DEPARTMENT OF INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Informatics

Single-Trajectory Identifiability of Controlled LTI Systems

Ayşe Aybüke Ulusarslan

DEPARTMENT OF INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Informatics

Single-Trajectory Identifiability of Controlled LTI Systems

Identifizierbarkeit von gesteuerten LZI-Systemen anhand einer einzelnen Trajektorie

Author:	Ayşe Aybüke Ulusarslan
Supervisor:	Nora Schneider, M. Sc.
Advisor:	Prof.Dr. Niki Kilbertus
Submission Date:	15. October 2025

I confirm that this master's thesis in informatics is my own work and I have documented all sources and material used.

Munich, 15. October 2025 Ayşe Aybüke Ulusarslan

Acknowledgments

To *Mom, Dad and Berkin*, for your patience and unwavering support.

To *Nora Schneider*, for giving and demanding nothing less then excellence, for the amount of work and care you put into this, for always steering me towards more precise questions and more precise answers.

Abstract

We study what can and cannot be learned about a controlled, fully observed linear time-invariant (LTI) system from a single trajectory. Given a dynamical system and an initial state, we define a *visible subspace*: the smallest subspace containing the initial state and the input directions, that is closed under the action of the state evolution matrix. We show that only the dynamics restricted to this visible subspace is recoverable, leaving the dynamics orthogonal to this visible subspace unconstrained. This yields explicit initial state-aware equivalence classes and a *partial identifiability theorem for controlled LTI systems* that extends recent autonomous single-trajectory results to the input-driven setting. We further treat input constraints via persistency of excitation: if the inputs fail to excite all directions of the visible subspace, then the amount of information that can be recovered from a single trajectory diminishes. We propose *computable identifiability criteria* that vanish exactly at structural initial-state-oriented identifiability. Empirically, we show that these criteria predict estimation error. In partially identifiable regimes, the projection of any consistent estimate onto the visible subspace converges to the true dynamics on that subspace, thereby collapsing error along the identifiable directions. Finally, we chart parameter regimes prone to unidentifiability (characterized by sparsity, underactuation, and ambient state dimension) and investigate the hypothesis that the order of input excitation tracks the dimension of the visible subspace more closely than the ambient state dimension.

Contents

Acknowledgments **iii**

Abstract **iv**

1. Introduction **1**

2. Problem Statement and Preliminaries **5**
2.1. Problem Statement 5
2.2. Key Objects 6
2.2.1. Definitions of Identifiability 6
2.2.2. The Visible Subspace 8
2.3. Preliminaries 9
2.3.1. Dynamical Systems 9
2.3.2. Reachability and Controllability 13
2.3.3. Linear Algebra 20
2.3.4. Signal Properties and Data Informativity 21
2.3.5. System Identification 24

3. Prior Work **26**
3.1. Classical View 26
3.2. Recent Works 27

4. Theoretical Analysis **32**
4.1. Identifiability under Sufficiently Rich Control Inputs 33
4.1.1. Characterization of Indistinguishable System Matrices 33
4.1.2. Partial Identifiability 36
4.1.3. Identifiability Criteria 37
4.1.4. Genericity of Identifiability as a Topological Property 42
4.2. Identifiability under Insufficiently Rich Control Signals 43
4.2.1. Pointwise Constraints 44
4.2.2. Nonlocal Constraints 45
4.2.3. Direct Richness (PE) Constraints 47
4.3. Summary and Extensions 49

5. Experiments **53**
5.1. Integrity of the Framework 54
5.1.1. Identifiability Criteria 56
5.1.2. Equivalence Classes 60
5.2. Parameter Regimes 68
5.2.1. (S1) Sparsity 69
5.2.2. (S2) State Dimension 70
5.2.3. (S3) Degree of Actuation 70
5.2.4. (D1) Sparsity vs. State Dimension 71
5.2.5. (D2) State Dimension vs. Underactuation 72
5.2.6. (D3) Underactuation vs. Sparsity 72
5.3. Input Sufficiency 73

6. Discussion and Conclusion **76**

A. Appendix **78**
A.1. Notation 78
A.2. Deferred Proofs 79
A.2.1. From: Chapter 2 79
A.2.2. From: Chapter 4 82
A.3. Additional Results and Experimental Details 92
A.3.1. Pseudo-Algorithms 92
A.3.2. Additional Results 96
A.3.3. Experimental Details 102

List of Figures **103**

List of Tables **105**

Bibliography **106**

1. Introduction

Dynamical systems are ubiquitous in science and engineering– from epidemic waves [1] and climate models [2] to flight control systems [3]. A common language for describing these is through differential equations. Data-driven approaches aim at learning these differential equations from observed trajectories of the system, such as classical system identification algorithms [4–9] to modern machine learning algorithms, like *Sparse Identification of Nonlinear Dynamics* (SINDy) [10] and Neural ODEs [11]. While all methods infer a dynamics model, they do not give any guarantees on how well the estimates reflect the true dynamics, raising a foundational question: *what parts of the true dynamics can—and cannot—be recovered from the data available?* Addressing these questions is particularly relevant, as multiple systems can fit the observed data equally well, while exhibiting divergent behaviors outside the observed data regime. This can lead not only to unreliable extrapolations but also to misleading mechanistic inferences about the underlying dynamics [12–14]. This study aims to help bridge the growing gap between the advances in ODE learners, often used in single-trajectory identification settings, and the absence of quantitative criteria that describe the intrinsic limits of what can be recovered from such observations.

Related Work. For *single-trajectory* identifiability in *autonomous* linear ODEs (without inputs), recent analyses show that what is recoverable is governed by the invariant subspace generated by the initial condition, yielding explicit equivalence classes and characterizations of non-identifiable models [15, 16]. More broadly, classical results in geometric systems theory [17, 18]—most notably the Kalman rank tests for controllability and observability—explain how having control inputs can, in principle, enlarge the portion of state space that experiments can excite and observe, thereby improving identifiability. However, these system-level tests speak to properties of (A, B) under sufficiently informative data, not to the *trajectory-level* question: given a *specific* initial state x_0 and a *single* realized input, which parts of the dynamics are determined by the data? Our work addresses this gap by extending single-trajectory identifiability from the autonomous setting to *input-driven* linear time-invariant (LTI) systems, making the roles of x_0 and input informativeness explicit and deriving x_0-aware equivalence classes and partial-identifiability guarantees.

Contributions. We ask: Given a *single trajectory* of *controlled systems* starting at a given initial state x_0, which parts of the dynamics are identifiable from that trajectory? We answer this in three parts. (i) We derive *closed-form* x_0-*equivalence classes*– the complete set of matrix pairs (A, B) that generate the same trajectory from x_0– both in an *idealized regime* without control constraints and in a *control-constrained regime* that makes informativeness explicit via persistency of excitation. (ii) We prove a *partial identifiability theorem for single-trajectory, controlled LTI systems*: only the restriction of (A, B) to the visible subspace $V(x_0)$– the minimum set of states that contains both the initial state x_0 and input directions, and is closed under the action of A– is recoverable. The components of (A, B) acting outside of $V(x_0)$ remain unidentifiable, with associated degrees of freedom governed by the difference in ambient state dimension and the dimension of the x_0-visible subspace $V(x_0)$. (iii) We propose *quantitative criteria* for the identifiability of (A, B, x_0) tuples and demonstrate empirically that these criteria predict estimation accuracy. Together, these results reveal intrinsic, input-aware limits of what can be recovered from single-trajectory data. Our findings complement prior identifiability results for autonomous linear ODEs [15, 16] by extending the analysis to input-driven systems.

Overview

Chapter 2 - Preliminaries. In this chapter, we first provide a formal description of our problem statement, as well as our assumptions (Section 2.1). Next, in Section 2.2 we introduce two concepts that are central to our analysis in Chapter 4: *(i)* Definitions of identifiability, and *(ii)* Characterization of the so-called *visible subspace*, that is, the system states that can be reached from an initial state x_0. We show that, for two dynamical systems to generate the same trajectory from a particular x_0, their "visible subspaces must agree. Finally, in Section 2.3, we present standard results from systems theory, input design, and system identification. To aid the analysis developed in Chapter 4.2., where the sufficient input richness assumption is lifted, we propose a separation of input admissibility constraints as *pointwise* and *nonlocal*, depending on whether the imposed constraint is a property of the whole input trajectory (nonlocal) or concerns only "snapshots" $u(t)$ of the control signal (pointwise).

Chapter 3 - Prior Work. This chapter surveys system identifiability through two lenses: a *classical view* and *recent works*. The *classical view* synthesizes foundational results from control and systems theory, making explicit their identifiability assumptions even when not all works featured in this section primarily focus on identifiability. In *recent works*, we give a detailed account of the two studies– *Identifiability of Linear and Linear in Parameters*

Dynamical Systems from a Single Trajectory by Stanhope et al. [15] and *Identifiability Analysis of Linear Ordinary Differential Equation Systems with a Single Trajectory* by Qiu et al. [16], both of which employ invariant subspace-based analyses to articulate what is identifiable in a single-trajectory setting, with quantitative criteria that issue directly from this geometric view. Their approach have shaped ours most, and their results are essentially equivalent to ours in the autonomous LTI setting. A detailed, formal review of their results is therefore important: Since we employ the same tools in our analysis and assume the same setting except for the addition of control, we expect our statements to reduce to theirs when we set $B = \mathbf{0}_{n\times m}$. We use this correspondence as a running consistency check throughout Chapter 4.

Chapter 4 — Theoretical analysis. This is where our core contribution lives: We pursue a double-tracked analysis, with the input sufficiency assumption serving as a branching point. In Section 4.1., we assume an idealized setting where control inputs are always persistently exciting of sufficient order. Leaning on the classical results reviewed in Chapter 3, we conclude that any nontrivial structure in a set containing systems x_0-equivalent to the original pair (A, B) must be related to its controllability structure. This insight largely informs the tools that we use in the pursuit of a closed-form characterization of this set, which we call the x_0-equivalence class.

The x_0-equivalence relationship which spans this set is further developed into a partial identifiability theorem, which extends [15]'s result on partial identifiability to the controlled LTI setting. The takeaway here is that, even when a system (A, B) isn't fully identifiable, we can still identify its dynamics *within the visible subspace* $V(\mathrm{x}_0)$, as defined earlier. The key insight that builds the bridge from controllability to identifiability in the controlled setting is that we can conceptualize the intial state x_0 as *initial excitation*, and the homogeneous response as part of the "forced dynamics". This helps us reuse many of the standing results from the system-theoretic view of controllability. In particular, by absorbing the initial state x_0 into the control matrix B, we obtain quantitative, single-trajectory identifiability criteria derived directly from classical controllability criteria. To complete the analysis, we formalize x_0-identifiability as an algebraic property on the domain of (A, B, x_0), and examine its genericity. The key takeaway is that unidentifiable system matrix-initial state tuples (A, B, x_0) form a proper algebraic variety, hence have Lebesgue measure zero under any distribution absolutely continuous w.r.t. a Lebesgue measure. This can be interpreted as the probability of a randomly selected tuple (A, B, x_0) being unidentifiable vanishes under the assumption that the probability measure governing the distribution of (A, B, x_0) is Lebesgue-continuous.

In Section 4.2., we lift the assumption of sufficiently rich input signals, and formalize the effects of *(i) pointwise* and *(ii) nonlocal* admissibility constraints on single-trajectory

identifiability. These restrictions determine the excitation actually delivered to the system and hence shrink what is recoverable from the a-priori visible subspace $V(x_0)$. Together, Sections 4.1 and 4.2 give a complete geometric picture of what is identifiable from a single trajectory: $V(x_0)$ sets the *ceiling* of identifiability, while input richness sets the *floor*. We wrap up the section with an executive summary of key results, and a few natural extensions of the framework that didn't fit into the main line of analysis.

Chapter 5 — Experiments. Our primary objective here is to empirically validate the theory developed in Chapter 4, by testing whether our single-trajectory identifiability criteria and x_0-equivalence classes can predict what can be recovered in practice. Concretely, we (i) correlate identifiability scores computed directly from (A, B, x_0) with estimation errors of a few selected learners such as *Dynamic Mode Decomposition with Control* (DMDc), and (ii) verify invariances on $V(x_0)$ predicted by the equivalence classes.

Apart from the theory validations, we use this equivalence class membership tests to assess the consistency of few identification algorithms: for system - initial state pairs that are only partially identifiable, do their estimates remain within the proposed class?

After confirming that estimation errors co-vary with identifiability scores, we use the latter to map regimes of (A, B) along the axes of sparsity, state dimension, and underactuation, that are especially prone to unidentifiability. Concretely, for each grid point, we synthesize an ensemble of matrix pairs (A, B) satisfying the property implied by that grid point. We then sample initial states uniformly from the unit sphere in $\mathbb{R}^n$, and compute a score for each system - initial state combination. This second branch is organized as follows: in §5.2.1 we vary one axis at a time (sparsity, state dimension n, underactuation) to isolate effects; in §5.2.2 we vary multiple axes jointly to probe interactions that may yield unexpected unidentifiability patterns.

Third, we test the claim that "visible-subspace dimension is the ceiling, PE order is the floor" by fixing a system with $n = 8$ and $k := \dim V(x_0) = 3$, varying the PE order between 1 and 5 using a PRBS signal, and comparing to a control system with $n = k = 8$ over the same grid. The leading hypothesis is that the estimation error "elbow" appears near $\dim(V(x_0))$.

Appendix. To improve readability, we include only proof *sketches* in the main text and refer to Appendix A.2 for complete arguments. The appendix also collects several standard proofs that inform our analysis and are reused in establishing results for non-standard objects such as the visible subspace and the x_0-enriched Kalman decomposition. Proofs original to this work are flagged with the symbol [This Work] to distinguish them from standard ones.

2. Problem Statement and Preliminaries

This section lays out the mathematical foundation for our study. First, we define our problem statement and clarify the standing assumptions– finite-dimensional, continuous-time, *linear time-invariant (LTI) systems* with known *control inputs* and *full state measurements* – tailored to the *single-trajectory identifiability* question. We then introduce formalisms which will help us state our notion of identifiability precisely, and relate it to other definitions used in prior works. Next, we review the linear-algebraic tools (with emphasis on *invariant subspaces*), the system-theoretic notions of *reachability and controllability*; and summarize *signal properties* and data informativity that co-determine what can be learned from a single experiment in practice. These preliminaries lay the groundwork for the analysis developed in Chapter 4.

2.1. Problem Statement

We study a *structural, single-trajectory* identifiability question for *controlled, continuous-time LTI systems* $\mathcal{S} \in \Sigma_{i/s}$ (see eq. (2.10)) with full state measurements and known control inputs. In contrast to practical identifiability, no noise or estimation procedure enters the definition. Unlike global identifiability, which requires injectivity on the entire parameter set regardless of the initial states or input properties, we fix x_0 and assess identifiability relative to an admissible control input class. Concretely, we work with the x_0-equivalence relation and its input-restricted variant (Eqs. (2.1), (2.3)).

Given a parameter domain $\Omega \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, an initial state $\mathrm{x}_0 \in \mathbb{R}^n$, and an admissible input class $\mathcal{U} \subseteq \mathcal{C}\left([0,\infty), \mathbb{R}^m\right)$, define:

(i) x_0-identifiability: there exists a control input u from the admissible input class $\mathcal{U}$ that makes the map $(A, B) \to \varphi(\cdot \mid A, B, \mathrm{x}_0, \mathrm{u})$ injective on Ω.

(ii) $(\mathrm{x}_0, \mathcal{U})-$identifiability: For $\mathcal{U}_c \subset \mathcal{U}$, the map $(A, B) \to \varphi(\cdot \mid A, B, \mathrm{x}_0, \mathrm{u})$ injective on Ω, for all $\mathrm{u} \in \mathcal{U}_c$.

When injectivity fails, we ask to *characterize the full equivalence class* $[A, B]_{\mathrm{x}_0}$ (or $[A, B]_{\mathrm{x}_0,\mathcal{U}_c}$) and to determine which *components of the system* remain identifiable. These objectives are the focus of Chapter 4.

Standing assumptions. All vector spaces and matrices are over $\mathbb{R}$ unless stated otherwise, we use $\mathbb{C}$ only for spectral arguments. We work with continuous-time LTI systems in state-space form and full-state measurement ($C = I$, $D = 0$). Under zero-order-hold (ZOH) sampling with period Δt and standard non-aliasing conditions, we set $(A_d, B_d) = \left(e^{At}, \int_0^{\Delta t} e^{As} B \, ds\right)$. For structural analyses, use the set of piecewise continuous functions $\mathcal{C}\left([0,\infty), \mathbb{R}^m\right)$ to denote the ambient space of control inputs. To facilitate later analyses involving operators, we switch to using $L^2_{\text{loc}}([0,\infty); \mathbb{R}^m)$.

Next, we establish the formal framework that makes our notion of identifiability precise, and distinguishes it from related definitions in the literature. We will then introduce the visible subspace, which is a key object in our analysis.

2.2. Key Objects

2.2.1. Definitions of Identifiability

Before speaking of identifiability, it is natural to first formalize indistinguishability, which requires a notion of similarity or equivalence. Fixing the class of experiments, e.g., initialization, control inputs, and time horizon, "global" observational equivalence between two systems requires that they generate identical output trajectories under every admissible experiment. We will weaken this condition by considering a single x_0, while other experiment axes are allowed to vary freely. We will formalize this weakened notion of x_0-equivalence as follows.

Definition 2.1. *(x_0-**equivalence.**) Fix a finite time horizon T, and allow all piecewise continuous signals on $[0,T]$ as control inputs. We say that (A', B') is x_0-equivalent to (A, B), if for all allowed inputs, their trajectories with initial state x_0 are identical over $[0,T]$. Formally:*

$$(A', B') \overset{\mathrm{x}_0}{\sim} (A, B) \iff \forall \mathrm{u} \in \mathcal{U} : \forall t \in [0,T] : \varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right) = \varphi\left(t \,|\, A', B', \mathrm{x}_0, \mathrm{u}\right) \tag{2.1}$$

We will define the corresponding equivalence class as

$$[A, B]_{\mathrm{x}_0} = (A', B') \;\; (A', B') \overset{\mathrm{x}_0}{\sim} (A, B). \tag{2.2}$$

If the admissible inputs are restricted to a subset $\mathcal{U}_c$ of $\mathcal{U} = \mathcal{C}([0,T], \mathbb{R}^m)$, we will denote the corresponding equivalence relationship as:

$$(A', B') \overset{\mathrm{x}_0, \mathcal{U}_c}{\sim} (A, B) \iff \forall \mathrm{u} \in \mathcal{U}_c : \forall t \in [0,T] : \varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right) = \varphi\left(t \,|\, A', B', \mathrm{x}_0, \mathrm{u}\right), \tag{2.3}$$

with the corresponding equivalence class

$$[A,B]_{\mathrm{x}_0,\,\mathcal{U}_c} = (A',B') \;\; (A',B') \overset{\mathrm{x}_0,\mathcal{U}_c}{\sim} (A,B). \tag{2.4}$$

With this definition, we can formalize x_0-identifiability as there being no $(\tilde{A},\tilde{B}) \neq (A,B)$ in the considered parameter class Ω which is x_0-equivalent to (A,B).

Definition 2.2. ***(x_0-identifiability).*** *We will say that a given system instance $\mathcal{S} = (A,B)$ is identifiable in $\Omega \subseteq \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, from an initial state x_0, if a piecewise continuous input signal $\mathrm{u}(\cdot) \in \mathcal{C}([0,\infty),\mathbb{R}^m)$ can be found that makes the trajectory $\varphi(\cdot\,|\,A,B,\mathrm{x}_0,\mathrm{u})$ unique:*

$$\begin{aligned}\mathcal{P}_{\mathrm{x}_0}(A,B\,|\,\Omega) \iff & \exists\, \mathrm{u}(\cdot) \in \mathcal{C}([0,\infty),\mathbb{R}^m) : \forall\, \Omega \ni (A',B') \neq (A,B) : \\ & \exists\, t \in \mathbb{R}_+ : \varphi(t\,|\,A',B',\mathrm{x}_0,\mathrm{u}) \neq \varphi(t\,|\,A,B,\mathrm{x}_0,\mathrm{u}).\end{aligned} \tag{2.5}$$

Using (2.1)*, we can also state this as follows:*

$$\mathcal{P}_{\mathrm{x}_0}(A,B\,|\,\Omega) \iff \neg\exists\, (A',B') \in \Omega,\; (A',B') \neq (A,B) : (A',B') \overset{\mathrm{x}_0}{\sim} (A,B). \tag{2.6}$$

In §4.1, we will look at "purely structural" identifiability, which is only concerned with system-intrinsic properties, excluding what can be generalized as *data sufficiency* properties. Thus, the standing assumption there is that the control inputs $\mathrm{u}(\cdot)$ can be chosen freely from a maximally admissible set of inputs. Thus, x_0-identifiability as defined above is suitable for our objective in Section 4.1, and will be the primary notion of identifiability we employ there.

In the second branch of the analysis, we lift this assumption of sufficiently rich control inputs. Rather than attempting an exhaustive taxonomy of signal-induced nonidentifiability, we focus on representative classes of restrictions. This requires a slightly revised version of identifiability where the set of admissible control inputs is a proper subset of the ambient signal space. We formalize this as "input-restricted x_0-", or $(\mathrm{x}_0,\mathcal{U}_c)-$identifiability.

Definition 2.3. ***(Input-restricted (x_0,$\mathcal{U}_c$-) identifiability).*** *We will say that a given system instance $\mathcal{S} = (A,B)$ is identifiable in $\Omega \subseteq \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, from an initial state x_0, if all signals $\mathrm{u}(\cdot)$ in a subset $\mathcal{U}_c$ of the ambient input space $\mathcal{C}([0,\infty),\mathbb{R}^m)$ induce unique trajectories:*

$$\begin{aligned}\mathcal{P}_{\mathrm{x}_0,\,\mathcal{U}_c}(A,B\,|\,\Omega) \iff & \forall\, \Omega \ni (A',B') \neq (A,B) : \forall\, \mathrm{u}(\cdot) \in \mathcal{U}_c : \\ & \exists\, t \in \mathbb{R}_+ : \varphi(t\,|\,A',B',\mathrm{x}_0,\mathrm{u}) \neq \varphi(t\,|\,A,B,\mathrm{x}_0,\mathrm{u}).\end{aligned} \tag{2.7}$$

Using (2.3)*, we can also write this as*

$$\mathcal{P}_{\mathrm{x}_0,\,\mathcal{U}_c}(A,B\,|\,\Omega) \iff \neg\exists\, (A',B') \in \Omega,\; (A',B') \neq (A,B) : (A',B') \overset{\mathrm{x}_0,\mathcal{U}_c}{\sim} (A,B) \tag{2.8}$$

Intuitively, this condition becomes increasingly difficult to fulfill, as the set of constrained-admissible inputs $\mathcal{U}_c$ becomes larger. Thus, we will look at scenarios where $\mathcal{U}_c$ has a compact representation, e.g., where $\mathcal{U}_c$ is simply a set consisting of a finite set of admissible inputs, or contains inputs with richness properties (e.g., persistency of excitation order (Definition 2.25)).

In single-trajectory identification one might expect the time horizon to play a central role. We make no such assumption here: for CT–LTI pairs (A, B) with full-state output and piecewise continuous inputs, structural identifiability is invariant to the horizon. In fact, if $\varphi(\cdot \mid A, B, \mathrm{x}_0, \mathrm{u}) = \varphi(\cdot \mid \tilde{A}, \tilde{B}, \mathrm{x}_0, \mathrm{u})$ on $[0, T]$ for every admissible u, then the equality persists for all $t \geq 0$ (Lemma 2.4). We therefore set $T = \infty$ without loss of generality.

Lemma 2.4. ***(Horizon-independence of x_0-equivalence for continuous-time LTIs).*** *Fix $\mathrm{x}_0 \in \mathbb{R}^n$. Let (A, B), $(\tilde{A}, \tilde{B})$ be two continuous-time LTI pairs with piecewise continuous inputs. Suppose there exists a finite $T > 0$ such that, for every admissible $\mathrm{u} : [0, T] \to \mathbb{R}^m$,*

$$\forall t \in [0, T] : \varphi(\cdot \mid A, B, \mathrm{x}_0, \mathrm{u}) = \varphi(\cdot \mid \tilde{A}, \tilde{B}, \mathrm{x}_0, \mathrm{u}).$$

Then, the same equality holds for all $t \geq 0$ and for all piecewise continuous inputs on $[0, \infty)$. Equivalently, the x_0-equivalence relation does not depend on the time horizon being finite.

Proof sketch. (Full proof in Appendix A.2, Appendix A.2.1) .

Remark 2.5 (Structural vs. practical)**.** *The trajectory length directly affects data informativity and estimator performance in practice. The lemma concerns structural (in-principle) identifiability only, for which the finite choice of horizon is immaterial.*

2.2.2. The Visible Subspace

Later, we will define *partial identifiability* which depends on the notion of a *visible subspace*– an extension of the controllable subspace that also accounts for the homogeneous response from a specific initial state x_0. Next, we formally define this subspace and briefly discuss its algebraic properties.

Definition 2.6. ***(A$-$invariant hull, visible subspace).*** *We define as $A-$invariant hull of $\mathrm{x}_0 \cup \mathrm{Im}(B)$ the vector sum of the $A-$cyclic subspace generated by x_0 and the controllable*

subspace:

$$\begin{aligned} V(\mathrm{x}_0) &:= \mathcal{K}(A, \mathrm{x}_0) + \mathcal{R}(0) \\ &= \operatorname{span}\{\mathrm{x}_0, A\mathrm{x}_0, \ldots, A^{n-1}\mathrm{x}_0\} + \sum_{j=0}^{n-1} A^j \operatorname{Im}(B) \\ &= \operatorname{span}\{A^k\mathrm{x}_0, A^k b_j : k \in [0, n-1], j \in [m]\}, \quad \textit{where } B = [b_1 \ldots b_m] \\ &= \operatorname{span}\{[\mathrm{x}_0\ B], A[\mathrm{x}_0\ B], \ldots, A^{n-1}[\mathrm{x}_0\ B]\} \\ &:= \mathcal{K}_n(A, [\mathrm{x}_0\ B]). \end{aligned} \tag{2.9}$$

This can be understood as the $A-$invariant hull of the controllable subspace $\mathcal{R}(0)$ translated by the homogeneous orbit $\{\varphi(t \,|\, A, B, \mathrm{x}_0, \mathbf{0}) : t \geq 0\} = \{e^{At}\mathrm{x}_0 : t \geq 0\}$.

We define the following properties of the visible subspace, which are directly relevant to the analysis we develop in Chapter 4.

Corollary 2.7. ***(Properties of the $A-$invariant hull).*** *Let V be as in Equation* (2.9)*. Then*

(a) *V is a subspace.*

(b) *V is $A-$invariant, i.e., $AV \subseteq V$.*

(c) *V is $[A, B]-$invariant.*

(d) *V is the smallest $[A, B]-$invariant subspace that contains both x_0 and the image of B, i.e.,*

$$V = \bigcap \{S \subseteq \mathbb{R}^n \textit{ a subspace} : \{\mathrm{x}_0 \cup \operatorname{Im}(B)\} \subseteq S \textit{ and } AS \subset S + \operatorname{Im}(B)\}$$

(e) *V is the smallest $A-$invariant subspace containing $\mathrm{x}_0 \cup \operatorname{Im}(B)$.*

Proof. See Appendix A.2.1. □

2.3. Preliminaries

2.3.1. Dynamical Systems

We model controlled dynamics via a transition map

$$\phi : \mathbb{R}_{\geq 0} \times \mathcal{X} \times \mathcal{U}^{[0,t]} \to \mathcal{X}, \qquad (t, x_0, u) \mapsto \phi(t; x_0, u),$$

which returns the state reached at time t from x_0 under input signal u on $[0, t]$. The system itself can be expressed as a tuple $(\mathcal{T} := \mathbb{R}_{\geq 0}, \mathcal{X} \subseteq \mathbb{R}^n, \mathcal{U} \subseteq \mathbb{R}^m, \phi)$, where $\mathcal{T}$ denotes the time axis, $\mathcal{X}$ the state space, $\mathcal{U}$ the control input space, and ϕ the transition function. It satisfies

(i) identity: $\phi(0; \mathrm{x}_0, \mathrm{u}) = \mathrm{x}$; and

(ii) semigroup with input concatenation: $\phi(t+s;\ \mathrm{x}_0,\ \mathrm{u}\cdot v) = \phi\big(t;\ \phi(s;\ \mathrm{x}\ , \mathrm{u}),\ v(\cdot - s)\big)$,

following Sontag's formalism [18]. For LTI systems $\dot{x} = Ax + Bu$, existence and uniqueness hold for all piecewise-continuous u, and

$$\phi(t; x_0, u) = e^{At} x_0 + \int_0^t e^{A(t-s)} Bu(s)\, ds.$$

If the system produces an output signal distinct from the inner state, we can encode this by extending the tuple from earlier by an output set $\mathcal{Y}$ and an output map $h : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$.

The concept of the dynamical system admits multiple representation modalities, all of which can be traced back to this abstract formal definition. The most commonly used modalities include the state-space representation, where the system state is expressed as a finite vector, that minimally encodes the entire information needed to compute future states of the system. Under that key assumption, the state space becomes a subset of $\mathbb{R}^n$, where n is the dimension of the state vector, the input value set becomes a subset of $\mathbb{R}^m$, where m is the dimension of the input signal, and the function f becomes a linear map from $\mathbb{R}^{n+m}$ into $\mathbb{R}^n$. Another commonly used representation uses the frequency domain, and primarily models the input-output map of the system.

A major part of our analysis involves invariant subspaces, and controllability, both of which have their most natural expression in the state-space representation. Thus, we will exclusively use state-space models in our analysis.

	Continuous: $t \in \mathbb{R}_{\geq 0}$	**Discrete:** $k \in \mathbb{N}$
Time-varying	$\dot{x}(t) = f(x(t), u(t); t)$ $y(t) = g(x(t), u(t); t)$	$x_{k+1} = f(x_k, u_k; k)$ $y_k = g(x_k, u_k; k)$
Time-Independent	$\dot{x}(t) = f(x(t), u(t))$ $y(t) = g(x(t), u(t))$	$x_{k+1} = f(x_k, u_k)$ $y_k = g(x_k, u_k)$

Table 2.1.: State-space formulations across discrete/continuous, time varying / time independent systems.

We will exclusively focus on unperturbed, continuous-time linear-time invariant (LTI) systems with full observability, meaning the measured outputs equal the system state at all times. To distinguish system classes and system instances more easily, we will

define the class of fully observed LTI systems as

$$\Sigma_{i/s} = \begin{cases} \dot{\mathrm{x}}(t) = A\mathrm{x}(t) + B\mathrm{u}(t) \\ \mathrm{y}(t) = \mathrm{x}(t) \\ \mathrm{x}(0) = \mathrm{x}_0 \in \mathbb{R}^n, \end{cases} \tag{2.10}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ denote the system matrices, $\mathrm{u}(\cdot) \in \mathcal{C}(,\mathcal{U})$ is the piecewise-continuous control input, and x_0 is the initial state. We refer to a specific system instance as $\mathcal{S} = (A, B) \in \Sigma_{i/s}$.

$\Sigma_{i/s}$ is a special case of the larger system class $\Sigma_{i/s/o}$,

$$\Sigma_{i/s/o} : \begin{cases} \dot{\mathrm{x}}(t) = A\mathrm{x}(t) + B\mathrm{u}(t) \\ \mathrm{y}(t) = C\mathrm{x}(t) + D\mathrm{u}(t) \\ \mathrm{x}(0) = \mathrm{x}_0 \in \mathbb{R}^n, \end{cases} \tag{2.11}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$.

At any given point in time $t \in \mathbb{R}_{\geq 0}$, for a given initial state x_0, and a piecewise-continuous input signal $u(\cdot) \in \mathcal{C}\left([0, t], \mathbb{R}^m\right)$ we can compute the state of the system by solving the system of equations given in (2.10). To do that, we use the following properties of the matrix exponential:

$$\frac{d}{dt}e^{At} = Ae^{At} \overset{(*)}{=} e^{At}A,$$

where $(*)$ holds since A and e^{At} commute. Multiplying the state evolution equation in (2.10) with e^{-At} on both sides gives:

$$\begin{aligned} e^{-At}\dot{\mathrm{x}}(t) &= e^{-At}A\mathrm{x}(t) + e^{-At}B\mathrm{u}(t) \quad \longleftrightarrow \\ \underbrace{e^{-At}\dot{\mathrm{x}}(t) - e^{-At}A\mathrm{x}(t)}_{=\frac{d}{dt}e^{-At}\mathrm{x}(t)} &= e^{-At}B\mathrm{u}(t) \Big| \int_0^t (\cdot)\, ds \quad \longleftrightarrow \\ \int_0^t \frac{d}{ds}e^{-As}\mathrm{x}(s)\, ds &= \int_0^t e^{-As}B\mathrm{u}(s)\, ds \quad \longleftrightarrow \\ e^{-As}\mathrm{x}(s)\Big|_0^t = e^{-At}\mathrm{x}(t) - I\mathrm{x}(0) &= \int_0^t e^{-As}B\mathrm{u}(s)\, ds \quad \longleftrightarrow \\ \mathrm{x}(t) = e^{At}\left(\mathrm{x}_0 + \int_0^t e^{-As}B\mathrm{u}(s)\, ds\right) &= e^{At}\mathrm{x}_0 + \int_0^t e^{A(t-s)}B\mathrm{u}(s)\, ds, \end{aligned} \tag{2.12}$$

The expression in (2.12), also known as the Variation-of-constants (Duhamel) formula,

is a combination of a homogeneous and the particular solution:

$$\mathrm{x}(t) := \varphi(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}) = \underbrace{e^{At}\mathrm{x}_0}_{:=\mathrm{x}^h(t)} + \underbrace{\int_0^t e^{A(t-s)} B\mathrm{u}(s) \; ds}_{\mathrm{x}^p(t)} . \tag{2.13}$$

The homogeneous solution $\mathrm{x}^h(t)$ can be understood as the "autonomous" system response, and the particular $\mathrm{x}^p(t)$ solution can be understood as the control response.

It is often the case that some aspect of the system (the dynamical equations if we're simulating the system on a computer, or the input signal, if the controller is digital) needs to be computed digitally, which requires a discrete-time representation. If the input $\mathrm{u}(\cdot)$ is generated digitally, followed by a digital-to-analog converter. Then, $\mathrm{u}(t)$ will be piecewise constant. It is frequently assumed that the input values only change equidistant discrete time instants, yielding

$$\mathrm{u}(t) = \mathrm{u}(k\Delta t) := \mathrm{u}[k] \;\; \text{for } t \in [k\Delta t, (k+1)\Delta t),$$

which is known as the Zero-Order Hold (ZOH) assumption. Under this assumption, we can derive an exact discretization of the system dynamics given in (2.10) as follows. We start by defining $k := \frac{t}{\delta t}$ equidistant time steps on the interval $[0, t)$, and discrete samples $x[k], u[k]$. The Variation-of-constants formula (2.13) gives

$$x[k+1] = e^{A\delta t}\mathrm{x}[k] + \left(\int_0^{\delta t} e^{A\tau} \, d\tau \right) B\mathrm{u}[k]$$

(See Appendix A.2.1 for a detailed derivation). Thus, if an input changes value only at discrete-time instants $k\delta t$ and if we compute only the responses at $t = k\delta t$, the state-space dynamical equation becomes

$$\mathrm{x}[k+1] = \underbrace{e^{A\delta t}}_{:=A_d} \, \mathrm{x}[k] + \underbrace{\left(\int_0^{\delta t} e^{A\tau} \, d\tau \right)}_{:=B_d} B \, \mathrm{u}[k], \tag{2.14}$$

which reflects the original dynamics exactly.
Let's take a detour and compute the discrete solution curve for this ZOH-discretized

pair (A_d, B_d) at an arbitrary time step $k \in \mathbb{N}$.

$$\begin{aligned}
\mathrm{x}[1] &= A_d\mathrm{x}[0] + B_d\mathrm{u}[0] \\
\mathrm{x}[2] &= A_d\mathrm{x}[1] + B_d\mathrm{u}[1] \\
&= A_d\left[A_d\mathrm{x}[0] + B_d\mathrm{u}[0]\right] + B_d\mathrm{u}[1] \\
&= A_d^2\mathrm{x}[0] + \left[A_dB_d \;\; B_d\right] \cdot \left[\mathrm{u}[0] \;\; \mathrm{u}[1]\right]^\top \\
&\;\;\vdots \\
\mathrm{x}[k] &= A_d^k\mathrm{x}[0] + \left[A_d^{k-1}B_d \;\ldots\; A_dB_d \;\; B_d\right] \cdot \left[\mathrm{u}[0] \;\ldots \mathrm{u}[k-2] \;\; \mathrm{u}[k-1]\right]^\top
\end{aligned}$$

Dropping the subscripts for the moment, we get the general formula

$$\begin{aligned}
\mathrm{x}[k] &= A^k\mathrm{x}[0] + \left[A^{k-1}B \;\ldots\; AB \;\; B\right] \cdot \begin{bmatrix} \mathrm{u}[0] \\ \vdots \\ \mathrm{u}[k-1] \end{bmatrix} \\
&= A^k\mathrm{x}[0] + \sum_{j=1}^{k} A^{k-j}B\mathrm{u}[j-1] := \phi(k \,|\, \mathcal{S}, \mathrm{x}_0, \mathrm{u}).
\end{aligned} \tag{2.15}$$

Since any numerical system identification ultimately computes such a discretized pair, this will become relevant in later discussions on practical identifiability.

As motivated earlier, the observational identifiability of a given system often comes down to whether, under a fixed experimental setting, it can be driven to a state that distinguishes it from all others in the same set of candidate systems. Thus, a defining tool in our analysis are the concepts of reachability and controllability, which we will discuss next.

2.3.2. Reachability and Controllability

In a single experiment starting from x_0, identifiability is geometry: can the input drive the system state into directions that distinguish (A, B) from other pairs? The central objects are the time-$t-$reachable set $\mathcal{R}^t(\mathrm{x}_0)$ and the origin-reachable subspace $\mathcal{R}^t(\mathbf{0})$. We start by fixing the basic reachability notions and then pass from these time-indexed sets to the subspace structure that will drive our analysis.

