NAZARBAYEV UNIVERSITY, PHYSICS DEPARTMENT

Assignment 1

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May 31, 2019

TASK GIVEN

1. Study about *temporal couple mode theory* for two waveguides and resonator system. Resonating frequency ω_0, decaying constants for each waveguide γ_1 and γ_2 respectively. Find the scattering matrix S, also study carefully special case where $\gamma_1 = \gamma_2$.

2. Reproduce Fig.2 from Science 343 (2014) [1]. For the system with three waveguides localized symmetrically and resonator, find a corresponding 3x3 scattering matrix S in terms ω, ω_R, ν, γ.

SOLUTION 1

From the book [2], temporal coupled-mode equations for the system with doubled waveguides

$$\frac{dA}{dt} = -i\omega_0 A - \gamma_1 A - \gamma_2 A + \sqrt{2\gamma_1}s_{1+} + \sqrt{2\gamma_2}s_{2+} \tag{0.1}$$

$$s_{1-} = -s_{1+} + \sqrt{2\gamma_1}A \tag{0.2}$$

$$s_{2-} = -s_{2+} + \sqrt{2\gamma_2}A \tag{0.3}$$

Scattering matrix S should connect the input and output waves by

$$\begin{pmatrix} s_{1-} \\ s_{2-} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} s_{1+} \\ s_{2+} \end{pmatrix} \tag{0.4}$$

To find a scattering matrix S, give some input from one of the waveguides $s_{1+} = e^{-i\omega t}$ and $s_{2+} = 0$. Then, inside resonator $A = A(0)e^{-i\omega t}$. By substituting this into equation (0.1)

$$-i\omega A(0) = -i\omega_0 A(0) - \gamma_1 A(0) - \gamma_2 A(0) + \sqrt{2\gamma}$$
$$A(0) = \frac{\sqrt{2\gamma_1}}{i(\omega_0 - \omega) + \gamma_1 + \gamma_2}$$

By inserting this to the equation (0.2) and (0.3)

$$S_{11} = \frac{s_{1-}}{e^{-i\omega}} = \frac{-(\omega - \omega_0) + i(\gamma_1 - \gamma_2)}{(\omega - \omega_0) + i(\gamma_1 + \gamma_2} \tag{0.5}$$

$$S_{21} = \frac{s_{2-}}{e^{-i\omega t}} = \frac{i2\sqrt{\gamma_1\gamma_2}}{(\omega - \omega_0) + i(\gamma_1 + \gamma_2)} \tag{0.6}$$

In similar fashion, we can find the other two elements by sending wave through other waveguide $s_{2+} = e^{-i\omega t}$ and $s_{1+} = 0$.

$$S_{12} = \frac{s_{1-}}{e^{-i\omega t}} = S_{21} \tag{0.7}$$

$$S_{22} = \frac{s_{2-}}{e^{-i\omega t}} = \frac{-(\omega - \omega_0) + i(\gamma_2 - \gamma_1)}{(\omega - \omega_0) + i(\gamma_1 + \gamma_2)} \tag{0.8}$$

For specific case $\gamma_1 = \gamma_2 = \gamma$, elements simplified as follows

$$S_{12} = S_{21} = \frac{i2\gamma}{(\omega - \omega_0) + i2\gamma} \tag{0.9}$$

$$S_{11} = S_{22} = \frac{\omega_0 - \omega}{(\omega - \omega_0) + i2\gamma} \tag{0.10}$$

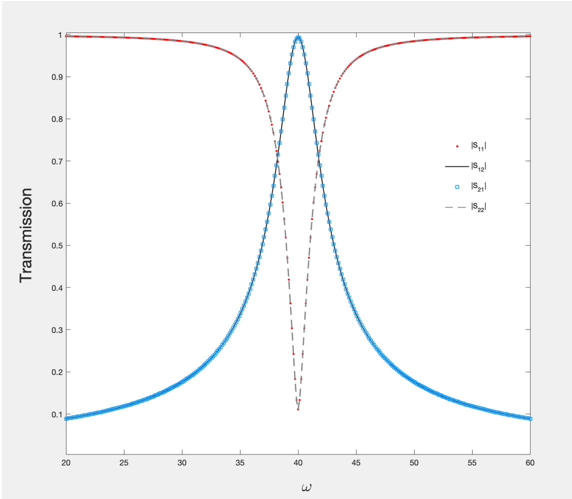

Figure 0.1: The value of the parameters; $\omega_0 = 40$, $\gamma_1 = 0.8$ and $\gamma_2 = 1$.

In Fig. (0.1) and Fig. (0.2) shown the value of the elements of scattering matrix with respect to the parameter ω (from 20 to 70). As it is clearly seen from the figures, at a specific case ($\gamma = \gamma_1 = \gamma_2$) $|S_{11}|$ and $|S_{22}|$ drops exactly to zero for $\omega = \omega_0$. From the equation (0.4), we get the relations ($s_{1-} = s_{2+}$ and $s_{2-} = s_{1+}$). It actually means that at resonating frequency waves flow freely in either direction (resonator does not affect at all!).

