DPT Exercise 3

github.com/aiulus/dpt-ss25

START: 12:15

Emphasis points

1) Formal modeling of random experiments; "proper" probability space construction
2) Law of total probability, Bayes' Theorem
3) Conditional probabilities

Other contents

Laplace distr., (Continuous) Uniform distr., pdf's / cdf's, random variables

Exercise 1. We consider a fair tetrahedron whose faces are labeled with the numbers $\{1, 2, 3, 4\}$. Assume the tetrahedron is rolled $k \geq 1$ times. Let E be the event that the same number appears in each roll. Furthermore, let F be the event that the product of the k rolled numbers is even.

(i) Compute the probability of E.

(ii) Compute the probability of F.

(iii) Compute the probability of $E \cup F$.

disjoint $\Rightarrow$ $P(E \cup F) = P(E) + P(F)$

not disj. $\Rightarrow$ $\ldots = \ldots - P(E \cap F)$

$|\Omega| = 4^k$

(i) $E = \{ w = (\underbrace{n \ldots n}_{\times k}) \mid n \in [4] \}$; $|E| = |[4]| = 4$

Laplace distr. $\Rightarrow \forall \omega_i, \omega_j \in \Omega: P(\{\omega_i\}) = P(\{\omega_j\}) = \frac{1}{|\Omega|}$ („fair")

$\Rightarrow P(E) = \frac{|E|}{|\Omega|} = \frac{4}{4^k} = \frac{1}{4^{k-1}}$

(ii) F: product $\prod_{i=1}^{k} w_i$ even $\Leftrightarrow$ $w = (w_1 \ldots w_k) : \exists w_i \in \{2, 4\}$

F^c: product odd $\Leftrightarrow$ $w : \forall w_i : w_i \in \{1, 3\}$ $\Rightarrow$ $|F^c| = |\{1,3\}^k| = 2^k$

$P(F) = 1 - P(F^c) \underset{\uparrow \text{ Laplace}}{=} 1 - \frac{|F^c|}{|\Omega|} = 1 - \frac{2^k}{4^k} = 1 - \left(\frac{1}{2}\right)^k$

(iii) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$E \cap F = \{ w = (n \ldots n) \mid n \in \{2, 4\} \}$ $\Rightarrow$ $|E \cap F| = |\{2, 4\}| = 2$

$\Rightarrow P(E \cup F) = \frac{4}{4^k} + 1 - \left(\frac{1}{2}\right)^k - \frac{|E \cap F|}{|\Omega|}$

$= \frac{4}{4^k} + 1 - \left(\frac{1}{2}\right)^k - \frac{2}{4^k} = 1 - \left(\frac{1}{2}\right)^k + \frac{2}{4^k}$

Definition 1.16 (measure, measure space). Let $(\Omega, \mathcal{A})$ be a measurable space. We call $\mu : \mathcal{A} \to [0, \infty]$ a *measure* if

~~(i)~~ $\mu(\emptyset) = 0$,

~~(ii)~~ it is non-negative, i.e., for all $A \in \mathcal{A}$ we have $\mu(A) \geq 0$, and

~~(iii)~~ it is *σ-additive*, i.e., for all countable collections $(A_i)_{i \in \mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$, we have

$$\mu\Big(\bigcup_{i \in \mathbb{N}} A_i\Big) = \sum_{i \in \mathbb{N}} \mu(A_i) \, .$$

The triple $(\Omega, \mathcal{A}, \mu)$ is then called a *measure space*.

Definition 1.18 (probability measure, probability space). Let $(\Omega, \mathcal{A})$ be a measurable space. A measure $P : \mathcal{A} \to [0, \infty]$ is called *probability measure* if $P(\Omega) = 1$. The triple $(\Omega, \mathcal{A}, P)$ is then called *probability space*. In this case, $\mathcal{A}$ is called the *event space* or *set of events of interest*.

Exercise 2. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $B \in \mathcal{A}$ with $P(B) > 0$ and I a countable index set. Show the following statements:

(i) The map

$$\widetilde{P} : \mathcal{A} \to [0,1]\,, \quad A \mapsto \widetilde{P}(A) := P(A \mid B)$$

is a probability measure on $(\Omega, \mathcal{A})$. The probability space $(B, \mathcal{A}|_B, \widetilde{P}|_B)$ is called *trace of* $(\Omega, \mathcal{A}, P)$ *on* B.

