Exercise 1. True or false? Justify your answers either by a proof or a counterexample.

(i) For a probability space $(\Omega, \mathcal{A}, P)$ we have $P(\Omega \setminus A) = 1 - P(A)$ for all $A \in \mathcal{A}$.

$$\begin{cases} \Omega = A \cup A^c \\ A^c := \Omega \setminus A \\ P(\Omega) = 1 \quad (**) \\ A \cap A^c = \emptyset \quad (*) \end{cases}$$

$$\underbrace{P(A \cup A^c)}_{= \Omega} \overset{(*)}{=} P(A) + P(A^c) \underset{(**)}{\overset{!}{=}} 1$$

$$\Leftrightarrow P(A^c) = 1 - P(A)$$

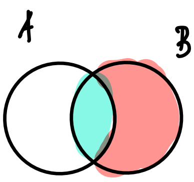
$$P : \mathcal{A} \mapsto [0,1]$$

- $\mu(B \setminus A) = \mu(B) - \mu(A)$, *if* $A \subseteq B$ *and* $\mu(A) < \infty$,

(ii) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $A, B \in \mathcal{A}$ such that $\mu(A \cap B) < \infty$. Then $\mu(B \setminus A) = \mu(B) - \mu(A \cap B)$.

$\rightarrow$ $\mu(B \setminus A) = \mu(B) - \mu(A)$, *if* $A \subseteq B$ *and* $\mu(A) < \infty$, $(*)$

$$\mu(B \setminus A) = \mu(B \setminus (B \cap A)) \overset{(*)}{=} \mu(B) - \mu(B \cap A) \quad \square$$



$(B \cap A) \subseteq B$

(iii) For a probability space $(\Omega, \mathcal{A}, P)$ and a sequence $(A_i)_{i\in\mathbb{N}}$ in $\mathcal{A}$, we have

$$P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sup_{i\in\mathbb{N}} P(A_i)\,.$$

Let $S \subseteq \mathbb{R}$. A real number $u \in \mathbb{R}$ is called an *upper bound* of S if $x \leq u$ for all $x \in S$. An upper bound s of S is called *supremum* or *least upper bound* if for all upper bounds u of S in $\mathbb{R}$ we have $s \leq u$. We then write $\sup(S) = s$. When $S = \{x_i \mid i \in \mathbb{N}\} \subseteq \mathbb{R}$ is at most countable, we also write $\sup_{i\in\mathbb{N}} x_i = s$. Suprema and infima can be defined for subsets of any partially ordered set.

$\Omega = \{0,1\}$

$\mathcal{A} := 2^{\{0,1\}} = \mathcal{P}(\{0,1\})$

$A_0 := \{0\}$, $A_1 := \{1\}$

$P(A_0) := 1/2$

$P(A_1) := 1/2$

LHS: $P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = P(\Omega) = 1$

$\neq$

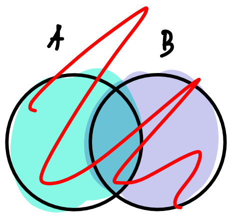
RHS: $\sup_{i\in\{0,1\}} P(A_i) = \sup_{i\in\{0,1\}} (1/2) = 1/2$

(false)

(iv) Let $(\Omega, \mathcal{A}, P)$ be a probability space and $A, B, C \in \mathcal{A}$. Then

I/E : Inclusion / Exclusion formula

$$P(A \cup \underbrace{B \cup C}_{:= X_{BC}}) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$



$P(A\cup B) = P(A) + P(B) - P(A\cap B)$ (Prop. 1.17, item 1)

$\Rightarrow P(A \cup X_{BC}) = P(A) + P(X_{BC}) - P(A \cap X_{BC})$

(v) The triple consisting of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, and the set function

$$\mu : \mathcal{P}(\mathbb{N}) \to [0, \infty],\ A \mapsto \begin{cases} 0 & A \text{ finite} \\ \infty & A \text{ infinite} \end{cases}$$

form a measure space.

Definition 1.16 (measure, measure space). Let $(\Omega, \mathcal{A})$ be a measurable space. We call $\mu : \mathcal{A} \to [0, \infty]$ a *measure* if

(i) $\mu(\emptyset) = 0$, ✓

(ii) it is non-negative, i.e., for all $A \in \mathcal{A}$ we have $\mu(A) \geq 0$, and ✓

(iii) it is *σ-additive*, i.e., for all countable collections $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$, we have

$$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mu(A_i)\,.$$

The triple $(\Omega, \mathcal{A}, \mu)$ is then called a *measure space*.

