NAZARBAYEV UNIVERSITY, PHYSICS DEPARTMENT

Assignment 2

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June 6, 2019

TASK GIVEN

1. Consider 1D periodic lattice with multiple waveguides and resonators. Construct a band gap for the propagation of waves along these structure

2. Now, consider 2D periodic lattice with multiple waveguides and circulators. Find a scattering matrix for one unit cell with four ports. Also, construct a band gap of wave propagation using different rotational velocities of circulator. Ideally, you should reproduce Fig. 14 in p. 51 [1].

SOLUTION 1

Even though it's a simple problem, there are many things to consider. From previous assignment, we know

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \tilde{S}_0 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{0.1}$$

where b_i and a_i - means outgoing and ingoing wave through i-th waveguide write write before resonator.

Since, inside waveguide sound wave travels with speed $c = 340 m/s$, there are also change of phase to consider. The set of equations is simply

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i k_0 \frac{l}{2}} \tag{0.2}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} e^{-ik_0 \frac{l}{2}} \tag{0.3}$$

where B_i and A_i - outgoing and ingoing wave through i-th waveguide in the middle of it, l physical length of a waveguide and $k_0 = \omega/c$. By substituting equations (0.2) and (0.3) into (0.1)

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = e^{i\omega l/c} \tilde{S}_0 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \tilde{S} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{0.4}$$

As well as from Bloch theorem [2] for periodic lattice, we can also write

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \tilde{B} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{-ik_B l} \\ e^{ik_B l} & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{0.5}$$

Then, from the equation (0.4) and (0.5) we get that $(\tilde{S} - \tilde{B})A = 0$. Nontrivial case happens when $|\tilde{S} - \tilde{B}| = 0$. It's a determinant of a simple 2x2 matrix (where $S_{11} = S_{22} = S_1$ and $S_{12} = S_{21} = S_2$), thus, it can easily be solved analytically. By solving it, one can find a characteristic equation:

$$\cos k_B l = \frac{(S_1^2 - S_2^2)e^{ik_0 l} - e^{-ik_0 l}}{2S_2 e^{ik_0 l}} \tag{0.6}$$

where right hand side of equation is a function of only ω. In Figure (0.1), bulk band structure for 1D structure:

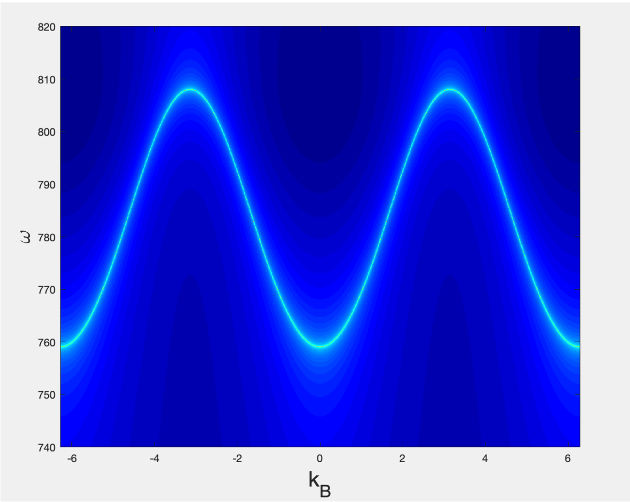

Figure 0.1: Values of $-\log(det(\tilde{S} - \tilde{B}))$ for $\omega - k_B$ plane.

From the tangent line of this curve, we get an information about the speed of wave throughout structure.

SOLUTION 2

From the second part of previous assignment

$$\begin{pmatrix} b_1 \\ b_2 \\ b_5 \end{pmatrix} = S_0 \begin{pmatrix} a_1 \\ a_2 \\ a_5 \end{pmatrix} \tag{0.7}$$

Phase change throughout structure can be written in matrix form as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_5 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_5 \end{pmatrix} e^{ik_0 l} \tag{0.8}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_5 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_5 \end{pmatrix} e^{-ik_0 l} \tag{0.9}$$

where $l = \frac{a}{2\sqrt{3}} = \frac{b}{2\sqrt{3}}$ stands for the unit length of a waveguide.
Then, final scattering matrix for the ports 1,2 and 5

$$\begin{pmatrix} B_1 \\ B_2 \\ B_5 \end{pmatrix} = e^{i2k_0 l} S_0 \begin{pmatrix} A_1 \\ A_2 \\ A_5 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{15} \\ S_{21} & S_{22} & S_{25} \\ S_{51} & S_{52} & S_{55} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_5 \end{pmatrix} \tag{0.10}$$

