2.1 The short-distance regime $L \ll L_c$

At *small space-time distances* $L \ll L_c$, because of asymptotic freedom, the properties of QCD are well described by a perturbative expansion in powers of the coupling g_s of the generating functional for the connected Green functions,

$$W[J,\eta,\overline{\eta}] \;=\; \int \mathcal{D}A^\mu \mathcal{D}\psi \mathcal{D}\overline{\psi}\, \det\mathcal{F}\; \exp\left\{ i \int d^4r \left(\mathcal{L}[A^\mu,\psi,\overline{\psi}] \;+\; J_{\mu,a}A^\mu_a \;+\; \overline{\psi}_i \eta \;+\; \overline{\eta}\psi_i \right)\right\} \;. \tag{4}$$

In the path integral, $\det\mathcal{F}$ denotes the Fadeev-Popov determinant and J, η, $\overline{\eta}$ are the generating currents for the gluon fields A^μ_a and the quark fields ψ, $\overline{\psi}$ (which are vectors in flavor space, $\psi \equiv (\psi_u, \psi_d, \ldots)$), and the QCD Lagrangian is

$$\mathcal{L}[A^\mu,\psi,\overline{\psi}] \;=\; -\frac{1}{4}\, F_{\mu\nu,a}F^{\mu\nu}_a \;+\; \overline{\psi}_i \left[\, (i\gamma_\mu\partial^\mu - m)\delta_{ij} - g_s\gamma_\mu A^\mu_a T^{ij}_a \right] \psi_j \;+\; \xi_a(A) \;, \tag{5}$$

where $F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g_s f_{abc} A^\mu_b A^\nu_c$ is the gluon field-strength tensor. The subscripts a, b, c label the color components of the gluon fields, and g_s denotes the color charge related to $\alpha_s = g_s^2/(4\pi)$. The T_a are the generators of the $SU(3)$ color group, satisfying $[T_a, T_b] = i f_{abc} T_c$ with the structure constants f_{abc}. The indices i, j label the color components of the quark fields and $m \equiv \mathrm{diag}(m_u, m_d, \ldots)$. Throughout, summation over the color indices a, b, c and i, j is understood. We recall that on setting the quark current masses m to zero, one has exact chiral symmetry. The gauge-fixing term is denoted by a general function $\xi_a(A)$ which, e.g., in covariant gauges is $\xi_a(A) \equiv -1/(2\alpha)(\partial_\mu A^\mu_a)^2$ with Lagrange multiplier $1/\alpha$. However, we will later consider a different (ghost-free) gauge that is more convenient for our purposes.

2.2 The long-distance regime $L \gg L_c$

The long-range physics of QCD at *large space-time distances* $L \gg L_c$, is known to be described well by an effective low-energy theory. Here we adopt the approach of Ref. [14] and define the corresponding generating functional as:

$$W[J_\chi, K_U, K_U^\dagger] \;=\; \int \mathcal{D}\chi \mathcal{D}U \mathcal{D}U^\dagger \;\; \exp\left\{ i \int d^4r \left(\mathcal{L}[\chi, U, U^\dagger] \;+\; J_\chi \chi \;+\; U^\dagger K_U \;+\; K_U^\dagger U \right)\right\} \;. \tag{6}$$

The field degrees of freedom are a *scalar gluon condensate field* χ and a *pseudoscalar quark condensate field* $U = f_\pi \exp\left(i \sum_{j=0}^{8} \lambda_j \phi_j / f_\pi \right)$ for the nonet of the meson fields ϕ_j ($f_\pi = 93$