$A_i := \{i\}\ \forall i \in \mathbb{N}$

$|A_i| = 1 < \infty$

$\Rightarrow \mu(A_i) = 0$

LHS: $\bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i\in\mathbb{N}} \{i\} = \mathbb{N}$

$\Rightarrow P\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = P(\mathbb{N}) = \infty$

RHS: $\sum_{i\in\mathbb{N}} 0 = 0$ $\quad \neq \quad$ (false)

(vi) We consider the two-time fair coin flip, where we interpret the two sides to count as 0 or 1. We are only interested in the sum of the two outcomes. Then the cdf is given by

$$F(x) = \begin{cases} 0 & x < 0\,, \\ \frac{1}{4} & x \in [0, 1)\,, \\ \frac{3}{4} & x \in [1, 2)\,, \\ 1 & x \geq 2\,. \end{cases}$$

$\Omega' = \{0, 1, 2\}$

$\Omega = \{(0,0), (0,1), (1,0), (1,1)\}$ $\leftarrow$ Uniformity: $\forall \omega \in \Omega: P(\omega) = 1/|\Omega|$

$A_1 := \{(0,1), (1,0)\} \Rightarrow P(A_1) = \frac{|A_1|}{|\Omega|} = \frac{2}{4} = 1/2$; $P(A_0) = P(A_2) = 1/4$

$$F(x) = P((-\infty, x]) = \sum_{n\in\{0,1,2\}} P(A_n) = \begin{cases} 0 & , x < 0 \\ 1/4 & , x \in [0,1) \\ 3/4 & , x \in [1,2) \\ 1 & , x \in [2, \infty) \end{cases}$$

Exercise 2. Let $\Omega = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$, representing the rolling of a fair die. Assume a uniform distribution over Ω and let $p \in [0,1]$ be the probability of obtaining at least one ⚃ when throwing six fair dice at once. Compute p (using rigorous mathematical notation, i.e., setting up the probability spaces properly).

» $(\Omega, \mathcal{A}, P)$

$\Omega' := \{1, \dots, 6\} := [6]$

$\Omega = \{ w = (w_1 \dots w_6) \mid \forall i \in [6]: w_i \in \Omega' \} = [6]^6$

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$|\Omega| = |[6]^6| = |[6]|^6 = 6^6$

$\forall w \in \Omega: P(w) = \frac{1}{|\Omega|} = 1/6^6$

$A^c :=$ „no '4's"

$A^c = \{ w = (w_1 \dots w_6) \mid \forall i \in [6]: w_i \in \Omega' \setminus \{4\} \} = (\Omega' \setminus \{4\})^6$

$|A^c| = |(\Omega' \setminus \{4\})^6| = 5^6$

$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|} = 1 - (5/6)^6 \approx 0\ 665$

Exercise 3 (*Inclusion-exclusion formula*)**.** Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(A_i)_{i\in[n]} \in \mathcal{A}^{[n]}$ for a given $n \in \mathbb{N}$. Then

$$P(A_1 \cup \ldots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1\le i_1<\ldots<i_k\le n} P(A_{i_1} \cap \ldots \cap A_{i_k})$$

In particular, it holds that

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)\,.$$

$n \in \mathbb{N}_0$

$A(0) \xrightarrow{IS} A(1) \xrightarrow{IS} A(2) \to \cdots$

IH: $A(n)$

IB: $A(0)$

IS: $A(n) \longrightarrow A(n+1)$

IH

Exercise 3 (*Inclusion-exclusion formula*). Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(A_i)_{i\in[n]} \in \mathcal{A}^{[n]}$ for a given $n \in \mathbb{N}$. Then

IH: $$P(A_1 \cup \ldots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{\cancel{1\le i_1<\ldots<i_k\le n}\ \mathcal{I}_k \subseteq [n]} P(A_{i_1} \cap \ldots \cap A_{i_k})$$

