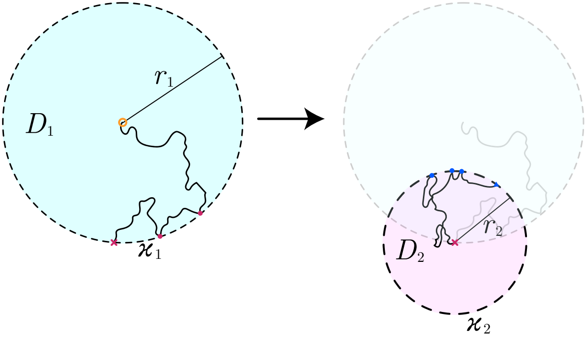
More on the annealed model of diffusion in semi-permeable domains

A. Brown — Hathcock group

January 26, 2026

The permeable domain limit



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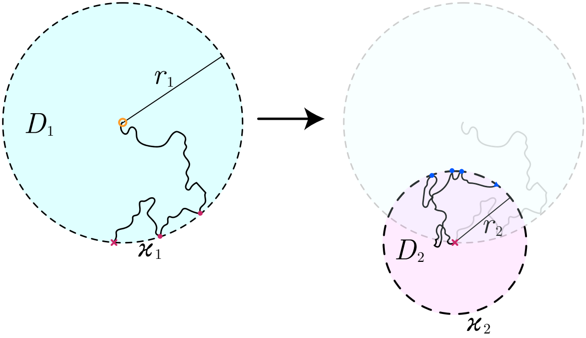

- Should generically expect that longer steps are penalized by higher time cost[1]
  - Necessitates coupling between the position of the walk and time

Joint density

- Let $\psi(\tau)$ denote the density of the **patch duration** τ and $\phi(x, t \mid \tau)$ be the propagator for the displacement after **elapsed time** $t \in [0, \tau]$ within that patch. We define the joint density

$$\rho(x, t, \tau) = \phi(x, t \mid \tau)\psi(\tau) \tag{1}$$

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- Let $\eta(x, t)$ be the density of being at (x, t) **immediately after** a patch completion. Then the full propagator $p(x, t)$ is a convolution of renewal points with the unfinished-patch density:

$$p(x, t) = \int_{\mathbb{R}^d} d^d x' \int_0^{t'} dt'\, \eta(x', t')\zeta(x - x', t - t') \tag{3}$$

Fourier-Laplace representation

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- Patch completions form a renewal process. The renewal density η can be written implicitly as

$$\eta(x,t) = \delta(t)\delta(x) + \int d^d x' \int dt'\, \eta(x',t')w(x-x',t-t') \tag{5}$$

  where $w(x,t)$ is the joint density of **completed** patch increments (displacement and duration) for one full patch
- Completion occurs when $t = \tau$, so

$$w(x,t) = \int_0^\infty d\tau\, \rho(x,t,\tau) = \phi(x,t \mid t)\psi(t). \tag{6}$$

Fourier-Laplace propagator

- Taking the Fourier-Laplace transform of Eq. (5) gives

$$\tilde{\eta}(k,s) = 1 + \tilde{\eta}(k,s)\tilde{w}(k,s),$$

hence

$$\tilde{\eta}(k,s) = \frac{1}{1-\tilde{w}(k,s)} \tag{7}$$

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- Substituting Eq. (7) into Eq. (4), we get a nice closed form expression for the FL-transformed propagator

$$\tilde{p}(k,s) = \frac{\tilde{\zeta}(k,s)}{1-w(k,s)} \tag{8}$$

Fourier-Laplace MSD

- The transformed propagator $\tilde{p}$ is the Laplace transform of the characteristic function. For an isotropic unbiased process, the Laplace-transformed MSD in $\mathbb{R}^d$ is obtained via the k-Laplacian:

$$\left\langle |x(s)|^2 \right\rangle = - \left. \Delta_k \tilde{p}(k,s) \right|_{k=0} \tag{9}$$

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- Differentiating Eq. (8) and using unbiasedness (so $\nabla_k \tilde{\zeta}(0,s) = \nabla_k \tilde{w}(0,s) = 0$), we obtain

$$\left\langle |x(s)|^2 \right\rangle = -\frac{\Delta_k \tilde{\zeta}(k,s)\big|_{k=0}}{1-\tilde{\psi}(s)} - \frac{\Delta_k \tilde{w}(k,s)\big|_{k=0}}{s\left(1-\tilde{\psi}(s)\right)}. \quad (10)$$

since $\tilde{w}(0,s) = \tilde{\psi}(s)$ and $\tilde{\zeta}(0,s) = (1-\tilde{\psi}(s))/s$

Reducing to within-patch second moments

- The remaining inputs in Eq. (10) are $\Delta_k \tilde{w}(k,s)|_0$ and $\Delta_k \tilde{\zeta}(k,s)|_0$. These can be written entirely in terms of the **within-patch characteristic function**, $\hat{\phi}(k,t \mid \tau)$

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- Completion kernel (one full patch):

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- Therefore, determining the MSD reduces to knowing the within-patch MSD

$$M_2(t \mid \tau) := \int_{\mathbb{R}^d} |x|^2\, \phi(x,t \mid \tau)\, d^d x = -\,\Delta_k \hat{\phi}(k,t \mid \tau)\Big|_{k=0}$$

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- For now, let's specify the full within-patch propagator itself. For Brownian motion on a patch with (possibly τ-dependent) diffusivity $D(\tau)$,[2]

$$\phi(x, t \mid \tau) = \frac{1}{(4\pi D(\tau)t)^{d/2}} \exp\left(-\frac{x^2}{4D(\tau)t}\right), \qquad t \leq \tau \quad (13)$$

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  - For fractional Brownian motion within a patch one analogously has $M_2(t \mid \tau) = 2dK_\beta(\tau)t^\beta$, where $K_\beta(\tau)$ is the generalized diffusion coefficient on a patch.