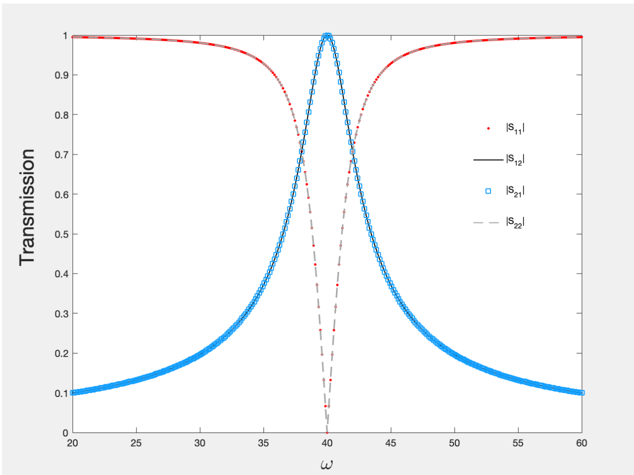

Figure 0.2: The value of the parameters; $\omega_0 = 40$ and $\gamma = 1$.

SOLUTION 2

The scattering matrix for the second problem will have additional elements. The relation between input and output:

$$\begin{pmatrix} s_{1-} \\ s_{2-} \\ s_{3-} \end{pmatrix} = S \begin{pmatrix} s_{1+} \\ s_{2+} \\ s_{3+} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} s_{1+} \\ s_{2+} \\ s_{3+} \end{pmatrix} \tag{0.11}$$

In this case we will have two waves inside resonator $a_{\pm}$. Then, our system of linear equations [3]

$$\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} -i\omega_+ - \gamma_+ & 0 \\ 0 & -i\omega_- - \gamma_- \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{2\gamma_+}{3}} & e^{i\frac{2\pi}{3}}\sqrt{\frac{2\gamma_+}{3}} & e^{i\frac{4\pi}{3}}\sqrt{\frac{2\gamma_+}{3}} \\ \sqrt{\frac{2\gamma_-}{3}} & e^{i\frac{4\pi}{3}}\sqrt{\frac{2\gamma_-}{3}} & e^{i\frac{2\pi}{3}}\sqrt{\frac{2\gamma_-}{3}} \end{pmatrix}\begin{pmatrix} s_{1+} \\ s_{2+} \\ s_{3+} \end{pmatrix} \tag{0.12}$$

$$\begin{pmatrix} s_{1-} \\ s_{2-} \\ s_{3-} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} s_{1+} \\ s_{2+} \\ s_{3+} \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{2\gamma_+}{3}} & \sqrt{\frac{2\gamma_-}{3}} \\ e^{i\frac{4\pi}{3}}\sqrt{\frac{2\gamma_+}{3}} & e^{i\frac{2\pi}{3}}\sqrt{\frac{2\gamma_-}{3}} \\ e^{i\frac{2\pi}{3}}\sqrt{\frac{2\gamma_+}{3}} & e^{i\frac{4\pi}{3}}\sqrt{\frac{2\gamma_-}{3}} \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \tag{0.13}$$

So, with the similar strategy as in problem 1 ($s_{1+} = e^{-i\omega t}$ and $s_{2+} = s_{3+} = 0$), while resonator ($a_{\pm} = \alpha_{\pm} e^{i\omega t}$). The elements in the first row

$$S_{11} = \frac{s_{1-}}{e^{-i\omega t}} = -1 + \frac{i\frac{2\gamma_+}{3}}{\omega - \omega_+ + i\gamma_+} + \frac{i\frac{2\gamma_-}{3}}{\omega - \omega_- + i\gamma_-} \tag{0.14}$$

$$S_{21} = \frac{s_{2-}}{e^{-i\omega t}} = e^{i\frac{4\pi}{3}} \frac{i\frac{2\gamma_+}{3}}{\omega - \omega_+ + i\gamma_+} + e^{i\frac{2\pi}{3}} \frac{i\frac{2\gamma_-}{3}}{\omega - \omega_- + i\gamma_-} \tag{0.15}$$

$$S_{31} = \frac{s_{1-}}{e^{-i\omega t}} = e^{i\frac{2\pi}{3}} \frac{i\frac{2\gamma_+}{3}}{\omega - \omega_+ + i\gamma_+} + e^{i\frac{4\pi}{3}} \frac{i\frac{2\gamma_-}{3}}{\omega - \omega_- + i\gamma_-} \tag{0.16}$$

Since, we can denote positive orientation as $(1 \to 3 \to 2)$, from the symmetry we found

$$S_{11} = S_{22} = S_{33} \tag{0.17}$$

$$S_{12} = S_{23} = S_{31} \tag{0.18}$$

$$S_{13} = S_{32} = S_{21} \tag{0.19}$$

So, with all these equations we can reproduce Fig2(a-c)[1].