$\mathcal{I}_k := \{i_1, \ldots, i_k\} \subseteq [n]$

$\mathcal{S}_n := (A_1 \cup \ldots \cup A_n)$

In particular, it holds that

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)\,.$$

IB: $(n=2)$ $$P(A_1 \cup A_2) = \sum_{k=1}^{2} (-1)^{(k-1)} \cdot \sum_{\mathcal{I}_k \subseteq [2]} P(A_{i_1} \cap \ldots \cap A_{i_k}) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$\square$ Prop. 1 17, Item 1

IS: $\underbrace{A(n)}_{\text{IH}} \rightarrow A(n+1)$

$$A(n+1): \quad P(\mathcal{S}_{n+1}) = \sum_{k=1}^{n+1} (-1)^{(k-1)} \cdot \sum_{\mathcal{I}_k \subseteq [n+1]} P(A_{i_1} \cap \ldots \cap A_{i_k})$$

$$P(\mathcal{S}_{n+1}) = P(A_{n+1} \cup \mathcal{S}_n) \overset{\text{IB}}{=} P(A_{n+1}) + P(\mathcal{S}_n) - P(A_{n+1} \cap \mathcal{S}_n)$$

IH

$$P(A_{n+1} \cap \mathcal{S}_n) = P(A_{n+1} \cap (A_1 \cup \ldots \cup A_n)) = P\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right)$$

$$A(n+1): \quad P(S_{n+1}) = \underline{\sum_{k=1}^{n+1} (-1)^{k-1} \cdot \sum_{I_k \subseteq [n+1]} P(A_{i_1} \cap \dots \cap A_{i_k})}$$

$$P(S_{n+1}) = P\left(A_{n+1} \cup S_n\right) \overset{B}{=} P(A_{n+1}) + \underbrace{P(S_n)}_{IH} - P(A_{n+1} \cap S_n)$$

$$-P(A_{n+1} \cap S_n) = -P\left(A_{n+1} \cap (A_1 \cup \dots \cup A_n)\right) = P\Big(\underbrace{\bigcup_{i=1}^{n} (A_i \cap A_{n+1})}_{\to IH}\Big)$$

$$= -\sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{I_k \subseteq [n]} P\left((A_{i_1} \cap A_{n+1}) \cap \dots \cap (A_{i_k} \cap A_{n+1})\right)$$

$$= -\sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{\substack{I_k \subseteq \cancel{[n]} \\ [n+1], \\ n+1 \in I_k}} P\left(A_{i_1} \cap \dots \cap A_{i_k} \cap A_{n+1}\right)$$

$$= \sum_{k=1}^{n} (-1)^{k} \cdot \sum_{\substack{I_k \subseteq [n+1], \\ n+1 \in I_k}} P\left(A_{i_1} \cap \dots \cap A_{i_k}\right)$$

$$P(S_{n+1}) = P(A_{n+1} \cup S_n) \overset{IB}{=} P(A_{n+1}) + \underbrace{P(S_n)}_{IH} - P(A_{n+1} \cap S_n)$$

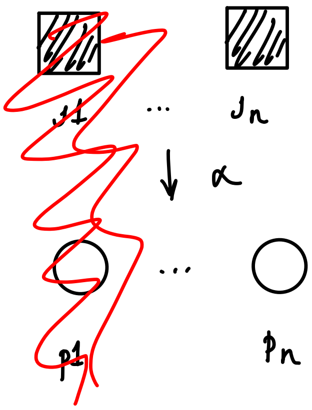
$$= \underbrace{\sum_{k=1}^{n} (-1)^{(k-1)} \cdot \sum_{I_k \subseteq [n]} P(A_{i_1} \cap \ldots \cap A_{i_k})}_{\text{all terms w/o } A_{n+1}} + \underbrace{P(A_{n+1})}_{\text{singleton terms with } A_{n+1}}$$

$$+ \underbrace{\sum_{k=2}^{n} (-1)^{k-1} \cdot \sum_{\substack{I_k \subseteq [n+1], \\ n+1 \in I_k}} P(A_{i_1} \cap \ldots \cap A_{i_k})}_{\text{terms of length} \geq 2 \text{ with } A_{n+1}}$$

$$= \sum_{k=1}^{n+1} (-1)^{(k-1)} \cdot \sum_{I_k \subseteq [n+1]} P(A_{i_1} \cap \ldots \cap A_{i_k}) \qquad \square$$