Fundamental Definitions

Definition 2.8. ***(Reachability).*** *Given a system $\mathcal{S}$, a state $\mathrm{x}_f \in \mathcal{X}$ is reachable from $\mathrm{x}_0 \in \mathcal{X}$ in time t, if there exists an admissible control input $\mathrm{u} \in \mathcal{U}^{[0,t]}$ such that $\mathrm{x}_f = \varphi\left(t \,|\, \mathcal{S}, \mathrm{x}_0, u\right)$.*

We will denote this as $\mathrm{x}_0 \underset{t}{\rightsquigarrow} \mathrm{x}_f$*:*

$$\mathrm{x}_0 \underset{t}{\rightsquigarrow} \mathrm{x}_f \iff \exists \mathrm{u} \in \mathcal{U}^{[0,t]} : \mathrm{x}_f = \varphi\left(t \,|\, \mathcal{S}, \mathrm{x}_0, u\right).$$

We will say that x_f *is reachable from* x_0*, if it is reachable in some* $t \in \mathcal{T}$ *:*

$$\mathrm{x}_0 \rightsquigarrow \mathrm{x}_f \iff \exists t \in \mathcal{T} : \mathrm{x}_0 \underset{t}{\rightsquigarrow} \mathrm{x}_f.$$

Corollary 2.9. *A system* $\mathcal{S}$ *is reachable, if any given state* $\mathrm{x}_f \in \mathcal{X}$ *can be reached from any given initial state* x_0*, i.e.* $\forall \mathrm{x}_0, \mathrm{x}_f : \exists t \in \mathcal{T} : \mathrm{x}_0 \underset{t}{\rightsquigarrow} \mathrm{x}_f$*.*

With the primitives in place, we now express the t-reachable set from a general initial condition and isolate the contribution of the origin-reachable part. We will denote the $t-$reachable set from x_0 as

$$\begin{aligned}
\mathcal{R}^t(\mathrm{x}_0) &\overset{\text{def.}}{=} \left\{\varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right) : \mathrm{u} \in \mathcal{C}\left([0,t], \mathcal{U}\right)\right\} \\
&= \left\{ e^{At}\mathrm{x}_0 + \int_0^t e^{A(t-s)} Bu(s)\, ds \,\middle|\, \mathrm{u} \in \mathcal{C}([0,t], \mathcal{U}) \right\} \\
&= e^{At}\mathrm{x}_0 + \left\{ \int_0^t e^{A(t-s)} Bu(s)\, ds \,\middle|\, \mathrm{u} \in \mathcal{C}([0,t], \mathcal{U}) \right\} \\
&= e^{At}\mathrm{x}_0 + \left\{\varphi\left(t \,|\, A, B, \mathbf{0}, \mathrm{u}\right) : \mathrm{u} \in \mathcal{C}([0,t], \mathcal{U})\right\} \\
&= e^{At}\mathrm{x}_0 + \mathcal{R}^t(\mathbf{0}). \qquad (2.16)
\end{aligned}$$

With this decomposition in hand, assembling all horizons gives a clean algebraic picture:

$$\begin{aligned}
\mathcal{R}(\mathrm{x}_0) &\overset{\text{def.}}{=} \left\{\mathcal{R}^t(\mathrm{x}_0) : t \in \mathbb{R}_{\geq 0}\right\} = \left\{ e^{At}\mathrm{x}_0 + \mathcal{R}^t(\mathbf{0}) : t \in \mathbb{R}_{\geq 0} \right\} \\
&\overset{\text{Lemma 2.12}}{=} \left\{ e^{At}\mathrm{x}_0 : t \in \mathbb{R}_{\geq 0} \right\} + \mathcal{R}(A, B) := \gamma(A, \mathrm{x}_0) + \mathcal{R}(A, B) \qquad (2.17) \\
&\subseteq \mathcal{K}(A, \mathrm{x}_0) + \mathcal{R}(A, B) \overset{\text{Theorem 2.14}}{=} \mathcal{K}(A, \mathrm{x}_0) + \mathcal{K}(A, B) \\
&:= \mathcal{K}\left(A, [\mathrm{x}_0 \; B]\right). \qquad (2.18)
\end{aligned}$$

This separates the homogeneous flow $e^{At}\mathrm{x}_0$ from the directions unlocked by control. To reason about the latter, we pass from sets to the subspaces they generate.

Sets and Subspaces

Definition 2.10. ***(Reachable set).*** *The set of states reachable from* $\mathrm{x}_0 \in \mathcal{X}$ *in time* t *is given by*

$$\mathcal{R}^t(\mathrm{x}_0) := \left\{ \mathrm{x}_f \in \mathcal{X} \,\middle|\, \mathrm{x}_0 \underset{t}{\rightsquigarrow} \mathrm{x}_f \right\} = \left\{ \mathrm{x}_f \in \mathcal{X} \,\middle|\, \exists \mathrm{u} \in \mathcal{U}^{[0,t]} : \mathrm{x}_f = \varphi\left(t \,|\, \mathcal{S}, \mathrm{x}_0, \mathrm{u}\right) \right\},$$

with $\mathcal{R}(\mathrm{x}_0) := \left\{ \mathrm{x}_f \in \mathcal{X} \,\middle|\, \mathrm{x}_0 \rightsquigarrow \mathrm{x}_f \right\} = \bigcup_{t \in \mathcal{T}} \mathcal{R}^t(\mathrm{x}_0)$*.*

Since, in case of LTI systems, $t-$reachability implies reachability, we will use $\mathcal{R}(\mathrm{x}_0)$ when talking about reachable sets. To make the reachable set into a reachable *space*, we need $\mathbf{0} \in \mathcal{R}(\mathrm{x}_0)$, which is guaranteed if we set $\mathrm{x}_0 = \mathbf{0}$. This is why, in analyses that involve reachability, it is often $\mathcal{R}(\mathbf{0})$ that is meant by "reachable subspace".

Definition 2.11. ***(Controllable subspace).*** *The set of states reachable from $\mathrm{x}_0 = \mathbf{0}$, called the controllable subspace, is defined as*

$$\mathcal{R}_0 := \mathcal{R}(0) = \left\{ \mathrm{x}_f \in \mathcal{X} \middle| \exists \mathrm{u} \in \mathcal{C}\left([0,\infty), \mathcal{U}\right), t \in [0,\infty) : \mathrm{x}_f = \varphi(t \,|\, \mathcal{S}, \mathbf{0}, \mathrm{u}) \right\},$$

with $\mathcal{R}_0^t := \left\{ \mathrm{x}_f \in \mathcal{X} \middle| \mathbf{0} \underset{t}{\rightsquigarrow} \mathrm{x}_f \right\}$.

Crucially, for continuous-time LTIs this subspace does not depend on the specific horizon once $t > 0$.

Lemma 2.12. ***(Time independence of the controllable subspace of continuous-time systems).*** *For any continuous-time LTI pair (A, B) and any $t > 0$,*

$$\begin{aligned} \mathcal{R}^t(0) &= \left\{ \int_0^t e^{A(t-s)} Bu(s)\, ds \middle| u \in \mathcal{C}([0,t], \mathcal{U}) \right\} \\ &= \operatorname{span}\left\{ A^k B : 0 \le k \le n-1 \right\} =: \mathcal{R}(A, B), \end{aligned} \tag{2.19}$$

i.e., the origin-reachable set at any positive time is the same fixed subspace (Full proof in Appendix A.2, Appendix A.2.1).

We can therefore speak of *the* controllable subspace. Next, we identify it with its algebraic (Krylov) description, which will later help us define the visible subspace $V(\mathrm{x}_0)$ as a unified algebraic object.

Lemma 2.13. *Let $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ and let b_j denote the $j-$th column of B. Then,*

$$\begin{aligned} \mathcal{R}(A, B) &\overset{\text{Lemma 2.12}}{=} \operatorname{span}\left\{ A^k B : 0 \le k \le n-1 \right\} \\ &= \operatorname{span}\left\{ A^k \left[b_1 \,|\, \ldots \,|\, b_m \right] : 0 \le k \le n-1 \right\} \\ &= \operatorname{span}\left\{ A^k b_j \,|\, k \in [0, n-1], j \in [m] \right\}. \end{aligned}$$

Equivalently, $\mathcal{R}(A, B)$ is the smallest $A-$invariant subspace of $\mathcal{X} = \mathbb{R}^n$ that contains the columns of B. In particular, (A, B) is controllable iff this span is all of $\mathcal{X}$.

We now record the formal version used later with proofs deferred to the appendix.

Theorem 2.14. *The reachable subspace $\mathcal{R}_0$ of the system $\mathcal{S} \in \Sigma_{s/i}$ coincides with the $A-$cyclic (Krylov) subspace generated by (the column vectors of) B [18]. Formally:*

$$\begin{aligned}\mathcal{R}_0 &\overset{def.}{=} \{\varphi\,(t\,|\,\mathcal{S}, \mathbf{0}, \mathrm{u}) : \mathrm{u} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathcal{U}), t \in \mathbb{R}_{\geq 0}\} \\ &= \mathcal{K}(A, B) \\ &\overset{def.}{=} \operatorname{span}\left\{A^k b_j :\ k \geq 0,\ j \in [m],\ b_j : j\text{-th column of } B\right\} \\ &= \sum_{k=0}^{n-1} A^k \operatorname{Im}(B)\end{aligned} \tag{2.20}$$

Geometrically, this means $\mathcal{R}_0$ is exactly the smallest A-invariant hull of $\operatorname{Im} B$

Corollary 2.15. *The reachable subspace, $\mathcal{R}_0 = \mathcal{K}(A, B)$ coincides with the minimal $A-$invariant subspace that contains the image of B [18].*

Remark 2.16. *In case of single-input systems ($m = 1$), Lemma 2.13 implies that the controllability of (A, B) is equivalent to the requirement that $B \in \mathbb{R}^{n\times 1}$ be cyclic for the matrix A:*

$$\begin{aligned}(A, B)\ \textit{controllable} &\iff \operatorname{rank}\begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix} = n\ , B \leftarrow b \in \mathbb{R}^n \\ &\iff \dim\left(\operatorname{span}\left(b, Ab, \dots, A^{n-1}\right)\right) = \dim\left(\mathcal{K}_n(A, B)\right) = n \\ &\overset{def.}{\iff} b\ \textit{ is cyclic for } A.\end{aligned}$$

Why powers stop at $n - 1$. By Cayley–Hamilton, A^n is a linear combination of $\{I, A, \dots, A^{n-1}\}$. Hence $\operatorname{span}\{A^k B\}_{k\geq 0} = \operatorname{span}\{A^k B\}_{k=0}^{n-1}$.

Having the span picture, we summarize standard controllability criteria used later in Chapter 4.

Theorem 2.17. ***(Controllability criteria). [19]*** *The following statements are equivalent.*

1. *The matrix pair (A, B) is controllable.*
2. *The controllability matrix has full rank:* $\operatorname{rank}\begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix} = n$.
3. *The matrix*

$$W_t = \int_0^t e^{sA} B B^\top e^{sA^\top}\, ds$$

is positive definite for every $t > 0$.

4. *The matrix pencil* $[A - \lambda I,\ B]$ *has full row rank for all* $\lambda \in \sigma(A)$*, where* $\sigma(A)$ *denotes the spectrum of* A*.*

5. *The minimal* $A-$*invariant subspace over* $\operatorname{Im} B$ *coincides with* $\mathbb{R}^n$*.*

6. *If, in addition, all eigenvalues of* A *have negative real parts, then the unique solution of the Lyapunov equation*

$$AW_c + W_c A^\top = -BB^\top$$

is positive definite. This unique solution is called the controllability Gramian, and is given by

$$W_c = \int_0^\infty e^{As} BB^\top e^{A^\top s}\, ds.$$

7. *No left eigenvector of* A *is orthogonal to* B*, i.e.,* $\{w \in \mathbb{R}^n \setminus \{0\} \,|\, \exists \lambda \in \mathbb{R} : w^\top A = \lambda w \wedge w^\top B = 0\} = \emptyset$*.*

Proposition 2.18. *Let* $T \in \mathrm{GL}(n)$*, and* $G \in \mathrm{GL}(m)$*. If* (A, B) *is controllable, then* $(T^{-1}AT, T^{-1}BG)$ *is controllable [18].*

This invariance justifies working in convenient coordinates—in particular, those induced by the Kalman decomposition.

Kalman Controllability Decomposition and ZOH Sampling

In Chapter 4, we develop a canonical identifiability form which is based on the Kalman decomposition. The Kalman decomposition gives a partitioning of the full state vector that correspond to *controllable and uncontrollable modes* of the system. We obtain a similar partitioning through our x_0-enriched Kalman decomposition, which helps us give a closed-form definition of the x_0-equivalence class under sufficiently rich input signals, as well as precisely state what is *partially identifiable*.

Lemma 2.19. ***(Kalman controllability decomposition).*** *Assume that* (A, B) *is not controllable with* $\dim \mathcal{R}(A, B) = k < n$*. Then, there exists an invertible linear transformation* $S \in \mathrm{GL}(n)$ *such that the matrices* $\tilde{A} := S^{-1}AS$ *and* $\tilde{B} := S^{-1}B$ *have an upper-triangular block structure*

$$\tilde{A} = \begin{array}{c} \overset{k}{\longleftrightarrow}\ \overset{n-k}{\longleftrightarrow} \\ \begin{bmatrix} A_c & A_* \\ 0 & A_{uc} \end{bmatrix} \end{array} \begin{array}{c} k \\ \updownarrow \\ n-k \end{array}, \qquad \tilde{B} = \begin{array}{c} \overset{m}{\longleftrightarrow} \\ \begin{bmatrix} B_c \\ 0 \end{bmatrix} \end{array} \begin{array}{c} k \\ \updownarrow \\ n-k \end{array}, \tag{2.21}$$

where (A_c, B_c) *form a controllable pair of size* $(k \times k)$ *and* A_{uc} *governs an autonomous subsystem of size* $(n-k) \times (n-k)$ *[18].*

Proof. **(1) Triangular form:** Pick any subspace $\mathcal{S}$ such that $\mathcal{S}$ and the controllable subspace $\mathcal{R}(A,B)$ partition the state space $\mathcal{X} = \mathbb{R}^n$, (i.e., $\mathcal{R}(A,B) \oplus \mathcal{S} = \mathbb{R}^n$, where '$\oplus$' denotes the direct sum– meaning $\mathcal{R}(A,B) + \mathcal{S} = \mathbb{R}^n$ and $\mathcal{R}(A,B) \cap \mathcal{S} = \{0\}$), and let $\{r_1, \dots, r_k\}$ be a basis of $\mathcal{R}(A,B)$ (e.g., any subset of k columns of the controllability matrix $\mathbf{R}(A,B)$ that are linearly independent), and let $\{s_1, \dots, s_{n-k}\}$ be a basis of $\mathcal{S}$ (e.g., non-pivot columns of the controllability matrix). Let us build the linear transformation T as $T := \begin{bmatrix} r_1 & \dots & r_k & s_1 & \dots & s_{n-k} \end{bmatrix}$. Because $\mathcal{R}(A,B)$ is $A-$invariant, we have $A\mathcal{R}(A,B) \subseteq \mathcal{R}(A,B)$, hence in the basis T, the matrix of A is block upper-triangular. Since $\mathrm{Im}(B) \subseteq \mathcal{R}(A,B)$, the last $n-k$ coordinates of $T^{-1}B$ vanish. **(2) Controllability of $(\mathbf{A_c}, \mathbf{B_c})$:** It remains to show (A_c, B_c) is controllable. Let $\mathbf{R}(A,B) = [B,\ AB,\ \dots\ A^{n-1}B]$ be the controllability matrix. The first k rows of $T^{-1}\mathbf{R}(A,B)$ span all of $\mathbb{R}^k$ because $\mathcal{R}(A,B) = \mathrm{span}\{A^j \,\mathrm{Im}\, B \,:\, j \geq 0\}$ projects isomorphically to the first k coordinates under T^{-1}. But $T^{-1}\mathbf{R}(A,B) = \begin{bmatrix} B_c & A_cB_c & \dots & A_c^{n-1}B_c \\ 0 & 0 & \dots & 0 \end{bmatrix}$, so the top block has rank k, which implies $\mathrm{rank}[B_c\ A_cB_c\ \dots\ A_c^{k-1}B_c] = k$ and hence (A_c, B_c) is controllable. □

For a continuous-time LTI system the Kalman decomposition can be interpreted as a basis transform which partitions the state vector into a part that corresponds to the controllable entries of the system matrix pair (A,B), and into a part that corresponds to the uncontrollable entries. To show that, let us first look at how T^{-1} transforms the entire dynamical equation:

$$
\begin{aligned}
\dot{\mathrm{x}}(t) &= A\mathrm{x}(t) + B\mathrm{u}(t)\ ,\ \text{dynamical equation} \mid\ T^{-1}(\cdot) \\
T^{-1}\dot{\mathrm{x}}(t) &= T^{-1}A\mathrm{x}(t) + T^{-1}B\mathrm{u}(t) = T^{-1}A\left(TT^{-1}\right)\mathrm{x}(t) + T^{-1}B\mathrm{u}(t) \\
&= T^{-1}AT\underbrace{\left(T^{-1}\mathrm{x}(t)\right)}_{:=z(t)} + T^{-1}B\mathrm{u}(t) \quad \Longleftrightarrow \\
\dot{z}(t) &= \tilde{A}z(t) + \tilde{B}\mathrm{u}(t) \\
&\overset{\text{def}}{=} \begin{bmatrix} A_c & A_* \\ 0 & A_{uc} \end{bmatrix} \begin{bmatrix} (z)_{1:k} := z_1 \\ (z)_{k+1:n} := z_2 \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} \mathrm{u}(t) \\
&= \begin{bmatrix} A_cz_1(t) + A_*z_2(t) + B_c\mathrm{u}(t) \\ A_{uc}z_2(t) \end{bmatrix}
\end{aligned}
$$

The $k-$th order reachable state-space model $\dot{z}_1(t) = A_cz_1(t) + A_*z_2(t) + B_c\mathrm{u}(t)$ is sometimes called the reachable subsystem of (A,B). The behavior of $z_2(t)$ is described by the autonomous state-space model $\dot{z}_2(t) = A_{uc}z_2(t)$.

Remark 2.20 (Block–preserving uniqueness of the Kalman controllability decomposition)**.** *Let (A, B) be a linear system over $\mathbb{R}$ (or $\mathbb{C}$), and let $k = \dim \mathcal{R}(A, B)$. Suppose $T_i \in \mathrm{GL}(n)$, $i = 1, 2$, are two similarity transformations whose first k columns form a basis of the reachable subspace $\mathcal{R}(A, B)$, so that in the corresponding coordinates*

$$T_i^{-1} A T_i = \begin{bmatrix} A_c^{(i)} & A_*^{(i)} \\ 0 & A_{uc}^{(i)} \end{bmatrix}, \qquad T_i^{-1} B = \begin{bmatrix} B_c^{(i)} \\ 0 \end{bmatrix},$$

with $(A_c^{(i)}, B_c^{(i)})$ controllable and $\dim A_c^{(i)} = k$. Then there exists an invertible block upper–triangular matrix

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad T_{11} \in \mathrm{GL}(k),\ T_{22} \in \mathrm{GL}(n-k),$$

such that $T_2 = T_1 T$ and, consequently,

$$\begin{aligned} A_c^{(2)} &= T_{11}^{-1} A_c^{(1)} T_{11}, & B_c^{(2)} &= T_{11}^{-1} B_c^{(1)}, \\ A_{uc}^{(2)} &= T_{22}^{-1} A_{uc}^{(1)} T_{22}, & A_*^{(2)} &= T_{11}^{-1} \big(A_*^{(1)} T_{22} + A_c^{(1)} T_{12} - T_{12} A_{uc}^{(1)} \big). \end{aligned}$$

In particular, the diagonal blocks A_c and A_{uc} are determined by (A, B) up to independent similarity on $\mathcal{R}(A, B)$ and its chosen complement. Hence their spectra and Jordan structures are invariants of (A, B), whereas the off–diagonal block A_* is not invariant (it changes with T_{12}).

Nonpathological sampling for (A, B) Let $\sigma_r(A, B) \subseteq \sigma(A)$ denote the *reachable spectrum* of A (the set of eigenvalues whose primary components have nontrivial projection from B). A sampling period $\Delta t > 0$ is *nonpathological for (A, B)* if

$$e^{\lambda \Delta t} \neq e^{\mu \Delta t} \ \text{ for all distinct } \lambda, \mu \in \sigma_r(A, B), \ \text{ and } \ e^{\lambda \Delta t} \neq 1 \ \text{ for all } \ \lambda \in \sigma_r(A, B).$$

Equivalently, $(\lambda - \mu)\Delta t \notin 2\pi \mathrm{i}\, \mathbb{Z}$ for $\lambda \neq \mu$ in $\sigma_r(A, B)$, and $\lambda \Delta t \notin 2\pi \mathrm{i}\, \mathbb{Z}$ for $\lambda \in \sigma_r(A, B)$.

This excludes only a discrete set of sampling periods ("almost all" Δt are admissible) and is standard in sampled-data controllability and observability (See [20–23]).

Theorem 2.21. ***(Invariants under ZOH sampling).*** *Fix a sampling period $\Delta t > 0$ and let $A_d = e^{A \Delta t}$, $B_d = \int_0^{\Delta t} e^{As} B \, ds$. Then:*

(i) (Exact sampled dynamics). *For any ZOH-sampled piecewise-constant input $\mathrm{u}(t)$ that equals $\mathrm{u}[k]$ on $[k\Delta t,\ (k+1)\Delta t)$, the sampled states satisfy*

$$\mathrm{x}[k+1] = A_d \mathrm{x}[k] + B_d \mathrm{u}[k],$$

and $\mathrm{x}(t) = \mathrm{x}[t/\Delta t]$ for all $t \in \{k\Delta t : k \in \mathbb{N}\}$.

(ii) ***(Reachable subspace is preserved).*** *Under the nonpathological sampling assumptions discussed above, the continuous-time reachable space and its ZOH-sampled counterpart are equual:*

$$\mathcal{R}_0^{dt} := \operatorname{span}\{A_d^k B_d : k \geq 0\} = \operatorname{span}\{A^k B : k \geq 0\} = \mathcal{R}_0.$$

(iii) ***(Visible subspace is preserved).*** *If, in addition, the map $\lambda \mapsto e^{\lambda \Delta t}$ is injective on $\sigma(A)$ (i.e., no aliasing on the whole spectrum), then for every $\mathrm{x}_0 \in \mathbb{R}^n$,*

$$V_{dt}(\mathrm{x}_0) := \operatorname{span}\left\{[\mathrm{x}_0\ B_d],\ A_d[\mathrm{x}_0\ B_d],\ \ldots\ , A_d^{n-1}[\mathrm{x}_0\ B_d]\right\} = V_{ct}(\mathrm{x}_0).$$

Proof sketch. See Appendix A.2.1.

Input constraints When control space is constrained to $\mathcal{U}_c \subseteq \mathcal{C}([0,\infty), \mathbb{R}^m)$, the same constructions apply with $\mathcal{U}$ replaced by $\mathcal{U}_c$, shrinking the reachable directions accordingly. We formalize this below and use it in §4.2.

Definition 2.22. ***(Input-restricted reachable set).*** *For a class $\mathcal{U}_c \subseteq \mathcal{C}([0,\infty), \mathbb{R}^m)$, the input-restricted reachable set from x_0 in time t is*

$$\mathcal{R}^t(\mathrm{x}_0 \,|\, \mathcal{U}_c) := \left\{\mathrm{x}_f \in \mathcal{X} \;\middle|\; \exists \mathrm{u} \in \mathcal{U}_c : \mathrm{x}_f = \varphi(t \,|\, A, B, \mathrm{x}_0, \mathrm{u})\right\}. \tag{2.22}$$

Analogously to Definition 2.10, we define $\mathcal{R}(\mathrm{x}_0 \,|\, \mathcal{U}_c) := \bigcup_{t \in \mathbb{R}_{\geq 0}} \mathcal{R}^t(\mathrm{x}_0 \,|\, \mathcal{U}_c)$.

2.3.3. Linear Algebra

We denote the field of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. For a linear transformation represented by a matrix $A : \mathbb{R}^n \to \mathbb{R}^n$, we denote its null space (kernel) as $\ker(A)$ and its range (image) as $\operatorname{Im}(A)$. The set of all eigenvalues (spectrum) of A is denoted by $\sigma(A)$.

A (linear) subspace $V \subseteq \mathbb{R}^n$ is a nonempty subset closed under linear combinations: $\forall v, u \in V;\ \alpha, \beta \in \mathbb{R} : \alpha v + \beta u \in V$. In particular, $\mathbf{0} \in V$, and V is an affine subspace passing through the origin.

We say that a subspace V is $A-$invariant if,

(i) $\forall v \in V : Av \in V$, or, equivalently,

(ii) $AV \subseteq V$, where $AV = \{Av : v \in V\}$.

We will frequently encounter invariant subspaces such as $\ker(A), \operatorname{Im}(A)$, and (generalized) eigenspaces.

Remark 2.23 (Intuition). *Once inside an A-invariant subspace V, iterating A any finite number of times never exits V; that is, $A^k v \in V$ for all $v \in V$ and $k \geq 0$.*

We also define $[A\ B]-$invariance, which extends this idea to linear transformations $T : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and can be understood as "controlled invariance" in the context of dynamical systems, as we will see later.

Theorem 2.24. ***($[A\ B]-$invariant subspaces).*** *Let V be a subspace of $\mathbb{R}^n$ and and $[A, B]$ a block matrix consisting of $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Then, the following are equivalent [24]:*

(i) *V is $[A\ B]-$invariant.*

(ii) *Let $P : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the canonical projection onto the first n coordinates. Then, there exists a subspace S of $\mathbb{R}^{n+m}$ with $V = PS = \{Ps : s \in S\}$, and an extension of $[A\ B]$, i.e.,*

$$T := \begin{bmatrix} A & B \\ E_1 & E_2 \end{bmatrix};\ E_1 \in \mathbb{R}^{m\times n},\ E_2 \in \mathbb{R}^{m\times m},$$

under which S is invariant, i.e., $TS \subseteq S$.

(iii) *The subspace V satisfies $AV \subset V + \mathrm{Im}(B)$. This characterization is also referred to as V being $A-$invariant modulo $\mathrm{Im}(B)$.*

(iv) *There is a transformation $F : \mathbb{R}^n \to \mathbb{R}^m$ such that $(A + BF)V \subset V$.*

The concept of $[A\ B]-$invariant subspaces is central to geometric control theory and will be critical in later sections when analyzing constraints on system identification.

2.3.4. Signal Properties and Data Informativity

In-principle (structural) identifiability questions can be answered using the tools we have reviewed so far. In practice, the success of estimators depends of whether the provided trajectory (or sets thereof) carries enough independent directions of information. In the controlled setting, this depends first and foremost on whether the control input is *persistently exciting* of sufficient order. Constraining inputs *(i)* pointwise through, e.g. hard actuation limits, or *(ii)* nonlocally, e.g. through energy or frequency specifications may lead to a loss of informativity, ultimatelty resulting in a control signal that fails to excite all modes of the system. In the following, we provide concrete definitions of what constitutes a sufficiently exciting signal.

Intuitively, a signal is persistently exciting, if, over a fixed window of time, it *spreads* energy in all directions of the state space, so that no direction is systematically missed.

In discrete time, PE of order L asks that the Hankel matrix built from L consecutive shifts of u have full row rank: the shifted copies $\{\mathrm{u}(k), \ldots, \mathrm{u}(k+L-1)\}$ are linearly independent and excite all directions.

Definition 2.25. ***(Persistence of Excitation).*** *[25] The control input sequence $(u(1), \ldots, u(T))$ with $\mathrm{u}(\cdot) \in \mathbb{R}^m$ is said to be persistently exciting of order L, if the Hankel matrix $\mathscr{H}_L(u)$ has full row rank:*

$$\mathscr{H}_L(u) = \begin{bmatrix} u(1) & u(2) & \cdots & u(T-L+1) \\ u(2) & u(3) & \cdots & u(T-L+2) \\ \vdots & \vdots & \ddots & \vdots \\ u(L) & u(L+1) & \cdots & u(T) \end{bmatrix}, \quad \operatorname{rank}(\mathscr{H}_L(u)) = mL.$$

In continuous time, this principle of *sustained multi-directional variation of* $\mathrm{u}(\cdot)$ is formalized as the windowed input Gramian $\int_t^{t+T_0} \mathrm{u}(s)\mathrm{u}(s)^\top ds$ being positive definite: every unit vector v sees a nontrivial projection energy $\int_t^{t+T_0} \langle v, u(s)\rangle^2 ds \geq \alpha > 0$ for all t.

Definition 2.26 (Continuous-time PE)**.** *A locally square-integrable input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is* PE on windows of length T_0 with level $\alpha > 0$ *if*

$$\int_t^{t+T_0} u(s)u(s)^\top ds \;\succeq\; \alpha I_m \qquad \textit{for all } t \geq 0.$$

Equivalently, for every unit vector v, the energy $\int_t^{t+T_0} \langle v, u(s)\rangle^2 ds$ is uniformly bounded below (See [26] and [27] for related formulations).

Remark 2.27. ***(Signal-only vs. system-relative PE).*** *In the classical rank-based definition , PE is a property of the input sequence alone (no algorithm or model enters the definition)—what changes with the task is how large L must be. In contrast, more recent works [28] have defined persistence of excitation more broadly as the uniqueness of the trajectory generated by the input-output map of the system from the given control input, in which case the system can be uniquely identified from the system response.*

Admissibility constraints on the input signal– such as pointwise limits, energy budgets, or bandlimits– can constrain the degree of PEness attainable in a given finite time horizon. To aid the theoretical analysis of how such constraints influence what is learnable from a single trajectory, we propose a partitioning of input constraints as *pointwise* and *nonlocal*. The motivation behind this is the fact that input admissibility criteria that can be expressed directly in terms of *admissible input directions* blend in very naturally with the framework introduced in Section 4.1, whereas input constraints that can only be formulated as properties of the whole input trajectory require a different analytical toolbox.

Definition 2.28. ***(Pointwise vs. nonlocal constraints).*** *Fix a signal space $\mathcal{U} \subseteq \mathcal{C}([0,T],\mathbb{R}^m)$. Let $K \subset \mathbb{R}^m$ be a measurable set with $0 \in K$. A pointwise constrained input set can be defined as*

$$\mathcal{U}_{pt}(K) := \{\mathrm{u} \in \mathcal{U} : \mathrm{u}(t) \in K \text{ a.e. on } [0,\infty)\}.$$

A nonlocal constraint, on the other hand, is any $\mathcal{U}_c \subset \mathcal{U}$ specified by trajectory-level properties (e.g., energy bounds, bandlimits, PE order).

We introduced persistency of excitation to facilitate the analysis of input-constrained LTI systems. In §4.2, we will relate this input richness property to the question of what is *partially* identifiable through a known result that links the order of PEness to how many Markov parameters can be identified from data.

Markov parameters. For CT–LTI systems the variation–of–constants formula

$$\mathrm{x}(T) = e^{AT}\mathrm{x}_0 + \int_0^T e^{A(T-s)}B\mathrm{u}(s)\,ds$$

combined with the power series $e^{A(T-s)} = \sum_{k=0}^{\infty} A^k \frac{(T-s)^k}{k!}$ and termwise integration yields the *moment expansion*

$$x(T) = e^{AT}x_0 + \sum_{k=0}^{\infty} A^k B\,\psi_k(u), \qquad \psi_k(u) := \int_0^T \frac{(T-s)^k}{k!}\,u(s)\,ds, \tag{2.23}$$

valid for all piecewise–continuous u. The matrices $\{A^kB\}_{k\geq 0}$ form the reachability sequence, which we also refer to as the state-Markov parameters (formed by setting $C = I$ in the original $M_k = CA^{k-1}B$) [18, 19].

Definition 2.29 (Moment–PE at horizon T)**.** *Let $\mathcal{U}_c \subset L^2([0,T];\mathbb{R}^m)$ be an admissible input class. For $r \in$, define*

$$\mathcal{M}_r : \mathcal{U}_c \to \mathbb{R}^{mr}, \qquad \mathcal{M}_r(u) := \big(\psi_0(u),\ldots,\psi_{r-1}(u)\big).$$

We say $\mathcal{U}_c$ is moment–PE of order r at T *if* span $\mathcal{M}_r(\mathcal{U}_c) = \mathbb{R}^{mr}$ *(equivalently, the associated Gram matrix is $\succeq \alpha I_{mr}$ for some $\alpha > 0$).*

Proposition 2.30 (Moment–PE $\Rightarrow$ low-order Markov-parameter identifiability)**.** *Fix $T > 0$ and suppose $0 \in \mathcal{U}_c$ and $\mathcal{U}_c$ is moment–PE of order r at T. If two CT–LTI pairs (A,B) and $(\tilde{A},\tilde{B})$ generate identical input–state maps on $\mathcal{U}_c$ over $[0,T]$, i.e.*

$$\int_0^T e^{A(T-s)}Bu(s)\,ds \;=\; \int_0^T e^{\tilde{A}(T-s)}\tilde{B}\,u(s)\,ds \quad \text{for all } u \in \mathcal{U}_c,$$

then their Markov parameters coincide up to order $r-1$:

$$A^k B \;=\; \tilde{A}^k \tilde{B}, \qquad k = 0, 1, \ldots, r-1.$$

Proof sketch. *Using* (2.23) *for both systems and subtracting gives* $\sum_{k\geq 0}(A^k B - \tilde{A}^k \tilde{B})\, \psi_k(u) = 0$ *for all* $u \in \mathcal{U}_c$. *By moment–PE of order* r, *one can choose inputs whose* $(\psi_0, \ldots, \psi_{r-1})$ *realize any vector in* $\mathbb{R}^{mr}$, *forcing* $A^k B = \tilde{A}^k \tilde{B}$ *for* $k = 0, \ldots, r-1$. □

2.3.5. System Identification

We consider a discrete-time LTI system without direct feedthrough (matrix D) as our base (state-space) model:

$$\mathrm{x}[k+1] = A_d \mathrm{x}[k] + B_d \mathrm{u}[k] + \eta_k, \quad \mathrm{y}[k] = C_d \mathrm{x}[k] + \varepsilon_k, \quad \mathrm{x}[0] = \mathrm{x}_0.$$

Let $(t_0, \ldots, t_N)$ be a sequence of points in time. The single-trajectory system identification problem generally asks/answers this question: Given a trajectory of input-output measurements of the system $\mathcal{D} = \{(\mathrm{y}_k, \mathrm{u}_k) : k \in [0, T-1]\}$, can we identify the system (A, B, C) which produced it?

In our problem setting, where we assume unperturbed, full state measurements, this question becomes: Given an input-state trajectory $\{(\mathrm{x}_k, \mathrm{u}_k) : k \in [0, T-1]\}$ generated by the system

$$\mathrm{x}[k+1] = A_d \mathrm{x}[k] + B_d \mathrm{u}[k], \quad \mathrm{x}[0] = \mathrm{x}_0,$$

can we uniquely determine (A_d, B_d)? Essentially, this is an inverse problem.
Let us defined the stacked data matrices X, X_+, U:

$$X = [\mathrm{x}_0, \ldots, \mathrm{x}_{T-1}], \quad X_+ = [\mathrm{x}_1, \ldots, \mathrm{x}_T], \quad U = [\mathrm{u}_0, \ldots, \mathrm{u}_{T-1}].$$

The dynamics equation $\mathrm{x}[k+1] = A_d \mathrm{x}[k] + B_d \mathrm{u}[k]$ implies

$$X_+ = A_d X + B_d U = [A_d \;\; B_d] \begin{bmatrix} X \\ U \end{bmatrix} := \Theta Z,$$

where Θ is the parameter of interest, and Z our regressor. Provided that Z is full-rank and right-invertible, We will be able to compute Θ uniquely as $\Theta = X_+ Z^\dagger$.

Formulations of the identification problem vary along several axes: *(i)* whether state and input dimensions (thus the partition $\Theta \mapsto (A_d, B_d)$) are known, *(ii)* full vs. partial state observations, *(iii)* noise models, *(iv)* closed vs. open-loop, *(v)* number and length of the trajectories. In this work, we fix the following setting: the dimensions (n, m) are known, observations are full state and noise-free on the provided trajectory,

the trajectory is sufficiently long to guarantee the required order of PEness under a fixed control signal class, and the goal of the estimator is solely to recover Θ from the single-trajectory $\mathcal{D} = \{(\mathrm{x}_k, \mathrm{u}_k) : k \in [0, T-1]\}$.

Under the same dataset and informativity, all consistent estimators identify the same model on the excited subspace. Algorithmic choices influence sample complexity, noise–robustness, and closed–loop bias/variance, not what is identifiable in principle from a single trajectory. Thus, the choice of the estimator is immaterial to structural identifiability analysis. Nevertheless, for completeness, we briefly review the system identification algorithms used to produce the results in Chapter 5.

- **DMDc (Continuous Dynamic Mode Decomposition with Control). [29]** With full state-control observations, estimate the block matrix $\Theta = [A_d \;\; B_d]$ via

  $$\hat{\Theta}_{\mathrm{LS}} = X_+ Z^{\dagger}, \quad \hat{\Theta}_{\mathrm{ridge}} = X_+ Z^{\top} \left(Z Z^{\top} + \lambda I \right)^{-1}, \quad \hat{\Theta}_{\mathrm{tSVD}} = X_+ \left(U_r \Sigma_r^{-1} V_r^{\top} \right),$$

  where $Z = U \Sigma V^{\top}$ and (U_r, Σ_r, V_r) denote the $r-$truncated SVD.

- **MOESP (Multivariable Output-Error State-Space). [30]** Subspace method estimating a minimal realization. In the full-state, noiseless specialization considered here, MOESP reduces to the same least squares fit on $X_+ = \Theta Z$.

- **SINDy (Sparse identification of nonlinear dynamics). [10]** Postulate a model class (library) $\phi(\mathrm{x}, \mathrm{u})$ and fit sparse coefficients Γ:

  $$\dot{\mathrm{x}}(t) \approx \phi(\mathrm{x}(t), \mathrm{u}(t))\Gamma \;\; \text{(cont. time)} \quad \mathrm{x}_{k+1} - \mathrm{x}_k \approx \phi(\mathrm{x}_k . \mathrm{u}_k)\Gamma \;\; \text{(discr. time)}.$$

  In the linear setting, choosing $\phi(\mathrm{x}, \mathrm{u}) = [\mathrm{x}^{\top} \; \mathrm{u}^{\top}]$ reduces to linear regression.

- **NODE (Neural ordinary differential equations). [11]** Parametrize a continuous-time vector field $\dot{\mathrm{x}} = f_{\theta}(\mathrm{x}, \mathrm{u}; t)$ and learn θ by minimizing a trajectory mismatch via adjoint/sensitivity methods. In our experiments, we use a linear instantiation, which is equivalent to DMDc on $X_+ = \Theta Z$.

3. Prior Work

Definitions of identifiability vary across the literature. In this chapter, we survey representative lines of work, with emphasis on invariant subspace-based analyses that most directly inform our approach.

3.1. Classical View

The literature mainly distinguishes between two types of identifiability, based on the specific assumptions. *Structural* identifiability is concerned with whether the map from the parameter space defining the system to input-output trajectories generated by it is injective. It is considered a property of the model's inherent structure, and represents a theoretical prerequisite to any successful parameter estimation. *Practical* identifiability, on the other hand, refers to whether a given set of actual experimental data (with noise, finite time horizon, discrete sampling, etc.) is sufficient to reliably estimate the parameters. Identifiability analysis is widely regarded as the first criticacal step before fitting a model to data, ensuring that the model is capable of yielding a unique parameter solution if perfect data were available. If a model is not identifiable, this may result in misleading parameter estimates.