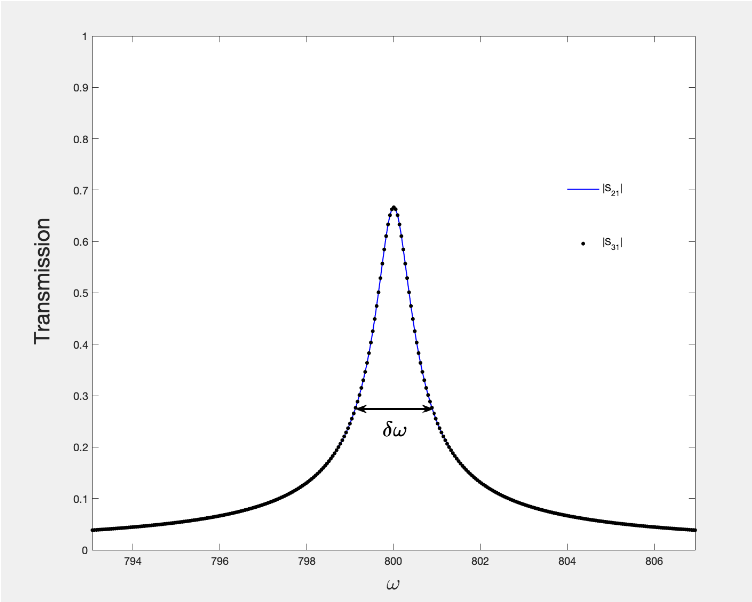

Figure 0.3: Case of no bias. The paramters are $\omega_R = 800$, $\gamma = 0.4$

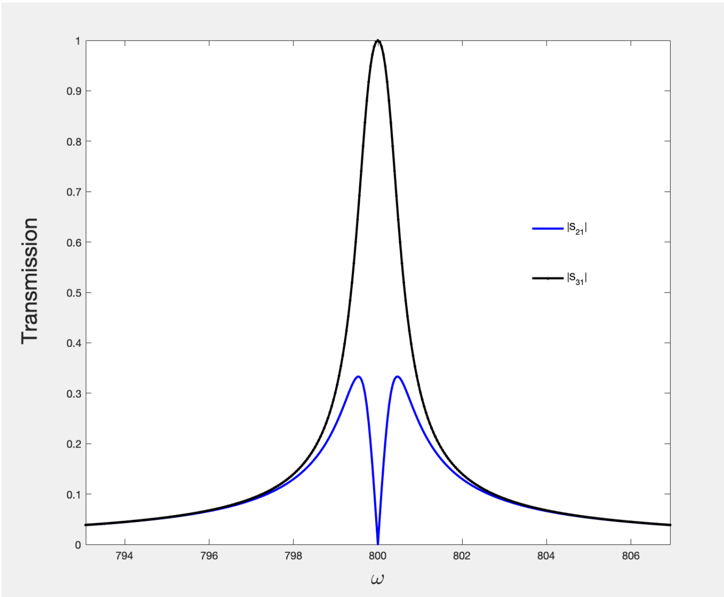

Figure 0.4: With optimal velocity $v_{opt} = /frac\gamma R\sqrt{3}$. Same parameters.

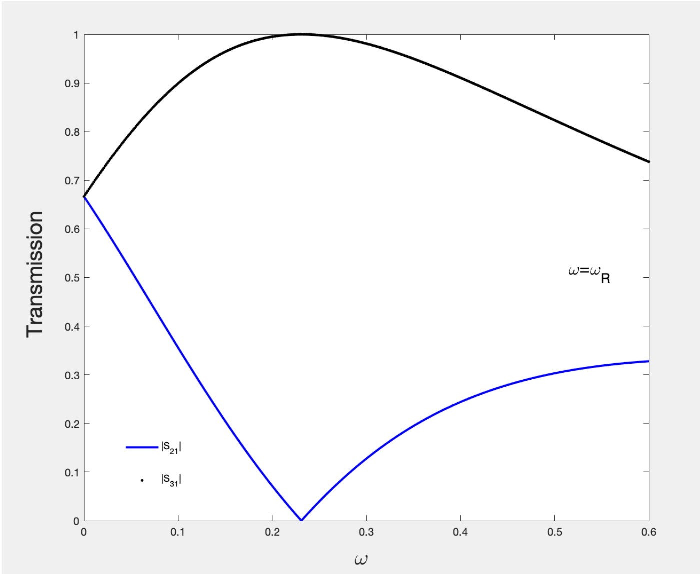

Figure 0.5: Fixed $\omega = \omega_R$. The velocity ranges from 0 to 0.6.

References

[1] Romain Fleury, Dimitrios L. Sounas, Caleb F. Sieck, Michael R. Haberman and Andrea Alu. *Sound Isolation and Giant Linear Nonreciprocity in a Compact Acoustic Circulator*, Science 343, 6170, 2014.

[2] John D. Joannopoulos, Steven G. Johnson, Joshua N. Winn, and Robert D. Meade. *Photonic Crystals: Molding the Flow of Light*, 1914.

[3] Z. Wang, S. Fan. *Magneto-optical defects in two-dimensional photonic crystals*, Appl. Phys. B 81, (369-375), 2005.