DPT Exercise 2

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DPT Exercises

Exercise sheet 2 — Niki Kilbertus

Exercise 1. Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be measurable spaces, $f : X \to Y$ a map, and $\mathcal{E}_Y \subseteq \mathcal{A}_Y$ arbitrary. Show that

$$\sigma(f^{-1}(\mathcal{E}_Y)) = f^{-1}(\sigma(\mathcal{E}_Y)) \,.$$

Further, show that

$$f \text{ is } \mathcal{A}_X\text{-}\sigma(\mathcal{E}_Y)\text{-measurable} \quad \Longleftrightarrow \quad f^{-1}(E') \in \mathcal{A}_X \text{ for all } E' \in \mathcal{E}_Y \,.$$

With that, finally show that for $\sigma(\mathcal{E}_Y) = \mathcal{A}_Y$ it holds that

$$f \text{ is } \mathcal{A}_X\text{-}\mathcal{A}_Y\text{-measurable} \quad \longleftrightarrow \quad f^{-1}(\mathcal{E}_Y) \subseteq \mathcal{A}_X \,.$$

This means that to show that a function is measurable, it is enough to show measurability on a generator.

Exercise 2. Let $\Omega \neq \emptyset$ be a set, $A \subseteq \Omega$, $A \neq \emptyset$, and let $\mathcal{E} \subseteq \mathcal{P}(\Omega)$. Then it holds that

$$\sigma(\mathcal{E}\big|_A) = \sigma(\mathcal{E})\big|_A \,.$$

Exercise 3. Let $(p_n)_{n\in\mathbb{N}} \subset [0, 1]$ with a fixed $\lambda := \lim_{n\to\infty} n\, p_n \in \mathbb{R}_{>0}$. Then for all $\omega \in \mathbb{N}$ we have that

$$\lim_{n\to\infty} p_{\mathrm{Bin}(n,p_n)}(\omega) = p_{\mathrm{Poi}(\lambda)}(\omega) \,.$$

Exercise 4. After a long game night, Agathe and Balthasar are standing alone at the subway station Marienplatz, where a train departs every 10 minutes. Unfortunately, the trains are so overcrowded that only one person can board each time. Every person on the platform has the same chance of boarding the train. Additionally, until the next train arrives, two new people arrive at the station. Determine the probability that Agathe will be able to board

(i) the n-th train,

(ii) at the latest by the n-th train,

where $n \in \mathbb{N}$.

Exercise 1. Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be measurable spaces, $f : X \to Y$ a map, and $\mathcal{E}_Y \subseteq \mathcal{A}_Y$ arbitrary. Show that

$$\sigma(f^{-1}(\mathcal{E}_Y)) = f^{-1}(\sigma(\mathcal{E}_Y)) .$$

(*) $\sigma(f^{-1}(\mathcal{E}_Y))$ is by definition the $\underline{\underline{\text{smallest}}}$ σ-algebra that includes $f^{-1}(\mathcal{E}_Y)$

Mutual inclusion: $(A \subseteq B) \cap (B \subseteq A) \Rightarrow A = B$

1. „$\subseteq$": $\underbrace{\sigma(f^{-1}(\mathcal{E}_Y))}_{(*)} \subseteq f^{-1}(\sigma(\mathcal{E}_Y))$

σ-alg. by def

$\rightarrow \mathcal{E}_Y \subseteq \sigma(\mathcal{E}_Y) \Rightarrow f^{-1}(\mathcal{E}_Y) \subseteq \underbrace{f^{-1}(\sigma(\mathcal{E}_Y))}$

$\uparrow$ properties of σ-algebras

$\uparrow$ „preimage preserves subsets" (Ex. 01, E2. (i))

σ-algebra via Ex-01 E6

$\Rightarrow$ $\underline{a}$ σ-algebra that includes $f^{-1}(\mathcal{E}_Y)$

$\Rightarrow \sigma(f^{-1}(\mathcal{E}_Y)) \subseteq f^{-1}(\sigma(\mathcal{E}_Y))$

2. „$\supseteq$": $f^{-1}(\sigma(\mathcal{E}_Y)) \subseteq \sigma(f^{-1}(\mathcal{E}_Y))$

Ansatz: Construct a set $\tilde{A} = \sigma(\mathcal{E}_Y)$ that „conveniently" relates to $\sigma(f^{-1}(\mathcal{E}_Y))$