In their study from 1982, Nguyen et al. [31] give a comprehensive review of various *linear* identifiability concepts that have been proposed in the literature up to that point. As they point out, initial condition-based identifiability definitions go as far back as 1964, to a book by Lee et al. [32], where they propose that a system $\mathcal{S} \in \Sigma_s^{dt}$,

$$\Sigma_s^{dt} = \begin{cases} \mathrm{x}(t+1) = A\mathrm{x}(t), \\ \mathrm{y}(t) = \mathrm{x}(t), \\ \mathrm{x}(0) = \mathrm{x}_0 \in \mathbb{R}^n, \end{cases}$$

is identifiable if the matrix $\left[\mathrm{x}_0 \,|\, A\mathrm{x}_0 \,|\, \ldots \,|\, A^{n-1}\mathrm{x}_0\right]$ is non-singular, with the physical interpretation that x_0 must excite all modes of the system.

Proposition 3.1. ***(Classical identifiability principle).*** *Consider a finite-dimensional, time-invariant LTI system with input* u *and (possibly unknown) order* n*. Suppose the system is minimal (controllable and observable) and the input is sufficiently exciting on a window of*

sufficient length (i.e., the relevant Hankel/Gramian matrices are full rank). Then the system's impulse response is uniquely determined by the data, and any minimal realization consistent with the data is unique up to similarity [33].

Corollary 3.2 (Full-state specialization)**.** *Under $C = I$ (and $D = 0$), observability is automatic, and minimality reduces to controllability. Since $C = I$ also fixes the state basis, for a sufficiently exciting input, (A, B) is uniquely identified.*

Takeaway

LTI models are uniquely identifiable from a single trajectory on a sufficiently long window provided that: *(i)* the input is persistently exciting of adequate order (full-rank Hankel/Gramian), *(ii)* measurements are noise-free, and *(iii)* the realization is minimal (controllable and observable). Under $C = I$ this reduces to controllability, and (A, B) is unique (up to the fixed state basis). This informs our branching in Chapter 4.

3.2. Recent Works

Some recent works [15],[16] on the identifiability of autonomous LTI systems have produced additional criteria based on the initial state. The system class considered is given as

$$\Sigma_s = \begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Essentially, Stanhope et al. have shown that, two systems with different dynamics matrices A, A' with shared initial state x_0, can produce the same trajectory given that this initial state x_0 is confined to a proper (i.e., with dimension less than that of the full $\mathcal{X}$) subspace of $\mathcal{X}$.
They say that the autonomous LTI model (3.1) is identifiable in a set $\Omega \subseteq \mathbb{R}^{n\times n}$ iff for all $A, A' \in \Omega$ with $A \neq A'$, an initial condition $x_0 \in \mathbb{R}^n$ can be found that differentiates the trajectories at some point in time $t \in \mathbb{R}_+$. Equating a system instance $\mathcal{S}$ with the dynamics matrix that defines it, we formalize this definition as:

$$\begin{aligned} \mathcal{P}_1^{\text{Stanhope}}\left(\Sigma_s \,|\, \Omega\right) \iff & \forall A, A' \in \Omega, A \neq A' : \\ & \exists x_0 \in \mathbb{R}^n : \exists t \in \mathbb{R}_+ : \varphi\left(t \,|\, A, x_0\right) \neq \varphi\left(t \,|\, A', x_0\right). \end{aligned}$$

They than go on to prove that the system class Σ_s is identifiable in all of $\mathbb{R}^{n\times n}$, so $\mathcal{P}_1^{\text{Stanhope}}\left(\Sigma_s \,|\, \mathbb{R}^{n\times n}\right)$.

To make this criterion more practical (in most practical applications, the initial state can't be chosen freely), they introduce model identifiability from a specific initial state x_0 as:

$$\mathcal{P}_2^{\text{Stanhope}}\left(\Sigma_s \,|\, \Omega, x_0\right) \iff \forall A, A' \in \Omega, A \neq A' : \varphi\left(t \,|\, A, x_0\right) \neq \varphi\left(t \,|\, A', x_0\right), \tag{3.2}$$

so the previous definition can also be formulated as

$$\mathcal{P}_1^{\text{Stanhope}}\left(\Sigma_s \,|\, \Omega\right) \iff \exists x_0 \in \mathbb{R}^n : \mathcal{P}_2^{\text{Stanhope}}\left(\Sigma_s \,|\, \Omega, x_0\right).$$

For completeness, they also introduce unconditional identifiability, which has less practical relevance. The call the autonomous LTI model (3.1) unconditionally identifiable in Ω, if for all pairs of system matrices in Ω, their inequality implies that for all possible initial states, at one time point their trajectories will diverge:

$$\begin{aligned}&\mathcal{P}_3^{\text{Stanhope}}\left(\Sigma_s \,|\, \Omega\right) \iff \\ &\forall A, A' \in \Omega : (A \neq A) \implies \left(\forall x_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\} : \exists t \in \mathbb{R}_+ : \varphi(t \,|\, A, x_0) \neq \varphi(t \,|\, A', x_0)\right).\end{aligned}$$

They define the largest open set, in which Σ_s is unconditionally identifiable as the following intersection:

$$\Omega_{\Sigma_c} := \bigcap_{x_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \Omega_{x_0}, \tag{3.3}$$

where Ω_{x_0} denotes the largest open set in which Σ_c is identifiable:

$$\mathcal{P}_2^{\text{Stanhope}}\left(\Sigma_c \,|\, \Omega_{x_0}, x_0\right) \wedge \neg\exists \Omega' \supseteq \Omega_{x_0} : \mathcal{P}_2^{\text{Stanhope}}\left(\Sigma_c \,|\, \Omega', x_0\right),$$

or just

$$\Omega_{x_0} = \left\{A \in \mathbb{R}^{n \times n} \middle| \dim\left({}_n(A, x_0)\right) = n\right\}.$$

Thus, also in any subset of Ω_{x_0}, Σ_s will be identifiable from x_0:

$$\forall\, \Omega' \subseteq \Omega_{x_0} : \mathcal{P}_2^{\text{Stanhope}}\left(\Sigma_c \,|\, \Omega', x_0\right).$$

A practical criterion for identifiability from a fixed initial state in the sense of (3.2) can be formulated equivalently as

$$\mathcal{P}_2^{\text{Stanhope}}\left(\Sigma_s \,|\, \Omega, x_0\right) \iff \forall A \in \Omega : \left\{x_0, Ax_0, \ldots, A^{n-1}x_0\right\} \text{ are linearly indep.} \tag{3.4}$$

$$\iff \forall A \in \Omega : \dim\left(\mathcal{K}_n(A, x_0)\right) = n, \tag{3.5}$$

$$\iff \forall A \in \Omega : \dim\left(\operatorname{Im}\left[x_0 \;\; Ax_0 \;\; \ldots \;\; A^{n-1}x_0\right]\right) = \operatorname{rank}\left(\left[x_0 \;\; Ax_0 \;\; \ldots \;\; A^{n-1}x_0\right]\right) = n, \tag{3.6}$$

where $\mathcal{K}_n(A, \mathrm{x}_0) = \operatorname{span}\left\{\mathrm{x}_0, A\mathrm{x}_0, \ldots, A^{n-1}\mathrm{x}_0\right\}$ denotes the $A-$cyclic (Krylov) subspace generated by x_0. One of the key contributions of Stanhope et al. [15] in this study come from the following two observations

$$\dim\left(\mathcal{K}_n(A, \mathrm{x}_0)\right) < n \iff \mathrm{x}_0 \in L \text{ with:} \tag{3.7}$$

1. L is a proper subspace of $\mathbb{R}^n$, i.e., $\dim L < n$.
2. L is $A-$invariant, i.e.: $\forall \mathrm{x} \in L : A\mathrm{x} \in L$, or, equivalently, $AL \subseteq L$,

and the orbit $\gamma(A, \mathrm{x}_0) = \{\varphi(t \,|\, A, \mathrm{x}_0) : t \in \mathbb{R}_{\geq 0}\}$ is confined to a proper subspace of $\mathbb{R}^n$ iff x_0 is contained in an $A-$invariant proper subspace of $\mathbb{R}^n$:

$$\exists\, V : AV \subseteq V, \dim V < n\,, \gamma(A, \mathrm{x}_0) \in V \iff \exists\, L : AL \subseteq L\,, \dim L < n\,, \mathrm{x}_0 \in L. \tag{3.8}$$

These two observations together give an alternative criterion for identifiability from a given initial state in the sense of (3.2):

$$\begin{aligned}
\mathcal{P}_2^{\text{Stanhope}}\left(\Sigma_s \,|\, \Omega, \mathrm{x}_0\right) &\overset{(3.5)}{\iff} \forall A \in \Omega : \dim\left(\mathcal{K}_n(A, \mathrm{x}_0)\right) = n && (3.9)\\
&\overset{(3.7)}{\iff} \forall A \in \Omega : \exists L : AL \subseteq L,\ \dim L < n,\ \mathrm{x}_0 \in L && (3.10)\\
&\overset{(3.8)}{\iff} \forall A \in \Omega : \exists V : AV \subseteq V,\ \dim V < n,\ \gamma(A, \mathrm{x}_0) \in L, && (3.11)
\end{aligned}$$

meaning the model (3.1) is identifiable in an open set Ω iff for all A (again, using interchangeably with $\mathcal{S}$,) in Σ_c, the orbits aren't confined to a proper $A-$invariant subspace of $\mathbb{R}^n$. If the orbit $\gamma(A, \mathrm{x}_0) = \{\varphi(t|A, \mathrm{x}_0) : t \geq 0\}$ is not contained in any proper A-invariant subspace, then the trajectory uniquely determines the parameter matrix A within the mode class (3.1).

All results discussed so far consider the whole system class, and not a specific instance defined by a particular system matrix A. Although not introduced as a standalone definition, they establish the identifiability of a system instance A as the existence of an initial state x_0 such that the matrix $[\mathrm{x}_0 \;\; A\mathrm{x}_0 \;\; \ldots \;\; A^{n-1}\mathrm{x}_0]$ has full rank:

$$\begin{aligned}
\mathcal{P}_4^{\text{Stanhope}}(A) &\iff \exists \mathrm{x}_0 \in \mathbb{R}^n : \{\mathrm{x}_0, \ldots, A^{n-1}\mathrm{x}_0\} \text{ lin. indep.}\\
&\iff \exists \mathrm{x}_0 \in \mathbb{R}^n : \dim \mathcal{K}_n(A, \mathrm{x}_0) = n\\
&\iff \exists \mathrm{x}_0 \in \mathbb{R}^n : A \text{ has only one Jordan bl. per eigenvalue (*)} && (3.12)\\
&\iff \neg\exists v \in \mathbb{R}^n, \lambda \in \mathbb{R} : (vA = \lambda v) \wedge (v^\top \mathrm{x}_0 = 0) \text{ (**)} && (3.13)
\end{aligned}$$

(*) rephrase after introducing Jordan decomposition.
(**) rephrase after introducing the Popov-Belevitch-Hautus controllability test for linear

systems. They use the result in (3.13) to derive an alternative characterization of Ω_{Σ_s}:

$$\Omega_{\Sigma_s} = \bigcap_{x_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \Omega_{x_0} =$$
$$\left\{ A \in \mathbb{R}^{n \times n} \middle| \, \forall x_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\} : \left(\exists \lambda \in \mathbb{R} : w^\top A = \lambda w \right) \implies w^\top x_0 \neq 0 \right\}.$$

Another important contribution from this study is the definition of partial identifiability, which seeks to answer the question: "If Σ_s isn't identifiable in all of $\mathbb{R}^{n \times n}$ for a specific initial state x_0 (meaning $\mathbb{R}^{n \times n} \setminus \Omega_{x_0} \neq \emptyset$), how much information about any given $A \in \mathbb{R}^{n \times n}$ can be obtained from a single trajectory starting from x_0?" This is formalized by the following theorem.

Theorem 3.3. ***(Partial identifiability [15]).*** *Let $A|_V$ denote the restriction of A to V, a proper linear $A-$invariant subspace of $\mathbb{R}^n$ with $\dim V = k < n$. Then, the following statements are equivalent:*

(i) *V is the minimal $A-$invariant subspace that contains x_0.*

(ii) *$\{x_0, \ldots, A^{k-1}x_0\}$ are linearly independent in V.*

(iii) *There exists no $A' \in \mathbb{R}^{n \times n}$ with a projection on V distinct from that of A ($A'|_V \neq A|_V$) such that the trajectories, starting from x_0 never diverge ($\forall\ t \in \mathbb{R}_{\geq 0} : \varphi(t \,|\, A', x_0) = \varphi(t \,|\, A, x_0)$).*

Especially, the last statement indicates that the projection of A on to the minimal $A-$invariant subspace containing x_0 can be uniquely identified.

Finally, they translate these results, obtained under the assumption that continuous-time, unperturbed measurements of the system are available, to a more realistic setting where only a set of noisy sample of data points on a trajectory are available instead of a perfect, continuous measurement of the true trajectory. They formulate the following criterion:

$$\neg \exists V : AV \subseteq V,\ \dim V < n,\ \gamma(A, x_0) \in V \iff \tag{3.14}$$
$$\exists \{t_i\}_{i=1}^n : \{x_j\}_{j=1}^n := \{\varphi\left(t_i \,|\, A, x_0\right)\}_{i=1}^n \text{ are linearly independent.} \tag{3.15}$$

They call an autonomous LTI globally identifiable in a subset $\Omega \subset \mathbb{R}^{n \times n}$, iff for all pairs of distinct state transition matrices in Ω, there exists an initial state $x_0 \in \mathbb{R}^n$ that lead to non-identical trajectories:

$$\mathcal{P}_1^{\text{Qiu}}(\Sigma_s \,|\, \Omega) \iff \forall A \neq A' \in \Omega : \exists x_0 \in \mathbb{R}^n : \varphi\left(t \,|\, A, x_0\right) \neq \varphi\left(t \,|\, A', x_0\right),$$

which is equivalent to $\mathcal{P}_1^{\text{Stanhope}}(\Sigma_s \,|\, \Omega)$.
Qiu et al. [16] revisit the same autonomous setting and adopt compatible definitions. They make two substantive additions:

1. **A closed-form description of unidentifiable classes at x_0.** Let $A = QJQ^{-1}$ be a real Jordan-like decomposition. Transform the initial state x_0 to $\tilde{x_0} = Q^{-1}x_0$ and group coordinates by invariant blocks. If some block receives no excitation through x_0, the corresponding diagonal selector I_0 flags that block. Their *Theorem 2.5* shows that the equivalence class of systems that share exactly the same trajectory from x_0 is

$$[A]_{x_0} = \left\{ A + Q\left(I_0 D I_0\right) Q^{-1} : D \in \mathbb{R}^{n\times n} \right\},$$

   i.e., degrees of freedom live precisely inside the unexcited invariant blocks. This sharpens the "partial identifiability" story by making the freedoms explicit.

2. **Computable single-trajectory criteria and practical scores.** Assuming distinct eigenvalues, their *Theorem 2.4* introduces the initial-condition identifiability score (ICIS) which they show to be nonzero exactly when (A, x_0) is identifiable. They also propose two trajectory-level practical metrics built from the full trajectory via two-stage (derivative/state) inner product quantities: a smoothed condition number and a practical identifiability score that approximates MSE. In simulations, SCN and PIS are reported to outperform Stanhope's κ under noise. (Stanhope's κ is the condition number of the problem of computing $\Phi(\Delta t) = X_2 X_1^{-1}$, thus the condition number $\kappa(X_1) = \|X_1\|\|(X_1)^{-1}\|$ of the data matrix $X_1 = [x_1 \cdots x_n]$.)

These additions directly inform out controlled-identifiability analysis.

Takeaway

The invariant subspace picture delivers two durable takeaways that we carry into the controlled case:

- **Visibility governs identifiability.** What the trajectory reveals is exactly the restriction of the dynamics to the $A-$invariant subspace excited by the "inputs" in a generalized sense.
- **Equivalence classes factor along unexcited directions.** In the autonomous case, Qiu's $[A]_{x_0} = A + Q(I_0 D I_0)Q^{-1}$ isolates freedoms in unexcited invariant blocks. In Chapter 4, we prove the controlled analogue, where the selector is built from the union of x_0 and $\mathrm{Im}(B)$, recovering Qiu when $B = 0$.

4. Theoretical Analysis

Core Question

For fully observed LTI systems, *controllability* together with a *persistently exciting* input ensures identifiability from any initial condition. This chapter asks what remains uniquely identifiable when one (or both) of these assumptions fails: *for which initial states* x_0 *does single–trajectory identification still succeed, and when it does not, which components of the system are nonetheless recoverable?* Our goal is to characterize these partially identifiable components via explicit *equivalence classes*.

It is a widely accepted fact that, LTI systems are uniquely identifiable from any trajectory, starting form any initial state, provided that *(i)* the control input that produced the trajectory is sufficiently rich, *(ii)* the measurements are noise-free, and *(iii)* the system itself is both controllable and observable (see Proposition 3.1). We omit the latter since our assumption of full observability ($C = I$) trivially fulfills this condition.

These conditions are often separated as *structural identifiability* and *practical identifiability*. Structural identifiability is concerned with system-intrinsic properties such as controllability and observability, whereas practical identifiability is ultimately concerned with data informativity, including factors such as control input richness or the presence of noise. For our purposes, we interpret this as a dichotomy that *(a)* separates potential sources of non-identifiability, and *(b)* informs our hypotheses on what shapes the "inner structure" of the equivalence classes we aim to develop. We formalize this through the following theorem.

Theorem 4.1. ***(Single-trajectory identifiability dichotomy).*** *Fix* x_0 *and an ambient input class* $\mathcal{C}([0,\infty),\mathbb{R}^m)$.

- *If* (A,B) *is controllable, under* $\mathcal{U} = \mathcal{C}([0,\infty),\mathbb{R}^m)$*, the controllable subspace* $\mathcal{R}(0)$ *(hence also the visible subspace) will be all of* $\mathbb{R}^n$ *which will collapse the* x_0-*equivalence class to a singleton:* $[A,B]_{x_0} = \{(A,B)\}$*. Thus, any nontrivial structure that might be present in* $[A,B]_{x_0}$ *must relate to the structure of the restricted control input class* $\mathcal{U}_c \subsetneq \mathcal{C}\left([0,\infty),\mathbb{R}^m\right)$*, resulting in a loss of input richness.*

- *If* (A, B) *is uncontrollable, then*

$$[A, B]_{x_0} = \left\{ (A', B') : B' = B,\ A'\big|_V = A\big|_V \right\},$$

with $V := \mathcal{K}(A, x_0) + \mathcal{K}(A, B)$ *(Definition 2.6). Thus, even with sufficiently rich data, only the* $V-$*subsystem is identifiable from* $(x_0, \mathcal{U})$*. The dynamics on any complementary subspace* W *with* $V \oplus W = \mathbb{R}^n$*) will be "invisible" from this particular* x_0*.*

Proof. See Appendix A.2.2 and Appendix A.2.2. □

This suggest that the structure of $[A, B]_{x_0}$ depends on whether (A, B) is controllable. Since the objective of this chapter is to develop explicit x_0-equivalence classes, we treat the controllable and uncontrollable cases separately.

4.1. Identifiability under Sufficiently Rich Control Inputs

In this section, we assume that the control input is persistently exciting of sufficient order and over a sufficiently long horizon. Thus, anything that makes $[A, B]_{x_0}$, not a singleton thus not fully identifiable, must be related to the system itself. We propose that any system matrix pair (A, B) can be identified up to its component on $V(x_0)$, the subspace "visible" from initial state x_0. Systems that span the same visible subspace from a particular x_0 will be indistinguishable on a single-trajectory basis, which we formalize as x_0-equivalence. We further characterize the matrix pairs spanning the same visible subspace from x_0 through equivalence *classes*, for which we derive a closed-form expression. Controllable systems (under sufficiently rich control inputs) are always identifiable, regardless of the choice of x_0. In the language of equivalence classes, this can be expressed as the equivalence class collapsing to a singleton, i.e., $[A, B]_{x_0} = \{(A, B)\}$.

Further, we propose quantitative x_0-identifiability criteria that approximate "how close" a given system (A, B) is to being unidentifiable from that particular x_0. These criteria are obtained by interpreting the homogeneous response as *"initial excitation"*, and extending classical controllability tests by replacing B with $[x_0\ B]$. The same augmentation principle is used to obtain an x_0-enriched Kalman decomposition: on $V(x_0)$, the dynamics are identifiable under sufficiently rich inputs, while the orthogonal complement is provably invisible form x_0.

4.1.1. Characterization of Indistinguishable System Matrices

For a single observed trajectory, different system matrices can produce the same trajectories. We characterize when a pair of distinct matrices $(\tilde{A}, \tilde{B}) \neq (A, B)$ can lead

to the same trajectory, making them indistinguishable. We formalize this through an x_0-equivalence class, defined as follows. For particular values of (A, B, x_0), if this equivalence class is a singleton, we can conclude that the system (A, B)'s trajectory from the given x_0 will be unique, making (A, B) single-trajectory identifiable for this x_0.

Proposition 4.2. ***(Structure of $[A, B]_{x_0}$ under lack of controllability.)*** *Let $[A, B]_{x_0}$ be as in* (2.2)*. Then:*

$$[A, B]_{x_0} = \left\{(A', B') \mid A'\big|_V = A|_V\,,\ B' = B\right\}, \tag{4.1}$$

where $V := \mathcal{K}(A, x_0) + \mathcal{K}(A, B)$ is the smallest $[A, B]-$invariant subspace that contains both x_0 and the image of B. We may also write: $[A, B]_{x_0} = \{\tilde{A} : \tilde{A}P_V = AP_V\} \times \{B\}$, where P_V is the orthogonal projector onto V.

Proof sketch. "$\subseteq$" (necessity). Admissibility of the constant zero input signal enforces the equality of homogeneous responses. The equality of homogeneous responses for all $t \in \mathbb{R}_{\geq 0}$ enforces equality of all derivatives, for every t where they exist. Evaluating an arbitrary order derivative at $t = 0$ gives us $A^k x_0 = \tilde{A}^k x_0$ for all $k \in \mathbb{N}_0$. Equality of the homogeneous responses implies equality between forced responses. Through uniqueness of kernels, we deduce $e^{At}B = e^{\tilde{A}}\tilde{B}$ for all $t \in \mathbb{R}_{\geq 0}$. Again, we consider arbitary-order derivatives, evaluated at $t = 0$ to derive that $A^k B = \tilde{A}^k \tilde{B}$ for all $k \in \mathbb{N}_0$. With $k = 0$, we get $B = \tilde{B}$. Finally, using the argument that, in order for the restrictions of two matrices on a subspace V to agree, it is sufficient that they agree on independent vectors which span V, we derive the subspace-based characterization $\tilde{A}\big|_V = A|_V$ and $\tilde{B} = B$.

"$\supseteq$" (sufficiency). In this direction, we want to show that $\tilde{A}\big|_V = A|_V$ and $\tilde{B} = B$ is sufficient to make the trajectories coincide. We first show that $A-$invariance of V and $\tilde{A}\big|_V = A|_V$ enforces V to be also $\tilde{A}-$ invariant. This helps us show that not only $\tilde{A}\big|_V = A|_V$, but $\tilde{A}^k\big|_V = A^k\big|_V$ for all $k \in \mathbb{N}_0$. Through a series expansion of the matrix exponential, this enforces $e^{\tilde{A}t}\Big|_V = e^{At}\big|_V$ for all $t \in \mathbb{R}_{\geq 0}$. Since $x_0 \in V$ by construction, this trivially implies $e^{\tilde{A}t}\Big|_V x_0 = e^{At}\big|_V x_0$. Following the same argument, we can deduce equality of the forced responses as well, which, with $e^{\tilde{A}t}\Big|_V x_0 = e^{At}\big|_V x_0$, implies that the trajectories coincide. **(See Appendix A.2.2 for the full proof.)**

$\tilde{B} = B$ and $\tilde{A}\big|_V = A|_V$ is a necessary and sufficient condition for $(\tilde{A}, \tilde{B}) \in [A, B]_{x_0}$.

Remark 4.3. ***(Edge cases).*** *If (A, B) is controllable, then $\mathcal{R}(0) = \mathbb{R}^n$, hence $V = \mathbb{R}^n$. The condition $A'|_V = A|_V$ becomes $A' = A$, and since we already have $B' = B$, the equivalence*

class collapses to a singleton: $[A, B]_{\mathrm{x}_0} = \{(A, B)\}$.

For the autonomous dynamics, i.e. $B = 0$*, we recover [15]:*

$$V = \mathcal{K}(A, \mathrm{x}_0) + \mathrm{span}\{A^k B : k \in \mathbb{N}_0\}\Big|_{\mathcal{U}=\{\mathbf{0}\}} = \mathcal{K}(A, \mathrm{x}_0), \text{ and}$$

$\tilde{A}|_{\mathcal{K}(A,\mathrm{x}_0)} = A|_{\mathcal{K}(A,\mathrm{x}_0)}$ *as the partial identifiability condition.*

Through the Kalman controllability decomposition (Lemma 2.19), one obtains a canonical form that splits the state into controllable and uncontrollable coordinates. Interpreting the homogeneous response as *initial excitation*, we analyze the augmented pair $(A, [\mathrm{x}_0\ B])$ and derive an x_0-enriched Kalman decomposition. This yields a basis that separates the state into x_0-visible and x_0--invisible coordinates. The takeaway is that two systems $(A, B) \neq (\tilde{A}, \tilde{B})$ will lead to identical trajectories from x_0, as long as their x_0-visible coordinates agree.

Proposition 4.4. ***(Block structure in a*** (V, W) ***basis).*** *Let* $V := \mathcal{K}(A, \mathrm{x}_0) + \mathcal{R}(0)$ *be the invariant hull (Equation* (2.9)*). Choose a subspace* W *such that* $W \cap V = \{\mathbf{0}\}$*, and* $V + W = \mathbb{R}^n$ *(i.e.,* $V \oplus W = \mathbb{R}^n$*). Let* $\{v_1, \ldots, v_k\}$*,* $k = \dim(V) \leq n$ *(conversely,* $\dim(W) = n - k \leq n$*) be a basis for* V*, and* $\{w_1, \ldots, w_{n-k}\}$ *be a basis for* W*. Build the basis matrix as* $T = \begin{bmatrix} v_1 & \ldots v_k & w_1 & \ldots & w_{n-k} \end{bmatrix}$ *(we know that* T *is invertible, since it has* n *columns form which together form a basis for* $\mathbb{R}^n$*). Build the state vector in this basis as* $z(t) := T^{-1}\mathrm{x}(t)$*, and split it as* $z = (z_V, z_W)$ *with* $z_V = P_V z$ *and* $z_W = P_W z$*. Then:*

1. ***Block forms.*** *Since* $AV \subseteq V$ *and* $\mathrm{Im}(B) \subseteq V$*,*

$$T^{-1}AT = \begin{bmatrix} A_V & A_* \\ 0 & A_W \end{bmatrix},\ T^{-1}B = \begin{bmatrix} B_V \\ 0 \end{bmatrix},\ T^{-1}\mathrm{x}_0 = \begin{bmatrix} \mathrm{x}_{0,V} \\ 0 \end{bmatrix}$$

2. ***Dynamics in*** (V, W) ***coordinates.***

$$\begin{cases} \dot{z}_V(t) = A_V z_V(t) + B_V \mathrm{u}(t) + A_* z_W \\ \dot{z}_W(t) = A_W z_W(t), \\ z_V(0) = \mathrm{x}_{0,V},\ z_W(0) = 0. \end{cases}$$

3. ***Solution curve (confinement).*** *For all* $t \geq 0$*,*

$$z_W(0) = 0, \quad z_V(t) = e^{A_V t}\mathrm{x}_{0,V} + \int_0^t e^{A_V(t-s)} B_V \mathrm{u}(s)\, ds.$$

Hence, $z(t) = [z_V(t),\ 0]^\top$*, and* $\mathrm{x}(t) = Tz(t) = T[z_V(t),\ 0]$*, which implies that* $\mathrm{x}(t)$ *is independent of* A_* *and* A_W*, so long as* x_0 *has a zero* $W-$*component, equivalently,* $\mathrm{x}_0 \in V$*.*

4. ***Pairs with the same restriction on*** V. *If* $(\tilde{A}, \tilde{B})$ *satisfies* $\tilde{A}|_V = A|_V$ *and* $\tilde{B} = B$, *then*

$$T^{-1}\tilde{A}T = \begin{bmatrix} A_V & \tilde{A}_* \\ 0 & \tilde{A}_W \end{bmatrix}, \text{ and } T^{-1}\tilde{B} = \begin{bmatrix} B_V \\ 0. \end{bmatrix}$$

See Appendix A.2.2 for a proof sketch.

Corollary 4.5. *Using the* x_0-*enriched Kalman decomposition developed above (Proposition 4.4), we can characterize the equivalence class from Proposition 4.2 also as:*

$$[A, B]_{\mathrm{x}_0} = \left\{ \begin{bmatrix} A_V & \Theta \\ 0 & \Psi \end{bmatrix} : \Theta \in \mathbb{R}^{k\times(n-k)}, \Psi \in \mathbb{R}^{(n-k)\times(n-k)} \right\} \times \{B\}. \tag{4.2}$$

This induces $n(n-k)$ *degrees of freedom, which can be interpreted as a size proxy for this class.*

4.1.2. Partial Identifiability

Having characterized the equivalence class of systems consistent with a single observed trajectory, we next examine which aspects of the system remain identifiable even when the system as a whole is not. Fix x_0 and a persistently exciting input, and let $V(\mathrm{x}_0)$ denote the smallest A–invariant subspace containing $\mathrm{x}_0 \cup \operatorname{Im} B$. The homogeneous response and the input action are uniquely determined on $V(\mathrm{x}_0)$, whereas any modification on $V(\mathrm{x}_0)^\perp$ leaves the observed trajectory unchanged. Thus identifiability is *partial* whenever $V(\mathrm{x}_0) \neq \mathbb{R}^n$. In the special case $V(\mathrm{x}_0) = \mathbb{R}^n$, the equivalence class collapses to a singleton and full identifiability is recovered. Echoing Stanhope et al. (Theorem 3.3), the next theorem formalizes this via three equivalent characterizations—minimal invariance, a finite Krylov rank condition, and trajectory indistinguishability.

Theorem 4.6. ***(Partial Identifiability).*** *Let* $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ *and fix* $\mathrm{x}_0 \in \mathbb{R}^n$. *We continue using*

$$V(\mathrm{x}_0) := \mathcal{K}(A, \mathrm{x}_0) + \mathcal{R}(0) = \operatorname{span}\{[\mathrm{x}_0\ B], A[\mathrm{x}_0\ B], \ldots, A^{n-1}[\mathrm{x}_0\ B]\}. \tag{4.3}$$

Let $S \subseteq \mathbb{R}^n$ *be a linear subspace with* $\{\mathrm{x}_0\} \cup \operatorname{Im}(B) \subseteq S$ *and* $AS \subseteq S$. *Let* $k := \dim(S)$. *The following statements are equivalent:*

(i) S *is the smallest* $A-$*invariant subspace containing* $\{\mathrm{x}_0\} \cup \operatorname{Im}(B)$ *(equivalently: smallest* $[A, B]-$*invariant subspace ...).*

(ii) *The columns of* $\big[[\mathrm{x}_0\ B], A[\mathrm{x}_0\ B], \ldots, A^{k-1}[\mathrm{x}_0\ B]\big]$ *span* S. *Equivalently,*

$$\operatorname{rank}\Big([\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots\ , A^{k-1}[\mathrm{x}_0\ B]\Big) = k.$$

(iii) *There is no pair $(\tilde{A}, \tilde{B})$ whose action on S is distinct from (A, B), but with trajectories identical to those of (A, B), for all $t \geq 0$ and $\mathrm{u}(\cdot)$. Formally:*

$$\neg\exists\left(\tilde{A}\big|_S, \tilde{B}\big|_S\right) \neq \left(A\big|_S, B\big|_S\right) \quad \textit{such that}$$
$$\forall t \geq 0, \mathrm{u} \in \mathcal{C}([0,\infty), \mathbb{R}^m) : \varphi\left(t \,|\, \tilde{A}, \tilde{B}, \mathrm{x}_0, \mathrm{u}\right) = \varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right).$$

Proof sketch. Let $V(\mathrm{x}_0) = \mathrm{span}\{[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\}$ and let $S \subseteq \mathbb{R}^n$ satisfy $\{\mathrm{x}_0\} \cup \mathrm{Im}(B) \subseteq S$ and $AS \subseteq S$ with $k := \dim(S)$. $(i) \implies (ii)$ Apply the Cayley-Hamilton theorem to $A|_S$ as a $k \times k$ operator. The Krylov columns up to order $k-1$ already span the $A-$invariant hull of $\{\mathrm{x}_0\} \cup \mathrm{Im}(B)$ inside S. minimality of S among $A-$invariant supspaces of $\{\mathrm{x}_0\} \cup \mathrm{Im}(B)$ is therefore equivalent to their rank being k. $(ii) \implies (iii)$ If $\varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right) = \varphi\left(t \,|\, \tilde{A}, \tilde{B}, \mathrm{x}_0, \mathrm{u}\right)$ for all $t \geq 0$ and inputs, variation of constants gives $e^{At}\mathrm{x}_0 = e^{\tilde{A}t}\mathrm{x}_0$ and $e^{At}B = e^{\tilde{A}t}\tilde{B}$ for all t. Expanding at $t = 0$ yields $A^j\mathrm{x}_0 = \tilde{A}^j\mathrm{x}_0$ and $A^jB = \tilde{A}^j\tilde{B}$ for all j. By (ii), the set $\{A^j[\mathrm{x}_0\ B] : 0 \leq j \leq k-1\}$ spans S, forcing (A, B) and $(\tilde{A}, \tilde{B})$ to coincide on S. $(iii) \implies (i)$ Let us assume that S is not minimal, and $H \subsetneq S$ be the minimal $A-$invariant hull of $\{\mathrm{x}_0\} \cup \mathrm{Im}(B)$. Modify A outside of H to get $\tilde{A} \neq A$ on S, and set $\tilde{B} = B$. Trajectories from x_0 remain in H and coincide for all inputs, contradicting (iii).

See Appendix A.2.2 for a full proof.

Remark 4.7. *If $\mathrm{x}_0 \in \mathcal{R}(0)$, then $\mathcal{K}(A, \mathrm{x}_0) \subseteq \mathcal{R}(0)$, thus $V = \mathcal{R}(0)$, meaning only the controllable part of (A, B) will be "visible" from x_0. In particular, the (V, W) basis transformation will yield the Kalman canonical form (Equation* (2.21)*).*

Takeaway

- **Visible subspace** $V \triangleq$ Any trajectory of (A, B) with initial state x_0 will be contained in this subspace. If V is a proper subspace of $\mathbb{R}^n$, two system can have identical components *within* V without being algebraically equal.
- **Hidden subspace** $W \triangleq$ Any complement of V. "Invisible" modes live entirely in W.
- Any single-trajectory identification can at best recover $(A|_V, B|_V)$, the restriction of (A, B) to the visible subspace V.

4.1.3. Identifiability Criteria

To translate this analysis into quantitative criteria suitable for a-priori analysis of system matrix - initial state tuples (A, B, x_0), we lean on algebraic characterizations of

controllability (see Theorem 2.17). The bridge from controllability to x_0-identifiability is achieved by replacing B with the column-concatenated $[\mathrm{x}_0\ B]$.

Theorem 4.8. ***(Controllability-derived characterizations of identifiability).*** *Consider the continuous-time system $\dot{\mathrm{x}}(t) = A\mathrm{x}(t) + B\mathrm{u}(t)$ with known initial condition $\mathrm{x}(0) = \mathrm{x}_0 \in \mathbb{R}^n$. Assume that the control input $\mathrm{u} : [0, T] \to \mathbb{R}^m$ is persistently exciting of identification-sufficient order. Then, the following statements are equivalent:*

- ***(a)*** *The pair (A, B) is identifiable from x_0 under sufficiently rich control inputs.*
- ***(b)*** ***(Krylov test).*** *The augmented controllability matrix has full row rank:*

$$\operatorname{rank}\Big[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\Big] = n,$$

  or equivalently, its columns span $\mathbb{R}^n$.

- ***(c)*** ***(Finite-horizon Gramian).*** *For some $t > 0$, the x_0-augmented finite-horizon Gramian is positive definite:*

$$W_t\left([\mathrm{x}_0\ B]\right) := \int_0^t e^{As}[\mathrm{x}_0\ B][\mathrm{x}_0\ B]^\top e^{A^\top s}\, ds \succ 0.$$

- ***(d)*** ***(PBH test).*** *No nonzero left eigenvector $w^\top A = \lambda w^\top$ satisfies $w^\top[\mathrm{x}_0\ B] = 0$. Equivalently,* $\operatorname{rank}[\lambda I - A, [\mathrm{x}_0\ B]] = n$ *for all $\lambda \in \sigma(A)$.*
- ***(e)*** ***(Invariant subspace test).*** *The smallest $A-$invariant subspace containing $\{\mathrm{x}_0\} \cup \operatorname{Im}(B)$ equals $\mathbb{R}^n$.*
- ***(f)*** ***(Test under stable A).*** *If A is Hurwitz, the unique solution $W \succ 0$ of the Lyapunov equation*

$$AW + WA^T = -[\mathrm{x}_0\ B][\mathrm{x}_0\ B]^\top$$

  exists, and equivalently, the infinite-horizon Gramian is positive definite:

$$W_t\left([\mathrm{x}_0\ B]\right) := \int_0^\infty e^{As}[\mathrm{x}_0\ B][\mathrm{x}_0\ B]^\top e^{A^\top s}\, ds \succ 0.$$

These criteria can be interpreted as quantitative scores, which can serve as proxies in evaluating "how close" a given tuple (A, B, x_0) is to being unidentifiable. We formalize this through the following corollary.

