So, the total energy $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{\mu\sqrt{r_0^2 - 2mr_0 + a^2}}{\sqrt{r_0^2 + a^2}} + \frac{eqr_0}{r_0^2 + a^2} + \frac{e^2 m}{2(r_0^2 + a^2)}. \tag{9}$$

For the charged object, its last state which is possible outside the outer horizon is defined by $\ell = R$. By assuming that $\ell$ is small, we deduce from (6) the asymptotic form

$$\ell \sim \frac{\sqrt{2}(r_+^2 + a^2)}{(m^2 - a^2)^{1/4}}\sqrt{r_0 - r_+} \quad \text{as} \quad r_0 \to r_+. \tag{10}$$

By substituting (10) into (9), we obtain the energy $\mathcal{E}_{last}$ of the last state

$$\mathcal{E}_{last} \sim \frac{\mu R\sqrt{m^2 - a^2}}{r_+^2 + a^2} + \frac{eqr_+}{r_+^2 + a^2} + \frac{e^2 m}{2(r_+^2 + a^2)} \quad \text{as} \quad R \to 0. \tag{11}$$

We are now in a position to apply thermodynamics of the Kerr-Newman black hole when the charged object falls infinitely slowly along the symmetry axis until the absorption inside the outer horizon which is the final state. This is the original method of Bekenstein [12] for a neutral object in the Schwarzschild black hole. The final state is again a Kerr-Newman black hole but with the new parameters

$$m_f = m + \mathcal{E}_{last} \quad , \quad j_f = j \quad \text{and} \quad q_f = q + e. \tag{12}$$

In the last state outside the outer horizon, the total entropy is $S_{BH}(m, j, q) + S$ but after the absorption the entropy is only $S_{BH}(m_f, j_f, q_f)$ in the final state. By virtue of the generalised second law of thermodynamics, we must have

$$S_{BH}(m_f, j_f, q_f) \geq S_{BH}(m, j, q) + S. \tag{13}$$

We can calculate $\triangle S_{BH} = S_{BH}(m_f, j_f, q_f) - S_{BH}(m, j, q)$ from expression (4) with increments (12) by keeping only linear terms in $\mathcal{E}_{last}$. We find

$$\triangle S_{BH} = \frac{2\pi}{\sqrt{m^4 - j^2}}\left[2m\left(m^2 + \sqrt{m^4 - j^2}\right)\mathcal{E}_{last} - \left(m^2 + \sqrt{m^4 - j^2}\right)\left(eq + \frac{1}{2}e^2\right)\right] \tag{14}$$

that we can rewrite under the form

$$\triangle S_{BH} = \frac{2\pi}{\sqrt{m^2 - a^2}}\left[(r_+^2 + a^2)\mathcal{E}_{last} - eqr_+ - \frac{1}{2}e^2 r_+\right]. \tag{15}$$

We simplify expression (15) by using $\mathcal{E}_{last}$ given by (11). Then, taking into account inequality (13), we thus obtain the desired entropy bound

$$S \leq 2\pi\left(\mu R - \frac{1}{2}e^2\right). \tag{16}$$

In conclusion, we have extended the works of Bekenstein and Mayo [1] and Hod [2] by employing thermodynamics of the Kerr-Newman black hole instead of the Reissner-Nordström black hole. These kinds of method for determining the entropy bound is without pretending to any rigour. However, they confirm the physical importance of the electrostatic self-force acting on a point charge in a background black hole although its physical relation with the entropy of a charged object is not clear.