Exercise 4. The DWT tutor Angelika supervises an exercise group with n sheets being submitted anonymously. After grading, Angelika randomly redistributes the sheets. What is the probability that no one gets their original sheet back? How does this probability behave as $n \to \infty$?
[Hint: Use the inclusion-exclusion formula and that $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$]

$\Big| -1$



$\alpha(j_i) = p_i$

(ii) Without replacement, ordered draws. *The space of possible outcomes is*

$$\to \quad \Omega = \left\{(\omega_1, \ldots, \omega_k) \in [n]^k \,\middle|\, \omega_i \neq \omega_j \text{ for } i \neq j\right\} \quad \text{with} \quad |\Omega| = \frac{n!}{(n-k)!}.$$

(iii) Without replacement, unordered draws. *The space of possible outcomes is*

$$\Omega = \{S \subseteq [n] \mid |S| = k\} \quad \text{with} \quad |\Omega| = \binom{n}{k}.$$

$A_i := (\alpha(j_i) = p_i)$ „i-th assignment correct"

$|A_i| = (n-1)!$

$$P(A_i) = \frac{|A_i|}{|\Omega|} = \frac{(n-1)!}{n!}$$

$$P(A_{i_1} \cap A_{i_2}) = \frac{(n-2)!}{n!}$$

$$P(A_{i_1} \cap \ldots \cap A_{i_k}) = \frac{(n-k)!}{n!}$$

$A :=$ „no correct assignments"

$\Rightarrow \quad A^c := A_1 \cup A_2 \cup \ldots \cup A_n$

$$P(A_1 \cup \dots \cup A_n) = \sum_{i=1}^{n} (-1)^{k-1} \cdot \sum_{I_k \in [n]} \underbrace{P(A_{i_1} \cap \dots \cap A_{i_k})}$$

$$I_k := \{ i_1, \dots, i_k \} \subseteq [n]$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{I_k \in [n]} \frac{(n-k)!}{n!}$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \cdot \binom{n}{k} \cdot \frac{(n-k)!}{n!} = \sum_{k=1}^{n} (-1)^{k-1} \cdot \frac{\cancel{n!}}{\cancel{(n-k)!} \cdot k!} \cdot \frac{\cancel{(n-k)!}}{\cancel{n!}} \quad \frac{1}{k!} = P(A^c)$$

$$\Rightarrow P(A) = 1 - P(A^c) = 1 - \sum_{k=1}^{n} (-1)^{k-1} \cdot \frac{1}{k!}$$

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad ; \quad \exp(-1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$$

$$\lim_{n \to \infty} 1 + \sum_{k=1}^{n} (-1)^k \cdot \frac{1}{k!} = \sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!} = e^{-1}$$

$$1 = (-1)^k \cdot \frac{1}{k!} \Big|_{k=0}$$

Exercise 5 ([Bonus exercise]). Show that the Borel σ-algebra $\mathcal{B}^n := \sigma(\mathcal{E}_1)$, generated by the set of open sets

$$\mathcal{E}_1 := \{A \subseteq \mathbb{R}^n : A \text{ is open}\}$$

can also be generated by the set of half-open intervals

$$\mathcal{E}_2 := \{[a, b) : a, b \in \mathbb{Q}^n, a < b\}$$

meaning that $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

[Hint: Show $\mathcal{E}_i \subseteq \sigma(\mathcal{E}_j)$ and use the fact that $\sigma(\mathcal{E}_i)$ is the smallest σ-algebra containing $\mathcal{E}_i$.]

Remark 1 (Other possible generator sets)**.** *The Borel σ-algebra $\mathcal{B}^n$ can also be generated by*

$$\mathcal{E}_3 := \{A \subseteq \mathbb{R}^n \mid A \text{ is compact}\} \qquad \mathcal{E}_4 := \{B_r(x) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_{>9}\}$$
$$\mathcal{E}_5 := \{A \subseteq \mathbb{R}^N \mid A \text{ is closed}\} \qquad \mathcal{E}_6 := \{(a, b) \mid a, b \in \mathbb{Q}^n, a < b\}$$
$$\mathcal{E}_7 := \{(a, b] \mid a, b \in \mathbb{Q}^n, a < b\} \qquad \mathcal{E}_8 := \{[a, b] \mid a, b \in \mathbb{Q}^n, a < b\}$$
$$\mathcal{E}_9 := \{(-\infty, b) \mid b \in \mathbb{Q}^n\} \qquad \mathcal{E}_{10} := \{(-\infty, b] \mid b \in \mathbb{Q}^n\}$$
$$\mathcal{E}_{11} := \{(a, \infty) \mid a \in \mathbb{Q}^n\} \qquad \mathcal{E}_{12} := \{[a, \infty) \mid a \in \mathbb{Q}^n\}$$

meaning it holds $\mathcal{B}^n = \sigma(\mathcal{E}_i)$ for all $i \in [12]$.