Corollary 4.9. *Under the assumptions of Theorem 4.8, define the (augmented) matrix pencil $\mathcal{H}(\lambda) := [\lambda I - A,\ B] \in \mathbb{C}^{n\times(n+m)}$($\mathcal{H}_{aug}(\lambda) := [\lambda I - A,\ [\mathrm{x}_0\ B]] \in \mathbb{C}^{n\times(n+m+1)}$). Then:*

(a) (Frobenius distance to PBH failure). *Since we are interested in the stability of identifiability for a fixed initial state* x_0*, we can only perturb* (A, B)*. As we established in Theorem 4.8, identifiability is tied to the matrix pencil* $[\lambda I - A, [\mathrm{x}_0\ B]]$*, which augments* B *with* x_0*. This represents a structured distance to singularity problem, where we are looking at perturbations of* $[\lambda I - A, [\mathrm{x}_0\ B]]$ *that don't touch* x_0*. Thus, we define an orthonormal basis* $Q \in \mathbb{R}^{n\times(n-1)}$ *whose columns span* $\mathrm{x}_0^{\perp}$*, so that we can obtain a closed-form minimum singular value expression that fixes the shape of the perturbations to leave* x_0 *untouched.*

$$d_{Frob.}(A, B \,|\, \mathrm{x}_0) = \inf_{\lambda\in\mathbb{C}} \sigma_{min}\left(Q^\top\,[\lambda I - A,\ B]\right) =: \inf_{\lambda\in\mathbb{C}} \sigma_{min}\left(Q^\top \mathcal{H}(\lambda)\right)$$

If $\mathrm{x}_0 = 0$*, then* $Q^\top = I_n$*, and* $d_{Frob.}(A, B \,|\, \mathrm{x}_0)$ *recovers the distance to uncontrollability. (See Appendix A.2.2 for the derivation.)*

(b) (Krylov generator condition). *Let* $K_n := \left[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\right] \in \mathbb{R}^{n\times(n+nm)}$ *with* $\mathrm{span}(K_n) = V$*. Use any of*

$$\sigma_{min}(K_n), \quad \kappa_2(K_n) = \frac{\lambda_{max}}{\lambda_{min}}, \quad \rho_n := \frac{\dim(V)}{n} = \frac{\mathrm{rank}(K_n)}{n}$$

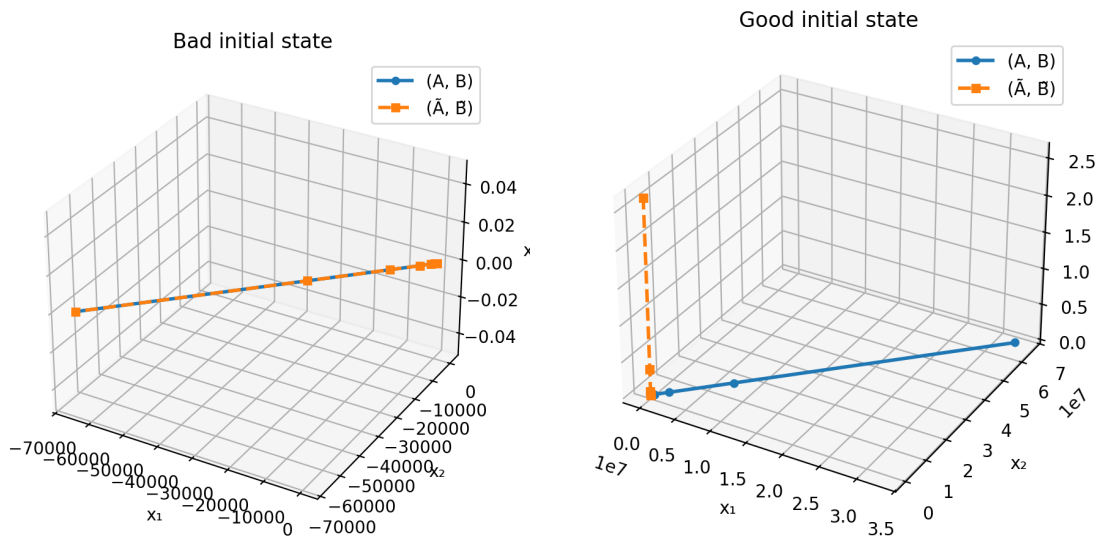
(respectively: minimum singular value, conditioning number, and rank of the Krylov generator matrix).

(c) (Gramian condition). *Define the augmented controllability Gramian as in Theorem 4.8. Use any of*

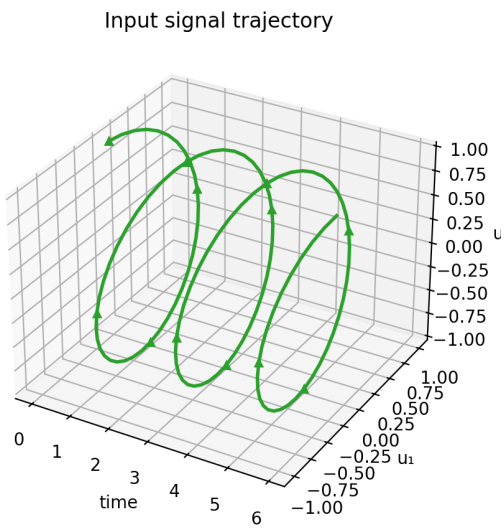
$$\lambda_{min}(W_t), \quad \kappa_2(W_t) = \frac{\lambda_{max}}{\lambda_{min}}$$

(d) (Left eigenvector criterion). *For each left eigenvector* $w_i^\top A = \lambda_i w_i^\top$*, define*

$$\mu_i(A, B \,|\, \mathrm{x}_0) := \frac{\|w_i^\top [\mathrm{x}_0\ B]\|_2}{\|w_i\|_2}, \quad \mu_{min} := \min_{i\in[|\sigma(A)|]} \mu_i.$$



(a) (b)



(c)

Figure 4.1.: (a) Primary and secondary trajectories for "bad" x_0, (b) Primary and secondary trajectories for "good" x_0, (c) Input signal used to generate the trajectories: $u(t) = \begin{bmatrix} a_1 \sin(\omega_1 t + \phi_1) \\ a_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}$

Example 4.10. ***(Single-trajectory*** x_0***−equivalence).*** *Let's have a look at the base pair*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}$$

and the initial states

$$\mathrm{x}_0^{(a)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{“}good\text{”}), \qquad \mathrm{x}_0^{(b)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad (\text{“}bad\text{”}).$$

A convenient left–eigenbasis of A is

$$w_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

for which

$$w_1^\top B = [-2, -2], \;\; w_2^\top B = [-2, -1], \;\; w_3^\top B = [0, 0], \qquad w_3^\top \mathrm{x}_0^{(a)} = 1 \neq 0, \;\; w_3^\top \mathrm{x}_0^{(b)} = 0.$$

Thus the third left–eigenvector is completely unexcited by B, exactly mirroring the m=1 base case.

Visible/hidden decomposition and the secondary system. *Let* $V := \mathcal{K}(A, [\mathrm{x}_0^{(b)} \;\; B])$ *and choose a complementary W so that* $\mathbb{R}^3 = V \oplus W$*. Because* $\mathrm{x}_0^{(b)}$ *has zero third component and the third row of B vanishes, one checks* $V = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ *and* $W = \mathrm{span}\{e_3\}$*. Define*

$$\tilde{A} := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \tilde{B} := B.$$

Then $\tilde{A}|_V = A|_V$ *and V is* $\tilde{A}$*–invariant. Consequently, for any input and any trajectory staying in V (in particular, those from* $\mathrm{x}_0^{(b)}$*), the responses of* (A, B) *and* $(\tilde{A}, \tilde{B})$ *coincide; while for initial conditions with a W–component (e.g.* $\mathrm{x}_0^{(a)}$*) the responses differ.*

Identifiability scores. *Let* $\mathcal{K}_3 := K\big(A, [\mathrm{x}_0 \;\; B]\big)$ *and let* $\mu_i := \|w_i^\top [\mathrm{x}_0 \;\; B]\|_2^2 / \|w_i\|_2^2$ *with* $\mu_{\min} = \min_i \mu_i$*. For our two initial states:*

initial state	$\mathrm{rank}(\mathcal{K}_3)$	$\mu_{\min}$	*PBH margin*	$\lambda_{\min}\big(W_T([\mathrm{x}_0 \; B])\big)$
good $\mathrm{x}_0^{(a)}$	3	1	≈ 0.77	≈ 8307.33
bad $\mathrm{x}_0^{(b)}$	2	0	$= 0$	$= 0$

Here "PBH margin" abbreviates $\inf_{\lambda\in\sigma(A)} \sigma_{\min}\big([\lambda I - A \;\; [\mathrm{x}_0 \; B]]\big)$*, and* W_T *is the finite-horizon augmented Gramian.*

How $(\tilde{A}, \tilde{B})$ **is picked.** The construction enforces $\tilde{B} = B$ and $\tilde{A}|_V = A|_V$ on the *visible* subspace

$$V = \mathcal{K}\big(A,\, [\mathrm{x}_0^{(b)} \;\; B]\big) = \{(x_1, x_2, 0)\},$$

while freely modifying the *hidden* block on $W = \mathrm{span}\{e_3\}$ (here: moving the eigenvalue $3 \to 4$ and zeroing the $V \to W / W \to V$ couplings). This preserves all single-trajectory data from $\mathrm{x}_0^{(b)}$ yet changes the responses from $\mathrm{x}_0^{(a)}$, illustrating the equivalence-class mechanism.

4.1.4. Genericity of Identifiability as a Topological Property

A property is merely a function $\mathcal{P} : \mathbb{R}^N \to \{0, 1\}$, where $\mathcal{P}(s) = 1$ means that $\mathcal{P}$ holds at $s \in \Omega$. Let $\Omega = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ and (A_0, B_0) a nominal (system) matrix pair in Ω. It is well-known that the property of controllability,

$$\mathcal{P}_{\mathrm{ctrl}}\left((A, B)\right) = 1 \iff \mathrm{rank}\left(\left[B \; AB \; \ldots \; A^{n-1}B\right]\right) = n$$

is generic [34], meaning that it is well-posed and holds in a neighborhood of the nominal pair (A_0, B_0), except only for (A, B) that live in some proper algebraic variety that corresponds to a zero measure. We now want to generalize this idea to identifiability from a fixed initial state x_0 as a property, which we define as

$$\mathcal{P}_{\mathrm{id}}\left((A, B, \mathrm{x}_0)\right) = 1 \iff \mathrm{rank}\left(\left[[\mathrm{x}_0 \; B],\; A[\mathrm{x}_0 \; B],\; \ldots,\; A^{n-1}[\mathrm{x}_0 \; B]\right]\right) = n,$$

and claim that this property is generic.

Theorem 4.11. ***(Genericity of structural identifiability from a fixed*** x_0***.)*** *Let* (A, B) *be a matrix pair with* $A \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^{n\times m}$*, and* $\mathrm{x}_0 \in \mathbb{R}^n$ *a fixed but arbitrary initial state. The property that* (A, B) *is identifiable from* x_0 *is generic, and well-posed at every nominal point* $p^* := (A^*, B^*, \mathrm{x}_0^*) \in \mathbb{R}^N$*,* $N = n(n + m + 1)$ *where it holds, i.e.,*

$$\mathcal{P}_{id}(p^*) \implies \exists\, \mathcal{N} \subset \mathbb{R}^N : \forall\, p \in \mathcal{N}(p^*) : \mathcal{P}_{id}(p),$$

where $\mathcal{N}(\cdot)$ *denotes a Euclidian neighborhood.*

Proof. We follow the line of argumentation in [34] to prove this claim. By listing entries of A and B, we regard $p = (A, B, \mathrm{x}_0)$ as a point in $\mathbb{R}^N$, where $N = n(n + m + 1)$. We

have previously shown that (A, B, x_0) is identifiable iff the Krylov generator matrix $K_n := \left[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\right]$ has rank n. Enumerate columns of K_n as $\left[k_1, \ldots, k_{n(m+1)}\right]$. Then, identifiability of (A, B, x_0) fails if and only every determinant formed by selecting n columns of K_n vanishes, that is,

$$\varphi_i(p) = \det\left[k_{i_1}, \ldots, k_{i_n}\right] = 0,$$

where the list of indices $\mathcal{I} := \{i_1, \ldots, i_n\} \subset [n(m+1)]$ can be selected in $C := \frac{(nm+n)!}{n!(nm)!}$ unique ways. The polynomials $\varphi_1(p), \ldots, \varphi_C(p)$ define an algebraic variety, i.e., a set that contains common zeros of the φ_i

$$V := \left\{p \in \mathbb{R}^N : \varphi_i(p) = 0\ \forall i \in [C]\right\} \subset \mathbb{R}^N.$$

As we argued earlier, controllable pairs (A, B) span all of $\mathbb{R}^n$, thus, maintaining the input sufficiency assumption, $\mathcal{P}_{\mathrm{id}}\left((A, B, \mathrm{x}_0)\right)$ will hold regardless of the choice of x_0, which makes $\mathcal{P}_{\mathrm{id}}(\cdot)$ proper (i.e., $V \neq \mathbb{R}^N$). Similarly, we argued that uncontrollable pairs (A, B), coupled with an initial state that leads to a homogeneous response that doesn't complement the uncontrollable section of $\mathbb{R}^n$ will be unidentifiable, which makes $\mathcal{P}_{\mathrm{id}}(\cdot)$ nontrivial (i.e., $V \neq \emptyset$). Thus, we conclude that $\mathcal{P}_{\mathrm{id}}(\cdot)$ is generic relative to V. □

This suggests that, for for continuous random distributions in the domain of (A, B, x_0), the probability of *unidentifiability* as we define it in Equation (2.5) vanishes.

4.2. Identifiability under Insufficiently Rich Control Signals

This section investigates how restrictions on the admissible input class affect single-trajectory identifiability. Our key organizing principle is that admissibility of control signals can be modeled either *pointwise in time* (local constraints on the instantaneous control vector) or *nonlocally* (global constraints on the whole function). These lead to distinct but ultimately compatible geometric pictures.

Definition 4.12. ***(Ambient spaces and admissibility).*** *Fix a horizon $T > 0$ and let $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = L^2_{loc}([0, \infty); \mathbb{R}^m)T]$.*

(i) *Pointwise admissibility. A measureable set $\mathbf{U} \subseteq \mathbb{R}^m$ with $0 \in \mathbf{U}$ determines*

$$\mathcal{U}_{pt}(\mathbf{U}) := \{\mathrm{u} \in \mathcal{U} \,|\, \forall t \in [0, T] : \mathrm{u}(t) \in \mathbf{U}\}. \tag{4.4}$$

We write $\mathbf{U} := \mathrm{span}(W)$ for the span of available directions.

(ii) *Nonlocal admissibility. A linear subspace $\mathcal{U}_c \subseteq \mathcal{U}$ describes global properties (bandlimited, energy-bounded, PE of given order, etc.).*

We will first assume that (A, B) is controllable to isolate the role of input richness. For uncontrollable pairs, we will reduce (A, B) to its projection onto the visible subspace $V = \mathcal{K}(A, \mathrm{x}_0) + \mathcal{K}(A, B)$ from the previous section in order to identify the maximal subsystem that can influence the trajectory from x_0. Then, we will apply the same admissibility model-dependent two-branch analysis to the reduced system, yielding a unified picture across both structural and input-constrained limitations on identifiability.

4.2.1. Pointwise Constraints

If the pair (A, B) is controllable, under persistency of excitation, all of $\mathbb{R}^n$ would be reachable. Thus, any structure present in $[A, B]_{\mathcal{U}}$ must come from properties of the control signal $\mathrm{u}(\cdot)$. These properties will be discussed in more detail later, but, in order to make the initial analysis simpler, we will define the restricted set of control inputs as a finite set $\mathcal{S}_u := \{\mathrm{u}_i \in \mathbb{R}^m : i \in [T]\} \subset \mathbb{R}^m$, and denote its span as $W := \mathrm{span}\{\mathcal{S}_u\}$, which encodes the restricted input space. Under this restriction, the set of reachable states becomes:

$$\begin{aligned}\mathcal{R}(\mathrm{x}_0|\mathrm{u} \in W) &= \gamma(A, \mathrm{x}_0) + \mathcal{R}(\mathbf{0}|\mathrm{u} \in W) \\ &= \gamma(A, \mathrm{x}_0) + \langle A \mid \mathrm{Im}(BW)\rangle \\ &\subseteq \mathcal{K}(A, \mathrm{x}_0) + \langle A \mid \mathrm{Im}(BW)\rangle, \end{aligned} \tag{4.5}$$

and we will define the restricted $A-$invariant hull $V_{\mathrm{pt}}(\mathrm{x}_0)$ as follows.

Definition 4.13. ***($A-$invariant hull with pointwise control constraints).*** *Defining $\mathcal{U}$ as the span of a finite set of vectors $W \subset \mathbb{R}^m$, we derive the input-constrained version of Definition 2.6 as follows:*

$$\begin{aligned}V_{pt}(\mathrm{x}_0) &:= \mathcal{K}(A, \mathrm{x}_0) + \langle A \mid \mathrm{Im}(BW)\rangle \\ &= \mathrm{span}\{A^k \mathrm{x}_0 : k \geq 0\} + \mathrm{span}\left\{A^k B\mathrm{u}_j : k \geq 0,\ j \in [T]\right\} \\ &= \mathrm{span}\left\{[\mathrm{x}_0 \ \ BW], \ldots, A^{n-1}[\mathrm{x}_0 \ \ BW]\right\} \\ &= \mathrm{span}\left\{[\mathrm{x}_0 \ \ B\mathrm{u}_1], \ldots, [\mathrm{x}_0 \ \ B\mathrm{u}_T], A[\mathrm{x}_0 \ \ B\mathrm{u}_1], \ldots, A^{n-1}[\mathrm{x}_0 \ \ B\mathrm{u}_T]\right\} \\ &= \mathcal{K}\left(A, [[\mathrm{x}_0 \ \ B\mathrm{u}_1], \ldots, [\mathrm{x}_0 \ \ B\mathrm{u}_T]]\right) \\ &= \mathcal{K}\left(A, [\mathrm{x}_0 \ \ BW]\right). \end{aligned} \tag{4.6}$$

$V_{pt}(\mathrm{x}_0)$ corresponds to the minimal $A-$invariant subspace that contains x_0 and the image of B on $\mathrm{Im}(W)$. Intuitively, it encodes the effective visible (reachable) space under input restrictions.

All properties of the $A-$invariant hull, discussed in Corollary 2.7 apply to $V_{\mathrm{pt}}(\mathrm{x}_0)$ after replacing B with BW.

Let $\mathcal{U}_c = \{\mathrm{u} \in \mathcal{C}([0,\infty), \mathbb{R}^m) : \mathrm{u}(t) \in \mathbf{U} \subseteq \mathbb{R}^m \ \forall t\}$ with $\mathbf{U} := \mathrm{span}(W)$. Define the restricted visible subspace as in Definition 4.13:

$$V_{\mathrm{pt}}(\mathrm{x}_0) := \mathcal{K}(A, [\mathrm{x}_0 \ BW]),$$

which coincides with the smallest $A-$invariant subspace that contains both x_0 and the image of B on $\mathbf{U}$.

Proposition 4.14. ***(Structure of controllable $[A,B]_{\mathrm{x}_0,\mathcal{U}_c}$ under pointwise input restrictions.)*** *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, fix $T > 0$, and let*

$$\mathcal{U}_c = \{\mathrm{u} \in L^1([0,T];\mathbb{R}^m) : \mathrm{u}(t) \in \mathrm{span}(W) \textit{ for a.e. } t\},$$

with a fixed subspace $W \subset \mathbb{R}^m$.
***(i) Case** $0 \in \mathcal{U}_c$. If the zero input is admissible), then the homogeneous response can be separated from the forced response, which allows us to write:*

$$[A,B]_{\mathrm{x}_0,\mathcal{U}_c} = \left\{(\tilde{A},\tilde{B}) \mid \ \tilde{A}|_{V_{pt}(\mathrm{x}_0)} = A|_{V_{pt}(\mathrm{x}_0)}, \ \tilde{B}W = BW\right\}.$$

***(ii) Case** $0 \notin \mathcal{U}_c$. Without the zero input, the equivalence class is*

$$[A,B]_{\mathrm{x}_0,\mathcal{U}_c} = \left\{(\tilde{A},\tilde{B}) \,|\, e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0, \ e^{\tilde{A}t}\tilde{B}w = e^{At}Bw \ \forall t \in [0,T], w \in W\right\}. \tag{4.7}$$

See Appendix A.2.2 for a proof sketch.

4.2.2. Nonlocal Constraints

Definition 4.15. ***(Controllability operator).*** *For $t \geq 0$, the controllability operator of an input-state LTI system (A,B) is defined as*

$$\mathcal{R}_t^{A,B} : \mathcal{U} \to \mathcal{X}, \quad (\mathcal{R}_t^{A,B}\mathrm{u}) := \int_0^t e^{A(t-s)}B\mathrm{u}(s)\, ds. \tag{4.8}$$

where $\mathcal{U}$ is the entire control input space. We recover the finite-time reachability Gramian by multiplying $\mathcal{R}_t^{A,B}$ with its adjoint:

$$\begin{aligned}
\mathcal{R}_t^{A,B}\left(\mathcal{R}_t^{A,B}\right)^\top &= \int_0^t e^{A(t-s)}B\left(e^{A(t-s)}B\right)^\top ds \\
&= \int_0^t e^{A(t-s)}BB^\top e^{A^\top(t-s)}\, ds \\
&= \int_0^t e^{A(t-s)}BB^\top e^{A^\top(t-s)}\, ds.
\end{aligned}$$

Remark 4.16. *We assume $\mathcal{U} = L^2_{loc}([0,\infty);\mathbb{R}^m)t]$. Let $\mathcal{U}_c \subset \mathcal{U}$ encode a nonlocal constraint such as a specific PE order or energy bound. We denote the restricted operator as $\mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c}$.*

We build the object that will replace (4.6) with its equivalent in the operator framework by considering the set of reachable states $\mathcal{R}(\mathrm{x}_0)$, like we did in the construction of the unconstrained $A-$invariant hull (2.9):

$$\mathcal{R}_t(\mathrm{x}_0\,|\,\mathcal{U}_c) \overset{\text{def.}}{=} \left\{ \mathrm{x}_f \in \mathcal{X} \;\middle|\; \exists \mathrm{u}(\cdot) \in \mathcal{U}_c : \mathrm{x}_0 \underset{t}{\rightsquigarrow} \mathrm{x}_f \right\} = e^{At}\mathrm{x}_0 + \mathrm{Im}\left(\mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c} \right) \tag{4.9}$$

Let $\mathcal{R}(\mathrm{x}_0\,|\,\mathcal{U}_c) := \bigcup_{t\geq 0} \mathcal{R}_t(\mathrm{x}_0\,|\,\mathcal{U}_c)$ be the all-times reachable set. Then:

$$\mathcal{R}(\mathrm{x}_0\,|\,\mathcal{U}_c) = \bigcup_{t\geq 0} \left\{ e^{At}\mathrm{x}_0 + \mathrm{Im}\left(\mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c} \right) \right\} \tag{4.10}$$

and

$$V_{\mathcal{U}_c}(\mathrm{x}_0) := \mathrm{span}\left(\bigcup_{t\geq 0} e^{At}\mathrm{x}_0 \cup \bigcup_{t\geq 0} \mathrm{Im}\left(\mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c} \right) \right). \tag{4.11}$$

(See Appendix A.2.2 for the derivation of the operator-based characterizations of the reachable set and visible subspace under nonlocal input restrictions.)

Definition 4.17. ***(Effective visible space under $\mathcal{U}_c$).*** *The effective visible space containing states reachable from a given x_0, using inputs from the constrained input class $\mathcal{U}_c$, can be expressed in operator notation as*

$$V_{\mathcal{U}_c}(\mathrm{x}_0) := \mathrm{span}\left\{ \left(\bigcup_{t\geq 0} e^{At}\mathrm{x}_0 \right) \cup \left(\bigcup_{t\geq 0} \mathrm{Im}\left(\mathcal{R}_t^{A,B}\big|_{\mathcal{U}_C} \right) \right) \right\}. \tag{4.12}$$

Any loss of identifiability will manifest itself as a shrinkage of $V_{\mathcal{U}_c}$. Depending on signal properties, two systems that have different subsystems $(\tilde{A}|_V, \tilde{B}|_V) \neq (A|_V, B|_V)$ might produce the same restricted controllability operators: $\mathcal{R}_t^{\tilde{A},\tilde{B}}\big|_{\mathcal{U}_c} = \mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c}$.

Proposition 4.18. ***(Structure of $[A,B]_{\mathrm{x}_0,\mathcal{U}_c}$ under nonlocal input constraints).*** *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, fix a horizon $T > 0$, and let $\mathcal{U}_c \subset L^1([0,T];\mathbb{R}^m)$ be a nonempty*

admissible class (e.g., bandlimited, bounded-energy, PE). For $t \in [0, T]$ define the (finite-horizon) controllability operator

$$\mathcal{R}_t^{A,B} : \mathcal{U}_c \to \mathbb{R}^n, \quad \mathcal{R}_t^{A,B}[u] := \int_0^t e^{A(t-s)} B\, u(s)\, ds$$

Then the $(x_0, \mathcal{U}_c)$–equivalence class is

$$[A, B]_{x_0,\mathcal{U}_c} = \Big\{ (\tilde{A}, \tilde{B}) : \; e^{\tilde{A}t} x_0 = e^{At} x_0, \;\; \mathcal{R}_t^{\tilde{A},\tilde{B}}\big|_{\mathcal{U}_c} = \mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c} \;\; \forall t \in [0, T] \Big\}. \tag{4.13}$$

Moreover, if $\alpha u \in \mathcal{U}_c$ for every $u \in \mathcal{U}_c$ and $\alpha \in [0, 1]$, then the homogeneous equality in (4.13) *is redundant:*

$$[A, B]_{x_0,\mathcal{U}_c} = \Big\{ (\tilde{A}, \tilde{B}) : \; \mathcal{R}_t^{\tilde{A},\tilde{B}}\big|_{\mathcal{U}_c} = \mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c} \; \forall t \in [0, T] \Big\}.$$

See Appendix A.2.2 for a proof sketch.

4.2.3. Direct Richness (PE) Constraints

Definition 4.19. ***(Moment-PE).*** *Fix $t_0 \in [0, T]$. For $k \geq 0$, define the linear functionals*

$$\psi_k(\mathrm{u}) := \int_0^{t_0} \frac{(t_0 - s)^k}{k!} \mathrm{u}(s)\, ds.$$

An input class $\mathcal{U}_c \subset \mathcal{C}\left([0, T], \mathbb{R}^m\right)$ is moment-PE of order r at t_0, if the linear map

$$\mathcal{M}_r : \mathcal{U}_c \to \mathbb{R}^{mr}, \quad \mathrm{u} \mapsto (\psi_0(\mathrm{u}), \; \ldots, \; \psi_{r-1}(\mathrm{u}))$$

has full range.

Proposition 4.20. ***(Recoverability of Markov parameters up to order $r - 1$).*** *[19, 35] Let $\mathcal{U}_c$ be moment-PE of order r at t_0 and assume $\mathbf{0} \in \mathcal{U}_c$. Then,*

$$\left(\mathcal{R}_t^{A,B}\Big|_{\mathcal{U}_c} \equiv \mathcal{R}_t^{\tilde{A},\tilde{B}}\Big|_{\mathcal{U}_c} \quad \forall t \in [0, t_0] \right) \implies A^k B = \tilde{A}^k \tilde{B} \;\; \forall k \in [0, r - 1].$$

To obtain a subspace-based characterization of this equivalence class, we construct the following input-restricted visible subspace:

$$V_{\mathrm{PE}}^r(\mathrm{x}_0) := \mathcal{K}(A, \mathrm{x}_0) + \operatorname{span}\Big\{ A^k B : \; 0 \leq k \leq r - 1 \Big\}, \tag{4.14}$$

with $k_r := \dim(V_{\mathrm{PE}}^r)$

Proposition 4.21. ***(x_0-equivalence class under PE-order-r signals).*** *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be the system matrices of $\mathcal{S} \in \Sigma_{s/i}$, and $\mathcal{U}_{PE}^r \subset L_{loc}([0,T];\mathbb{R}^m)$ be the class of inputs that is moment-PE of order r and contains the origin. Fix $\mathrm{x}_0 \in \mathbb{R}^n$ and let $V_{PE}^r(\mathrm{x}_0)$ be defined as in (4.14). Assume that V_{PE}^r is $A-$invariant and $\mathrm{Im}(B) \subseteq V_{PE}^r$. Then the set of systems $(\tilde{A},\tilde{B})$ that are $(\mathrm{x}_0,\mathcal{U}_{PE}^r)-$ equivalent to (A,B) is*

$$[A,B]_{\mathrm{x}_0,\mathcal{U}_{PE}^r} = \left\{(\tilde{A},\tilde{B}):\ \tilde{A}\big|_{V_{PE}^r} = A\big|_{V_{PE}^r},\ \tilde{B} = B\right\}. \tag{4.15}$$

Moreover, letting $k_r := \dim(V_{PE}^r)$ and choosing a basis $T = [P\ Q] \in \mathrm{GL}(n)$ adapted to $V_{PE}^r \oplus W = \mathbb{R}^n$ (so that $\mathrm{Im}(P) = V_{PE}^r$ and $\mathrm{Im}(Q) = W$), this class can be written in block form as

$$[A,B]_{\mathrm{x}_0,\mathcal{U}_{PE}^r} = \left\{(\tilde{A},\tilde{B}):\ T^{-1}\tilde{A}T = \begin{bmatrix} A_{V_{PE}^r} & \Theta \\ 0 & \Psi \end{bmatrix},\ \tilde{B} = B,\ [\Theta\ \ \Psi]^\top \in \mathbb{R}^{n\times(n-k_r)}\right\}, \tag{4.16}$$

where $A_{V_{PE}^r} = (T^{-1}AT)\big|_{V_{PE}^r}$.

See Appendix A.2.2 for a proof sketch.

Remark 4.22. *If one requires $(\mathrm{x}_0,\mathcal{U}_{PE}^r)-$equivalence for arbitrary $\mathrm{x}_0 \in \mathbb{R}^n$ (not necessarily in V_{PE}^r), then the same description holds with the additional constraint $\Theta = 0$ in the block form.*

Remark 4.23. ***(Substitution rule to recover former criteria).*** *Fix x_0 and an input class that is moment-PE of order r at some $t_0 > 0$. Define the PE-visible subspace as*

$$V_{PE}(\mathrm{x}_0) := \mathcal{K}(A,\mathrm{x}_0) + \left\{A^k B : 0 \le k \le r-1\right\}.$$

Then, every structural identifiability metric stated in Theorem 4.8 remains valid after performing the following substitutions:

- *Replace the visible subspace V by V_{PE}.*
- *Replace the Krylov generator by the truncated one*

$$K_{n,r} := \left[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots, A^{r-1}[\mathrm{x}_0\ B],\ A^r\mathrm{x}_0,\ldots,A^{n-1}\mathrm{x}_0\right].$$

To relate this back to previous results, we define the unified generator

$$K(\mathcal{U};\mathrm{x}_0) := \begin{cases} \left[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\right], & \text{unrestricted inputs,} \\ \left[[\mathrm{x}_0\ \ BW]\ \ A\,[\mathrm{x}_0\ \ BW]\ \ldots, A^{n-1}\,[\mathrm{x}_0\ \ BW]\right], & \text{ptw. admissibility,} \\ \left[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots, A^{r-1}[\mathrm{x}_0\ B],\ A^r\mathrm{x}_0,\ldots,A^{n-1}\mathrm{x}_0\right], & \text{PE of order } r < n. \end{cases} \tag{4.17}$$

4.3. Summary and Extensions

We conclude by distilling the main insights and giving a compact, algorithmic view of the framework, followed by extensions to i/s/o systems and a brief excursion into data-level equivalence.

Algorithmic view. The following pseudo-algorithms give a unified picture of the analysis we developed here.

Algorithm 1 EquivalenceClass

Inputs : System (A, B) with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$; initial state x_0; PE-order r
Optional: $D = \{\mathrm{u}(t), \mathrm{x}(t)\}_{t=1}^{T}$

Outputs: Eqv. cl. $\mathcal{E}_{x_0} \leftarrow [A, B]_{\mathrm{x}_0}$, score $d(A, B \,|\, \mathrm{x}_0)$.

$(\mathcal{E}_{x_0},\ (A|_V, B|_V), d) \leftarrow$ **EquivalenceClass**(A, B, x_0)
$(\mathcal{E}^r_{x_0},\ d^r) \leftarrow$ **PracticalIdentifiability**$((A|_V, B|_V),\ \mathrm{x}_0,\ r)$
return $\mathcal{E}_{x_0},\ d,\ \mathcal{E}^r_{x_0},\ d^r$

Algorithm 2 StructuralIdentifiability(A, B, x_0)

Inputs : $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$, r

Outputs: $\mathcal{E}_{x_0} \leftarrow [A, B]_{\mathrm{x}_0}$

$K \leftarrow [\,[x_0 \;\; B],\ A[x_0 \;\; B],\ \ldots,\ A^{n-1}[x_0 \;\; B]\,]$ `// Generator matrix of` V.
$(V_{\text{basis}},\ k) \leftarrow$ `svd_rank`(K) `// Rank-revealing SVD.`
Pick any W such that $V \oplus W = \mathbb{R}^n$ `// Pick a complement of` V.
$(W_{\text{basis}},\ n-k) \leftarrow$ `svd_rank`(W)
$T \leftarrow [\,V_{\text{basis}} \;\; W_{\text{basis}}\,] \in \mathrm{GL}(n)$ `// Basis matrix.`
$\tilde{A} \leftarrow T^{-1}AT = \begin{bmatrix} A_V & A_* \\ 0 & A_W \end{bmatrix},\quad \tilde{B} \leftarrow T^{-1}B = \begin{bmatrix} B_V \\ 0 \end{bmatrix}$ `// Block form.`

$$\mathcal{E}_{x_0} = \left\{ (A', B') :\ B' = B,\ A' = T \begin{bmatrix} A_V & \Theta \\ 0 & \Psi \end{bmatrix} T^{-1},\ \Theta \in \mathbb{R}^{k\times(n-k)},\ \Psi \in \mathbb{R}^{(n-k)\times(n-k)} \right\}.$$

return $\mathcal{E}_{x_0},\ (A|_V, B|_V)$

Algorithm 3 PRACTICALIDENTIFIABILITY($A|_V$, $B|_V$, x_0, `admissibility`)

Inputs : Reduced pair $(A|_V, B|_V)$; $x_0 \in V$;
`input_constr` $\in \{$`ptw`$(\mathbf{U}$: adm. set), `nonlocal`$(\mathcal{P}$: property), PE$(r$: `order`$)\}$

Outputs: $\mathcal{E}_{x_0,\mathcal{U}} \leftarrow [A, B]_{x_0,\mathcal{U}}$

```
if input_constr = ptw(U) then
    ptwise ← True
    K ← [ x_0  BW, A[x_0  BW], ..., A^{n-1}[x_0  BW] ]
else if input_constr = PE(r) then
    PE ← True
    K ← [ x_0  B, AB, ..., A^{r-1}B, A^r x_0, ..., A^{n-1} x_0 ]
else if input_constr = nonloc(P) then
    // Practically infeasible
    return ∅
V_eff(x_0) ← span(K),   k_eff ← svd_rank(K)
T = [V  P] ∈ GL(n), Ã ← T^{-1}AT,   B̃ ← T^{-1}B
E_{x_0,U} ← {(A', B') : A'|_{V_eff} = A|_{V_eff}, B'P = BP}
return E_{x_0,U}
```

Extension to input/state/output systems

Let us first define the class of input/state/output LTI systems [1]:

$$\Sigma_{i/s/o}: \quad \dot{\mathrm{x}}(t) = A\mathrm{x}(t) + B\mathrm{u}(t), \;\; \mathrm{y}(t) = C\mathrm{x}(t), \;\; \mathrm{x}(0) = \mathrm{x}_0 \in \mathbb{R}^n. \tag{4.18}$$

Let us denote the output-reachable set from x_0 as $\mathcal{Y}(\mathrm{x}_0)$, and examine how it relates to $\mathcal{R}(0)$:

$$\begin{aligned}
\mathcal{Y}(\mathrm{x}_0) &:= \{y(t \,|\, A, B, C, \mathrm{x}_0, \mathrm{u}) : \; \mathrm{u} \in \mathcal{C}\left([0,t], \mathbb{R}^m\right), \; t \geq 0\} \\
&= \left\{ Ce^{At}\mathrm{x}_0 + \int_0^t Ce^{A(t-s)}B\mathrm{u}(s)\, ds \;\middle|\; \mathrm{u} \in \mathcal{C}\left([0,t], \mathbb{R}^m\right), \; t \geq 0 \right\} \\
&= C\left\{ e^{At}\mathrm{x}_0 : \; t \geq 0 \right\} + C\left\{ \int_0^t e^{A(t-s)}B\mathrm{u}(s)\, ds : \; \mathrm{u} \in \mathcal{C}\left([0,t], \mathbb{R}^m\right), \; t \geq 0 \right\} \\
&= C\gamma(A, \mathrm{x}_0) + C\mathcal{R}(0) \\
&= C\mathcal{R}(\mathrm{x}_0).
\end{aligned} \tag{4.19}$$

[1] We deliberately leave out the feedthrough matrix D since it doesn't add inherent dynamical structure. It can be incorporated into the $[A, B, C]_{\mathrm{x}_0}$ equivalence class by additionally demanding $D' = D$.

The same relationship carries over to the (output-) visible subspace ($A-$invariant hull, Definition 2.6):

$$\begin{aligned} V_{\text{out}}(\mathrm{x}_0) &:= \operatorname{span}\left\{CA^k\mathrm{x}_0 :\ k \geq 0\right\} + \operatorname{span}\left\{CA^k b_j :\ k \geq 0,\ j \in [m],\ b_j : j\text{-th col. of } B\right\} \\ &= C\mathcal{K}\left(A, [\mathrm{x}_0\ B]\right) = CV(\mathrm{x}_0). \end{aligned} \tag{4.20}$$

For injective C (with column rank n), the output mapping essentially just represents a basis change. Thus, structural identifiability properties of (A, B) discussed in Section Section 4.1 will be retained.

Data-level equivalence

The principal barrier to practical use lies in translating the structural analysis into data-centric, trajectory-level criteria. While we do not develop this in full, we nevertheless conclude with a brief introduction of high-level objects that could serve as definitional primitives and first steps towards this effort.

Setup. Given a single ZOH-sampled trajectory $\mathcal{D} := \{(\mathrm{x}_i, \mathrm{u}_i) : i \in [0, T]\}$, where $\mathrm{x}_{i+1} = A_d\mathrm{x}_i + B_d\mathrm{u}_i$, and $(A_d, B_d) = \left(e^{Adt},\ \int_0^{dt} e^{As}B\, ds\right)$, we define

$$X_0 = (\mathrm{x}_0, \ldots, \mathrm{x}_{T-1}),\ \ X_1 = (\mathrm{x}_1, \ldots, \mathrm{x}_T),\ \ U = (\mathrm{u}_0, \ldots, \mathrm{u}_{T-1}),\ \ Z := \begin{bmatrix} X_0 \\ U \end{bmatrix}.$$

Let $\Theta := [A_d\ \ B_d] \in \mathbb{R}^{n\times(n+m)}$, so that $X_1 = [A_d\ \ B_d]\begin{bmatrix} X_0 \\ U \end{bmatrix} = \Theta Z$.