» push-fw. of A under f:

$$f_*(\mathcal{A}) := \{B \subseteq \Omega_2 \mid f^{-1}(B) \in \mathcal{A}\}$$

where $f: \Omega_1 \to \Omega_2$ and A, B σ-algebras.

$$\tilde{A} = \{E \in \sigma(\mathcal{E}_Y) \mid f^{-1}(E) \in \sigma(f^{-1}(\mathcal{E}_Y))\}$$

$$= \{E \in \mathcal{A}_Y \mid f^{-1}(E) \in \sigma(f^{-1}(\mathcal{E}_Y))\} \cap \sigma(\mathcal{E}_Y)$$

$$= f_*(\sigma(f^{-1}(\mathcal{E}_Y))) \cap \sigma(\mathcal{E}_Y)$$

1. $\tilde{A}$ is a σ-algebra via (i) intersection of σ-algebras
(ii) push-fw. of —〃—〃—

2. $\tilde{A} \subseteq \sigma(\mathcal{E}_Y)$; $\tilde{A} \subseteq f_*(\sigma(f^{-1}(\mathcal{E}_Y)))$

2. „2": $f^{-1}(\sigma(\mathcal{E}_Y)) \subseteq \sigma(f^{-1}(\mathcal{E}_Y))$

Ansatz: Construct a set $\tilde{A} \subseteq \sigma(\mathcal{E}_Y)$ that „conveniently" relates to $\sigma(f^{-1}(\mathcal{E}_Y))$

» push-fw. of $\mathcal{A}$ under f:

$$f_*(\mathcal{A}) := \{B \subseteq \Omega_2 \mid f^{-1}(B) \in \mathcal{A}\}$$

where $f: \Omega_1 \to \Omega_2$ and $\mathcal{A}, \mathcal{B}$ σ-algebras.

$$\tilde{A} = \{E \in \sigma(\mathcal{E}_Y) \mid \underline{f^{-1}(E)} \in \sigma(\overbrace{f^{-1}(\mathcal{E}_Y)}^{\mathcal{E}_X})\}$$

$$= \{E \in \mathcal{A}_Y \mid f^{-1}(E) \in \sigma(f^{-1}(\mathcal{E}_Y))\} \cap \sigma(\mathcal{E}_Y)$$

$$= f_*(\sigma(f^{-1}(\mathcal{E}_Y))) \cap \sigma(\mathcal{E}_Y)$$

1. $\tilde{A}$ is a σ-algebra via (i) intersection of σ-algebras
   (ii) push-fw. of — " — " —

2. $\tilde{A} \subseteq \sigma(\mathcal{E}_Y)$; $\boxed{\tilde{A} \subseteq f_*(\sigma(f^{-1}(\mathcal{E}_Y)))}$ (*)

Claim: $\tilde{A} = \sigma(\mathcal{E}_Y)$ 1. $\tilde{A} \subseteq \sigma(\mathcal{E}_Y)$

2. $\boxed{\sigma(\mathcal{E}_Y) \subseteq \tilde{A}}$ (Bonus)

$$f^{-1}(\sigma(\mathcal{E}_Y)) = f^{-1}(\tilde{A}) \subseteq \cancel{f^{-1}}(\cancel{f_*}(\sigma(f^{-1}(\mathcal{E}_Y)))) = \sigma(f^{-1}(\mathcal{E}_Y))$$

$\tilde{A} \subseteq f_*(\sigma(f^{-1}(\mathcal{E}_Y)))$

$\Rightarrow$ $f^{-1}(\tilde{A}) \subseteq f^{-1}(f_*(\sigma(f^{-1}(\mathcal{E}_Y))))$

[Ex. 0, E2, (i)]

Measurability can be verified on an arbitrary generating set $\mathcal{E}$ instead of $\mathcal{A}_Y$.

Further, show that

$$f \text{ is } \mathcal{A}_X\text{-}\sigma(\mathcal{E}_Y)\text{-measurable} \iff f^{-1}(E') \in \mathcal{A}_X \text{ for all } E' \in \mathcal{E}_Y\,.$$

$$f \text{ is } \mathcal{A}_X - \sigma(\mathcal{E}_Y)\text{-measurable} \overset{\text{def.}}{\iff} \forall A \in \sigma(\mathcal{E}_Y) : f^{-1}(A) \in \mathcal{A}_X \underset{1.2.}{\iff} f^{-1}(E) \in \mathcal{A}_X \ \forall E \in \mathcal{E}_Y$$

1. ("$\Rightarrow$"): $\forall A \in \sigma(\mathcal{E}_Y): f^{-1}(A) \in \mathcal{A}_X \Rightarrow f^{-1}(E) \in \mathcal{A}_X \ \forall E \in \mathcal{E}_Y$

$\forall A \in \sigma(\mathcal{E}_Y): f^{-1}(A) \in \mathcal{A}_X$ | $A \leftarrow \mathcal{E}_Y$ since $\mathcal{E}_Y \in \sigma(\mathcal{E}_Y)$ by def.