Definition 1.10 (measurable space, measurable sets). A *measurable space* is a tuple $(\Omega, \mathcal{A})$, where Ω is a set and $\mathcal{A}$ is a *σ-algebra* on Ω. The elements of $\mathcal{A}$ are then called *measurable sets*.

Definition 1.16 (measure, measure space). Let $(\Omega, \mathcal{A})$ be a measurable space. We call $\mu : \mathcal{A} \to [0, \infty]$ a *measure* if

(i) $\mu(\emptyset) = 0$,

(ii) it is non-negative, i.e., for all $A \in \mathcal{A}$ we have $\mu(A) \geq 0$, and

(iii) it is *σ-additive*, i.e., for all countable collections $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$, we have

$$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mu(A_i)\,.$$

The triple $(\Omega, \mathcal{A}, \mu)$ is then called a *measure space*.

Proposition 1.17 (basic properties of measures). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $A, B \in \mathcal{A}$, and $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}$. Then*

- $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$,
- $\mu(B \setminus A) = \mu(B) - \mu(A)$, *if* $A \subseteq B$ *and* $\mu(A) < \infty$,
- μ *is* monotone, *meaning that whenever* $A \subseteq B$ *we have* $\mu(A) \leq \mu(B)$,
- μ *is* continuous from below, *i.e.,* $\mu(A_k) \uparrow \mu\left(\bigcup_{i\in\mathbb{N}} A_i\right)$ *for* $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$,
- μ *is* continuous from above, *i.e.,* $\mu(A_k) \downarrow \mu\left(\bigcap_{i\in\mathbb{N}} A_i\right)$ *for* $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ *with* $\mu(A_0) < \infty$,
- μ *is* σ-subadditive, *meaning that*

$$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \leq \sum_{i\in\mathbb{N}} \mu(A_i)\,.$$

$\mu(A \cup B) = \mu(A \cup (B \setminus A))$

$= \mu(A) + \mu(B) - \mu(B \cap A)$

Definition 1.18 (probability measure, probability space). Let $(\Omega, \mathcal{A})$ be a measurable space. A measure $P : \mathcal{A} \to [0, \infty]$ is called *probability measure* if $P(\Omega) = 1$. The triple $(\Omega, \mathcal{A}, P)$ is then called *probability space*. In this case, $\mathcal{A}$ is called the *event space* or *set of events of interest*.

Lemma 1.33 (classical urn models). *We consider an urn with $n \in \mathbb{N}_{>0}$ distinct objects and $k \in \mathbb{N}_{>0}$ draws.*

(i) With replacement, ordered draws. *The space of possible outcomes is*

$$\Omega = [n]^k \quad \textit{with} \quad |\Omega| = n^k\,.$$

(ii) Without replacement, ordered draws. *The space of possible outcomes is*

$$\Omega = \Big\{(\omega_1, \ldots, \omega_k) \in [n]^k \,\Big|\, \omega_i \neq \omega_j \textit{ for } i \neq j\Big\} \quad \textit{with} \quad |\Omega| = \frac{n!}{(n-k)!}\,.$$

(iii) Without replacement, unordered draws. *The space of possible outcomes is*

$$\Omega = \{S \subseteq [n] \,|\, |S| = k\} \quad \textit{with} \quad |\Omega| = \binom{n}{k}\,.$$

(iv) With replacement, unordered draws. *The space of possible outcomes is*

$$\Omega = \left\{(a_1, \ldots, a_n) \in \mathbb{N}_0^n \,\middle|\, \sum_{i=1}^n a_i = k\right\} \quad \textit{with} \quad |\Omega| = \binom{n+k-1}{k}\,,$$

where a_i counts how often object i is drawn.