---

1. $\tilde{P}(\emptyset) = 0$

$$\tilde{P}(\emptyset) \overset{\text{def.}}{=} P(\emptyset \mid B) \overset{\text{def.}}{=} \frac{P(\emptyset \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} \overset{(*)}{=} \frac{0}{P(B)} \overset{(**)}{=} 0$$

$(*)$ $P(\emptyset) = 0$

$(**)$ $P(B) > 0$

2. t.s.: $\forall A \in \mathcal{A}: \tilde{P}(A) \geq 0$

$$\tilde{P}(A) \overset{\text{def.}}{=} \frac{\overbrace{P(A \cap B)}^{\geq 0}}{\underbrace{P(B)}_{>0}} \geq 0$$

(1) $P(A \cap B) \geq 0$ since: $A, B \in \mathcal{A} \Rightarrow (A \cap B) \in \mathcal{A} \Rightarrow P(A \cap B) \geq 0$

(2) $P(B) > 0$

3. t.s.: $\tilde{P}\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \tilde{P}(A_n)$

$$\tilde{P}\left(\bigcup_{n \in \mathbb{N}} A_n\right) = P\left(\bigcup_{n \in \mathbb{N}} A_n \,\Big|\, B\right) = \frac{P\left(\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap B\right)}{P(B)} = \frac{P\left(\bigcup_{n \in \mathbb{N}} (A_n \cap B)\right)}{P(B)} \quad (***)$$

$$= \frac{\sum_{n \in \mathbb{N}} P(A_n \cap B)}{P(B)} = \sum_{n \in \mathbb{N}} \frac{P(A_n \cap B)}{P(B)} = \sum_{n \in \mathbb{N}} P(A_n \mid B) = \sum_{n \in \mathbb{N}} \tilde{P}(A_n)$$

$(***)$ $P\left(\bigcup_{n \in \mathbb{N}} (A_n \cap B)\right) = \sum_{n \in \mathbb{N}} P(A_n \cap B)$ since P is a prob. measure and A_n disjoint $\Rightarrow$ $(A_n \cap B)$

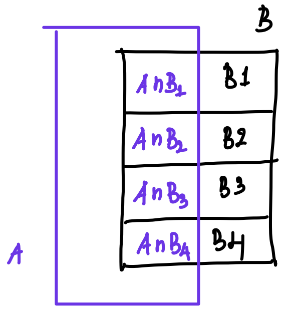
4. $\tilde{P}(\Omega) = 1$

$$\tilde{P}(\Omega) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \quad ; \; P(B) > 0$$

Exercise 2. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $B \in \mathcal{A}$ with $P(B) > 0$ and I a countable index set. Show the following statements:

(ii) Let $\{B_i \mid i \in I\} \subseteq \mathcal{A}$ be a measurable *partition* of B, i.e., $\bigcup_{i\in I} B_i = B$ and $B_i \cap B_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. For all $A \in \mathcal{A}$ we then have the *law of total probability*

$$P(A \cap B) = \sum_{i\in I} \underbrace{P(A \mid B_i)}_{= P(A\cap B_i)} P(B_i)\,.$$



(ii) $B = \bigcup_{i\in I} B_i$

$$P(A\cap B) = P\left(A\cap\left(\bigcup_{i\in I} B_i\right)\right) = P\left(\bigcup_{i\in I}(A\cap B_i)\right) = \sum_{i\in I} P(A\cap B_i) = \sum_{i\in I} \frac{P(A\cap B_i)}{P(B_i)}\cdot P(B_i)$$

(*)

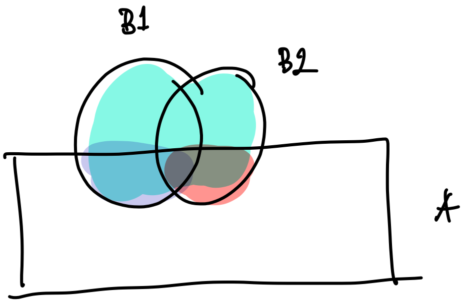
$$= \sum_{i\in I} P(A\mid B_i)\cdot P(B_i)$$

$$A\cap\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I}(A\cap B_i)$$

σ-additivity of P;

B_i disjoint $\Rightarrow (A\cap B_i)$ disjoint

(*) $P(B_i) \neq 0$



special case; $B \supseteq A$: $P(A) = P(A\cap B) = \sum_{i\in I} P(A\mid B_i)\cdot P(B_i)$

Exercise 2. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $B \in \mathcal{A}$ with $P(B) > 0$ and I a countable index set. Show the following statements:

