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TECHNICAL NOTE

An Efficient Method for Solving Linear Goal Programming Problems

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Abstract. This note proposes a solution algorithm for linear goal programming problems. The proposed method simplifies the traditional solution methods. Also, the proposed method is computationally efficient.

Key Words. Goal programming, weighted goal programming.

1. Introduction

The methodology known as goal programming first appeared in Charnes and Cooper (Ref. 1). Other texts on goal programming have been prepared by Lee (Ref. 2), Ignizio (Ref. 3), and Romero (Ref. 4). The overall purpose of goal programming is to minimize the deviations between the achievement of the goals and their aspiration levels. A goal programming problem is formulated below:

$$\text{(P1)} \quad \min \quad \sum_{i=1}^{n} |f_i(x) - g_i|,$$

$$\text{s.t.} \quad x \in F, x \geq 0, g_i \geq 0,$$

where $f_i(x)$ = linear function of the ith goal, g_i = aspiration level of the ith goal, F = feasible or constraint set defined by linear equations or inequalities.

Letting

$$f_i(x) - g_i = d_i^+ - d_i^- \quad \text{and} \quad d_i^+, d_i^- \geq 0,$$

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Problem (P1) can be reformulated easily as the following equivalent problem (see Refs. 2–4):

$$\begin{aligned}
\text{(P2)} \quad \min \quad & \sum_{i=1}^{n} (d_i^+ + d_i^-), \\
\text{s.t.} \quad & f_i(x) - d_i^+ + d_i^- - g_i = 0, \qquad i = 1, 2, \ldots, n, \\
& d_i^+, d_i^- \geq 0, \qquad i = 1, 2, \ldots, n, \\
& x \in F, x \geq 0,
\end{aligned}$$

where d_i^+, d_i^- are positive and negative deviations of the ith goal from its aspiration level.

2. Equivalent Formulations

Problem (P2) can be solved by introducing artificial variables to the model and using the two-phase method or big M method (Ref. 5). The big M method leads to the following Problem (P3), which is an equivalent formulation of the goal programming model:

$$\begin{aligned}
\text{(P3)} \quad \min \quad & \sum_{i=1}^{n} (d_i^+ + d_i^-) + M \sum_{i=1}^{n} s_i, \\
\text{s.t.} \quad & f_i - d_i^+ + d_i^- + s_i = g_i, \qquad i = 1, 2, \ldots, n, \\
& d_i^+, d_i^-, s_i \geq 0, \qquad i = 1, 2, \ldots, n, \\
& x \in F, x \geq 0.
\end{aligned}$$

Observing the constraints in the Problem (P2), we have

$$d_i^- = -f_i(x) + g_i + d_i^+ \geq 0.$$

Substituting this into the objective function and constraints, denoting d_i^+ as d_i, we obtain the following Problem (P4), another equivalent formulation of Problem (P2):

$$\begin{aligned}
\text{(P4)} \quad \min \quad & \sum_{i=1}^{n} (2d_i - f_i(x)), \\
\text{s.t.} \quad & -f_i(x) + d_i + g_i \geq 0, \qquad i = 1, 2, \ldots, n, \\
& d_i \geq 0, \qquad i = 1, 2, \ldots, n, \\
& x \in F, x \geq 0.
\end{aligned}$$

Problem (P4) can be directly extended to formulate a weighted goal programming problem. In Problem (P4), the positive deviation of the goal ith is d_i, and the negative deviation of the goal ith is $-f_i(x)+g_i+d_i$. Denote the weighted positive and negative deviations from the goal ith as ω_i^+ and ω_i^-, respectively; then a weighted goal programming problem can be formulated below:

$$\text{(P5)} \quad \min \quad \sum_{i=1}^{n} [(\omega_i^+ + \omega_i^-)d_i - \omega_i^- f_i(x)],$$

$$\text{s.t.} \quad -f_i(x)+d_i+g_i \geq 0, \qquad i=1,2,\ldots,n,$$

$$d_i \geq 0, \qquad i=1,2,\ldots,n,$$

$$x \in F,\ x \geq 0.$$

3. Computational Efficiency

The computational efficiency of the above equivalent formulations of the goalprogramming problem [Problem (P1)] can be compared by observing the number of variables actually used when solving the problem. Obviously, in solving Problem (P2) by the simplex method, usually one uses Problem (P3) or the two-phase method. In both cases, $3n$ variables are required in the solution process. Problem (P4) requires only $2n$ variables when it is solved by the simplex method. There are n deviation variables d_i, $i=1,2,\ldots,n$, and n slack variables for the n constraints

$$f_i(x)-d_i-g_i \leq 0, \qquad i=1,2,\ldots,n.$$

Consequently, Problem (P4) is more computationally efficient.

4. Example

Consider a weighted goal programming problem which appeared in Hillier and Lieberman (Ref. 6). Suppose that a company is considering to produce three products x_1, x_2, x_3. The goals of the company are: (i) achieving a long-run profit of at least 125 million dollars from these products; (ii) maintaining the current employment level of 40 thousand employees; (iii) holding the capital investment to less than 55 million dollars. Probably, it will not be possible to attain all of these goals simultaneously; hence, the company assigns penalty weights of 5 for missing the profit goal, 2 for going over the employment goal, 4 for going under the same goal, and 3 for exceeding the capital investment goal. The related functions and parameters are specified below:

$$f_1(x) = 12x_1 + 9x_2 + 15x_3 \geq 125 \qquad \text{(profit goal)},$$

$$f_2(x) = 5x_1 + 3x_2 + 4x_3 = 40 \qquad \text{(employment goal)},$$

$$f_3(x) = 5x_1 + 7x_2 + 8x_3 \geq 55 \qquad \text{(investment goal)},$$

$$\omega_1^+ = 0, \quad \omega_1^- = 5, \quad \omega_2^+ = 2, \quad \omega_2^- = 4, \quad \omega_3^+ = 3, \quad \omega_3^- = 0.$$

By the form of Problem (P5), this problem is formulated as the following program:

$$\begin{aligned} \min \quad & D = -5(12x_1 + 9x_2 + 15x_3 - d_1) + 6d_2 - 4(5x_1 + 3x_2 + 4x_3) + 3d_3, \\ \text{s.t.} \quad & -f_1(x) + 125 + d_1 \geq 0, \\ & -f_2(x) + 40 + d_2 \geq 0, \\ & -f_3(x) + 55 + d_3 \geq 0, \\ & x_1, x_2, x_3 \geq 0, \qquad d_1, d_2, d_3 \geq 0. \end{aligned}$$

Applying the simplex method to solve this example yields an optimal solution

$$x_1 = 25/3, \qquad x_2 = 0, \qquad x_3 = 5/3,$$

with positive deviations

$$d_1 = 0, \qquad d_2 = 25/3, \qquad d_3 = 0.$$

The negative deviations are computed as

$$-f_1(x) + 125 = 0, \qquad -f_2(x) + 40 + 25/3 = 0, \qquad -f_3(x) + 55 = 0.$$

So, the first and third goals are fully satisfied, but the employment level goal of 40 is exceeded by 25/3 (833) employees. The solution is the same as found by Hillier and Lieberman. The proposed method, however, uses a smaller number of variables to reach the optimal solution.

5. Conclusions

This paper proposes a new method of transforming a linear goal program into a linear program. The proposed method is more efficient than the traditional one, in the sense that it uses a smaller number of variables in computation.

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