$\Rightarrow f^{-1}(\mathcal{E}_Y) \in \mathcal{A}_X$

$$\mathcal{E}_Y := \bigcup_{\alpha \in \mathcal{I}} E_\alpha \text{ ; where } \mathcal{I} \text{ is an index set}$$

$\Rightarrow f^{-1}(\{E_\alpha \mid \alpha \in I\}) \in \mathcal{A}_X \iff \forall E \in \mathcal{E}_Y : f^{-1}(E) \in \mathcal{A}_X$ $\square$

2. ("$\Leftarrow$"): $\underbrace{f^{-1}(E) \in \mathcal{A}_X \ \forall E \in \mathcal{E}_Y}$ $\Rightarrow$ $\boxed{\forall A \in \sigma(\mathcal{E}_Y): f^{-1}(A) \in \mathcal{A}_X}$

$\boxed{f^{-1}(\mathcal{E}_Y) \subseteq \mathcal{A}_X}$ | $\mathcal{E}_Y \subseteq \sigma(\mathcal{E}_Y) \Rightarrow f^{-1}(\mathcal{E}_Y) \subseteq f^{-1}(\sigma(\mathcal{E}_Y))$

$\Rightarrow$ "preimage f^{-1} preserves subsets" (Ex. 0, E2, (i))"

(*) $\sigma(f^{-1}(\mathcal{E}_Y)) \subseteq \sigma(\mathcal{A}_X) = \mathcal{A}_X$

$\mathcal{A}_X$ already a σ-algebra.

$\Rightarrow f^{-1}(\mathcal{E}_Y) \subseteq f^{-1}(\sigma(\mathcal{E}_Y)) \subseteq \mathcal{A}_X$

$f^{-1} \circ \sigma(\mathcal{E}) = \sigma \circ f^{-1}(\mathcal{E})$ $\hat{=}$

$\Rightarrow \forall A \in \sigma(\mathcal{E}_Y): f^{-1}(A) \in \mathcal{A}_X$

(1.2.)

Further, show that

$$f \text{ is } \mathcal{A}_X\text{-}\sigma(\mathcal{E}_Y)\text{-measurable} \quad \Longleftrightarrow \quad f^{-1}(E') \in \mathcal{A}_X \text{ for all } E' \in \mathcal{E}_Y\,.$$

$\widehat{=}$

With that, finally show that for $\sigma(\mathcal{E}_Y) = \mathcal{A}_Y$ it holds that

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Exercise 2. Let $\Omega \neq \emptyset$ be a set, $A \subseteq \Omega$, $A \neq \emptyset$, and let $\mathcal{E} \subseteq \mathcal{P}(\Omega)$. Then it holds that

$$\sigma(\mathcal{E}\big|_A) = \sigma(\mathcal{E})\big|_A . \qquad (*)$$

$$\mathcal{E}\big|_A := \{E \cap A \mid E \in \mathcal{E}\} \quad ; \quad \sigma(\mathcal{E})\big|_A := \{B \cap A \mid B \in \sigma(\mathcal{E})\}$$

» Natural injection: $\iota_A : A \to \Omega$, $\iota_A(a) = a \;\; \forall a \in A$, $A \in \Omega$

$$\iota_A^{-1}(B) := \{a \in A \mid \iota_A(a) \in B\} = A \cap B$$

„ elements of A that also map to B"

$$\sigma(\{E \cap A \mid E \in \mathcal{E}\}) = \sigma(\{\iota_A^{-1}(E) \mid E \in \mathcal{E}\})$$
$$= \sigma(\iota_A^{-1}(\mathcal{E})) = \iota_A^{-1}(\sigma(\mathcal{E})) \overset{\text{def.}}{=} \sigma(\mathcal{E})\big|_A \qquad \square$$

$f^{-1} \circ \sigma = \sigma \circ f^{-1}$

$$\iota_A^{-1}(\sigma(\mathcal{E})) = \{a \in A \mid \iota_A(a) \in \sigma(B)\} = \sigma(\mathcal{E})\big|_A$$