Definition 4.24. ***(Data-level*** x_0-***equivalence).*** *Two parameters* $\Theta, \tilde{\Theta}$ *are data-*x_0-*equivalent, written* $\Theta \sim^{\mathcal{D}}_{\mathrm{x}_0} \tilde{\Theta}$*, if* $\Theta Z = \tilde{\Theta} Z$.

Theorem 4.25. ***(Data-level*** x_0-***equivalence class.)*** *With* $\mathcal{S}_{emp} := \operatorname{span}\left(\begin{bmatrix} X_0 \\ U \end{bmatrix}\right)$ *and projector* $\Pi_{\mathcal{S}_{emp}}$*, the class of all models consistent with* $\mathcal{D}$ *is*

$$[\Theta]_{\mathcal{D}} = \{\tilde{\Theta} : \tilde{\Theta}\Pi_{\mathcal{S}_{emp}} = \Theta\Pi_{\mathcal{S}_{emp}}\}.$$

Hence, Θ *is uniquely determined iff* $\operatorname{rank}(Z) = n + m$.

So the objective would be exploring under what assumptions x_0-equivalence relationships imply data-level x_0-relationships and vice versa.

Key summary

For a single controlled trajectory starting at x_0, all model information lives in the *visible subspace*

$$V(x_0) \;=\; \mathcal{K}(A, x_0) \;+\; \mathcal{R}(A, B) \;=\; \mathrm{span}\{A^k[x_0 \;\; B] : k = 0, \ldots, n-1\}$$

(i) **Structural:** The restriction $(A|_V, B|_V)$ (of (A, B) to the visible subspace $V(x_0)$) is the *partially identifiable* component. Outside of $V(x_0)$ the dynamics are indistinguishable from single trajectory data.
(ii) **Signal richness:** With a control input persistently exciting of order r, the data uniquely pins down the first r state–Markov parameters $[B, \; AB, \; \ldots \; A^{r-1}B]$; hence the practically visible slice is

$$V_r^{\mathrm{PE}}(x_0) \;=\; \mathcal{K}(A, x_0) \;+\; \mathrm{span}\{A^k B\}_{k=0}^{r-1},$$

which expands to $V(x_0)$ as r grows.

5. Experiments

This chapter is divided into three sections:

1. **Integrity of the Framework.** The objective here is to validate the core objects that issue from the theoretical analysis we developed in Section 4.1: *(i) identifiability criteria* and *(ii) equivalence classes*. Concretely, we first test that the criteria that we proposed co-vary with estimation errors of consistent estimators such as DMDc and MOESP, and thus can predict estimation error on randomly sampled systems and initial conditions. Next, we hypothesize that for partial identifiable systems, identification methods can at best return one element from the equivalence class. We validate this by comparing estimation errors in standard basis with estimation errors in a basis adapted to $V(\mathrm{x}_0)$, which is an invariant of the equivalence class. We evaluate different identification algorithms, including proven-consistent estimators and ML-methods (SINDy, NODE)

2. **Parameter Regimes.** After we have confirmed that our identifiability criteria co-vary with estimation errors, we use the identifiability criteria to map parameter ($\Theta = (A, B)$) regimes that are particularly prone to unidentifiability. To that end, we use three characteristics that are known to be prone to unidentifiability: (i) Sparsity, (ii) Underactuation, and (iii) (Higher) state dimension. In the first half of this chapter, we vary, on a grid, one property at a time and chart the corresponding (mean) identifiability scores. In the second half, we combine to axes to (a) compare their relative effects / which one "grows" nonidentifiability faster, or, more generally, to (b) investigate if there are nonlinearities in the growth of the scores (e.g., what if sparsity and underactuation interact in a way that when both are increased, the scores become smaller unexpectedly fast).

3. **Input Sufficiency.** Here, we demonstrate that the minimum order of persistent excitation needed for guaranteed identification success correlates more strongly with the dimension of the visible subspace then with state dimension.

5.1. Integrity of the Framework

Identifiability analysis is ultimately theoretical, so are the objects that issue from it. In other words, there is no "ground truth" we can test against, so what we do here is: we pick a consistent estimator that is guaranteed to recover the true system under certain conditions such as the data matrix having full rank and the absence of disturbances in the measured trajectory; we then measure estimation errors against identifiability scores we proposed.
All experiments share certain procedural elements, which we will review before going into the individual experiments.
Notation. We denote the matrix pair learned by the estimator at hand as $(\hat{A}, \hat{B})$.
Identifiability criteria. We continuously report on the following identifiability criteria, proposed in Theorem 4.8:

$$\begin{aligned}
\textbf{(PBH)} \quad & d_{\text{Frob.}}(A, B \,|\, \mathrm{x}_0) = \inf_{\lambda \in \mathbb{C}} \sigma_{\min}\left(Q^\top \left[\lambda I - A,\ B\right]\right) =: \inf_{\lambda \in \mathbb{C}} \sigma_{\min}\left(Q^\top \mathcal{H}(\lambda)\right) \\
\textbf{(Krylov)} \quad & \sigma_{\min}(K_n) = \sigma_{\min}\left(\left[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\right]\right) \\
\textbf{(Left eigenv.)} \quad & \mu_i(A, B \,|\, \mathrm{x}_0) := \frac{\|w_i^\top [\mathrm{x}_0\ B]\|_2}{\|w_i\|_2}, \quad \mu_{\min} := \min_{i \in [|\sigma(A)|]} \mu_i.
\end{aligned}$$

For each of the criteria listed above, the "binary" structural unidentifiability occurs exactly when they are zero.

$$\boxed{(A, B) \text{ is } \mathrm{x}_0 - \text{unidentifiable} \iff c(A, B, \mathrm{x}_0) = 0;\ \ c \in \{\text{PBH, Krylov, Left eigenvec.}\}}$$

Visible subspace basis and projectors. We construct an orthonormal basis $P_V \in \mathbb{R}^{n \times k}$ for the visible subspace $V(\mathrm{x}_0)$ by a rank reveal on the unified Krylov generator $K_n = \left[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\right]$. Concretely, we

1. compute the thin singular value decomposition (SVD) $K_n = U \Sigma V^\top$,
2. set $k := \sum_{i=1}^{n} \mathbf{1}\{\sigma_i > \tau\}$ with $\tau = \epsilon \cdot \|K_n\|_F$, with $\epsilon = 10^{-10}$ near machine precision.
3. We then define $P_V := U_{:,[k]}$.

The orthonormal projector onto $V(\mathrm{x}_0)$ is $\Pi_V := P_V P_V^\top$ and $\Pi_W := I = \Pi_V$ projects onto $V(\mathrm{x}_0)^\perp$
Relative estimation error. We use normalized Frobenius distances between "true" and estimated matrices to measure estimation error:

$$\mathcal{E}_A := \frac{\|\hat{A} - A\|_F}{\|A\|_F}, \quad \mathcal{E}_B := \frac{\|\hat{B} - B\|_F}{\|B\|_F}, \quad \mathcal{E} := \frac{1}{2}\left(\mathcal{E}_A + \mathcal{E}_B\right).$$

Equivalently, we denote the visible subspace-basis errors as

$$\mathcal{E}_{A|_V} := \frac{\|\hat{A}|_V - A|_V\|_F}{\|A|_V\|_F}, \quad \mathcal{E}_{B|_V} := \frac{\|\hat{B}|_V - B|_V\|_F}{\|B|_V\|_F}, \quad \mathcal{E}_V := \frac{1}{2}\left(\mathcal{E}_{A|_V} + \mathcal{E}_{B|_V}\right),$$

where $(A|_V, B|_V)$ denotes the projection of (A, B) onto the visible subspace $V(\mathrm{x}_0)$. We will occasionally use the superscripted versions $\mathcal{E}^{\mathrm{std}}$ for $\mathcal{E}$ and its matrix-specific versions to better distinguish between standard basis errors and $V-$basis errors.
The comparison of $\mathcal{E}$ and $\mathcal{E}|_V$ builds the backbone of validating the equivalence class proposed in §4.1 (see Definition 2.1), since any basis adapted to $V(\mathrm{x}_0) \oplus V(\mathrm{x}_0)^{\perp}$ can be considered an invariant of the equivalence class.
Random matrix ensembles. We use the *Ginibre* ensemble, where, for a given state and input dimensions $(n.m)$ we construct $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ by drawing each entry independently form a zero-centered Gaussian with standard deviation $\sigma = 1/n$:

$$A_{ij} \sim \mathcal{N}(0, 1/n), \quad B_{ij} \sim \mathcal{N}(0, 1/n);$$

and the *Sparse-Gaussian* ensemble, where we (i) fix density parameters $p_A, p_B \in [0, 1]$, which are used to create "masks" $M^A_{ij} \sim \mathrm{Ber}(p_A), M^B_{ij} \sim \mathrm{Ber}(p_B)$; and (ii) draw $G_A \sim \mathcal{N}(\mathbf{0}_n, I_{n\times n})$, $G_A \sim \mathcal{N}(\mathbf{0}_n, I_{n\times n})$ from the multi-variate standard Gaussian; and (iii) build $(A, B) = (M_A \odot G_A, M_B \odot G_B)$. Further, matrix pairs (A, B) with target visible subspace dimension $r = \dim(V(\mathrm{x}_0))$ are computed by using a Kalman block construction and random Haar embedding.
$(\mathbf{A}, \mathbf{B}, \mathrm{x}_0)$-ensemble generation with target $\dim(\mathbf{V}(\mathrm{x}_0))$ using Haar embeddings. We construct a pair (A, B) that has the target "visibility" order $k = \dim(V(\mathrm{x}_0))$ as its controllability matrix rank. We then choose x_0 inside the reachable subspace such that

$$\begin{aligned}\dim(V(\mathrm{x}_0)) &= \dim\left(\mathcal{K}_n(A, [\mathrm{x}_0\ B])\right)\\ &= \dim\left(\mathcal{K}_n(A, \mathrm{x}_0) + \mathcal{K}_n(A, B)\right)\\ &\overset{\mathrm{x}_0\in\mathcal{R}(A,B)}{=} \dim\left(\mathcal{K}_n(A, B)\right) = k.\end{aligned}$$

To construct pairs (A, B) with target controllability matrix, we

1. draw a random matrix pair (A_V, B_V) with $A_V \in \mathbb{R}^{k\times k}$ and $B_V \in \mathbb{R}^{k\times m}$, and make sure that it is controllable via rejection sampling.

2. Randomly select an autonomous block $A_W \in \mathbb{R}^{(n-k)\times(n-k)}$, set $B_W = \mathbf{0}_{n-k}$, and build the block matrices as

$$A_{blk} = \begin{bmatrix} A_V & 0 \\ 0 & W \end{bmatrix}, \quad B_{blk} = \begin{bmatrix} B_V \\ \mathbf{0}. \end{bmatrix}$$

3. To achieve rotational genericity, we sample Haar matrices $Q \in \mathrm{O}(n)$ and set

$$A = QA_{blk}Q^\top, \quad B = QB_{blk}.$$

Then, the reachable subspace is $\mathcal{R}(A, B) = \mathrm{span}(Q_{:,[k]})$.

5.1.1. Identifiability Criteria

Objective. *Assess whether our proposed identifiability criteria co-vary with estimation erros of consistent estimators from single input-state trajectories.*

Let $D_S = \{(A_i, B_i) : i \in [N_S]\}$ denote the matrix ensemble used for the following experiment. Let us further denote the set of initial states used as $D_{X_0} := \{x_0^j \sim \mathrm{Unif}(\mathbb{S}^{n-1}) : j \in [N_{x_0}]\}$. For each of the $N_S \cdot N_{x_0}$ tuples $(A, B, x_0) \in D_S \times D_{X_0}$, we compute

$$\texttt{score}^{(\ell)} := c^{(\ell)}\left(A^{(\ell)}, B^{(\ell)}, x_0^{(\ell)}\right); \;\; c \in \{\text{PBH, Krylov, Left eigenvec.}\},$$

as well as

$$\texttt{err}^{(\ell,\texttt{algo})} := \texttt{REE}\left(\left(\hat{A}^{(\ell,\texttt{algo})}, \hat{B}^{(\ell,\texttt{algo})}\right), \left(A^{(\ell)}, B^{(\ell)}\right)\right); \;\; \texttt{algo} \in \{\text{MOESP, DMDc}\},$$

where

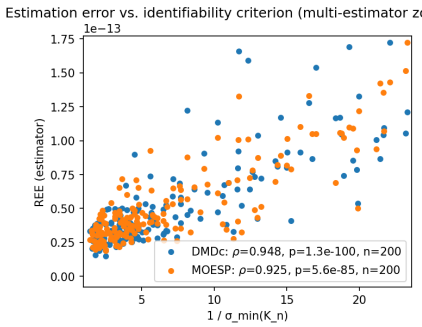
$$\texttt{REE}\left((\hat{A}, \hat{B}), (A, B)\right) = \frac{1}{2}\left(\frac{\|\hat{A} - A\|_F}{\|A\|_F} + \frac{\|\hat{B} - B\|_F}{\|B\|_F}\right).$$

Hypothesis. Using this, we can formulate our hypothesis as follows:

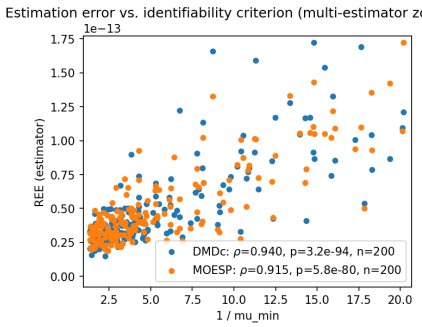
$$\boxed{H_1^{corr} : \rho(1/\texttt{score}, \texttt{err}) > 0.}$$

Here, $1/\texttt{score}$, under abuse of notation, contains all $1/\texttt{score}^{(i)}$, and $\texttt{err}$ contains all $\mathrm{err}^{(i)}$ (a total of $N_S \cdot N_{x_0}$ data points). We use $1/\texttt{score}$ instead of $\texttt{score}$, since smaller scores indicate unidentifiability, as argued earlier, and we expect estimation errors to grow as identifiability scores decrease. We report *Pearson correlations* $\rho(\cdot)$.

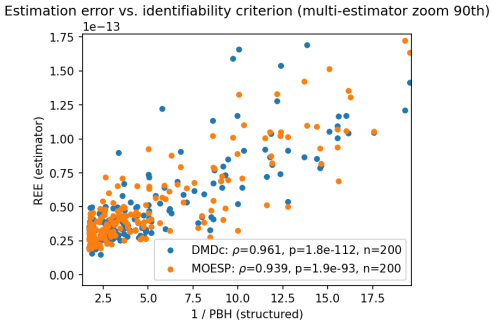
Procedure 1 (Single system). The results in Figure 5.1 were produced with a single system with state/input dimensions $(n, m) = (10, 10)$, and $\dim(\mathcal{R}(A, B)) = 9$, randomly produced with the draw-with-controllability-rank procedure described earlier. We used horizon $T = 100$, sampling step $dt = 0.01$, initial state set size $N = 200$. The initial states are sampled uniformly at random from the unit sphere $\mathbb{S}^{n-1}$. To produce input-state trajectories, we used a PRBS input with sufficient persistency of excitation order.



(a) $1/\sigma_{\min}(K_n)$



(b) $1/\mu_{\min}$



(c) $1/d_{\text{pbh}}$

Figure 5.1.: DMDc and MOESP estimation errors vs. identifiability scores.

For this arbitrary but fixed system, we measure the following Pearson correlations

Table 5.1.: Single-system pearson correlations (ρ) between identifiability criteria and errors.

	Krylov	Left eigv.	PBH
DMDc	0.9483	0.9397	0.9610
MOESP	0.9247	0.9149	0.9386

Procedure 2 (Multiple systems). We fix $n = 5, m = 5, N_S = 200, N_{x_0} = 200, T = 100, dt = 0.01$.

1. **Stratification.** For each $d \in \{0, n-1\}$, we randomly select $\left\lfloor \frac{N_S}{n} \right\rfloor$ matrix pairs (A, B) with controllability matrix dimension $\dim(\mathcal{R}(A, B)) = d$ using the procedure Algorithm 7.

2. **Initial states.** For each system $(A^{(i)}, B^{(i)})$, we draw multiple initial states $\{x_0^{(i)} \sim \mathbb{S}^{n-1} : i \in [N_{x_0}]\}$ from the unit sphere $\mathbb{S}^{n-1}$.

3. **Input signals and trajectory simulation.** We generate a PRBS input $U \in \mathbb{R}^{m\times(T-1)}$, simulate the discrete trajectories

$$X_{t+1}^{(i)} = A_d^{(i)} X_t^{(i)} + B_d^{(i)} U_t, \quad X_0 = x_0^{(i)},$$

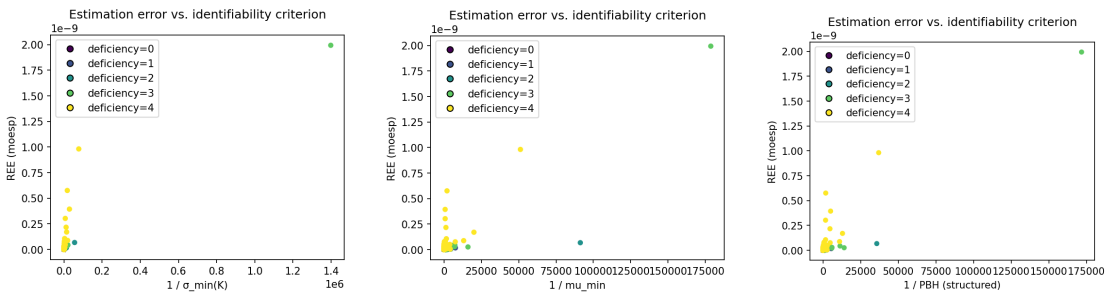
and set $X = [X_0, \ldots, X_{T-2}]$, $X_+ = [X_1, \ldots, X_{T-1}]$.

4. **System Identification.** Provide (X, X_+, U) to MOESP and DMDc, and obtain the estimates $(\hat{A}^{(i)}, \hat{B}^{(i)})$.

5. **Obtain scores and estimation errors.** For each matrix pair - initial state combination (and for `algo` $\in$ {DMDc, MOESP}), compute the identifiability scores (and estimation errors).

6. **Compute correlations and report.** Compute the Pearson correlation between inverted identifiability scores $\{1/d_{\text{pbh}}, 1/\sigma_{\min}(K_n), 1/\mu_{\min}\}$ and estimation errors.

Below are scatter plots that display the full range of the results.



(a) $1/\sigma_{\min}(K_n)$ (b) $1/\mu_{\min}$ (c) $1/d_{\text{pbh}}$

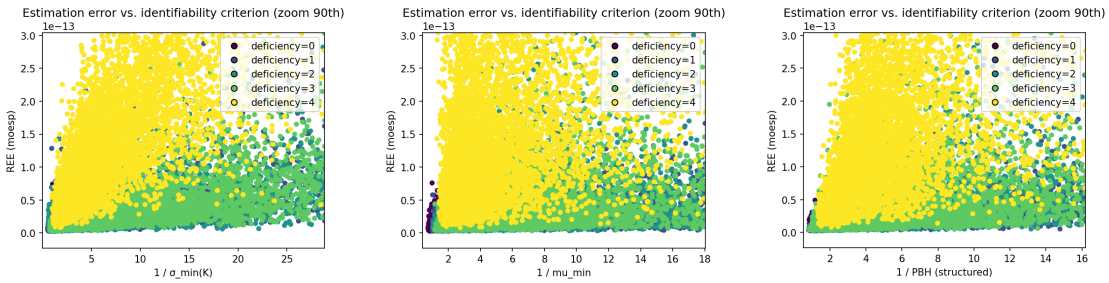
Figure 5.2.: MOESP estimation errors vs. identifiability scores.

We measure the following correlation values.

Table 5.2.: Pearson correlations (ρ) between identifiability criteria and errors (second dataset, ensvol=200, $x_0 = 200$).

	Krylov	Left eigv.	PBH
DMDc	0.9539	0.9405	0.9637
MOESP	0.9300	0.9177	0.9509

When we "zoom in" to the 90% quantile, the following pictures emerge:



(a) $1/\sigma_{\min}(K_n)$ (b) $1/\mu_{\min}$ (c) $1/d_{\text{pbh}}$

Figure 5.3.: MOESP estimation errors vs. identifiability scores.

What the pictures say. Across both algorithms (MOESP and DMDc) and for each of the three identifiability criteria, the *sample* Pearson correlation between the inverted identifiability scores and the estimation errors is numerically close to 1 (i.e., $r \approx 1$).

Geometrically, the scatter clouds concentrate along principal axes running from the origin toward the upper-right corner, with the axes tilting further away from the x−axis as the state dimension-visible subspace dimension gap $n - \dim(V(\mathrm{x}_0))$ increases. This indicates stronger covariance between the scores and estimation errors, as the identification becomes more "difficult".

What the pictures don't say. The hyperparameters (T, dt, n, m) and other design choices were held fixed at arbitrarily selected values. With the current data, we cannot rule out these settings as confounders or mediators of the observed association. Consequently, the evidence supports $H_1^{\text{corr}} : \rho(1/\texttt{score}, \texttt{err}) > 0$ only *conditionally* on the chosen parameter vector. To claim a more general statement, one would need experiments that systematically vary these settings (e.g., via stratification or regression/partial-correlation analyses) and still observe $\rho(1/\texttt{score}, \texttt{err}) > 0$.

We also note that, across all experiments, the *absolute* estimation error was extremely small—the mean was on the order of 10^{-13}-despite the strong correlations reported above. This supports the view that controlled identifiability is generically present in our setting, as formalized in Theorem 4.11. For a sharper follow-up to the second procedure, it would be preferable to stratify not the pairs (A, B) by controllability rank, but the tuples (A, B, x_0) directly by the dimension of their visible subspace (this can be achieved with the help of Algorithm 7) in order to obtain a closer empirical fit to the theory developed in Chapter 4.

Takeaway

Under the fixed experimental configuration, we observe near-perfect positive Pearson correlations between inverted identifiability scores and estimation error for both MOESP and DMDc, consistent with H_1^{corr}. Because key hyperparameters (T, dt, n, m) were held constant, this conclusion is conditional on those settings. Broader validation under varied configurations is needed to claim generality.

5.1.2. Equivalence Classes

Objective. *We aim to further validate the single-trajectory framework developed in Chapter 4 by checking whether the restriction of $(\hat{A}, \hat{B})$ to the visible subspace $V(\mathrm{x}_0)$ is recovered, even when the "true" system (A, B) is only partially identifiable from x_0. Here, $V(\mathrm{x}_0)$ is defined as the $A-$invariant hull of $\mathrm{x}_0 \cup \operatorname{Im} B$ Definition 2.6:*

$$V(\mathrm{x}_0) = \operatorname{span}\{\Big[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\Big]\}.$$

Additionally, we aim to explore how estimation errors vary with decreasing $\dim(V(\mathrm{x}_0))$ in both standard and $V(\mathrm{x}_0)-$adapted bases.

Hypotheses. *The estimation errors can be arbitrarily large when measured in standard basis with only partially identifiable (A, B, x_0) tuples, but they will vanish when converted to a basis adapted to $V(\mathrm{x}_0) \oplus V(\mathrm{x}_0)^{\perp}$.* To use the terminology from earlier:

$$\boxed{H_1^{\text{val}} : (\mathcal{E}^{\text{std}} > 0) \wedge (\mathcal{E}_V \approx 0).}$$

Additionally, we expect the estimation errors to increase as the gap between the state dimension n and the dimension of the visible subspace $k := \dim(V(\mathrm{x}_0))$ increases, since the number of degrees of freedom (see Corollary 4.5) increase with $n - k$, and so does the "size" of the equivalence class. Since, according to the theory developed in §4.1, the entirety of what is identifiable within the system (A, B) already lives in $V(\mathrm{x}_0)$, we don't expect to see an increase in estimation errors after the conversion to the basis $(P_V, I - P_V)$. On the contrary, an argument can be made that estimation errors, after conversion to the $(P_V, I - P_V)$ basis, should actually decrease, since the identification task becomes smaller (see the discussion in [16] on how state dimension effects identifiability). We informally solidify this monotonicity hypothesis as:

$$\boxed{H_1^{\text{mon}} : \mathcal{E}^{\text{std}} \text{ mon. increasing in } n - k, \text{ and } \mathcal{E}_V \text{ const. in } n - k}$$

Procedure. A full pseudo-algorithm of the computation that produced the following plots can be found in Appendix A.3.1. Here, we give a brief overview of the key steps therein. For a each $d = n - \dim(V(\mathrm{x}_0))$ that is in the sweep grid, we perform the following steps:

1. **Ensemble generation.** Sample $N = 200$ pairs $D := \{(A_i, B_i) \sim \texttt{ensemble_type} : i \in [N]\}$. Discretize under a constant time step dt to obtain (A_i^{dt}, B_i^{dt}).

2. **Reachable basis.** Compute an orthonormal basis $R^{(i)} \in \mathbb{R}^{n \times r}$ of the discrete-time reachable subspace (see Theorem 2.21) via SVD from the controllability matrix $[B_i^{dt}, \dots (A_i^{dt})^{n-1} B_i^{dt}]$ using SVD.

3. **Initial state construction.** Work in $A_i^r := R^\top A_i^{dt} R$ and choose $\mathrm{y}^{(i)} \in \mathbb{R}^n$ so that $\{\mathrm{y}^{(i)}, \dots, (A_i^r)^{k-1}\mathrm{y}^{(i)}\}$ are linearly independent. Set $\mathrm{x}_0^{(i)} := R\mathrm{y}^{(i)} / \|R\mathrm{y}^{(i)}\|$. This ensures almost surely, i.e., except on a measure-zero set where eigenvalue multiplicities force extra linear dependencies.

4. **Visible basis.** Build an orthonormal basis $(P, I - P)$ for $V(\mathrm{x}_0)$ using SVD and $\big[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \dots,\ A^{n-1}[\mathrm{x}_0\ B]\big]$ (indices are omitted for better readability, bit these computations are performed separately for all tuples $(A_i^{dt}, B_i^{dt}, \mathrm{x}_0^{(i)})$.

5. **Control input & Trajectory simulation.** Generate pseudo-random binary signal (PRBS) as input, and simulate $X_{t+1}^{(i)} = A_i^{dt} X_t^{(i)} + B_i^{dt} \mathrm{u}_t$ for $t = 0 : 1 : T - 1$. Form (X_0, X_1, U), where

$$X_0 = [\mathrm{x}_0^{(i)}, \dots, \mathrm{x}_{T-2}],\ X_1 = [\mathrm{x}_1, \dots, \mathrm{x}_{T-1}],\ U = [\mathrm{u}_0, \dots, \mathrm{u}_{T-2}].$$

   Run a persistency of excitation diagnostic to verify that the control input is sufficiently rich.

6. **Identification.** Provide each estimator `algo` $\in \{$DMDc, MOESP, SINDy, NODE$\}$ estimator with the shifted state and input trajectories $(X_0^{(i)}, X_1^{(i)}, U^{(i)})$ and obtain the estimates $(\hat{A}^{(i)}, \hat{B}^{(i)})$.

7. **Evaluation.** Compute the relative estimation errors in the standard basis, obtaining

$$\mathcal{E}_A^{(i)} = \frac{\|\hat{A}^{(i)} - A_i^{dt}\|_F}{\|A_i^{dt}\|_F}, \quad \mathcal{E}_B^{(i)} = \frac{\|\hat{B}^{(i)} - B_i^{dt}\|_F}{\|B_i^{dt}\|_F}, \quad \mathcal{E}_V^{(i)} = \frac{1}{2}\left(\mathcal{E}_A^{(i)} + \mathcal{E}_B^{(i)}\right)$$

   and re-compute after the change of basis to obtain $\mathcal{E}_{A|_V}^{(i)}, \mathcal{E}_{B|_V}^{(i)}, \mathcal{E}_V^{(i)}$.

Configuration details. All results presented in the following (DMDc, MOESP, SINDy, NODE) used a discretization step of $dt = 0.1$, trajectory length of $T = 200$ time steps, a PRBS signal of sufficient persistency of excitation order.

Dynamic Mode Decomposition with Control

We consider the one-step input-state regression $X_+ = AX + BU, \quad Z := \begin{bmatrix} X \\ U \end{bmatrix}$, where $X = [X_0, \ldots, X_{T-2}]$, $X_+ = [X_1, \ldots, X_{T-1}]$, $U = [U_0, \ldots, U_{T-2}]$. The DMDc estimator is the least-squares solution $\hat{\Theta} := [\hat{A} \;\; \hat{B}] = X_+ Z^\dagger = X_+ Z^\top \left(ZZ^\top\right)^{-1}$, whenever Z has full row rank. We call DMDc a *consistent estimator*, because, under standard assumptions (absence of noise, persistently exciting inputs), the least-squares fit converges in probability to the true parameter $\Theta^* = [A \;\; B]$ as $T \to \infty$.

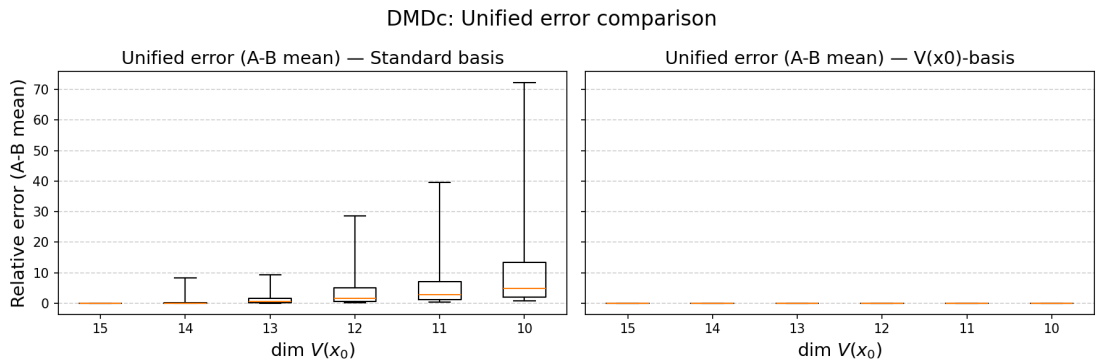
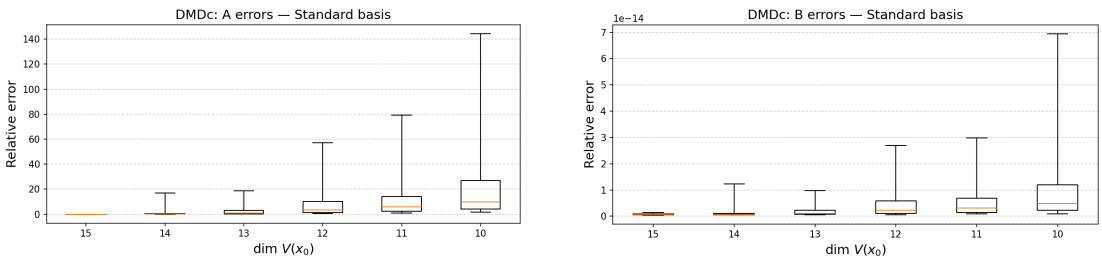


Figure 5.4.

Error collapse in the V–basis. Switching to the $V(\mathrm{x}_0)$-adapted basis collapses the errors to numerical precision: whereas in the standard basis the 10–90th percentile band spans $[0, , 20]$, in the V–basis the errors sit at $\sim 10^{-13}$ (Section 5.1.2). This is exactly the equivalence-class prediction: restricting to $V(\mathrm{x}_0)$ removes the unidentifiable directions, so consistent estimators recover $(A|_V, B|_V)$ up to machine precision. In both bases the error increases with the deficit $n - \dim V(\mathrm{x}_0)$, but in the V–basis any residual growth is negligible and attributable to conditioning (Figure 5.5).

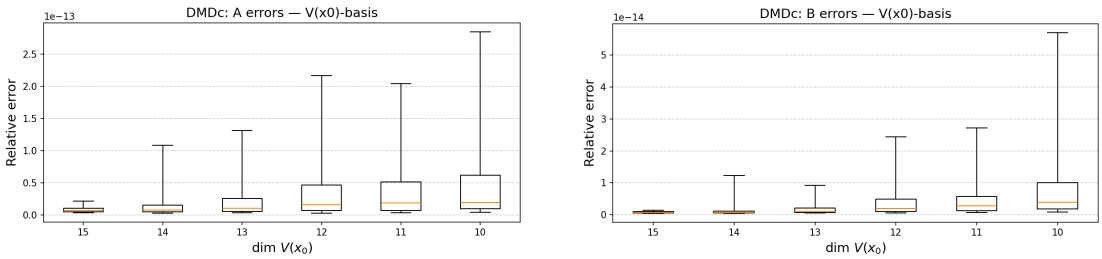
The dominating error term. To localize sources, we decompose the aggregate error into $\mathcal{E}_A$ and $\mathcal{E}_B$. In the standard basis (Figure 5.5), $\mathcal{E}_A$ dominates, while $\mathcal{E}_B$ is essentially at the numerical floor ($\sim 10^{-14}$). In the $V(\mathrm{x}_0)$-basis (Figure 5.6), both $\mathcal{E}_A$ and $\mathcal{E}_B$ collapse to machine-precision levels, consistent with recovery of the visible block and with B being directly constrained within $V(\mathrm{x}_0)$ by construction.



(a) $\hat{A}$ error

(b) $\hat{B}$ error

Figure 5.5.: Relative errors for DMDc standard-basis recovery of A and B.



(a) $\hat{A}$ error in $V(x_0)$ basis

(b) $\hat{B}$ error in $V(x_0)$ basis

Figure 5.6.: Relative errors for DMDc in the $V(x_0)$ basis.

Multivariable Output-Error State-Space (MOESP)

MOESP is a subspace identification method and, under standard assumptions (noise-free data, correct model order, and sufficiently exciting inputs), is consistent. We observe the same qualitative phenomenon as with DMDc: in the $V(\mathrm{x}_0)$-adapted basis the estimation error collapses to (near) numerical precision, further corroborating the equivalence-class claim

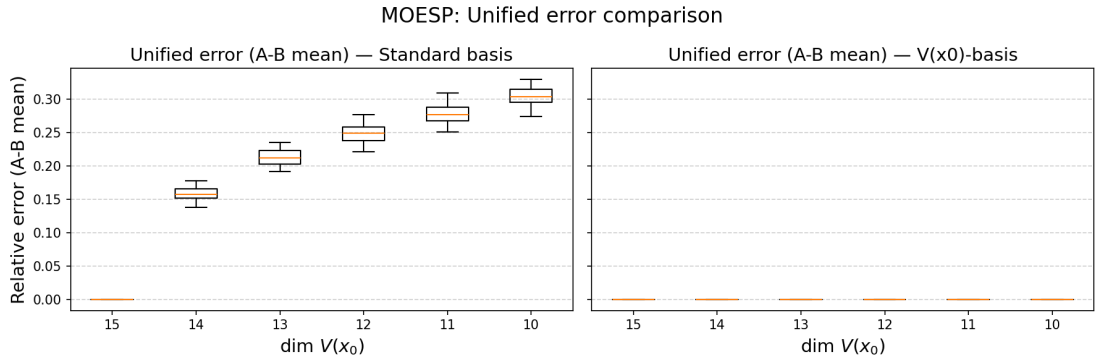


Figure 5.7.

A notable difference emerges in how errors vary with the visibility deficit $n - \dim(V(\mathrm{x}_0))$. For MOESP, the error in the standard basis still grows monotonically with this deficit, but in the $V(\mathrm{x}_0)-$basis it drecreases monotonically. A plausible explanation is that, as $V(\mathrm{x}_0)$ shrinks, the signal subspace extracted by MOESP separates more cleanly from its orthogonal complement.

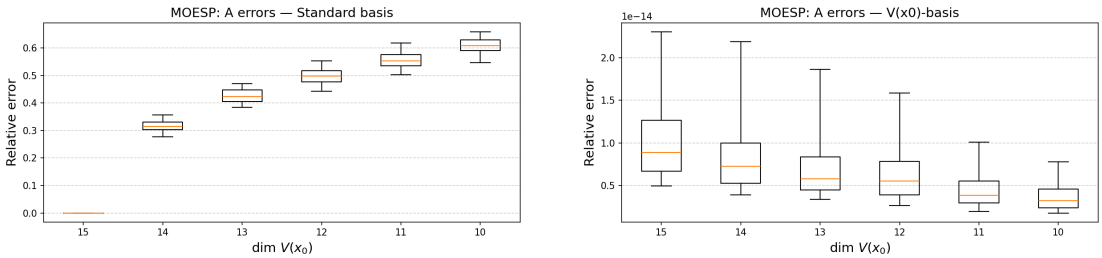
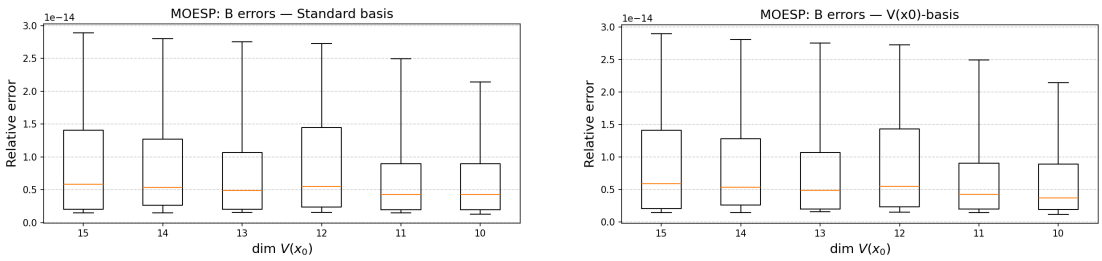


Figure 5.8.: $A-$ specific estimation errors for MOESP in standard and $V(\mathrm{x}_0)-$basis.



(a) $\hat{A}$ error

(b) $\hat{B}$ error

Figure 5.9.: $B-$specific estimation errors for MOESP in standard and $V(x_0)-$basis.

As with DMDc, the dominating error term is $\mathcal{E}_A = \|\hat{A} - A\|_F / \|A\|_F$.

Sparse Identification of Nonlinear Dynamics (SINDy)

SINDy posits a sparse linear model in a feature library $\Phi(x, u)$ and solves $X_1 \approx \Xi\,\Phi(X_0, U)$ with sparsity-promoting regression (STLSQ/Lasso). We use PySINDy in discrete time with a polynomial library (degree 2) on (x, u) and its default sparsification. The code forms $X_{\text{all}} = [x_0, x_1, \ldots, x_{T-1}]$, fits, and then extracts $(\hat{A}, \hat{B})$ by partitioning the learned coefficients.