(iii) For each $A \in \mathcal{A}$ with $P(A) > 0$ and every measurable partition $\{B_i \mid i \in I\}$ of Ω with $P(B_i) > 0$ for all $i \in I$, we have the *Bayes rule*

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_{j \in I} P(A \mid B_j)P(B_j)} \quad \text{for all } i \in I\,.$$

(iii)

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{\underset{(*)}{P(A)}} = \frac{P(A \mid B_i) \cdot P(B_i)}{\sum_{j \in I} P(A \mid B_j) \cdot P(B_j)}$$

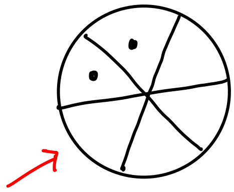
$$(*) \; P(A) = P(A \cap \Omega) = P\left(A \cap \left(\bigcup_{j \in \mathbb{N}} B_j\right)\right) = P\left(\bigcup_{j \in \mathbb{N}} (A \cap B_j)\right) = \sum_{j \in \mathbb{N}} P(A \cap B_j)$$

$$= \sum_{j \in \mathbb{N}} P(A \mid B_j) \cdot P(B_j)$$

(Law of prob., see part (ii))

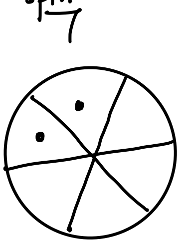
Exercise 3. For each of the following settings provide all modeling details and rigorous computations for your results.

(i) You play Russian roulette with a six shooter revolver and there are precisely two bullets in neighboring chambers. Your opponent goes first, spins the barrel, pulls the trigger and the gun does not fire. Now it is your turn. Should you spin the barrel again or pull the trigger right away? What is the probability of survival in each case?



$$P(\text{„survival"} \mid X_1 = \text{'nb'}) = P(X_2 = \text{'nb'} \mid X_1 = \text{'nb'}) \quad (*)$$

<u>spin:</u>



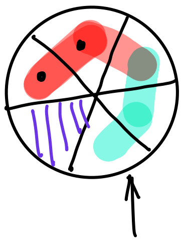
$P(\text{„survival"} \mid X_1 = \text{'nb'}) =$

$P(X_2 = \text{'nb'} \mid X_1 = \text{'nb'})$

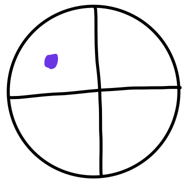
$= P(X_2 = \text{'nb'})$

$= 2/3$

<u>don't spin</u>



$\hat{=}$



$P(\text{„survival"} \mid X_1 = \text{'nb'}) =$

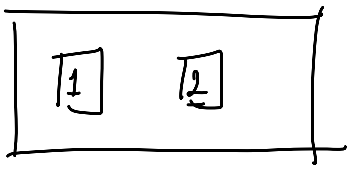
$P(X_2 = \text{'nb'} \mid X_1 = \text{'nb'})$

$= P(X_2 = \text{'nb'})$

$= 3/4$

Exercise 3. For each of the following settings provide all modeling details and rigorous computations for your results.

(ii) I have two cards in a closed opaque box, one is red on both sides and the other is red on one and blue on the other side. I pull out one card and place it on the table and a red side faces up. What's the probability that the other side is also red?



1: both sides red

2: red / blue

$\Omega = \{(1,r),(1,r),(2,r),(2,b)\}$

$X \sim \mathrm{Lap}_\Omega$; A: „upper side red" $= \{(1,r),(1,r),(2,r)\}$

B: „lower side red" $= \{(1,r),(1,r),(2,b)\}$

$$\Rightarrow P_X(B|A) = \frac{P_X(A \cap B)}{P_X(A)} = \frac{2/4}{3/4} = 2/3$$

$X_{card} : \Omega \mapsto \{1,2\}$; $X_{col} : \Omega \mapsto \{\text{red}, \text{blue}\}$

$$P(\text{„other side red"} \mid X_{col} = \text{„red"}) = P(X_{card} = 1 \mid X_{col} = \text{„red"})$$

$$= \frac{P(X_{col} = \text{„red"} \mid X_{card} = 1) \cdot P(X_{card} = 1)}{\sum\limits_{n = \{1,2\}} P(X_{col} = \text{„red"} \mid X_{card} = n) \cdot P(X_{card} = n)}$$

$$= \frac{P(X_{col} = \text{„red"} \mid X_{card} = 1) \cdot P(X_{card} = 1)}{P(X_{col} = \text{„red"} \mid X_{card} = n) \cdot P(X_{card} = n) + P(X_{col} = \text{„red"} \mid X_{card} = n) \cdot P(X_{card} = n)}$$

$$= \frac{1 \cdot 1/2}{\frac{1}{2} \cdot \left(1 + 1/2\right)} = \frac{1/2}{\frac{1}{2} \cdot 3/2} = 2/3$$

Exercise 4. We uniformly choose a real number from $[0, 1]$, square it, and represent the result by a real-valued random variable X.