Poisson distribution as the limit of the binomial distribution

Exercise 3. Let $(p_n)_{n\in\mathbb{N}} \subset [0,1]$ with a fixed $\lambda := \lim_{n\to\infty} n\,p_n \in \mathbb{R}_{>0}$. Then for all $\omega \in \mathbb{N}$ we have that

$$\lim_{n\to\infty} p_{\mathrm{Bin}(n,p_n)}(\omega) = p_{\mathrm{Poi}(\lambda)}(\omega)\,.$$

$$\lim_{n\to\infty} \binom{n}{\omega}\cdot p_n^{\omega}\cdot (1-p_n)^{n-\omega} \overset{\text{t.s.}}{=} \frac{\lambda^{\omega}}{\omega!}\cdot e^{-\lambda}$$

$$\text{(P1)} \quad \lim_{n\to\infty} \frac{\binom{n}{\omega}}{n^{\omega}/\omega!} = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} \binom{n}{\omega} = \frac{n^{\omega}}{\omega!}$$

$$\text{(P2)} \quad \lim_{n\to\infty} \Big(1 - \underbrace{\frac{x}{n}}_{\to 0}\Big)^{-\omega} = 1$$

$$\text{(P3)} \quad e^{x} = \lim_{n\to\infty}\Big(1+\frac{x}{n}\Big)^{n} \quad \text{bzw.} \quad e^{-x} = \lim_{n\to\infty}\Big(1-\frac{x}{n}\Big)^{n}$$

$$\lim_{n\to\infty} \binom{n}{\omega}\cdot p_n^{\omega}\cdot (1-p_n)^{n-\omega} = \lim_{n\to\infty} \underbrace{\frac{\binom{n}{\omega}}{n^{\omega}/\omega!}}_{\text{(P1): } =1} \cdot \frac{1}{\omega!}\cdot \underbrace{(n\cdot p_n)^{\omega}}_{\to \lambda^{\omega} \text{ (given)}} \cdot \underbrace{\Big(1 - \overbrace{\frac{n\cdot p_n}{n}}^{\lambda/n}\Big)^{n}}_{\to e^{-\lambda} \text{ via (P3)}} \cdot \underbrace{\Big(1 - \frac{n\cdot p_n}{n}\Big)^{\omega}}_{\to 1 \text{ via (P2)}}$$

$$= \frac{\lambda^{\omega}}{\omega!}\cdot e^{-\lambda} \qquad \square$$

Exercise 4. After a long game night, Agathe and Balthasar are standing alone at the subway station Marienplatz, where a train departs every 10 minutes. Unfortunately, the trains are so overcrowded that only one person can board each time. Every person on the platform has the same chance of boarding the train. Additionally, until the next train arrives, two new people arrive at the station. Determine the probability that Agathe will be able to board

(i) the n-th train, $=: A_n$

(ii) at the latest by the n-th train, A_n^c

where $n \in \mathbb{N}$.

(i)

people , # trains

$t=1$: 2 , 1

-1: train boarded / left

$+2$: people incoming

$+1$: train incoming

$t=2$: 3 , 1 | $P(E_2) = 1/3$

⋮

$t=n-1$: n, 1

$t=n$: $n+1$, 1 | $P(E_n) = 1/_{n+1}$

$E_n :=$ „ n-th train boarded irrespective of history"

$$P(A_n) = P(E_1^c \cap E_2^c \cap \dots \cap E_{n-1}^c \cap E_n)$$

$$= P(E_n) \cdot \prod_{k=1}^{n-1} P(E_k^c)$$

$$= \frac{1}{n+1} \cdot \prod_{k=1}^{n-1} \frac{n}{n+1}$$

$$= \frac{1}{n+1} \cdot \left(\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \dots \cdot \frac{\cancel{n-2}}{\cancel{n-1}} \cdot \frac{n-1}{n} \right)$$

$$= \frac{1}{n(n+1)}$$

(ii) $$P\left(\bigcup_{k=1}^{n} A_n\right) = \sum_{k=1}^{n} P(A_n) = \sum_{k=1}^{n} \frac{1}{k\cdot(k+1)}$$

$$= \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \dots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - 1/_{n+1} = n/_{n+1}$$

Exercise 1. Let $\Omega \neq \emptyset$ and let $\{A_\alpha \in \mathcal{P}(\Omega) \mid \alpha \in I\}$ be an arbitrary set of subsets of Ω, I the indexing set. Then the following two identities hold (commonly referred to as *de Morgan's law*):

$$\left(\bigcap_\alpha A_\alpha\right)^\mathsf{c} = \bigcup_\alpha A_\alpha^\mathsf{c}, \qquad \left(\bigcup_\alpha A_\alpha\right)^\mathsf{c} = \bigcap_\alpha A_\alpha^\mathsf{c}$$

Exercise 2. Let Ω_1, Ω_2 be non-empty sets, $f : \Omega_1 \to \Omega_2$ an arbitrary mapping and $\mathcal{C} \subseteq \mathcal{P}(\Omega_2)$ an arbitrary collection of subsets of Ω_2. Then the following statements hold.