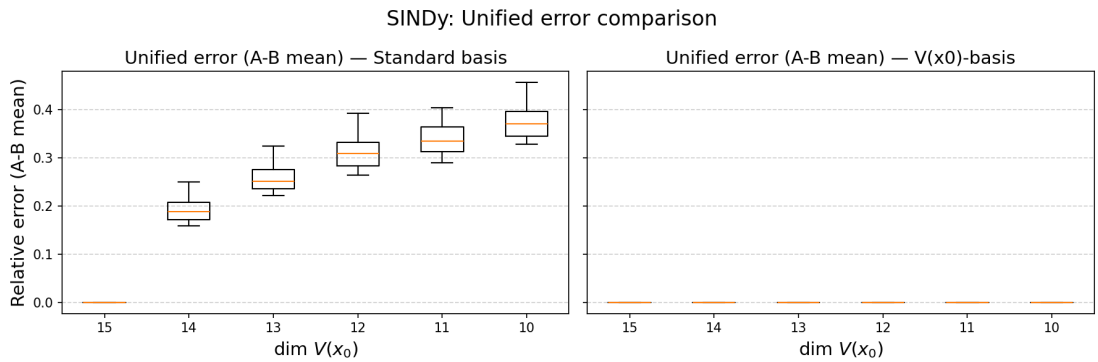
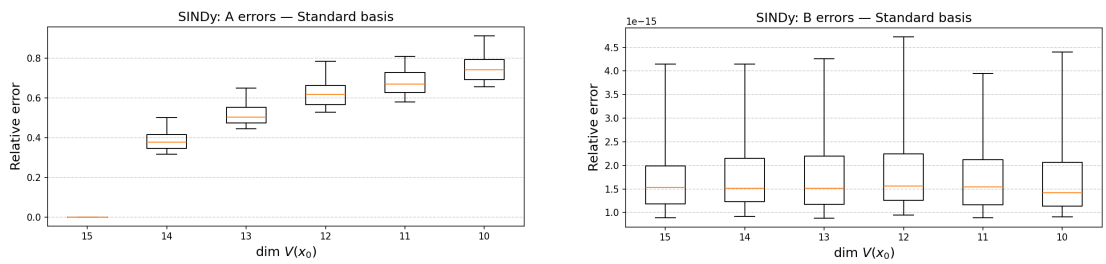


Figure 5.10.

As with MOESP and DMDc, moving from the standard basis to the $V(\mathrm{x}_0)-$adapted basis collapses the estimation error to numerical precision, supporting the equivalence-class prediction that $(A|_V, B|_V)$ is recoverable from a single trajectory.



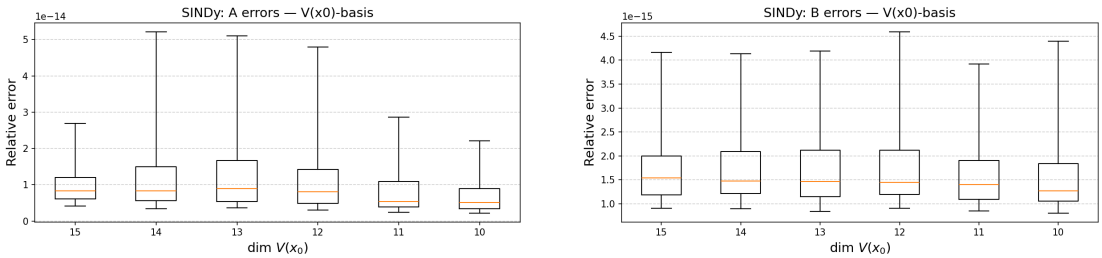
(a) $\hat{A}$ error

(b) $\hat{B}$

Figure 5.11.: Relative errors for SINDy in the standard basis.

Again, the aggregate error comes predominantly from the estimation error in A. An

unexpected trend is that standard-basis estimation errors actually seem to decrease monotonically starting at $\dim(V(\mathrm{x}_0)) = n - 2$ as the number of "invisible" dimensions increases.



(a) $\hat{A}$ error (b) $\hat{B}$ error

Figure 5.12.: Relative errors for SINDy in the $V(x_0)$ basis.

Neural Ordinary Differential Equations (NODE)

Although we call this estimator a Neural ODE, strictly speaking, we implement a *continuous-time linear state-space model with trainable matrices* (A, B) fit by one-step prediction after exact zero-order-hold (ZOH) discretization. The loss is MSE on one-step next state prediction, using the same trajectories (X, X_+, U) as the other estimators:

$$\mathcal{L}^k := \frac{1}{T}\sum_{k=0}^{T-2} \|\mathrm{x}_{k+1} - \hat{\mathrm{x}}_{k+1}\|_2^2 = \frac{1}{T}\sum_{k=0}^{T-2} \|\mathrm{x}_{k+1} - (A_d\mathrm{x}_k + B_d\mathrm{u}_k)\|_2^2.$$

Further configuration details can be found in Appendix A.3.3.

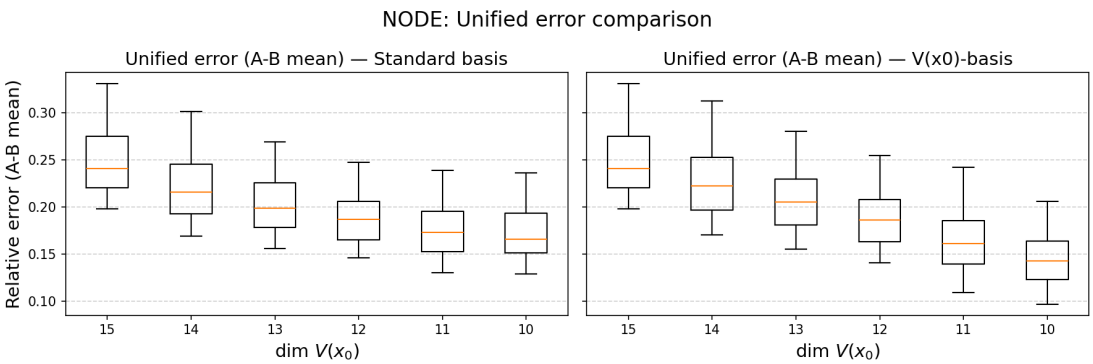
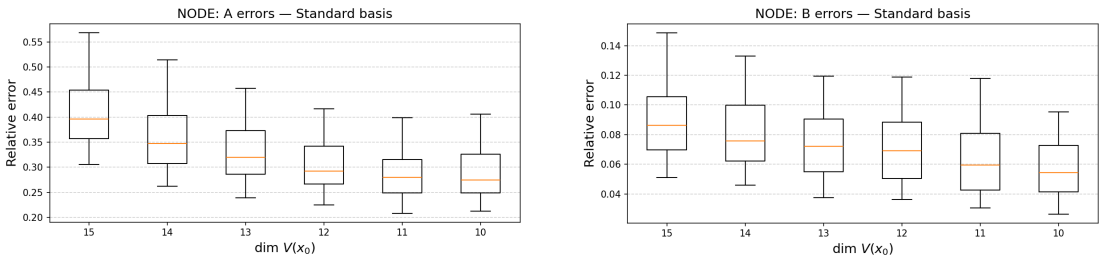


Figure 5.13.

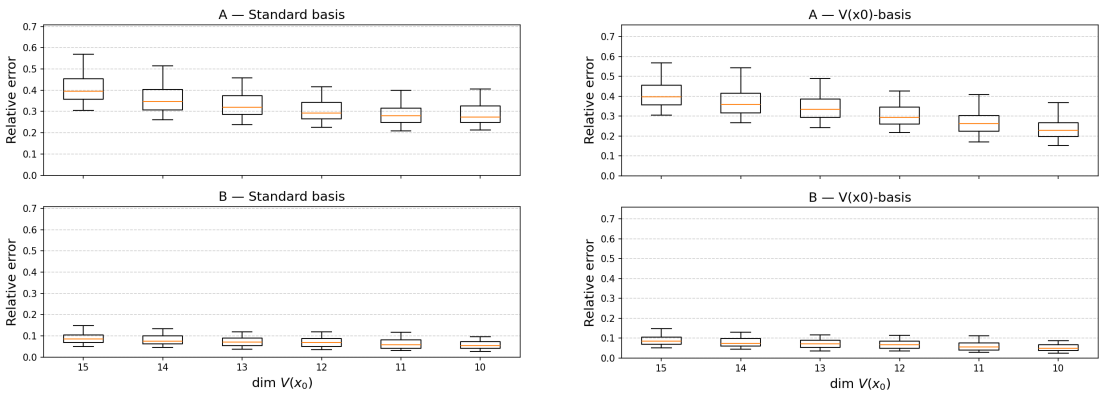
The estimation errors in the standard and $V(\mathrm{x}_0)$-bases don't show the same dramatic difference we have seen with the estimators we have reviewed so far. But, estimation errors measured in the $V(\mathrm{x}_0)-$ basis do remain smaller, with the difference to standard basis evaluations becoming smaller as the difference between state dimension and $\dim(V(\mathrm{x}_0))$ increases.



(a) $\hat{A}$ error (b) $\hat{B}$

Figure 5.14.: Relative errors for NODE in the standard basis.

As with other estimators, $A-$only estimation errors are larger than $B-$ only estimation errors, but the difference is not as dramatic as with previously reviewed estimators. Estimation errors in the standard basis remain large compared to other estimators we have surveyed. It can't be ruled out that this is due to the specific configuration Appendix A.3.3 under which the model was trained.



(a) $\hat{A}$ error (b) $\hat{B}$ error

Figure 5.15.: Relative errors for NODE in the $V(x_0)$ basis.

Estimation errors decrease monotonically in the $V-$basis, which can be attributed to the projection onto $V(\mathrm{x}_0)$ essentially removing the "invisible" dimensions, thus

reducing the effective dimension of the problem. Informed by earlier results on how the ambient state dimension effects identifiability Section 5.2.2, we can expect this to reduce estimation error.

All of the estimators we have tested so far, with the exception of DMDc had this property: in the $V-$basis, estimation error decreased as $\dim(V(\mathrm{x}_0))$ decreased. We could say that DMDc had $V-$basis errors in the vicinity of $1e-13$ and that this can be attributed to numerical interferences, but MOESP also has "$V-$errors" in the vicinity of $1e-13$, but still consistent in the sense that $V-$errors decrease as $\dim(V(\mathrm{x}_0))$ decreases.

Takeaway

Across consistent estimators (DMDc, MOESP) and SINDy, we observe that

(i) under the partially identifiable setting with $\dim(V(\mathrm{x}_0)) < n$, the estimation error can be nontrivial, when measured in the standard basis, but, as the theory predicted,

(ii) this error vanishes when measured in the basis adapted to $V(\mathrm{x}_0) \oplus V(\mathrm{x}_0)^{\perp}$.

With NODE, estimation errors in the visible subspace basis do remain smaller, but only slightly so, with the difference in measured errors across bases increasing as the gap between the dimension of the state space and that of the visible subspace increases.
For all methods, the dominating error term is the $A-$only relative estimation error.

Now that we have sufficient evidence that supports the analysis we developed in Chapter 4, we utilize the identifiability criteria proposed there to map certain parameter regimes that are most prone to unidentifiability.

5.2. Parameter Regimes

Objective. Map regions in the parameter space ($(A, B) \in \Omega,\ \Omega = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$) where single-trajectory identifiability is fragile. We do so using the x_0-based identifiability criteria from Corollary 4.9 without running any estimator, i.e., purely structurally from (A, B, x_0).
Procedure. For every grid point we draw N_S many (A, B) pairs, and, for each (A, B), we draw N_{x_0} many initial state samples uniformly from the unit sphere $\mathbb{S}^{n-1}$. Each point on the line plots, as well as each cell in the heat maps thus aggregate $N_{pt} = N_S \cdot N_{\mathrm{x}_0}$

score evaluations.

1. **Ensemble generation.** Unless stated otherwise, (A, B) are drawn from the sparse-continuous ensemble. By default, both A and B are sparsified. For sweeps not actively varying sparsity, the baseline density is $p = p_A = p_B = 0.3$. Initial states are drawn once from the unit sphere, as stated earlier.

2. **Scores.** For each (A, B, x_0) in the ensemble, we compute $\sigma_{\min}(K_n)$, $\mu_{\min} = \min_i \frac{\|w_i^\top [x_0\ B]\|}{\|w_i\|_2}$, and $d_{\text{pbh}} = \inf_{\lambda \in \sigma(A)} \sigma_{\min}\left(Q^\top [\lambda I - A, [x_0\ B]]\right)$. We report cellwise means.

Axes and grids. We sweep three "levers"

$$\begin{array}{rlrl} \textit{(Sparsity)} & p = p_A = p_B \in \{0.0, 0.1, \ldots, 1.0\} & \textit{(Default:)} & p = 0.3 \\ \textit{(State dim.)} & n \in \{2i : i \in [10]\} & \textit{(Default:)} & n = 10 \\ \textit{(Actuation)} & m \in \{1, \ldots, n\} & \textit{(Default:)} & m = 2 \end{array}$$

When an axis is not explicitly swept, the default values stated above are used.

5.2.1. (S1) Sparsity

We use $p = p_A = p_B \in \{0.0, 0.1, \ldots, 0.9\}$ as our sparsity grid. p_A and p_B are used to produce sparse-continuous matrices as described earlier. For each $p \in \{0.0, 0.1, \ldots, 0.9\}$, we produce a sparse-continuous ensemble containing 100 independently sampled (A, B) pairs. $\{x_0^{(i)} : i \in [100]\}$ are sampled uniformly from $\mathbb{S}^{n-1}$.

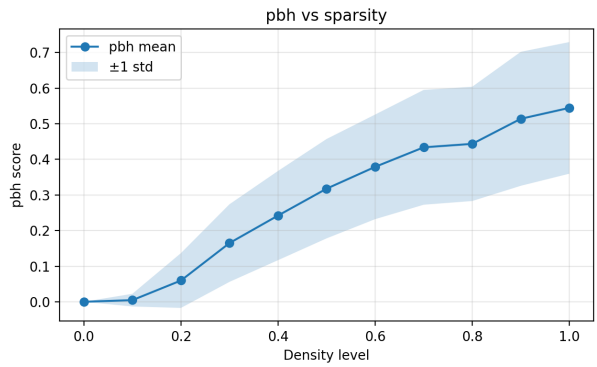


Figure 5.16.: Identifiability metrics vs. sparsity.

The empirical mean of the PBH score increases with density, as predicted by the intuition that the rank conditions that govern our identifibility criteria will become more easier to satisfy, as the matrices (A, B) become more dense. The PBH and left eigenvector curves show a saturating rise. The Krylov curve, denoting the minimum singular value of the generator matrix $\big[[\mathrm{x}_0\ B],\ A[\mathrm{x}_0\ B],\ \ldots,\ A^{n-1}[\mathrm{x}_0\ B]\big]$, exhibits a sharper turn-up once a critical density (ca. 0.7 by visual inspection) is crossed, with a corresponding jump in spread. (Please refer to Appendix A.3.2 for left-eigenvector and Krylov curves.)

5.2.2. (S2) State Dimension

We vary the state dimension as $n \in \{2, 4, \ldots, 20\}$. For each n we keep m fixed (default $m = 2$ unless stated) and draw from the baseline sparse ensemble at density $p = 0.3$.

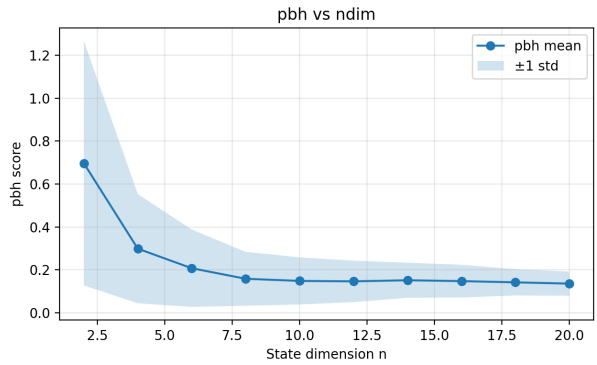


Figure 5.17.: PBH score vs. state dimension.

The PBH margin deteriorates with growing n at fixed m, in line with the intuition that rank conditions become harder in higher ambient dimension at a fixed input budget; dispersion also widens with n.

5.2.3. (S3) Degree of Actuation

As the last item in the single-axis part of this section, we look at varying degrees of actuation, since, intuitively, for matrices with $n \gg m$, the rank conditions we laid out in Chapter 4 become more difficult to fulfill. To measure this, we fix $n = 10$, and vary m between 1 and 10.

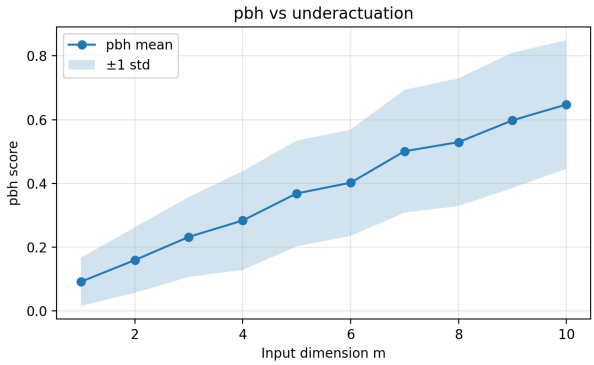


Figure 5.18.: PBH score vs. state dimension.

As $(n - m)$ grows (i.e., fewer inputs), the PBH distance drops monotonically; nevertheless, at the present density level the cellwise means remain strictly positive, indicating typical draws are structurally identifiable in the x_0–sense (though increasingly ill-conditioned).

For each heatmap, the color is the cellwise mean of the chosen score. A companion "descent" panel visualizes the negative gradient field (arrows), indicating directions along which the score decreases fastest. In **(D2)** we overlay the diagonal $n = m$ (full actuation) in red.

5.2.4. (D1) Sparsity vs. State Dimension

We vary sparsity $p \in [0, 1]$ on 0.1 steps, and state dimension on $n \in \{2i : i \in [10]\}$. We use $m = \min(2, n)$ and keep constant all "non-active" sweep parameters such as trajectory length T, and signal properties.

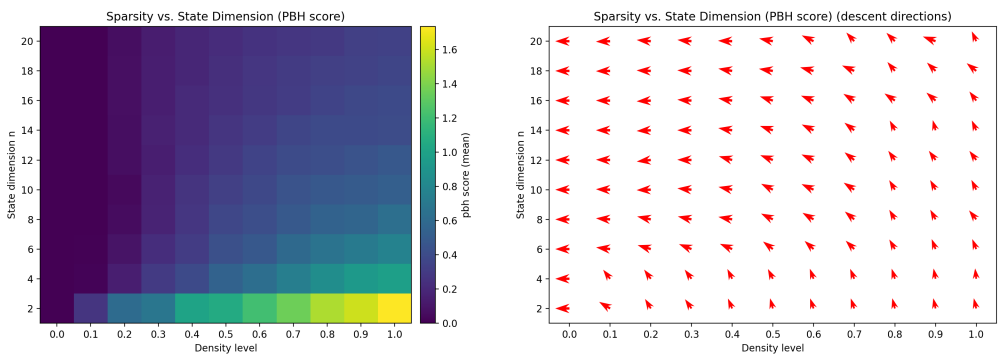
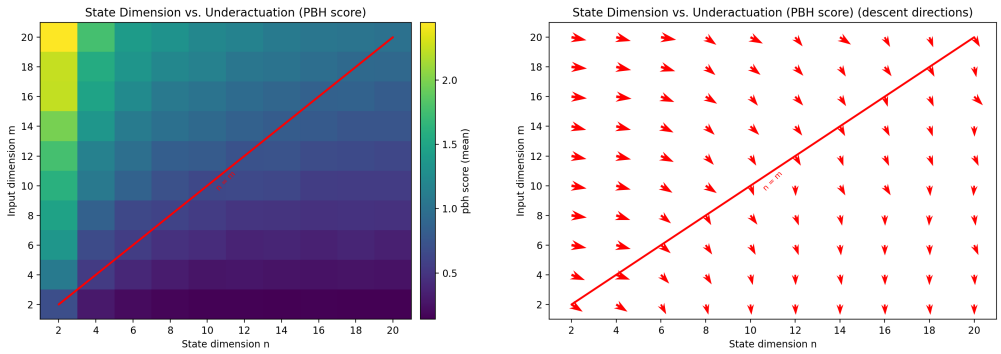


Figure 5.19.: Sparsity vs. state dimension.

Moving along the *sparsity* axis (toward higher densities) improves scores more strongly than comparable changes in n degrade them; the descent field is mostly horizontal, indicating that density dominates this interaction in the tested range. It is also interesting to note that the magnitude of change, indicated by the size of the arrows, monotonically drops in this direction.

5.2.5. (D2) State Dimension vs. Underactuation

We vary state dimension on $n \in \{2i : i \in [10]\}$, and sweep m on a grid covering $\{1, \ldots, n\}$.



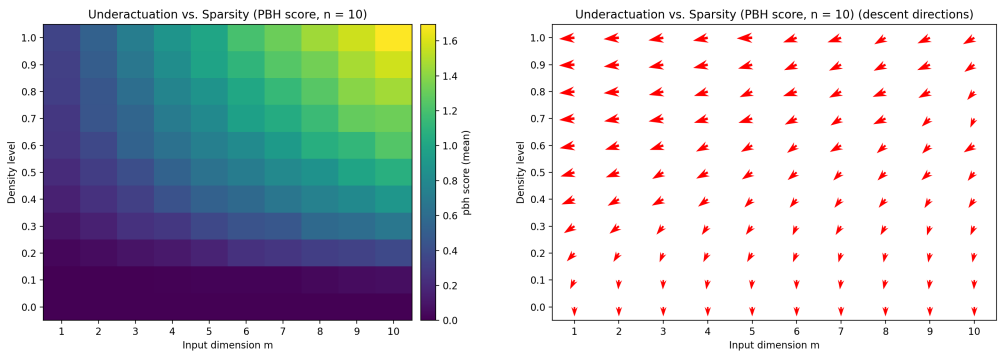
(a) PBH (b) PBH

Figure 5.20.: State dimension vs. underactuation.

It has been described in the literature that identification becomes increasingly difficult in higher state dimensions [16], thus, we expect the identifiability scores to drop as we move to the right, increasing n. Also, it is intuitive that our rank conditions become increasingly more difficult to satisfy as the relative number of input channels m/n sinks. Thus, we expect measured identifiability scores to drop as we move downwards, decreasing m. This is confirmed by the negative gradient field we obtain: overall, the flow is from the upper left corner to the bottom right. It is also interesting to note that the magnitude of change, indicated by the size of the arrows, monotonically drops in this direction.

5.2.6. (D3) Underactuation vs. Sparsity

We vary sparsity $p \in [0, 1]$ on 0.1 steps, fix the state dimension as $n = 10$ and sweep $m = \{1, \ldots, 10\}$.



(a) PBH (b) PBH

Figure 5.21.: Underactuation vs. sparsity.

Both increasing state-to-input-dimension ratios (*underactuatedness*) and increasing levels of sparsity are expected to make the identification problem more difficult. Thus, we expect our measured mean identifiability scores to sink as we move towards the bottom left corner. This is confirmed by both the heat map and the descent plot.

Takeaway

Across all sweeps, the three scores respond in the expected directions: increasing density and actuation improve distances to structural unidentifiability, whereas increasing ambient dimension degrades them; interactions are mostly monotone. Because ensemble type, baseline density, and x_0 sampling policy were held fixed, these statements are conditional on the present pipeline and should be re–checked under alternative ensembles and input–energy constraints.

5.3. Input Sufficiency

Objective. Quantify how the order of persistency of excitation required for single trajectory recovery depends on the visible subspace dimension $k = \dim V(x_0)$. We compare a partially identifiable case ($k = 3$) against a fully identifiable case ($k = n = 8$) under a fixed data generation and system identification pipeline (DMDc).

We fix $n = 8$, $m = 2$, $dt = 0.05$, and evaluate PE orders $s \in \{1, \ldots, 5\}$. For each scenario, $k \in \{3, 8\}$, we collect $N = 200$ (A, B, x_0) tuples with $k = \dim(V(x_0))$ using Algorithm 7. As input, we use a pseudo-binary random sequence (PRBS) with exact PE order s: for each s, we choose the minimal horizon compatible with order s and

system dimensions and repeatedly sample PRBS until a numerical rank test certifies order $s+1$ exactly. As with earlier runs, we report the relative estimation error.

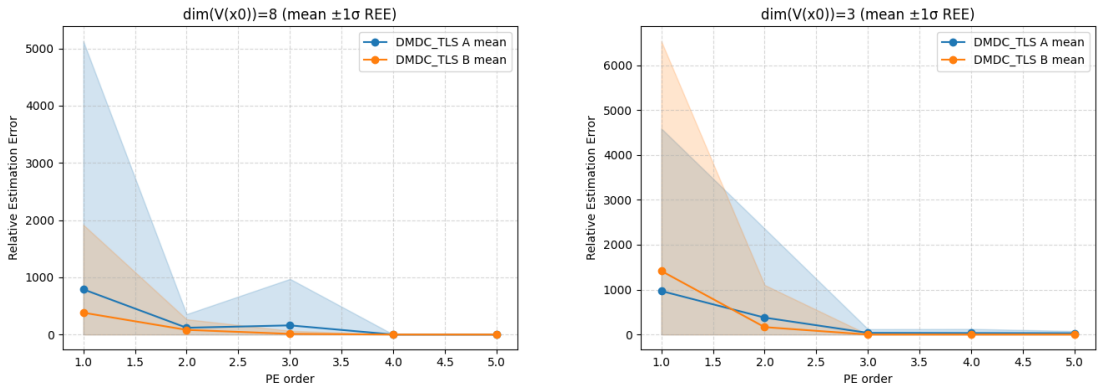


Figure 5.22.: Relative estimation errors for $(n, k := \dim(V(\mathrm{x}_0))) = (8, 8)$ (left figure) and $(n, k) = (8, 3)$ (right figure).

Procedure. For each $k \in \{3, 8\}$,

- draw (A, B, x_0) with $\dim(V(\mathrm{x}_0))$ via Algorithm 7.
- For each $s \in \{1, \ldots, 5\}$, generate a single PRBS with certified PE order s at a minimal time horizon $T = s(m+1)$.
- Simulate the trajectories, obtain (X, X_+, U) and feed them to the estimator (DMDc).
- Obtain estimates $(\hat{A}, \hat{B})$ and record the relative estimation error.

Each point on any of the lines in Figure 5.22 corresponds to the average over 200 estimation errors, obtained from 200 independently sampled (A, B, x_0) tuples, using the PRBS signal computed for that particular PE order s.

What the plots say. In the fully identifiable case ($k = 8$), exact recovery ($\mathcal{E} \approx 0$) is achieved already at $s = 4$. It is expected that identification can be achieved before the minimum sufficient PE order $s = n$ is reached, since the presence of this criterion ensures theoretical guarantees for identifiability [1] but is not required for identifiability (sufficient but not necessary condition). The decisive elbow in estimation errors occurs going from $s = 3$ to $s = 5$ for the fully identifiable system, and from $s = 2$ to $s = 3$ for

[1] Under noiseless conditions.

the partially identifiable system with $k = 3$. In the fully identifiable case, variability is large at smaller PE values, but collapses at $k \geq 4$. In the partially identifiable case, variability collapses at $s \geq 3$.

What the plots don't say. The results should be read with care: there are plausible confounding factors such as trajectory length and the choice of dimensions. The scope of this particular experiment is very limited—a single data point rather than a systematic analysis—so the figure serves as a motivator for subsequent, controlled study rather than a standalone diagnostic or claim of causality.

Findings

(i) *Identifiability criteria.* Across consistent estimators (DMDc and MOESP), we find $\rho(1/\texttt{score}, \texttt{error}) \approx 1$ for PBH, Krylov and left eigenvector criteria.

(ii) *Equivalence classes.* For partially identifiable matrix pair - initial state tuples, the standard estimation error grows with $n - \dim(V(\mathrm{x}_0))$, while the restriction to the visible subspace $V(\mathrm{x}_0)$ collapses the error to near-zero.

(iii) *Parameter regimes.* Distance to structural unidentifiability improves with system matrix density and actuation, and degrades with ambient state dimension. *(iv) Input sufficiency.* Under the conditions detailed in §5.3, the PE order appears to track $k = \dim(V)\mathrm{x}_0)$ more closely than state dimension n. Follow-up experiments that systematically sweep over potential confounding parameters with a higher data volume are needed for a conclusive assessment on the hypothesis that "$V(\mathrm{x}_0)$ is the ceiling, *PE order is the floor*".

6. Discussion and Conclusion

Instead of recapping the earlier chapters, we close with a geometric picture that summarizes the main ideas with an analogy.

Geometric interpretation. A single trajectory illuminates only part of the state space– starting from the initial condition and distributing energy across the input directions, the dynamics carve out a "lit stage" of reachable directions. We call this the visible subspace and formalize it as the minimal subspace which contains *both the initial state and all input directions*, while being closed under the action of the system matrix. All measurements live on this stage– *a single trajectory therefore can only determine the dynamics within the visible subspace*, while any purely off-stage behavior (on the orthogonal complement of the visible subspace) remains unidentifiable.

Resolution within the lit section is governed by persistency of excitation: inputs that distribute energy broadly make the illumination across the visible subspace more uniform, whereas weak excitation leaves some facets in shadow, blurring parameters along those directions. On finite horizons, this shows up as control over only the early response moments (Markov parameters)– higher order features become resolvable only when the input achieves higher persistency of excitation order. Designing better experiments therefore means both *enlarging the stage* and *brightening it uniformly*. If the visible subspace spans the whole room and the illumination is rich and uniform, the dynamics are essentially determined.

Limitations and Future Work. Our analysis is focused on algebraic reasoning with true system matrices, while practice sees only a finite, noisy trajectory. The missing link is a *data-centric translation* of the structural objects and criteria we proposed. In the future, we aim to develop *trajectory-level counterparts* which map the visible subspace and equivalence class statements to properties of the trajectories, or Hankel matrices issuing from these trajectories; and investigate if and how the structural equivalence class relates to the data-level equivalence class.
Among other avenues, such practical criteria would aid efforts to develop:

- *Identifiability-aware estimators.* Develop estimators that "know what is knowable" from a single trajectory. Concretely, such an estimator would: (a) *project-then-learn* on an estimated data-visible subspace V_{emp}, returning point estimates for $(A|_{V_{\text{emp}}}, B|_{V_{\text{emp}}})$ and a symbolic set for the invisible block; (b) *set-valued outputs*

that expose an admissible class of estimates consistent with the data rather than forcing a point estimate.

- *Experimental design rules.* that actively enlargen V_{emp}.

Empirical scope and next experiments. Within the examined regimes, our empirical findings align with the theory: identifiability scores co-vary with estimation error, and projection onto the visible subspace collapses error. We refrain from broad generalizations because several secondary parameters were held fixed and may act as confounders.[1] To address this, we plan to run a *confounder sweep* to isolate the effect of each secondary parameters on the individual experimental objectives that are the subject of this study. In particular, we will repeat the study in §5.3 with larger cohorts and an expanded grid to test more decisively whether the required PE order tracks the dimension of the visible subspace more closely than that of the ambient state space, beyond the specific configurations reported here.

[1]There is no single canonical list of confounders. Rather, we refer to a selected subset of parameters that were held constant within a given experiment and were not in a direct relationship to the swept factor.

A. Appendix

A.1. Notation

Table A.1.: Frequently reused symbols and aliases.

Symbol(s)	**Meaning / convention**
	Time, spaces, signals
$\mathcal{T}_{\mathrm{ct}} = \mathbb{R}_{\geq 0}$; $\mathcal{T}_{\mathrm{dt}} = \mathbb{N}$	Time axes (continuous / discrete).
$\mathcal{X} = \mathbb{R}^n$; $\mathcal{U} \subseteq \mathbb{R}^m$; $\mathcal{Y} \subseteq \mathbb{R}^p$	State, input, output value spaces. When $C = I$, $y(t) = x(t)$ so $\mathcal{Y} = \mathbb{R}^n$.
$\mathcal{PC}([0,T];\mathbb{R}^m)$; $\mathcal{C}([0,T];\mathbb{R}^m)$; $L^2_{\mathrm{loc}}([0,\infty);\mathbb{R}^m)$	Signal spaces (piecewise–continuous, continuous, locally square–integrable). We use $\mathcal{C}$ while allowing piecewise continuity; later operator arguments use L^2_{loc}.
	Systems, trajectories, operators
$\mathcal{S} = (A, B)$	A system instance in the fully observed LTI class.
$\varphi(t \mid \mathcal{S}, \mathrm{x}_0, \mathrm{u})$	Solution operator: $\mathrm{x}(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds := \varphi(t\|\mathcal{S}, \mathrm{x}_0, \mathrm{u})$.
$\gamma(A, x_0) = \{e^{At}x_0 : t \geq 0\}$	Homogeneous orbit from x_0.
	Reachability and visible subspaces
$\mathcal{R}^t(x_0)$; $\mathcal{R}(x_0) = \bigcup_{t\geq 0} \mathcal{R}^t(x_0)$	Set of states reachable from x_0 in time t; Set of states reachable from x_0 in arbitrary time.
$\mathcal{R}_0$; $\mathcal{R}(0)$; $\mathcal{R}(A, B)$; $\mathcal{K}(A, B)$	*Controllable (origin–reachable) subspace.* Canonical: $\mathcal{K}(A, B) = \mathrm{span}\{A^k B : 0 \leq k \leq n-1\}$.
V, $V(x_0)$; $\mathcal{K}_n(A, [x_0\ B])$; $\mathcal{K}(A, x_0) + \mathcal{K}(A, B)$	*Visible subspace* (the A–invariant hull of $\{x_0\} \cup \mathrm{Im}\, B$): $V(x_0) = \mathrm{span}\{[x_0\ B], A[x_0\ B], \ldots, A^{n-1}[x_0\ B]\}$.
	Invariance and algebra

Symbol(s)	Meaning / convention
$[A\ B]$–invariant	Controlled invariance: $AV \subseteq V + \operatorname{Im} B$.
$\sigma(A)$; $\operatorname{Im}(\cdot)$; $\ker(\cdot)$; $\operatorname{rank}(\cdot)$; $\dim(\cdot)$	Spectrum, image, kernel, rank, dimension.
$\mathbf{0}$; I_n	Zero vector (size given by context); identity matrix of size n.
	Transforms and concatenation
$\mathrm{GL}(n)$	General linear Lie group.
$[x_0\ B]$, $[A\ B]$	Column / row block concatenation.
	Signal richness (PE)
PE (discrete, order L)	Hankel rank condition: $\operatorname{rank}\left(\mathscr{H}_L(u)\right) = mL$.
PE (continuous)	Windowed energy bound: $\int_t^{t+T_0} u(s)u(s)^\top ds \succeq \alpha I_m$ for all $t \geq 0$.

A.2. Deferred Proofs

Proofs that are tailored to this study (i.e., non-standard proofs) are marked with "[This Work]" after the proof heading, though they may employ standard techniques as intermediate steps. Proofs that aren't marked with "[This Work]" are the author's own take on standard proofs, but will inevitably show similarity to others in the literature.

A.2.1. From: Chapter 2

Proof (Corollary 2.7) : Properties of the visible subspace [This Work]

Proof. **(a)** $V = \mathcal{K}_n(A, [\mathrm{x}_0\ B])$ is a span, thus a subspace.

(b) Using $V = \mathcal{K}_n(A, [\mathrm{x}_0\ B])$ from (2.9), we see that membership of $v \in \mathbb{R}^n$ in V implies the existence of a vector $\lambda \in \mathbb{R}^{n+nm}$ such that:

$$v = \begin{bmatrix}[\mathrm{x}_0\ B] & A[\mathrm{x}_0\ B] & \dots A^{n-1}[\mathrm{x}_0\ B]\end{bmatrix} \underbrace{\begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_n\end{bmatrix}}_{=\lambda} := K_n\lambda \quad \text{where } \lambda_i \in \mathbb{R}^{1+m} \quad \text{for all } i \in [n].$$

Thus, we can write

$$Av = AK_n\lambda = \begin{bmatrix} A[\mathrm{x}_0\ B] & A[\mathrm{x}_0\ B] & \dots A^n[\mathrm{x}_0\ B] \end{bmatrix}\ \lambda.$$

According to the Cayley-Hamilton Theorem we can always write A^n as a linear combination of $A^0, \dots, A^{n-1}$, thus, we can write $A^n[\mathrm{x}_0\ B] = K_n\phi$ for some $\phi \in \mathbb{R}^{n+nm}$. Thus, Av breaks down as

$$K_n \begin{bmatrix} 0_{1+m} \\ \lambda_1 \\ \dots \\ \lambda_{n-1} \end{bmatrix} + \lambda_n K_n \phi = K_n \underbrace{\begin{bmatrix} \lambda_n\phi_1 \\ \lambda_1 + \lambda_n\phi_2 \\ \lambda_2 + \lambda_n\phi_3 \\ \vdots \\ \lambda_{n-1} + \lambda_n\phi_n \end{bmatrix}}_{:=\hat{\lambda}\in\mathbb{R}^{n+nm}},$$

thus $Av = K_n\hat{\lambda}$ and $Av \in V$.

(c) This follows directly from $A-$invariance: $AV \subseteq V \subseteq V + \mathrm{Im}(B)$.

(d) Let $S \subseteq \mathbb{R}^n$ be any subspace such that **(i)** $AS \subset S + \mathrm{Im}(B)$ and **(ii)** $\{\mathrm{x}_0\} \cup \mathrm{Im}(B) \subseteq S$– this is the entirety of the subspaces over which we build the intersection. From **(ii)**, we can trivially derive that **(iii)** $\{\mathrm{x}_0\} \subseteq S$ and $\mathrm{Im}(B) \subseteq S$. Through $\mathrm{Im}(B) \subseteq S$ and $[A\ B]-$invariance, we get $AS \subset S + \mathrm{Im}(B) \subseteq S + S \subseteq S$, with the outermost relation, $AS \subset S$ implying **(iv)** $A-$invariance. $A-$invariance, together with $\mathrm{x}_0 \in S$ give: $A^k\mathrm{x}_0 \in S$ for all $k \in \mathbb{N}$, thus $\mathcal{K}(A, \mathrm{x}_0) = \mathrm{span}\{A^k\mathrm{x}_0 : k \in [0, k-1]\} \subseteq S$. Similarly, $\mathrm{Im}(B) \subseteq S$ together with $A-$invariance implies $A^k\,\mathrm{Im}(B) \subseteq S$ for all $k \in \mathbb{N}$, and therefore $\sum_{j=0}^{n-1} A^j\,\mathrm{Im}(B) \subseteq S$. Due to closer of subspaces under addition, we get $\mathcal{K}(A, \mathrm{x}_0) + \mathcal{R}(0) \overset{\text{def.}}{=} V \subseteq S$, which makes V minimal.