With the same logic, we can write the scattering matrix for the ports 3, 4 and 6

$$\begin{pmatrix} B_6 \\ B_3 \\ B_4 \end{pmatrix} = e^{i2k_0 l} S_0 \begin{pmatrix} A_6 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} S_{66} & S_{63} & S_{64} \\ S_{36} & S_{33} & S_{34} \\ S_{46} & S_{43} & S_{44} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_5 \end{pmatrix} \tag{0.11}$$

However, points 5 and 6 refers to the inner ports. For them relation $B_5 = A_6$ and $B_6 = A_5$ are valid. Using this, one can construct whole scattering matrix for the unit cell.

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \tilde{\sigma} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{0.12}$$

this $\tilde{\sigma}$ takes the form [3]

$$\tilde{\sigma} = \begin{pmatrix} S_{11} + \triangle(S_{15}S_{51}S_{66}) & S_{21} + \triangle(S_{15}S_{52}S_{66}) & \triangle(S_{15}S_{63}) & \triangle(S_{15}S_{64}) \\ S_{21} + \triangle(S_{25}S_{51}S_{66}) & S_{22} + \triangle(S_{25}S_{52}S_{66}) & \triangle(S_{25}S_{63}) & \triangle(S_{25}S_{64}) \\ \triangle(S_{51}S_{36}) & \triangle(S_{52}S_{36}) & S_{33} + \triangle(S_{36}S_{63}S_{55}) & S_{34} + \triangle(S_{36}S_{64}S_{55}) \\ \triangle(S_{51}S_{46}) & \triangle(S_{52}S_{46}) & S_{43} + \triangle(S_{46}S_{63}S_{55}) & S_{44} + \triangle(S_{46}S_{64}S_{55}) \end{pmatrix} \tag{0.13}$$

where $\triangle = (1 - S_{55}S_{66})^{-1}$
Also, Bloch theorem for this case

$$
\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \tilde{B} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{0.14}
$$

with

$$
\tilde{B} = \begin{pmatrix} 0 & 0 & e^{-i\mathbf{k_b}\cdot\mathbf{a}} & 0 \\ 0 & 0 & 0 & e^{-i\mathbf{k_b}\cdot\mathbf{b}} \\ e^{i\mathbf{k_b}\cdot\mathbf{a}} & 0 & 0 & 0 \\ 0 & e^{i\mathbf{k_b}\cdot\mathbf{b}} & 0 & 0 \end{pmatrix} \tag{0.15}
$$

Bulk mode which satisfy equations (0.12) and (0.14), also need to satisfy

$$
det(\tilde{\sigma} - \tilde{B}) = 0 \tag{0.16}
$$

The bulk band structure for the unit cell

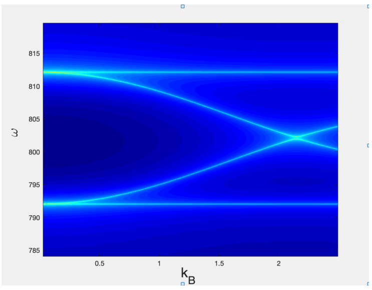

Figure 0.2: $v = 0m/s$

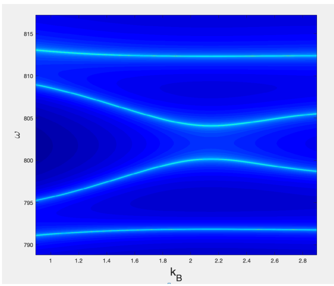

Figure 0.3: $v = 2m/s$

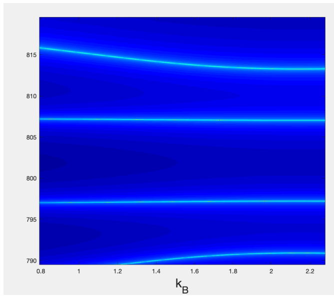

Figure 0.4: $v = 5m/s$

References

[1] Romain Fleury, MS. *Breaking Temporal Symmetries in Metamaterials and Metasurfaces*, August 2015.

[2] Charles Kittel. *Introduction to Solid State Physics*, 1914.

[3] Alexander B. Khanikaev, Romain Fleury, S. Hossein Mousavi and Andrea Alu. *Topologically robust sound propagation in an angular-momentum-biased graphene-like resonator lattice*, Nat. Comm. 2015