(i) If $A, B \subseteq \Omega_2$ and $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.

(ii) The inverse image of the union is equal to the union of the inverse images, meaning

$$f^{-1}\Big(\bigcup_{A \in \mathcal{C}} A\Big) = \bigcup_{A \in \mathcal{C}} f^{-1}(A) \ .$$

(iii) Given a subset $A \subseteq \Omega_2$, the inverse-image of the complement is equal to the complement of the inverse image, meaning

$$f^{-1}(A^\mathsf{c}) = \left(f^{-1}(A)\right)^\mathsf{c}$$

Exercise 3. Suppose $\mathcal{A}$ is a σ-algebra over $\Omega \neq \emptyset$, let $A_0, A_1, A_2, \ldots$ be a countable collection of sets in $\mathcal{A}$ and let $m \in \mathbb{N}$. Then the sets

$$\Omega, \quad A_0 \setminus A_1, \quad \bigcup_{i=0}^m A_i, \quad \bigcap_{i=0}^m A_i, \quad \bigcap_{i \in \mathbb{N}} A_i$$

are also in $\mathcal{A}$.
[Hint: Use de Morgan's law for the intersection properties.]

Exercise 4. True or false? Justify your answers either by a proof or a counterexample.

(i) The set $\mathcal{A} = \{\emptyset, \{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3,4\}\} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra over $\Omega = \{1,2,3,4\}$.

(ii) If $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{P}(\mathcal{A})$ are σ-algebras over Ω, then so is $\mathcal{A}_1 \cup \mathcal{A}_2$.

(iii) The set $\mathcal{A} := \{A \subseteq \Omega \mid A \text{ or } A^\mathsf{c} \text{ is countable}\}$ is a σ-algebra over Ω.

(iv) The set $\mathcal{A} := \{A \subseteq \Omega \mid A \text{ or } A^\mathsf{c} \text{ is finite}\}$ is a σ-algebra over Ω if and only if Ω is finite.

[Hint: You can use the following facts without further justification: The countable union of countable sets is again countable. Subsets of countable sets are again countable.]

Exercise 5. Let $\Omega \neq \emptyset$ be a set, I be a nonempty index set (finite, countable or even uncountable) and let $\mathcal{A}_\alpha$ be a σ-algebra over Ω for each $\alpha \in I$ in the index set. Then $\mathcal{A} := \bigcap_{\alpha \in I} \mathcal{A}_\alpha$, the intersection of all the sigma-algebras is also a σ-algebra over Ω.

Remark 1. *Let Ω be a nonempty set and let $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ also be nonempty. Then by Exercise 5, we have that*

$$\mathcal{A}_\sigma(\mathcal{S}) := \bigcap \{\mathcal{A} \subseteq \mathcal{P}(\Omega) \mid \mathcal{A} \subseteq \mathcal{S},\ \mathcal{A} \text{ is a } \sigma\text{-algebra over } \Omega\}$$

is well-defined, a σ-algebra over Ω and by definition the smallest σ-algebra over Ω containing $\mathcal{S}$. We call $\mathcal{A}_\sigma(\mathcal{S})$ the σ-algebra generated by $\mathcal{S}$.

Exercise 6. Let Ω_1, Ω_2 be nonempty sets, $f : \Omega_1 \to \Omega_2$ an arbitrary mapping and let $\mathcal{A}, \mathcal{B}$ be σ-algebras over Ω_1 and Ω_2 respectively. Then the *inverse image of $\mathcal{B}$ under f* and the *image or push-forward of $\mathcal{A}$ under f* defined as

$$f^{-1}(\mathcal{B}) := \{f^{-1}(B) \mid B \in \mathcal{B}\} \quad \text{and} \quad f_*(\mathcal{A}) := \{B \subseteq \Omega_2 \mid f^{-1}(B) \in \mathcal{A}\}$$

are σ-algebras over Ω_1 and Ω_2 respectively.
[Hint: Use statements from the second exercise.]