Tauberian theorems

- Suppose we are to follow the main text and let $\psi(\tau) \sim \tau^{-\alpha-1}$ and $D \sim \tau^{-1/\gamma}$. For Brownian motion, the Laplacian of the Fourier transform can be read off from Eq. (11)

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  - Issue: this integral only converges when $\Re(\rho) < 1$
  - Can determine its asymptotic properties more generally via the Tauberian theorem[3]: Let $f(t)$ be a real-valued function of bounded variation. For $\rho \in (0, 1)$, the following statements are equivalent of f and its Laplace transform F,

$$F(s) \sim s^{-\rho}, \quad s \to 0; \qquad f(t) \sim t^{\rho-1}/\Gamma(\rho), \quad t \to \infty \quad (14)$$

Applications of the Tauberian theorems I

- Since ψ is a probability distribution function, it permits the asymptotic expansion via the Tauberian theorems[3]

$$\tilde{\psi}(s) \sim 1 - cs^{\alpha} + o(s^{\alpha}), \qquad s \to 0 \tag{15}$$

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- If $\rho = \alpha + 1/\gamma < 1$, we can calculate the transforms in the numerators explicitly[4],

$$-\Delta_k \tilde{w}(k,s)\big|_{k=0} \sim s^{\rho-1}, \qquad -\Delta_k \tilde{\zeta}(k,s)\Big|_{k=0} \sim s^{\rho-2}$$

so both terms in Eq. (10) give rise to the same exponent,

$$\left\langle |x(s)|^2 \right\rangle \sim s^{-2+1/\gamma} \quad \Longrightarrow \quad \left\langle |x(t)|^2 \right\rangle \sim t^{1-1/\gamma}, \tag{16}$$

via an inversion Tauberian of the Tauberian theorem

Applications of the Tauberian theorems II

- On the other hand, when $\rho > 1$, the integral decays sufficiently fast at infinity such that the limit exists and is finite,

$$-\Delta_k \tilde{w}(k,s)\Big|_{k=0} \xrightarrow[s\to 0]{} C_w := \int_0^\infty \psi(t) M_2(t \mid t)\, dt \tag{17}$$

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- Meanwhile $1-\tilde{\psi}(s) \sim c\, s^\alpha$ still holds. Insert (17) into Eq. (10): the second term behaves as

$$\frac{-\Delta_k \tilde{w}|_0}{s\,(1-\tilde{\psi}(s))} \sim \frac{C_w}{s \times cs^\alpha} = \text{const.} \times s^{-1-\alpha}.$$

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- A similar piece of argumentation follows for the second term. Completing a Tauberian inversion we find

$$\left\langle |x(s)|^2 \right\rangle \sim s^{-1-\alpha} \quad \Longrightarrow \quad \left\langle |x(t)|^2 \right\rangle \sim t^\alpha, \tag{18}$$

What about FBM?

- If the within-patch motion is instead that of FBM with $K_\beta(\tau) \sim \tau^{-1/\gamma}$, for $\delta = \alpha + \frac{1}{\gamma} + 1 - \beta < 1$,

$$\left\langle |x(t)|^2 \right\rangle \sim t^{\beta - 1/\gamma} \tag{19}$$

and subsequently for $\delta > 1$, the analysis remains the same as above and is dominated by the waiting time,

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- FBM and per-patch scaling of diffusivity can obscure each other!

References

[1] J. Klafter and I. M. Sokolov, *First Steps in Random Walks* (Oxford University Press, 2011).

[2] P. Massignan, C. Manzo, J. A. Torreno-Pina, M. F. García-Parajo, M. Lewenstein, and G. J. Lapeyre, Phys. Rev. Lett. **112**, 150603 (2014).

[3] W. Feller, *An introduction to probability theory and its applications*, Vol. 2 (John Wiley and Sons, 1971).

[4] W. Magnus, F. Oberhettinger, and F. Tricomi, *Tables of Integral Transforms*, edited by A. Erdéyi, Vol. 1 (McGraw-Hill, 1954).

Fourier-Laplace notation

Throughout, the Laplace transform of a function $f(t)$ is written

$$F(s) = \int_0^\infty dt\, e^{-st} f(t) \tag{21}$$

Likewise, the Fourier transform of a function $f(x)$ is written

$$\hat{f}(k) = \int d^d x\, e^{-ik\cdot x} f(x) \tag{22}$$

and its inverse transform

$$f(x) = \int \frac{d^d k}{(2\pi)^d}\, e^{ik\cdot x} \hat{f}(k) \tag{23}$$

Laplace transform of an "exponential power law"

Consider the function

$$f(t) = e^{-at^w},$$

for constants $a > 0, \Re(w) > -1$. Our interest lies in its Laplace transform, as defined in Eq. (21). We can expand the exponential as a power series,

$$e^{-at^w} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a^n t^{wn}$$

and use the identity[4]

$$\int_0^\infty e^{-st} t^\nu \, dt = \frac{\Gamma(\nu+1)}{s^{\nu+1}},$$

for $\Re\nu > -1$, we can write the transform as a power series,

$$F(s) = \sum_{n=0}^{\infty} \frac{(-1)^n a^n}{n!} \frac{\Gamma(wn+1)}{s^{wn+1}} \tag{24}$$