(i) What is the cumulative distribution function of X?

(ii) What is the probability density function of X?

(iii) What is the probability that $X > \frac{1}{4}$. Try to find an answer in your head intuitively first.

$U \sim \text{Unif}([0,1])$

$$\begin{cases} f_U(w) = \begin{cases} 1 & \text{if } w \in [0,1] \\ 0 & \text{othw.} \end{cases} = \mathbb{1}\{w \in [0,1]\} \\ F_U(w) = w \cdot \mathbb{1}\{w \in [0,1]\} \end{cases}$$

$X = f(U) = U^2$

$U \in [0,1]$

(i) $F_X(w) = P(X \le w) = P(U^2 \le w) = P(|U| \le \sqrt{w}) = P(U \le \sqrt{w}) = \dots$

$$\dots = F_U(\sqrt{w}) = \begin{cases} 0, & \text{if } w < 0 \\ \sqrt{w}, & \text{if } w \in [0,1] \\ 1, & \text{if } w \geq 1 \end{cases}$$

$\frac{\partial}{\partial w} 1.)$

(ii) $f_X(w) = \frac{\partial}{\partial w} F_X(w) = \frac{\partial}{\partial w} F_U(\sqrt{w}) = f_U(\sqrt{w}) \cdot \frac{\partial}{\partial w} \sqrt{w} \cdot \mathbb{1}\{w \in [0,1]\}$

$$= 1 \cdot \frac{1}{2\sqrt{w}} \cdot \mathbb{1}\{w \in [0,1]\} = \begin{cases} \frac{1}{2\sqrt{w}} & \text{if } w \in [0,1] \\ 0 & \text{othw.} \end{cases}$$

(iii) $P(X > 1/4) = 1 - P(X \le 1/4) = 1 - F_X(1/4) = 1 - \sqrt{1/4} = 1 - 1/2 = 1/2$

Definition 1.10 (measurable space, measurable sets). A *measurable space* is a tuple $(\Omega, \mathcal{A})$, where Ω is a set and $\mathcal{A}$ is a *σ-algebra* on Ω. The elements of $\mathcal{A}$ are then called *measurable sets*.

Definition 1.16 (measure, measure space). Let $(\Omega, \mathcal{A})$ be a measurable space. We call $\mu : \mathcal{A} \to [0, \infty]$ a *measure* if

(i) $\mu(\emptyset) = 0$,

(ii) it is non-negative, i.e., for all $A \in \mathcal{A}$ we have $\mu(A) \geq 0$, and

(iii) it is *σ-additive*, i.e., for all countable collections $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$, we have
$$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mu(A_i)\,.$$

The triple $(\Omega, \mathcal{A}, \mu)$ is then called a *measure space*.

Proposition 1.17 (basic properties of measures). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $A, B \in \mathcal{A}$, and $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}$. Then*

- $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$,
- $\mu(B \setminus A) = \mu(B) - \mu(A)$, *if* $A \subseteq B$ *and* $\mu(A) < \infty$,
- μ *is* monotone, *meaning that whenever* $A \subseteq B$ *we have* $\mu(A) \leq \mu(B)$,
- μ *is* continuous from below, *i.e.,* $\mu(A_k) \uparrow \mu\left(\bigcup_{i\in\mathbb{N}} A_i\right)$ *for* $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$,
- μ *is* continuous from above, *i.e.,* $\mu(A_k) \downarrow \mu\left(\bigcap_{i\in\mathbb{N}} A_i\right)$ *for* $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ *with* $\mu(A_0) < \infty$,
- μ *is* σ-subadditive, *meaning that*
$$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \leq \sum_{i\in\mathbb{N}} \mu(A_i)\,.$$

Definition 1.18 (probability measure, probability space). Let $(\Omega, \mathcal{A})$ be a measurable space. A measure $P : \mathcal{A} \to [0, \infty]$ is called *probability measure* if $P(\Omega) = 1$. The triple $(\Omega, \mathcal{A}, P)$ is then called *probability space*. In this case, $\mathcal{A}$ is called the *event space* or *set of events of interest*.