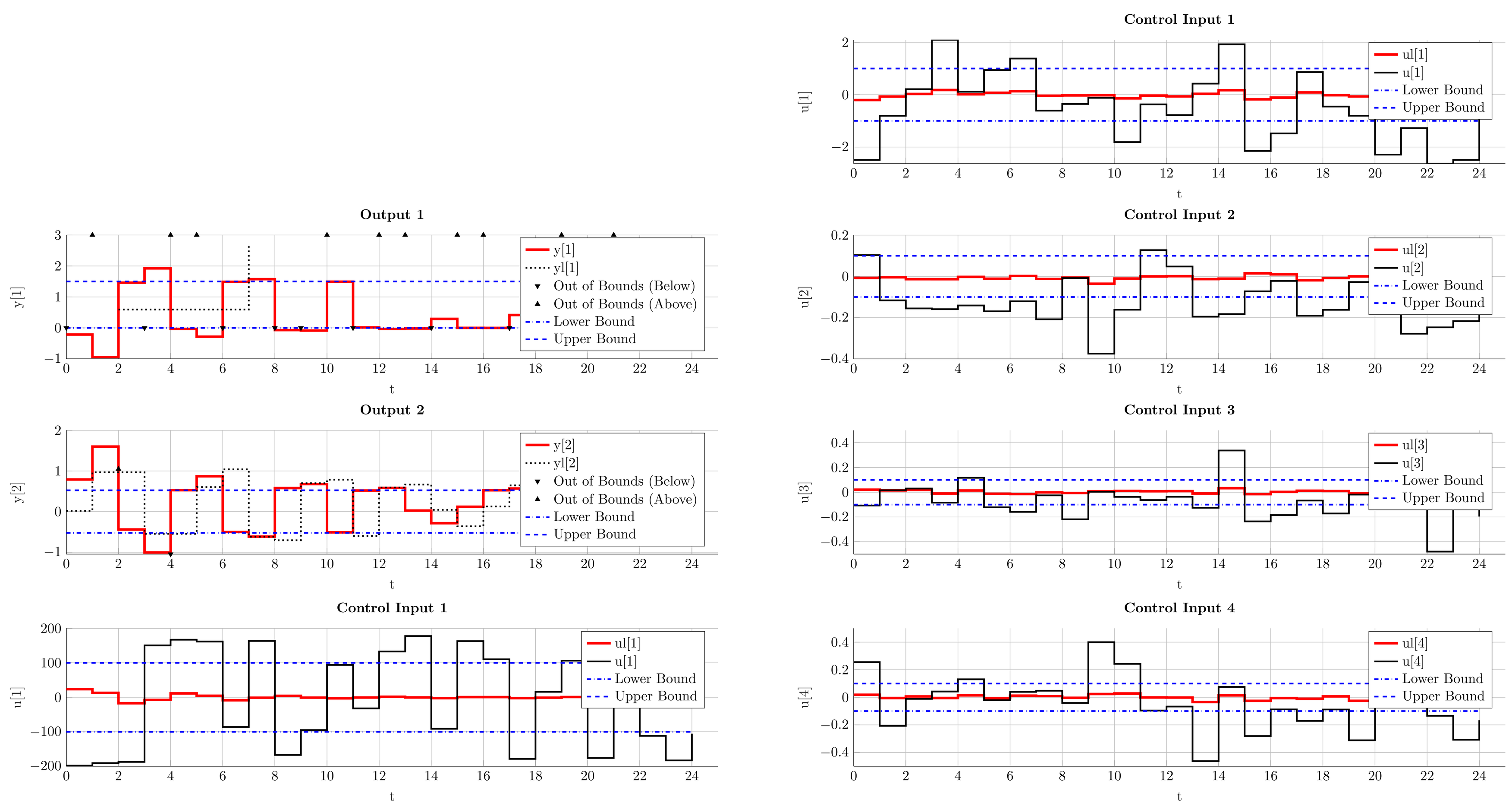


$$\vec{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}; u = F \in \mathbb{R}$$

$$y = [1 \quad 0] \cdot \vec{x} = x \in \mathbb{R}$$

$$x = [\phi,\ \theta,\ \psi,\ p,\ q,\ r,\ u,\ v,\ w,\ x,\ y,\ z]^\top,$$
$$u = [u_1,\ u_2,\ u_3,\ u_4]^\top,$$
$$y = [\phi,\ \theta,\ \psi,\ x,\ y,\ z]^\top$$

▶ **Simulation Results**



Key Observations

- Overall, the DDSF effectively enforced safety limits
- The conservatism of the filter behavior varies with system complexity.

Outlook

- **Feasibility** Translating classical algorithms into the Behavioral System Theory proved achievable
- **Limitations**
  - Neither of the algorithms was tested extensively on nonlinear systems
  - Constrained by numerical stability issues
  - Continuing dependency on the state-space representation in the computation of system lag and equilibria
- **Future Work**
  - Adress numerical challenges
  - Extend the framework to nonlinear systems
  - Facilitate truly data-driven methodologies
  - Conduct robustness tests under progressingly increasing levels of perturbation and time delay

KEY REFERENCES

[1] Mohammad Bajelani and Klaske van Heusden. *Data-Driven Safety Filter: An Input-Output Perspective*. 2023.
[2] Jeremy Coulson, John Lygeros, and Florian Dörfler. *Data-Enabled Predictive Control: In the Shallows of the DeePC*. 2019.
[3] Jan C. Willems et al. "A note on persistency of excitation". In: *Systems Control Letters* 54.4 (2005), pp. 325–329.