(e) Follows immediately from (d) and the fact that $S + \mathrm{Im}(B) = S$ due to $\mathrm{Im}(B) \subseteq S$:

$$\begin{aligned} V &= \bigcap \{S \subseteq \mathbb{R}^n \text{ a subspace} : \{\mathrm{x}_0 \cup \mathrm{Im}(B)\} \subseteq S \text{ and } AS \subset S + \mathrm{Im}(B)\} \\ &= \bigcap \{S \subseteq \mathbb{R}^n \text{ a subspace} : \{\mathrm{x}_0 \cup \mathrm{Im}(B)\} \subseteq S \text{ and } AS \subset S\}. \end{aligned}$$

□

Derivation: One-step ZOH formula

$$
\begin{aligned}
x[k+1] = \mathrm{x}(t+\delta t) &= e^{A(t+\delta t)}\mathrm{x}_0 + \int_0^{t+\delta t} e^{A(t+\delta t-s)}B\mathrm{u}(s)\, ds \\
&= e^{A(k+1)\delta t}\mathrm{x}_0 + \int_0^{(k+1)\delta t} e^{A((k+1)\delta t-s)}B\mathrm{u}(s)\, ds \\
&= e^{A(k+1)\delta t}\mathrm{x}_0 + \int_0^{k\delta t} e^{A((k+1)\delta t-s)}B\mathrm{u}(s)\, ds + \int_{k\delta t}^{(k+1)\delta t} e^{A((k+1)\delta t-s)}B\mathrm{u}(s)\, ds \\
&= e^{A\delta t}\left[e^{Ak\delta t}\mathrm{x}_0 + \int_0^{k\delta t} e^{A(k\delta t-s)}B\mathrm{u}(s)\, ds\right] + \left[\int_{k\delta t}^{k\delta t+\delta t} e^{A(k\delta t+\delta t-s)}\, ds\right] B\mathrm{u}[k] \\
&= e^{A\delta t}\mathrm{x}[k] + \left(\int_0^{\delta t} e^{A\tau}\, d\tau\right) B\mathrm{u}[k],
\end{aligned}
$$

using $\tau := (k+1)\delta t - s$ in the last step.

(Lemma 2.4) Horizon-independence of x_0-equivalence for CT-LTI

Proof. Set $\mathrm{u} \equiv 0$. Equality of trajectories on $[0,T]$ gives $\left(e^{At} - e^{\tilde{A}t}\right)\mathrm{x}_0 = 0$ for $t \in [0,T]$. Since $t \mapsto e^{At}\mathrm{x}_0$ (and $t \mapsto e^{\tilde{A}t}\mathrm{x}_0$) are real-analytic, equality on an interval implies equality for all $t \geq 0$. Fix $t \in (0,T]$. For all piecewise continuous u supported in $[0,t]$, $\int_0^t \left(e^{A(t-s)}B - e^{\tilde{A}(t-s)}\tilde{B}\right)\mathrm{u}(s)\, ds = 0$. Let $K_t(s) := e^{A(t-s)}B - e^{\tilde{A}(t-s)}\tilde{B}$. Each entry of $K_t(\cdot)$ is continuous on $[0,t]$. If $\int_0^t K_t(s)\mathrm{u}(s)\, ds = 0$ for all piecewise-continuous u, the fundamental lemma of the calculus of variations implies $K_t(s) \equiv 0$ on $[0,t]$. Equivalently, $e^{As}B = e^{\tilde{A}s}\tilde{B}$ for all $s \in [0,t]$. Since $t \leq T$ was arbitrary, $e^{As}B = e^{\tilde{A}s}\tilde{B}$ for all $s \in [0,T]$. Because $s \mapsto e^{As}B$ and $s \mapsto e^{\tilde{A}s}\tilde{B}$ are real-analytic matrix-valued functions, equality on $[0,T]$ implies equality on all of $\mathbb{R}_{\geq 0}$. □

(Theorem 2.21) Proof Sketch: Invariants under ZOH Sampling

Proof. **(i) Exact ZOH dynamics.** By variation of constants, for $u(t) \equiv u[k]$ on $[k\Delta t, (k+1)\Delta t)$,

$$
x((k+1)\Delta t) = e^{A\Delta t}x(k\Delta t) + \Big(\int_0^{\Delta t} e^{As}\, ds\Big)Bu[k],
$$

so $x[k+1] = A_d x[k] + B_d u[k]$ with $A_d = e^{A\Delta t}$ and $B_d = \int_0^{\Delta t} e^{As}B\, ds$ (e.g., [23]).

Two basic ingredients: Cayley–Hamilton implies $A_d^j = e^{jA\Delta t}$ is a polynomial in A; hence $\mathcal{K}_n(A_d, w) \subseteq \mathcal{K}_n(A, w)$ for any w. Also $B_d = g(A)B$ with $g(z) = (e^{z\Delta t} - 1)/z$ (and $g(0) = \Delta t$). **(ii) Reachable space is preserved.** From $\operatorname{span}\{A_d^k B_d\} \subseteq \operatorname{span}\{A^k B_d\}$ with $w = B_d$, $\operatorname{span}\{A_d^k B_d\} \subseteq \operatorname{span}\{A^k B_d\}$. If Δt is nonpathological on the *reachable spectrum* $\sigma_r(A, B)$, then $g(\lambda) \neq 0$ for all $\lambda \in \sigma_r(A, B)$, so $g(A)$ is invertible on $\mathcal{R}_0^{ct} := \operatorname{span}\{A^k B\}$ and

$$\operatorname{span}\{A^k B_d\} = \operatorname{span}\{A^k g(A)B\} = \operatorname{span}\{A^k B\} = \mathcal{R}_0^{ct},$$

giving $\mathcal{R}_0^{dt} \subseteq \mathcal{R}_0^{ct}$. For the reverse inclusion, restrict attention to $\mathcal{R}_0^{ct}$. On this subspace, the eigenvalues of A are precisely $\sigma_r(A, B)$, which remain distinct under $z \mapsto e^{\Delta t z}$ by Assumption **??**. Since $f(z) = e^{\Delta t z}$ is analytic with $f'(\lambda) = \Delta t e^{\lambda \Delta t} \neq 0$, the Jordan block sizes for A and $A_d = f(A)$ on $\mathcal{R}_0^{ct}$ coincide ([**Higham2008**]). Consequently, for every polynomial p there exists a polynomial q such that $p(A)\big|_{\mathcal{R}_0^{ct}} = q(A_d)\big|_{\mathcal{R}_0^{ct}}$. Applying both sides to $B_d \in \mathcal{R}_0^{ct}$ yields $\operatorname{span}\{A^k B_d\} \subseteq \operatorname{span}\{A_d^k B_d\}$, hence $\mathcal{R}_0^{ct} \subseteq \mathcal{R}_0^{dt}$. Combining the two inclusions gives $\mathcal{R}_0^{dt} = \mathcal{R}_0^{ct}$ (see also [20–22] for closely related criteria).

(iii) Visible subspace is preserved. From $\operatorname{span}\{A_d^k B_d\} \subseteq \operatorname{span}\{A^k B_d\}$ with $w \in \{x_0\} \cup \operatorname{col}(B)$ and $B_d = g(A)B$ we get the easy direction $V_{dt}(x_0) \subseteq V_{ct}(x_0)$.

For the reverse inclusion on the *autonomous* part $\mathcal{K}_n(A, x_0)$, assume $\lambda \mapsto e^{\lambda \Delta t}$ is injective on all of $\sigma(A)$. By the same analytic/Jordan-block argument as above (now on the full space), for each polynomial p there is a q with $p(A)x_0 = q(A_d)x_0$, hence $\mathcal{K}_n(A, x_0) \subseteq \mathcal{K}_n(A_d, x_0)$. Together with (ii), which gives $\mathcal{K}_n(A, B) = \mathcal{K}_n(A_d, B_d)$, we conclude $V_{ct}(x_0) \subseteq V_{dt}(x_0)$, and thus equality. □

A.2.2. From: Chapter 4

Proof (Proposition 4.2) : Structure of the x_0-equivalence class, the uncontrollable case [This Work]

Proof. We will prove this via mutual inclusion of (4.1) with the axiomatic x_0-equivalence class, which we will denote $[A, B]^*_{\mathrm{x}_0}$:

$$[A, B]^*_{\mathrm{x}_0} := \left\{ (A', B') \;\middle|\; \forall \mathrm{u} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) : \forall t \in \mathbb{R}_{\geq 0} : \varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right)) = \varphi\left(t \,|\, A', B', \mathrm{x}_0, \mathrm{u}\right) \right\}$$

(1) " $\subseteq$: " to show: $[A, B]^*_{\mathrm{x}_0} \subseteq \left\{ (A', B') \,|\, A'\big|_V = A\big|_V,\ B' = B \right\}$

$$\iff (\tilde{A}, \tilde{B}) \overset{\mathrm{x}_0}{\sim} (A, B) \implies \tilde{A}\big|_V = A\big|_V \wedge \tilde{B} = B \quad \textbf{(necessity)} \tag{A.1}$$

Assume that $(\tilde{A}, \tilde{B}) \in [A, B]_{\mathrm{x}_0}$. Since $\mathrm{u} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, the "for all u" condition admits the constant zero signal $\mathrm{u}(t) = 0$, which enforces the equality of the homogeneous response:

$$\forall t \in \mathbb{R}_{\geq 0} : \; e^{At}\mathrm{x}_0 = e^{\tilde{A}t}\mathrm{x}_0.$$

Equality for all t enforces equality of all derivatives, for every t where they exist. With $t = 0$, we get:

$$\begin{aligned} \forall k \in \mathbb{N}_0 : \; \frac{d^k}{dt^k} e^{At}\mathrm{x}_0 \Big|_{t=0} &= \frac{d^k}{dt^k} e^{\tilde{A}t}\mathrm{x}_0 \Big|_{t=0} \iff \\ \forall k \in \mathbb{N}_0 : \; A^k e^{At}\mathrm{x}_0 \Big|_{t=0} &= \tilde{A}^k e^{\tilde{A}t}\mathrm{x}_0 \Big|_{t=0} \iff \\ \boxed{\forall k \in \mathbb{N}_0 : A^k \mathrm{x}_0 = \tilde{A}^k \mathrm{x}_0} &. \end{aligned} \tag{A.2}$$

Since thereby $e^{At}\mathrm{x}_0 = e^{\tilde{A}t}\mathrm{x}_0$ is a necessary condition for $(\tilde{A}, \tilde{B}) \in [A, B]_{\mathrm{x}_0}$, the equality of trajectories becomes the equality of the forced responses:

$$\begin{aligned} \forall t \in \mathbb{R}_{\geq 0}, \; \mathrm{u} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) : \; \varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right)) &= \varphi\left(t \,|\, \tilde{A}, \tilde{B}, \mathrm{x}_0, \mathrm{u}\right) \iff \\ e^{At}\mathrm{x}_0 + \int_0^t e^{A(t-s)} B\mathrm{u}(s)\, ds &= e^{\tilde{A}t}\mathrm{x}_0 + \int_0^t e^{\tilde{A}(t-s)} \tilde{B}\mathrm{u}(s)\, ds \; ; \overset{e^{At}\mathrm{x}_0 = e^{\tilde{A}t}\mathrm{x}_0}{\iff} \\ \int_0^t e^{A(t-s)} B\mathrm{u}(s)\, ds &= \int_0^t e^{\tilde{A}(t-s)} \tilde{B}\mathrm{u}(s)\, ds. \end{aligned}$$

We define the kernels $K(\tau) := e^{A\tau}B$, $\tilde{K}(\tau) := e^{\tilde{A}\tau}\tilde{B}$, and use the following lemma to show the equivalence $K \equiv \tilde{K}$.

Lemma A.1. ***(Kernel uniqueness).*** *Let $K, \; \tilde{K} : [0, T] \to \mathbb{R}^{n \times m}$ be continuous. If for all $\mathrm{u} \in \mathcal{C}\left([0, T], \mathbb{R}^m\right)$ and all $t \in [0, T]$*

$$\int_0^t K(t-s)\mathrm{u}(s)\, ds = \int_0^t \tilde{K}(t-s)\mathrm{u}(s)\, ds,$$

then $K \equiv \tilde{K}$ on $[0, T]$.

$$\begin{aligned} \int_0^t K(t-s)\mathrm{u}(s)\, ds = \int_0^t \tilde{K}(t-s)\mathrm{u}(s)\, ds &\iff \\ \forall t \in \mathbb{R}_{\geq 0}, \; \mathrm{u} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) : \; (K * u)(t) = (\tilde{K} * u)(t) &\overset{\text{Lemma A.1}}{\Longrightarrow} \\ \forall t \in \mathbb{R}_{\geq 0} : \; K(t) = \tilde{K}(t) &\iff \\ \forall t \in \mathbb{R}_{\geq 0} : e^{At}B = e^{\tilde{A}t}\tilde{B}. \end{aligned}$$

Again, equality for all t enforces equality of all derivatives, including $t = 0$:

$$\begin{aligned}
\forall t \in \mathbb{R}_{\geq 0}: \; e^{At}B = e^{\tilde{A}t}\tilde{B} \iff \\
\forall k \in \mathbb{N}_0: \; \frac{d^k}{dt^k} e^{At}B\Big|_{t=0} = \frac{d^k}{dt^k} e^{\tilde{A}t}\tilde{B}\Big|_{t=0} \iff \\
\forall k \in \mathbb{N}_0: \; A^k e^{At}B\Big|_{t=0} = \tilde{A}^k e^{\tilde{A}t}\tilde{B}\Big|_{t=0} \iff \\
\boxed{\forall k \in \mathbb{N}_0: \; A^k B = \tilde{A}^k \tilde{B}}, \qquad \text{(A.3)}
\end{aligned}$$

i.e., the state-Markov parameters agree for all $k \in \mathbb{N}_0$. Especially, with $k = 0$, we get:

$$\forall k \in \mathbb{N}_0: \; A^k B = \tilde{A}^k \tilde{B} \implies A^0 B = \tilde{A}^0 \tilde{B} \iff \boxed{B = \tilde{B}}.$$

So far, we have:

$$\begin{aligned}
(\tilde{A}, \tilde{B}) \in [A, B]_{\mathrm{x}_0} \implies \forall k \in \mathbb{N}_0 : A^k \mathrm{x}_0 = \tilde{A}^k \mathrm{x}_0 \text{ and } A^k B = \tilde{A}^k \tilde{B} \\
\overset{\text{to show}}{\implies} A|_V = \tilde{A}|_V \text{ (and } B = \tilde{B}\text{), where } V = \mathcal{K}(A, \mathrm{x}_0) + \mathcal{R}(0).
\end{aligned}$$

Let us use the span expression of V (Definition 2.6):

$$V = \operatorname{span}\left(\left\{A^k \mathrm{x}_0 : k \geq 0\right\} \cup \left\{A^k b_j : k \geq 0, \; b_j := j\text{th column of } B\right\}\right) := \operatorname{span}(\mathrm{S}).$$

The following lemma provides the last link in deriving $A|_V = \tilde{A}|_V$.

Lemma A.2. *Let $M, M' : \mathbb{R}^n \to \mathbb{R}^n$ be linear and let $W \subseteq \mathbb{R}^n$ be a subspace spanned by a set U. If $Mu = M'u$ for all $u \in U$, then the restrictions of M, M' to the domain W agree: $M|_W = M'|_W$.*

Thus, if we can show that $As = \tilde{A}s$ for all $s \in S$, this will imply $A|_V = \tilde{A}|_V$. For any $s \in S$, we have either **(i)** $s = A^k \mathrm{x}_0$ for some $k \in \mathbb{N}_0$, or **(ii)** $s = A^k b_j$ for some $k \in \mathbb{N}_0$ and $j \in [m]$.

(i) to show: $As = \tilde{A}s$ in case $s = A^k \mathrm{x}_0$ for some $k \in \mathbb{N}_0$

$$As = A\left(A^k \mathrm{x}_0\right) = A^{k+1}\mathrm{x}_0 \overset{\text{(A.2)}}{=} \tilde{A}^{k+1}\mathrm{x}_0 = \tilde{A}(\tilde{A}^k \mathrm{x}_0) \overset{\text{(A.2)}}{=} \tilde{A}\left(A^k \mathrm{x}_0\right) = \tilde{A}s.$$

(ii) to show: $As = \tilde{A}s$ in case $s = A^k b_j$ for some $k \in \mathbb{N}_0$ and $j \in [m]$. With $B = B'$ and (A.3), we have:

$$As = A\left(A^k b_j\right) = \tilde{A}^{k+1}\tilde{b}_j = \tilde{A}\left(\tilde{A}^k \tilde{b}_j\right) = \tilde{A}\left(A^k b_j\right) = \tilde{A}s.$$

This concludes the first part of our proof.

$\tilde{B} = B$ and $\tilde{A}\big|_V = A\big|_V$ is a necessary condition for $(\tilde{A}, \tilde{B}) \in [A, B]_{x_0}$. In other words, the condition $\tilde{B} = B$ and $\tilde{A}\big|_V = A\big|_V$ gives us all pairs that yield the same trajectory from x_0, regardless of the control input.

We will next look at whether $\tilde{B} = B$ and $\tilde{A}\big|_V = A\big|_V$ also represents a sufficient condition.

(2) " $\supseteq$: " to show: $\{(A', B') \mid A'\big|_V = A\big|_V,\ B' = B\} \subseteq [A, B]^*_{x_0}$

$$\iff \tilde{A}\big|_V = A\big|_V \wedge \tilde{B} = B \implies (\tilde{A}, \tilde{B}) \overset{x_0}{\sim} (A, B) \quad \textbf{(sufficiency)} \tag{A.4}$$

Let $(\tilde{A}, \tilde{B})$ be any block matrix in $\mathbb{R}^{n\times(n+m)}$ such that $\tilde{A}\big|_V = A\big|_V$ and $\tilde{B} = B$. With that, we are assuming that the actions of $\tilde{A}$ and A on V are identical. Intuitively, since both the homogeneous part and the forced part are confined to V, equality of actions $\tilde{A}$ and A will force the whole trajectories to be identical. We will formally confirm this intuition with the help of a few lemmas.

Lemma A.3. ***(V is $\tilde{A}$-invariant).*** *Since $\tilde{A}\big|_V = A\big|_V$, and V is $A-$invariant **(*)** we have:*

$$\forall v \in V:\ \tilde{A}v = Av \overset{(*)}{\in} V \iff \tilde{A}V \subseteq V \overset{def.}{\iff} V \text{ is } \tilde{A}-\text{invariant}.$$

Lemma A.4. ***(Powers agree on V).*** *If V is $A-$ and $\tilde{A}-$invariant, and the actions of A and $\tilde{A}$ are identical on V, then*

$$\forall l \in \mathbb{N}_0:\ \tilde{A}^l\Big|_V = A^l\Big|_V.$$

Proof. We show this by induction:

(IB) $l = 1:$ $\tilde{A}\big|_V = A\big|_V$ by assumption. This implies:$\forall v \in V:\ \tilde{A}v = Av$.

(IS) $k \implies k+1:$ Suppose that $\tilde{A}^k\Big|_V = A^k\Big|_V$ for some fixed k (IH). Then:

$$\forall v \in V:\ \tilde{A}^k v = A^k v \implies \tilde{A}^{k+1}v = \tilde{A}\underbrace{\left(A^k v\right)}_{:=v'} = \tilde{A}v'.$$

Since V is $A-$invariant, we know that $v' \in V$. Thus, we can use **(IB)** to get: $\tilde{A}v' = Av$. Substituting back $v' = A^k v$ gives : $\tilde{A}v' = A^{k+1}v$. Overall, we get: $\forall v \in V:\ \tilde{A}^k v = A^k v \implies \tilde{A}^{k+1}v = \tilde{A}v' = A^{k+1}v$, which, by definition, implies $\tilde{A}^{k+1}\Big|_V = A^{k+1}\Big|_V$. □

Using the series expansion of the matrix exponential and Lemma A.4, we derive the equality of the exponentials $e^{\tilde{A}t}, e^{At}$ on V:

$$e^{\tilde{A}t}\Big|_V = \sum_{k=0}^{\infty} \frac{\tilde{A}^k t^k}{k!}\Bigg|_V = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, \tilde{A}^k\Big|_V \overset{\text{Lemma A.4}}{=} \sum^{\infty} \frac{t^k}{k!}\, A^k\Big|_V = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\Bigg|_V = e^{At}\Big|_V .$$

Since $\mathrm{x}_0 \in V$, this implies, for all $t \geq 0$:

$$\left(e^{\tilde{A}t}\Big|_V\right)\mathrm{x}_0 = \left(e^{At}\Big|_V\right)\mathrm{x}_0 \iff \left(e^{\tilde{A}t}\mathrm{x}_0\Big|_V\right) = \left(e^{At}\mathrm{x}_0\Big|_V\right) \iff e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0,$$

which means we have pointwise equality for the homogeneous response. Using the same argument, we can show that the kernels in the forced response coincide:

$$e^{\tilde{A}(t-s)}\tilde{B} = e^{A(t-s)}B.$$

Since $\mathrm{u}(\cdot)$ is (piecewise) continuous by assumption, the forced responses coincide:

$$\int_0^t e^{\tilde{A}(t-s)}\tilde{B}\mathrm{u}(s)\, ds = \int_0^t e^{A(t-s)}B\mathrm{u}(s)\, ds.$$

Overall, we get

$$\begin{aligned} &t \geq 0 \text{ and } \mathrm{u} \in \mathcal{C}([-,\infty),\mathbb{R}^m) : \\ &\quad e^{\tilde{A}t}\mathrm{x}_0 + \int_0^t e^{\tilde{A}(t-s)}\tilde{B}\mathrm{u}(s)\, ds = e^{At}\mathrm{x}_0 + \int_0^t e^{A(t-s)}B\mathrm{u}(s)\, ds \\ &\qquad \varphi(t \,|\, \tilde{A}, \tilde{B}, \mathrm{x}_0, \mathrm{u}) = \varphi(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}) \\ &\qquad\qquad (\tilde{A}, \tilde{B}) \overset{\mathrm{x}_0}{\sim} (A, B). \end{aligned}$$

□

Proof (Proposition 4.4): Block structure in a (V, W) basis [This Work]

Proof. Let us observe the dynamical equation in this new basis:

$$\begin{aligned} \underbrace{T^{-1}\dot{\mathrm{x}}(t)}_{:=\dot{z}(t)} &= T^{-1}A\mathrm{x}(t) + T^{-1}B\mathrm{u}(t) \\ &= \underbrace{T^{-1}AT}_{:=A'}\left(T^{-1}\mathrm{x}(t)\right) + \underbrace{T^{-1}B}_{:=B'}\mathrm{u}(t) \\ &= A'z(t) + B'\mathrm{u}(t). \end{aligned}$$

Because $V = \mathcal{K}(A, \mathrm{x}_0) + \mathcal{R}(0)$ is $A-$ invariant and $\mathrm{Im}(B) \subseteq V$, in this new basis defined by T, the objects x_0, A and B will have a particular shape: *(i)* Since x_0 lies in

V by definition, the coordinates of x_0 in the (V, W) basis will have a zero component in W: $T^{-1}\mathrm{x}_0 = \begin{bmatrix} \mathrm{x}_0|_V \\ 0 \end{bmatrix}$. *(ii)* Also by definition, we have $\mathrm{Im}(B) \subseteq V$, thus B will have a zero projection on W as well: $T^{-1}B = \begin{bmatrix} B|_V \\ 0 \end{bmatrix}$. *(iii)* $A-$invariance of V gives us the familiar block upper-triangular form: $T^{-1}AT = \begin{bmatrix} A_V & A_* \\ 0 & A_W \end{bmatrix}$. *(iv)* Since the actions of A, $\tilde{A}$ agree on V (i.e., $A'|_V = A|_V$), we get $T^{-1}\tilde{A}T = \begin{bmatrix} \tilde{A}_V & \tilde{A}_* \\ 0 & \tilde{A}_W \end{bmatrix} = \begin{bmatrix} A_V & \tilde{A}_* \\ 0 & \tilde{A}_W \end{bmatrix}$, i.e., the upper left blocks agree: $\tilde{A}_V = A_V$.

In this new basis, we get

$$\dot{z}(t) = T^{-1}\dot{\mathrm{x}}(t) = \begin{bmatrix} A_V & A_* \\ 0 & A_W \end{bmatrix} z(t) + \begin{bmatrix} B_V \\ 0 \end{bmatrix} \mathrm{u}(t).$$

Let us set $z_V := (z_1, \dots, z_k)$ and $z_W = (z_{k+1}, \dots, z_n)$. Then:

$$\begin{bmatrix} \dot{z}_V \\ \dot{z}_W \end{bmatrix}(t) = \begin{bmatrix} A_V & A_* \\ 0 & A_W \end{bmatrix} \begin{bmatrix} z_V \\ z_W \end{bmatrix}(t) + \begin{bmatrix} B_V \\ 0 \end{bmatrix} \mathrm{u}(t) \iff$$
$$\begin{cases} \dot{z}_V(t) = A_V z_V(t) + A_* z_W(t) + B_V \mathrm{u}(t) \\ \dot{z}_W(t) = A_W z_W(t), \end{cases}$$

so we get for the system response:

$$\varphi_W(t|A_W, z_W(0) = e^{A_W} z_W(0)$$
$$\varphi_V(t \,|\, A_V, B_V, z_V(0), \mathrm{u}) = e^{A_V t} z_V(0) + \int_0^t e^{A_V(t-s)} B_V \mathrm{u}(s)\, ds + z_W(0) \int_0^t e^{A_V(t-s)} A_* e^{A_W s}\, ds.$$

□

Proof (Theorem 4.6): Partial identifiability theorem [This Work]

Proof. Let $S \subseteq \mathbb{R}^n$ satisfy $\{\mathrm{x}_0\} \cup \mathrm{Im}\, B \subseteq S$ and $AS \subseteq S$, and let $k = \dim S$.
(i) $\implies$ (ii). With $G = \mathrm{span}\{\mathrm{x}_0\} + \mathrm{Im}\, B$, the minimal A-invariant supspace of G equals

$$\mathsf{Hull}_A(G) = \mathrm{span}\{[\mathrm{x}_0\, B], A[\mathrm{x}_0\, B], \dots\} \subseteq S.$$

By Cayley–Hamilton on $A|_S$,

$$\mathsf{Hull}_A(G) = \mathrm{span}\{[\mathrm{x}_0\, B], A[\mathrm{x}_0\, B], \dots, A^{k-1}[\mathrm{x}_0\, B]\}.$$

Thus S is minimal $\iff$ these k block-columns span S (i.e. have rank k) [1].
(ii) $\implies$ (iii). Assume (ii) and suppose

$$\varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right) = \varphi\left(t \,|\, \tilde{A}, \tilde{B}, \mathrm{x}_0, \mathrm{u}\right) \quad \forall t \geq 0,\ \forall \mathrm{u} \in \mathcal{C}([0,\infty), \mathbb{R}^m).$$

Taking $\mathrm{u} \equiv 0$ yields $e^{At}\mathrm{x}_0 = e^{\tilde{A}t}\mathrm{x}_0$. Subtracting forced parts and applying Lemma A.1 gives $e^{At}B = e^{\tilde{A}t}\tilde{B}$ for all $t \geq 0$. Analyticity implies

$$A^j\mathrm{x}_0 = \tilde{A}^j\mathrm{x}_0, \quad A^jB = \tilde{A}^j\tilde{B} \quad (j \in \mathbb{N}_0),$$

so $B = \tilde{B}$. Let $\mathcal{G} = \{A^j\mathrm{x}_0 : 0 \leq j \leq k-1\} \cup \{A^jb_i : 0 \leq j \leq k-1\}$. By (ii), $\operatorname{span}\mathcal{G} = S$. For every $g \in \mathcal{G}$, $Ag = \tilde{A}g$, hence $A|_S = \tilde{A}|_S$ and $(B|_S = \tilde{B}|_S)$; this is (iii).
(iii) $\implies$ (i). Assume (iii) but S not minimal. Let $H \subsetneq S$ be the minimal A-invariant hull of $\{\mathrm{x}_0\} \cup \operatorname{Im} B$. Take $\tilde{B} = B$ and any $\tilde{A}$ with $\tilde{A}|_H = A|_H$ and $\tilde{A}|_S \neq A|_S$. Then for all u and $t \geq 0$, trajectories from x_0 coincide (they lie in H), contradicting (iii). Hence S is minimal. □

Proof (Corollary 4.9): Frobenius distance to PBH failure [This Work]

Proof. (Frobenius distance to PBH failure).

$$\begin{aligned}
d_{\text{Frob.}}(A, B \,|\, \mathrm{x}_0) &:= \inf\{\|\Delta A\ \Delta B\|_F :\ (A + \Delta A,\ B + \Delta B)\ \text{ is unidentifiable}\} \\
&= \inf\{\|\Delta A\ \Delta B\|_F : \exists \lambda \in \sigma(A + \Delta A) : \ldots \\
&\qquad \ldots \operatorname{rank}[\lambda I - (A + \Delta A),\ [\mathrm{x}_0\ \ (B + \Delta B)]] < n\} \\
&= \inf\{\|\Delta A\ \Delta B\|_F :\ \exists \lambda \in \sigma(A + \Delta A) : \ldots \\
&\qquad \ldots \operatorname{rank}\left[Q^\top(\lambda I - (A + \Delta A)),\ Q^\top[\mathrm{x}_0\ \ (B + \Delta B)]\right] < n - 1\}, \\
&= \inf\{\|\Delta A\ \Delta B\|_F :\ \exists \lambda \in \sigma(A + \Delta A) : \ldots \\
&\qquad \ldots \operatorname{rank}\left[Q^\top(\lambda I - (A + \Delta A)),\ \mathbf{0},\ Q^\top(B + \Delta B)]\right] < n - 1\}, \\
&= \inf\{\|\Delta A\ \Delta B\|_F :\ \exists \lambda \in \sigma(A + \Delta A) : \ldots \\
&\qquad \ldots \operatorname{rank}\left[Q^\top(\lambda I - (A + \Delta A)),\ Q^\top(B + \Delta B)]\right] < n - 1\}, \\
&= \inf_{\lambda \in \mathbb{C}} \sigma_{\min}\left(Q^\top\,[\lambda I - A,\ B]\right) =: \inf_{\lambda \in \mathbb{C}} \sigma_{\min}\left(Q^\top \mathcal{H}(\lambda)\right)
\end{aligned}$$

□

[1] If one writes "$[A\ B]$-invariant" for $AS \subseteq S + \operatorname{Im} B$, then with $\operatorname{Im} B \subseteq S$ this reduces to A-invariance.

Proof (Equation (4.9)): Functionally constrained reachable space [This Work]

Proof.

$$
\begin{aligned}
\mathcal{R}_t(\mathrm{x}_0 \,|\, \mathcal{U}_c) &\overset{\text{def.}}{=} \left\{ \mathrm{x}_f \in \mathcal{X} \;\middle|\; \exists \mathrm{u}(\cdot) \in \mathcal{U}_c : \mathrm{x}_0 \underset{t}{\rightsquigarrow} \mathrm{x}_f \right\} \\
&= \left\{ \mathrm{x}_f \in \mathcal{X} \mid \exists \mathrm{u}(\cdot) \in \mathcal{U}_c : \mathrm{x}_f = \varphi\left(t \,|\, A, B, \mathrm{x}_0, \mathrm{u}\right) \right\} \\
&= \left\{ \mathrm{x}_f \in \mathcal{X} \;\middle|\; \exists \mathrm{u}(\cdot) \in \mathcal{U}_c : \mathrm{x}_f = e^{At}\mathrm{x}_0 + \int_0^t e^{A(t-s)} B\mathrm{u}(s)\, ds \right\} \\
&= \left\{ \mathrm{x}_f \in \mathcal{X} \;\middle|\; \exists \mathrm{u}(\cdot) \in \mathcal{U}_c : \mathrm{x}_f = e^{At}\mathrm{x}_0 + \mathcal{R}_t^{A,B}\mathrm{u} \right\} \\
&= \left\{ e^{At}\mathrm{x}_0 + \mathcal{R}_t^{A,B}\mathrm{u} \;\middle|\; \mathrm{u}(\cdot) \in \mathcal{U}_c \right\} \\
&= e^{At}\mathrm{x}_0 + \left\{ \mathcal{R}_t^{A,B}\mathrm{u} \;\middle|\; \mathrm{u}(\cdot) \in \mathcal{U}_c \right\} \\
&= e^{At}\mathrm{x}_0 + \left\{ \mathcal{R}_t^{A,B}\Big|_{\mathcal{U}_c} : \mathrm{u} \in \mathcal{C}([0,t], \mathbb{R}^m) \right\} \\
&= e^{At}\mathrm{x}_0 + \operatorname{Im}\left(\mathcal{R}_t^{A,B}\Big|_{\mathcal{U}_c} \right)
\end{aligned}
$$

Let $\mathcal{R}(\mathrm{x}_0 \,|\, \mathcal{U}_c) := \bigcup_{t \geq 0} \mathcal{R}_t(\mathrm{x}_0 \,|\, \mathcal{U}_c)$ be the all-times reachable set. Then:

$$
\mathcal{R}(\mathrm{x}_0 \,|\, \mathcal{U}_c) = \bigcup_{t \geq 0} \left\{ e^{At}\mathrm{x}_0 + \operatorname{Im}\left(\mathcal{R}_t^{A,B}\Big|_{\mathcal{U}_c} \right) \right\}
$$

and

$$
\begin{aligned}
V_{\mathcal{U}_c}(\mathrm{x}_0) &:= \operatorname{span}\left(\mathcal{R}\left(\mathrm{x}_0 \,|\, \mathcal{U}_c\right)\right) \\
&= \operatorname{span}\left(\bigcup_{t \geq 0} \left\{ e^{At}\mathrm{x}_0 + \operatorname{Im}\left(\mathcal{R}_t^{A,B}\Big|_{\mathcal{U}_c} \right) \right\} \right) \\
&= \operatorname{span}\left(\bigcup_{t \geq 0} e^{At}\mathrm{x}_0 \cup \bigcup_{t \geq 0} \operatorname{Im}\left(\mathcal{R}_t^{A,B}\Big|_{\mathcal{U}_c} \right) \right).
\end{aligned}
$$

□

Proof (Proposition 4.21): x_0-equivalence class under PE-order-r signals [This Work]

Proof. (i) "$\subseteq$" (Necessity). If $(\tilde{A}, \tilde{B})$ and (A, B) are $(\mathrm{x}_0, \mathcal{U}_{\mathrm{PE}}^r)-$equivalent, then, taking $\mathrm{u} \equiv 0$ forces equality of the homogeneous responses, hence $e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0$ for all t, which implies $A^k\mathrm{x}_0 = \tilde{A}^k\mathrm{x}_0$ for all k. By moment-PE of order r, equality of the

forced responses for all $\mathcal{U}^r_{\mathrm{PE}}$ implies equality of Markov parameters up to order $r-1$: $A^k B = \tilde{A}^k \tilde{B}$ for $k = 0, \ldots, r-1$. In particular, $B = \tilde{B}$. Thus, we get $A(A^j B) = A^{j+1} B = \tilde{A}^{j+1}\tilde{B} = \tilde{A}(A^j B)$ for $j = 0, \ldots, r-1$, so A and $\tilde{A}$ agree on $\mathrm{span}\{A^j B : 0 \le j \le r-1\}$. Together with the homogeneous part, they agree on $V^r_{\mathrm{PE}}(\mathrm{x}_0)$. Thus, $\tilde{A}|_{V^r_{\mathrm{PE}}(\mathrm{x}_0)} = A|_{V^r_{\mathrm{PE}}(\mathrm{x}_0)}$ and $\tilde{B} = B$. *(ii) "$\supseteq$" (Sufficiency).* Conversely, if $\tilde{A}|_{V^r_{\mathrm{PE}}(\mathrm{x}_0)} = A|_{V^r_{\mathrm{PE}}(\mathrm{x}_0)}$ and $\tilde{B} = B$, then $e^{\tilde{A}t}$ and e^{At} coincide on $V^r_{\mathrm{PE}}(\mathrm{x}_0)$. Since $\mathrm{x}_0 \in V^r_{\mathrm{PE}}(\mathrm{x}_0)$ and $\mathrm{Im}(B) \subset V^r_{\mathrm{PE}}(\mathrm{x}_0)$, both the homogeneous and the forced response stay in $V^r_{\mathrm{PE}}(\mathrm{x}_0)$ for all admissible inputs, hence the two trajectories coincide for all $\mathrm{u} \in \mathcal{U}^r_{\mathrm{PE}}$. *(iii) Block form.* In a basis $T = [P\ Q]$ adapted to $V^r_{\mathrm{PE}} \oplus W = \mathbb{R}^n$, we get $T^{-1}\tilde{A}T = \begin{bmatrix} A_{V^r_{\mathrm{PE}}(\mathrm{x}_0)} & \Theta \\ \mathbf{0} & \Psi \end{bmatrix}$, $\tilde{B} = B$, with $A_{V^r_{\mathrm{PE}}(\mathrm{x}_0)} = (T^{-1}\tilde{A}T)|_{V^r_{\mathrm{PE}}(\mathrm{x}_0)}$, as per the visible-subspace-block argument. Dimensions are as follows: $A_{V^r_{\mathrm{PE}}(\mathrm{x}_0)} \in \mathbb{R}^{k_r \times k_r}, \Theta \in \mathbb{R}^{k_r \times (n-k_r)}, \Psi \in \mathbb{R}^{(n-k_r)\times(n-k_r)}$. □

Proof (Proposition 4.14): Structure of controllable $[A, B]_{\mathrm{x}_0, \mathcal{U}_c}$ under pointwise input restrictions [This Work]

Proof. **(i) Case $0 \in \mathcal{U}_c$.** *"$\subseteq$" (Necessity). Homogeneous part.* Taking $\mathrm{u} \equiv 0$ gives $e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0$ for all $t \in [0, T]$. By the previously used cyclicity argument, this implies that A and $\tilde{A}$ agree on $\mathcal{K}(A, \mathrm{x}_0)$. *Forced part.* Fix $w \in W$. For any interval $[s, s+h] \subset [0, T]$, take the simple input $\mathrm{u}(t) := w\, \chi_{[s,s+h]}(t) \in \mathcal{U}_c$. Equality of trajectories yields $\int_0^h e^{A\theta} B w\, d\theta = \int_0^h e^{\tilde{A}\theta}\tilde{B} w\, d\theta$ for all $h \in [0, T-s]$. By strong continuity, differentiating at $h = 0$ gives $Bw = \tilde{B}w$, and $A^k B w = \tilde{A}^k \tilde{B} w$ for all $k \ge 1$. Consequently, A and $\tilde{A}$ agree on the $A-$cyclic span of $\mathrm{Im}(BW)$ and $\tilde{B}W = BW$. Combined with the homogeneous part, we get $\tilde{A}|_{V_{\mathrm{pt}}(\mathrm{x}_0)} = A|_{V_{\mathrm{pt}}(\mathrm{x}_0)}$ and $\tilde{B}W = BW$. *"$\supseteq$" (Sufficiency).* Assume $\tilde{A}|_{V_{\mathrm{pt}}(\mathrm{x}_0)} = A|_{V_{\mathrm{pt}}(\mathrm{x}_0)}$ and $\tilde{B}W = BW$. Since $V_{\mathrm{pt}}(\mathrm{x}_0)$ is $A-$invariant and contains x_0 and $\mathrm{Im}(BW)$, and $\tilde{A}|_{V_{\mathrm{pt}}(\mathrm{x}_0)} = A|_{V_{\mathrm{pt}}(\mathrm{x}_0)}$, it follows that $V_{\mathrm{pt}}(\mathrm{x}_0)$ is also $\tilde{A}-$invariant and $e^{\tilde{A}t}\mathrm{x}_0\Big|_{V_{\mathrm{pt}}(\mathrm{x}_0)} = e^{At}\mathrm{x}_0\big|_{V_{\mathrm{pt}}(\mathrm{x}_0)}$. For any admissible $\mathrm{u}(\cdot)$, we have $B\mathrm{u}(s) \in \mathrm{Im}(BW) \subset V_{\mathrm{pt}}(\mathrm{x}_0)$, hence both the homogeneous and the forced responses remain in $V_{\mathrm{pt}}(\mathrm{x}_0)$. Therefore, $e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0$, $\int_0^t e^{\tilde{A}(t-s)}\tilde{B}\mathrm{u}(s)\, ds = \int_0^t e^{A(t-s)} B\mathrm{u}(s)\, ds$, hence trajectories coincide for all $t \in [0, T]$. **(ii) Case $0 \notin \mathcal{U}_c$.** *"$\subseteq$" (Necessity).* If $e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0$ and $e^{\tilde{A}t}\tilde{B}w = e^{At}Bw$ for all $t \in [0, T]$ and $w \in W$, then for any $\mathrm{u}(\cdot)$ with $\mathrm{u}(s) \in \mathrm{span}(W)$, we can write $\mathrm{u}(s) = \sum_i \alpha_i(s) w_i$ with $w_i \in W$. Linearity gives $\int_0^t e^{\tilde{A}(t-s)}\tilde{B}\mathrm{u}(s)\, ds = \sum_i \int_0^t \alpha_i(s) e^{\tilde{A}(t-s)}\tilde{B} w_i\, ds = \sum_i \int_0^t \alpha_i(s) e^{A(t-s)} B w_i\, ds = \int_0^t e^{A(t-s)} B\mathrm{u}(s)\, ds$, hence trajectories match for all admissible u. *"$\supseteq$" (Sufficiency).* Assume trajectories coincide for all $\mathrm{u} \in \mathcal{U}_C$ and on $[0, T]$. Since $0 \in \mathcal{U}_c$, equality for $\mathrm{u} \equiv 0$ implies the equality of homogeneous responses. □

Proof (Proposition 4.18): Structure of $[A, B]_{\mathrm{x}_0,\mathcal{U}_c}$ under nonlocal input constraints [This Work]

Proof. For any $u \in \mathcal{U}_c$ and $t \in [0, T]$, variation of constants gives

$$\mathrm{x}(t) = e^{At}\mathrm{x}_0 + \mathcal{R}_t^{A,B}[u], \qquad \tilde{\mathrm{x}}(t) = e^{\tilde{A}t}\mathrm{x}_0 + \mathcal{R}_t^{\tilde{A},\tilde{B}}[u].$$

Thus if $e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0$ and $\mathcal{R}_t^{\tilde{A},\tilde{B}}\big|_{\mathcal{U}_c} = \mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c}$ for all $t \in [0, T]$, then $\mathrm{x}(t) = \tilde{\mathrm{x}}(t)$ for all admissible u (sufficiency). Conversely, assume $\mathrm{x}(t) = \tilde{\mathrm{x}}(t)$ for all $u \in \mathcal{U}_c$, $t \in [0, T]$. Then

$$\Delta_t(u) := \mathcal{R}_t^{\tilde{A},\tilde{B}}[u] - \mathcal{R}_t^{A,B}[u] = e^{At}\mathrm{x}_0 - e^{\tilde{A}t}\mathrm{x}_0$$

is independent of u. If $0 \in \mathcal{U}_c$, taking $u \equiv 0$ yields $\Delta_t(0) = 0$, hence $\Delta_t \equiv 0$ and both $\mathcal{R}_t^{\tilde{A},\tilde{B}}\big|_{\mathcal{U}_c} = \mathcal{R}_t^{A,B}\big|_{\mathcal{U}_c}$ and $e^{\tilde{A}t}\mathrm{x}_0 = e^{At}\mathrm{x}_0$. If instead if $\alpha u \in \mathcal{U}_c$ for every $u \in \mathcal{U}_c$ and $\alpha \in [0, 1]$, fix $u \in \mathcal{U}_c$ and $\alpha \in (0, 1]$ so that $\alpha u \in \mathcal{U}_c$. Linearity of $\mathcal{R}_t^{\cdot,\cdot}$ gives $\Delta_t(\alpha u) = \alpha\, \Delta_t(u)$, while independence of u gives $\Delta_t(\alpha u) = \Delta_t(u)$, hence $\Delta_t(u) = 0$ and the same conclusions follow. □

A.3. Additional Results and Experimental Details

A.3.1. Pseudo-Algorithms

Algorithm 4 (E0) Identifiability Score vs. Estimation Errors

Inputs : n, m (dims), T (trajectory length), dt (sample step), trials N, density q; $u_{\max}$ (PRBS amplitude); MOESP horizons (p, f); rng seed.

Outputs: Pearson correlations CSV; scatter plots; metadata.

```
filter ← rank(R(A,B)) == k
(A,B) ← sample_filter(n, m, filter)        // A_ij = Z_ij X_ij,  Z_ij ~ Ber(p), X_ij ~ P_X
(A_d, B_d) ← cont2discrete_zoh(A, B, dt)
errors ← {"dmdc": [], "moesp":[]}    // Dict. containing arrays of length N
scores ← {"pbh": [], "kryl":[], "eigv":[]}
ρ ← {}                               // Reserved for Pearson correlation results
for i ← 1 to N do
    x_0 ← sample(N((0,I)); x_0 ← x_0/||x_0||_2
    U ← prbs(m, T−1, u_max, dwell=d)          // standard PRBS generator
    X[:,0] ← x_0
    for t = 0 to T−2 do
        X[:,t+1] ← A_d X[:,t] + B_d U[:,t]
    X_+ ← X[:,1:T],   X ← X[:,0:T−1],   Z ← [X; U]
    Θ̂_dmdc = [Â_dmdc, B̂_dmdc] ← X_+ Z^†
    (Â_moesp, B̂_moesp) ← MOESP(X, U; p, f)
    errors.dmdc[i] ← ||Â_dmdc − A_d||_F / ||A_d||_F;  errors.moesp[i] ← ||Â_moesp − A_d||_F / ||A_d||_F
    d_pbh, σ_min(K_n), μ_min ← compute_criteria(A, B, x_0)
    scores.pbh[i], scores.kryl[i], scores.eigv[i] ← d_pbh, σ_min(K_n), μ_min
for (e,s) in (errors, scores) do
    plot(ax_x: 1/s, ax_y: e)
    ρ[e][s] ← pearson_corr(e,s)
return ρ, metadata
```

Algorithm 5 compute_criteria(A, B, x_0)

Inputs : $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$
Outputs: d_{pbh}, $\sigma_{\min}(K_n)$, μ_i

```
// Dist.  to PBH failure
```
1 $H(\lambda) = [\,\lambda I - A \quad [x_0 \;\; B]\,]$
2 $d_{\text{pbh}} \leftarrow \min\limits_{\lambda\in\sigma(A)} \sigma_{\min}\big(H(\lambda)\big)$

```
// Minimum singular value of the Krylov generator matrix
```
3 $K_n \leftarrow \big[\,[x_0 \;\; B],\; A[x_0 \;\; B],\; \ldots,\; A^{n-1}[x_0 \;\; B]\,\big]$
4 $\sigma_{\min}(K_n) \leftarrow \texttt{svd}(K_n)$

```
// Left eigenvector criterion
```
5 $E_\lambda^\ell(A) \leftarrow \{w_i \in \mathbb{R}^n \,|\, w_i^\top A = \lambda_i w_i^\top\}$
6 $\mu_{\min} \leftarrow \min\limits_i \frac{\|w_i^\top [x_0 \;\; B]\|_2}{\|w_i\|_2}$
7 **return** d_{pbh}, $\sigma_{min}(K_n)$, μ_{min}

Algorithm 6 x_0-Equivalence-Class Membership

Inputs : $\dim(\mathcal{X})$, $\dim(\mathcal{U})$, $\dim(V(\mathrm{x}_0)) \in \mathcal{K} \subseteq [n-1]$; ensemble size N per $k \in \mathcal{K}$; discretization time step Δt; horizon T; amplitude $u_{\max}$; estimator set ALGOs; tolerance ϵ; seed s

Outputs: Estimation errors in standard basis, custom basis adapted to $V(\mathrm{x}_0)$

```
rng ← PRNG(s);  logs ← {}
foreach k ∈ K do                                   // target visible dimension
    count ← 0
    while count < N do                             // build N trials at this k
        (A, B) ← draw_with_ctrb_rank(n, m, k, rng, ...)
        (A_d, B_d) ← cont2discrete_zoh(A, B, Δt)  R ← orth_basis(R(A_d, B_d))
        A_r ← R^⊤ A_d R
        y_0 ← unit_random(ℝ^k)
        K_k ← [ y_0, A_r y_0, ..., A_r^{k-1} y_0 ]
        Q_r ← qr(K_k)
        y ← Q_r[:, k]
        x_0 ← Ry / ||Ry||_2
        Q ← [ Q_V  Q_{V^⊥} ] ← build_visible_basis_dt(A_d, B_d, x_0)
        d ← max{1, ⌊γ/k⌋}
        U ← prbs(T, m, u_max, dwell=d, rng)
        L̂ ← max{ L ≤ L_max : rank(H_L(U)) = mL }
        X[:, 0] ← x_0
        for t = 0 to T−2 do                        // Simulate the trajectory.
            X[:, t+1] ← A_d X[:, t] + B_d U[:, t]
        X_0 ← [x_0, ..., x_{T−1}];  X_1 ← [x_1, ..., x_T];  U_tr ← U^⊤
        foreach algo ∈ ALGOs do
            (Â, B̂) ← algo.fit(X_0, X_1, U_tr)
            ε_A ← ||Â − A_d||_F / ||A_d||_F;  ε_B ← ||B̂ − B_d||_F / ||B_d||_F
            Ã ← Q^⊤ A_d Q;  B̃ ← Q^⊤ B_d;  Ã ← Q^⊤ Â Q;  B̃ ← Q^⊤ B̂
            ε_{A,V} ← ||Ã_{1:k,1:k} − Ã_{1:k,1:k}||_F / ||Ã_{1:k,1:k}||_F  ε_{B,V} ← ||B̃_{1:k,:} − B̃_{1:k,:}||_F / ||B̃_{1:k,:}||_F
            ε̄_stdCanal ← 1/2 (ε_A + ε_B)
            ε̄_vis ← 1/2 (ε_{A,V} + ε_{B,V})
            logs.append({ dim_visible= k, algo= algo, ε_A, ε_B, ε_{A,V}, ε_{B,V}, ε̄_std, ε̄_vis, ... })
        count ← count +1
```

Algorithm 7 (E2) TargetVisibility$(n, m, k;$ `base-ensemble`, `max_tries`$)$

Inputs : $n, m, k \in \mathbb{N}$ with $0 \leq k \leq n$; base ensemble for blocks (e.g. Ginibre); `max_tries`.

Outputs: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$ with $\dim V(x_0) = k$.

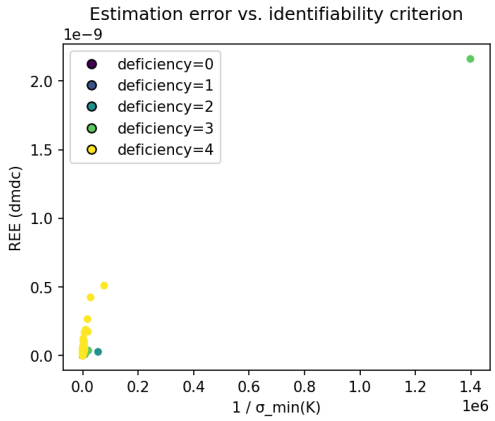
(B1) Controllable block of size k.
if $k = 0$ **then** set A_V, B_V empty and **goto** (B2)
for $t \leftarrow 1$ **to** *max_tries* **do** `// rejection sampling`
  $(A_V, B_V) \leftarrow$ `EnsembleSampler`$(k, m;$ `base-ensemble`$)$
  Ctrb $\leftarrow [\, B_V,\ A_V B_V,\ \ldots,\ A_V^{k-1} B_V \,]$
  if rank(Ctrb) $= k$ **then break**
if rank(Ctrb) $\neq k$ **then error:** failed to draw controllable (A_V, B_V)
(B2) Autonomous complement and block assembly.
Draw $A_W \in \mathbb{R}^{(n-k)\times(n-k)}$ arbitrarily (e.g. same ensemble or stable);
set $B_W \leftarrow 0_{(n-k)\times m}$.
Set

$$A_{\text{blk}} = \begin{bmatrix} A_V & 0 \\ 0 & A_W \end{bmatrix}, \qquad B_{\text{blk}} = \begin{bmatrix} B_V \\ 0 \end{bmatrix}.$$

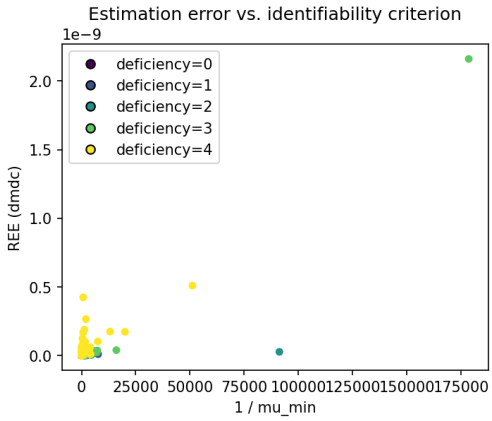
(B3) Haar embedding (rotational genericity).
Draw $G \sim \mathcal{N}(0,1)^{n\times n}$; compute the QR factorization $G = QR$
Set $D = \operatorname{diag}(\operatorname{sign}(\operatorname{diag} R))$ with zeros mapped to $+1$; $Q \leftarrow QD$
(Optional) if $\det Q < 0$, flip last column: $Q_{:,n} \leftarrow -Q_{:,n}$
Set $A \leftarrow Q A_{\text{blk}} Q^\top$, $B \leftarrow Q B_{\text{blk}}$
(B4) Choose $x_0 \in \mathcal{R}(A, B)$ so that $\dim V(x_0) = k$.
if $k = 0$ **then** set $x_0 \leftarrow 0$; **return** (A, B, x_0)
Let $R \leftarrow Q_{:,1:k}$ (orthonormal basis of $\mathcal{R}(A, B)$) and $A_r \leftarrow R^\top A R$
for $t \leftarrow 1$ **to** *max_tries* **do** `// pick an` A_r`-cyclic vector`
  draw $y \sim \mathcal{N}(0, I_k)$; set $\mathcal{K} \leftarrow [\, y,\ A_r y,\ \ldots,\ A_r^{k-1} y \,]$
  if rank$(\mathcal{K}) = k$ **then break**
if rank$(\mathcal{K}) \neq k$ **then error:** failed to draw cyclic y
Set $x_0 \leftarrow Ry / \|Ry\|_2$
return (A, B, x_0)

A.3.2. Additional Results

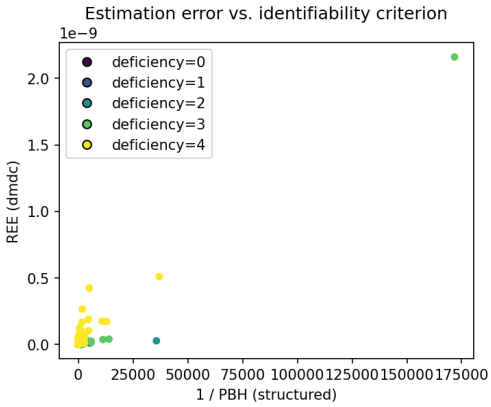
DMDc: Criterion - REE correlations



(a) Krylov criterion.



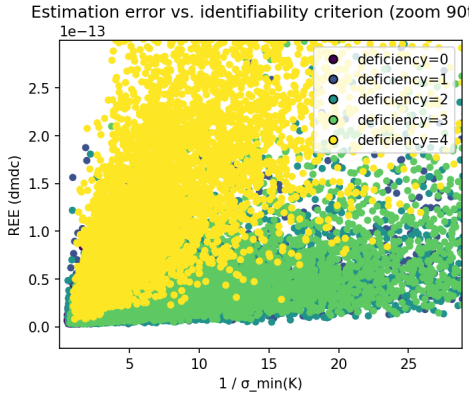
(b) Left eigenvector criterion.



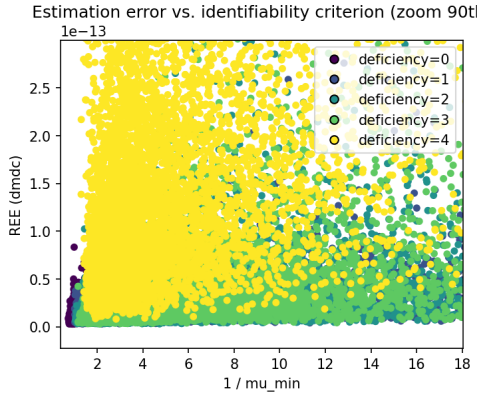
(c) PBH criterion.

Figure A.1.: DMDc estimation errors vs. identifiability scores.

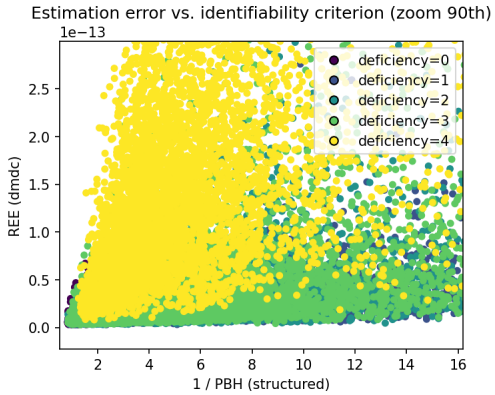
DMDc: Criterion - REE correlations



(a) Krylov criterion.



(b) Left eigenvector criterion.

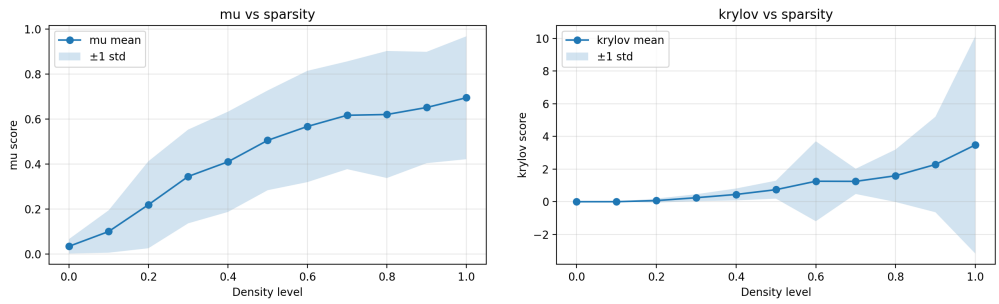


(c) PBH criterion.

Figure A.2.: DMDc estimation errors vs. identifiability scores.

(S1, S2, S3) Additional Criteria

Sparsity

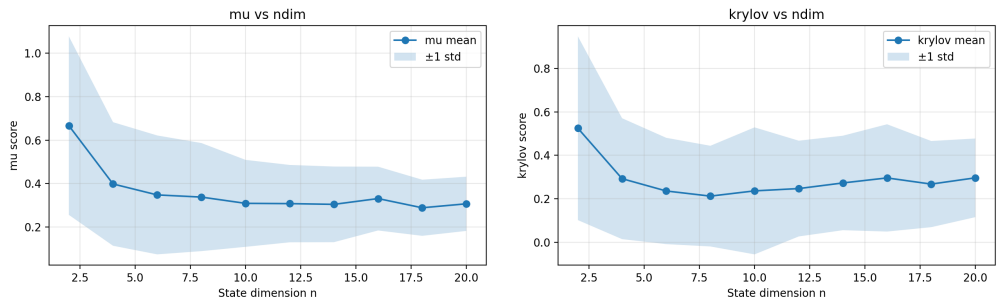


Left eigenvector. Krylov.

Figure A.3.: Identifiability metrics vs. sparsity.

Beyond the parameter–regime effects visible in the plots, several consistent trends emerge across the three criteria. As the entrywise density increases, all three scores rise—consistent with the intuition that denser matrices lie farther from the non-identifiable set and thus enjoy a larger robustness margin. For the PBH-based score and the left-eigenvector score, the growth is saturating: gains are rapid at low densities and then taper with diminishing returns. By contrast, the Krylov score exhibits a threshold-like profile—nearly flat at very low densities, followed by a sharp increase once the density crosses a small positive level a, a, reminiscent of an exponential ramp on $(-\infty, a]$. The dispersions differ as well: PBH and left-eigenvector scores show a small but steady rise in standard deviation with density, whereas the Krylov score is almost variance-free initially and then jumps to a spread of about ± 8.

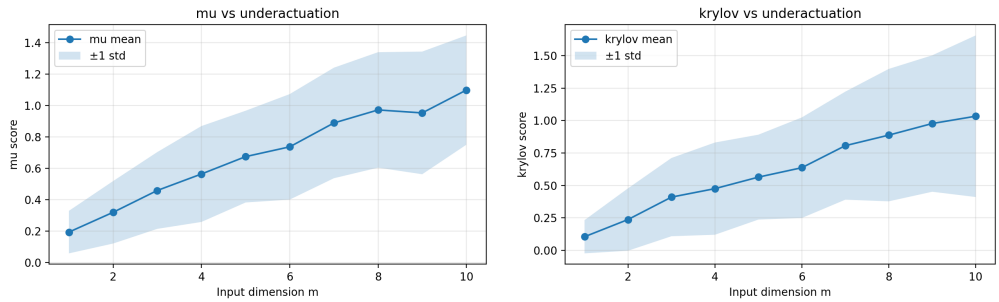
State Dimension



Left eigenvector. Krylov.

Figure A.4.: Identifiability metrics vs. state dimension.

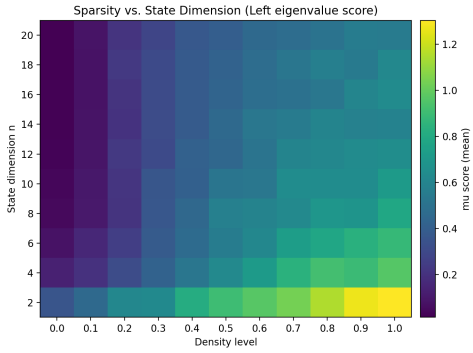
Underactuation



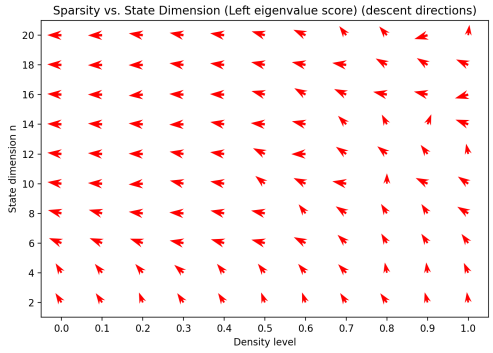
Left eigenvector. Krylov.

Figure A.5.: Identifiability metrics vs. state dimension.

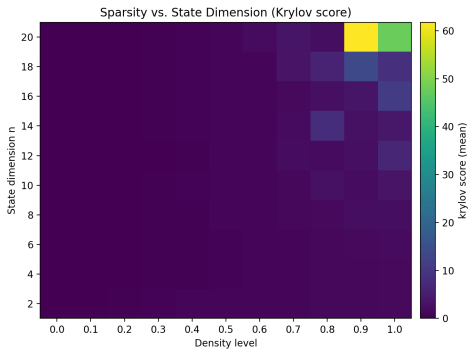
(D1) Sparsity vs. Underactuation



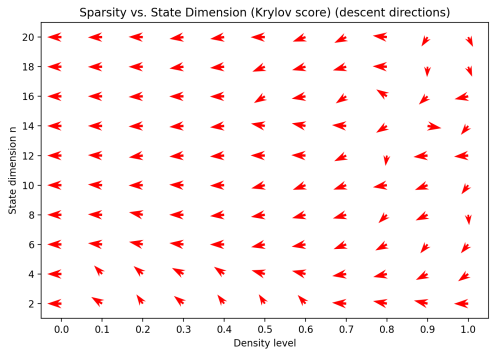
Left eigenvector.



Left eigenvector.



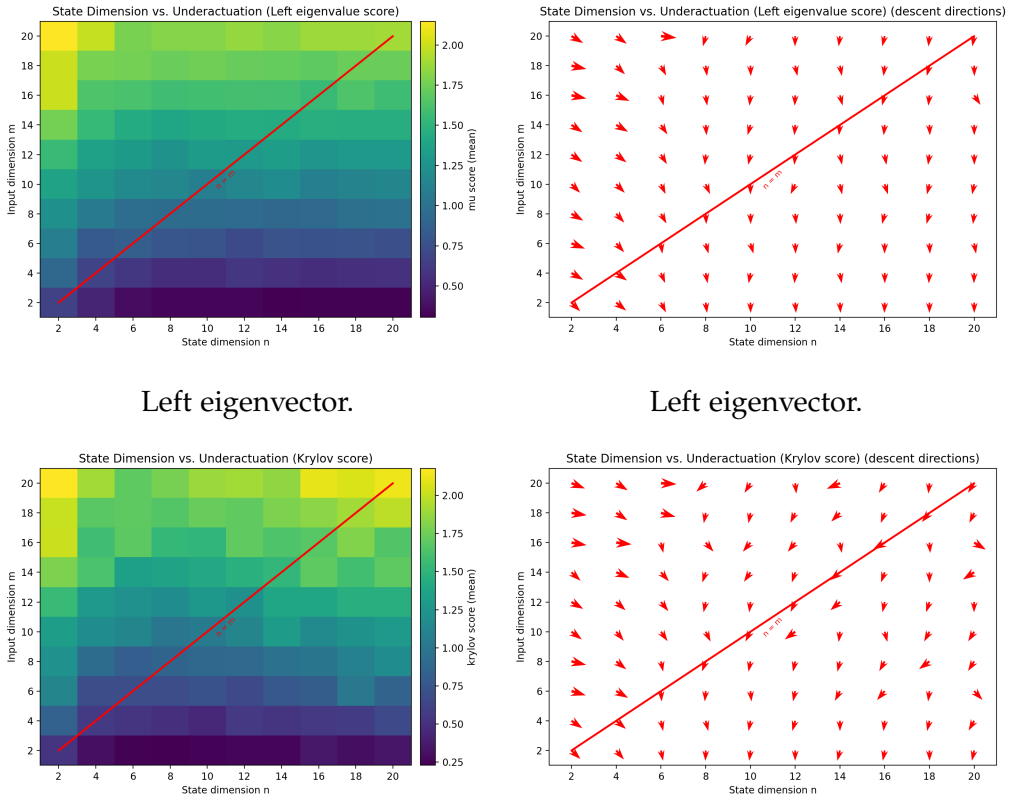
Krylov.



(Krylov.

Figure A.6.: Sparsity vs. state dimension.

(D2) State Dimension vs. Underactuation

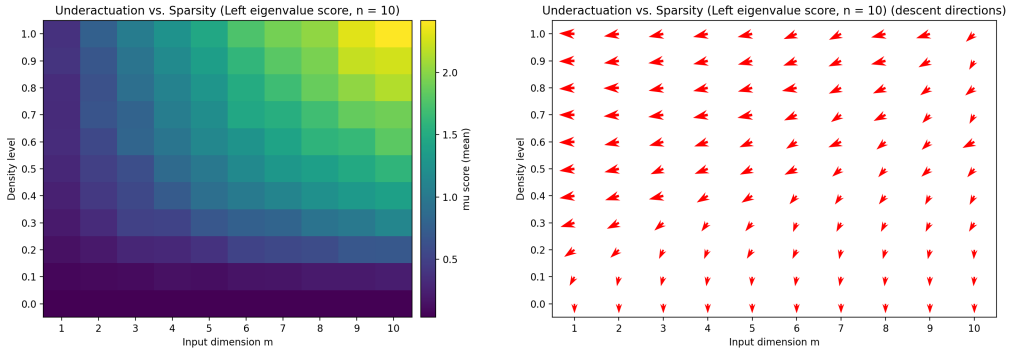


Left eigenvector. Left eigenvector.

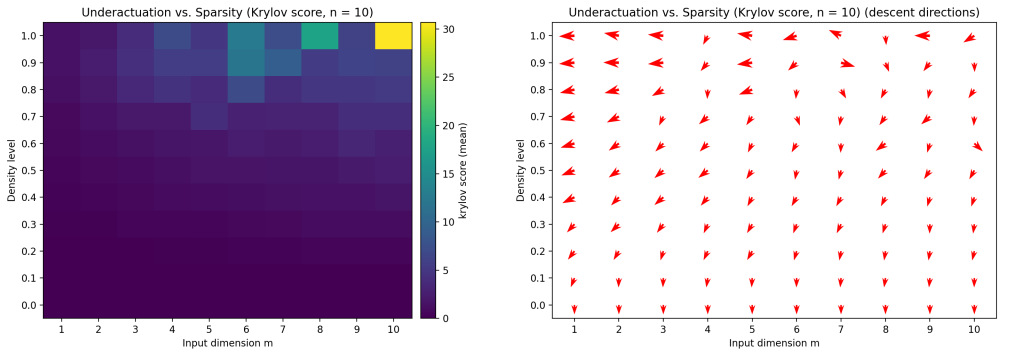
Krylov. Krylov.

Figure A.7.: State Dimension vs. Underactuation

(D3) Underactuation vs. Sparsity



Left eigenvector. Left eigenvector.



Krylov. Krylov.

Figure A.8.: Underactuation vs. sparsity.

A.3.3. Experimental Details

Reproducibility. All results presented in Chapter 5 can be reproduced by following the instructions on `https://github.com/aiulus/pyident`.

NODE configuration details. We optimize the continuous matrices (A, B) with Adam at a learning rate of 10^{-2}, without weight decay or scheduling. We perform full-batch updates. In each epoch, we form (A_d, B_d) from the fitted (A, B) via sampling, compute the loss $\mathcal{L}^k$, and backpropagate through $e^{(\cdot)}$. We use two early-termination criteria: (i) training-loss early stopping with patience 25 and improvement threshold $1e-8$ (ii) gradient-norm convergence with threshold $1e-8$. The epoch budget is 500.

List of Figures

4.1. (a) Primary and secondary trajectories for "bad" x_0, (b) Primary and secondary trajectories for "good" x_0, (c) Input signal used to generate the trajectories: $u(t) = \left[a_1 \sin(\omega_1 t + \phi_1)\, a_2 \sin(\omega_2 t + \phi_2)\right]$ 40

5.1. DMDc and MOESP estimation errors vs. identifiability scores. 57
5.2. MOESP estimation errors vs. identifiability scores. 58
5.3. MOESP estimation errors vs. identifiability scores. 59
5.4. 62
5.5. Relative errors for DMDc standard-basis recovery of A and B. 63
5.6. Relative errors for DMDc in the $V(x_0)$ basis. 63
5.7. 64
5.8. $A-$ specific estimation errors for MOESP in standard and $V(x_0)-$basis. 64
5.9. $B-$specific estimation errors for MOESP in standard and $V(x_0)-$basis. 64
5.10. 65
5.11. Relative errors for SINDy in the standard basis. 65
5.12. Relative errors for SINDy in the $V(x_0)$ basis. 66
5.13. 66
5.14. Relative errors for NODE in the standard basis. 67
5.15. Relative errors for NODE in the $V(x_0)$ basis. 67
5.16. Identifiability metrics vs. sparsity. 69
5.17. PBH score vs. state dimension. 70
5.18. PBH score vs. state dimension. 71
5.19. Sparsity vs. state dimension. 71
5.20. State dimension vs. underactuation. 72
5.21. Underactuation vs. sparsity. 73
5.22. Relative estimation errors for $(n, k := \dim(V(x_0))) = (8, 8)$ (left figure) and $(n, k) = (8, 3)$ (right figure). 74

A.1. DMDc estimation errors vs. identifiability scores. 96
A.2. DMDc estimation errors vs. identifiability scores. 97
A.3. Identifiability metrics vs. sparsity. 98
A.4. Identifiability metrics vs. state dimension. 99
A.5. Identifiability metrics vs. state dimension. 99

A.6. Sparsity vs. state dimension. 100
A.7. State Dimension vs. Underactuation 101
A.8. Underactuation vs. sparsity. 102

List of Tables

2.1. State-space formulations across discrete/continuous, time varying / time independent systems. 10

5.1. Single-system pearson correlations (ρ) between identifiability criteria and errors. 57

5.2. Pearson correlations (ρ) between identifiability criteria and errors (second dataset, ensvol=200, $x_0 = 200$). 58

A.1. Frequently reused symbols and aliases. 78

Bibliography

[1] G. Perakis, D. Singhvi, O. Skali Lami, and L. Thayaparan. "COVID-19: A multi-wave SIR-based model for learning waves." In: *Production and Operations Management* 32.5 (2023), pp. 1471–1489.

[2] A. Somacal, Y. Barrera, L. Boechi, M. Jonckheere, V. Lefieux, D. Picard, and E. Smucler. "Uncovering differential equations from data with hidden variables." In: *Physical Review E* 105.5 (2022), p. 054209.

[3] R. C. Nelson et al. *Flight stability and automatic control*. Vol. 2. WCB/McGraw Hill New York, 1998.

[4] B.-L. Ho. *On effective construction of realizations from input-output descriptions*. Stanford University, 1966.

[5] T. Bohlin. "On the maximum likelihood method of identification." In: *IBM Journal of research and development* 14.1 (1970), pp. 41–51.

[6] M. Verhaegen and P. Dewilde. "Subspace model identification part 2. Analysis of the elementary output-error state-space model identification algorithm." In: *International journal of control* 56.5 (1992), pp. 1211–1241.

[7] P. Van Overschee and B. De Moor. "N4SID: Subspace algorithms for the identification of combined deterministic-stochastic systems." In: *Automatica* 30.1 (1994), pp. 75–93.

[8] H. Pohjanpalo. "System identifiability based on the power series expansion of the solution." In: *Mathematical biosciences* 41.1-2 (1978), pp. 21–33.

[9] L. Ljung and T. Glad. "On global identifiability for arbitrary model parametrizations." In: *automatica* 30.2 (1994), pp. 265–276.

[10] S. L. Brunton, J. L. Proctor, and J. N. Kutz. "Discovering governing equations from data by sparse identification of nonlinear dynamical systems." In: *Proceedings of the national academy of sciences* 113.15 (2016), pp. 3932–3937.

[11] R. T. Chen, Y. Rubanova, J. Bettencourt, and D. K. Duvenaud. "Neural ordinary differential equations." In: *Advances in neural information processing systems* 31 (2018).

[12] K. Beven. "A manifesto for the equifinality thesis." In: *Journal of hydrology* 320.1-2 (2006), pp. 18–36.

[13] R. N. Gutenkunst, J. J. Waterfall, F. P. Casey, K. S. Brown, C. R. Myers, and J. P. Sethna. "Universally sloppy parameter sensitivities in systems biology models." In: *PLoS computational biology* 3.10 (2007), e189.

[14] M. C. Eisenberg and M. A. Hayashi. "Determining identifiable parameter combinations using subset profiling." In: *Mathematical biosciences* 256 (2014), pp. 116–126.

[15] S. Stanhope, J. E. Rubin, and D. Swigon. "Identifiability of linear and linear-in-parameters dynamical systems from a single trajectory." In: *SIAM Journal on Applied Dynamical Systems* 13.4 (2014), pp. 1792–1815.

[16] X. Qiu, T. Xu, B. Soltanalizadeh, and H. Wu. "Identifiability analysis of linear ordinary differential equation systems with a single trajectory." In: *Applied Mathematics and Computation* 430 (2022), p. 127260.

[17] W. M. Wonham. *Linear multivariable control*. Vol. 101. Springer, 1974.

[18] E. D. Sontag. *Mathematical control theory: deterministic finite dimensional systems*. Vol. 6. Springer Science & Business Media, 2013.

[19] C.-T. Chen. *Linear system theory and design*. Oxford university press, 1999.

[20] G. Kreisselmeier. "On sampling without loss of observability/controllability." In: *IEEE Transactions on Automatic Control* 44.5 (2002), pp. 1021–1025.

[21] J. H. Braslavsky et al. *Frequency domain analysis of sampled-data control systems*. University of Newcastle, 1995.

[22] T. Hagiwara. "Preservation of reachability and observability under sampling with a first-order hold." In: *IEEE Transactions on Automatic Control* 40.1 (1995), pp. 104–107.

[23] T. Chen and B. A. Francis. *Optimal sampled-data control systems*. Springer Science & Business Media, 2012.

[24] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant subspaces of matrices with applications*. SIAM, 2006.

[25] I. Markovsky, J. C. Willems, S. Van Huffel, and B. De Moor. *Exact and approximate modeling of linear systems: A behavioral approach*. SIAM, 2006.

[26] P. A. Ioannou and J. Sun. *Robust adaptive control*. Vol. 1. PTR Prentice-Hall Upper Saddle River, NJ, 1996.

[27] S. Shankar. "Adaptive Control-Stability, Convergence, and Robustness." In: *Prentice Hall Information and System Sciences Series, Englewood Cliffs* (1989).

[28] M. Petreczky and L. Bako. "On the notion of persistence of excitation for linear switched systems." In: *Nonlinear Analysis: Hybrid Systems* 48 (2023), p. 101308.

[29] J. L. Proctor, S. L. Brunton, and J. N. Kutz. "Dynamic Mode Decomposition with Control." In: *SIAM Journal on Applied Dynamical Systems* 15.1 (2016), pp. 142–161.

[30] I. W. Jamaludin, N. A. Wahab, N. S. Khalid, S. Sahlan, Z. Ibrahim, and M. F. Rahmat. "N4SID and MOESP subspace identification methods." In: *2013 IEEE 9th International Colloquium on Signal Processing and its Applications*. 2013, pp. 140–145.

[31] V. Nguyen and E. Wood. "Review and unification of linear identifiability concepts." In: *SIAM review* 24.1 (1982), pp. 34–51.

[32] R. C. Lee. "Optimal estimation, identification, and control." In: *(No Title)* (1964).

[33] R. E. Kalman. "Mathematical description of linear dynamical systems." In: *Journal of the Society for Industrial and Applied Mathematics, Series A: Control* 1.2 (1963), pp. 152–192.

[34] W. Wonham and A. Morse. "Feedback invariants of linear multivariable systems." In: *Automatica* 8.1 (1972), pp. 93–100.

[35] A. C. Antoulas. *Approximation of large-scale dynamical systems*. SIAM, 2005.