Multiplicative noise and the interpretation problem

When multiplicative noise is introduced to an SDE, i.e.,

$$dX_t = \mu(X_t)\,dt + \sigma(X_t) \cdot dW_t, \tag{1}$$

for $\sigma(X_t) \neq \mathrm{const.}$, the manner in which the integral over a process Y_t,

$$I = \int_0^t Y_t \cdot dW_t \tag{2}$$

is interpreted influences the resulting FPE[1]

The interpreation problem

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1. Itô integral (left endpoint)

$$\int_0^t Y_t \, dW_t = \lim_{||P|| \to 0} \sum_{i=0}^{n-1} Y_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{3}$$

2. Statonovich (midpoint)

$$\int_0^t Y_t \circ dW_t = \lim_{||P|| \to 0} \sum_{i=0}^{n-1} \frac{Y_{t_{i+1}} - Y_{t_i}}{2}(W_{t_{i+1}} - W_{t_i}) \tag{4}$$

3. Hänggi-Klimontovich/ isothermal (right endpoint)

$$\int_0^t Y_t \diamond dW_t = \lim_{||P|| \to 0} \sum_{i=0}^{n-1} Y_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \tag{5}$$

Which to choose? : The Wong-Zakai theorem

Theorem. If $W^{(\varepsilon)}$ is a smooth ε-approximation to a Brownian motion B, for smooth functions μ and σ, the solution to the ODE,

$$\dot{X}^{(\varepsilon)} = \mu(X^{(\varepsilon)}) + \sigma(X^{(\varepsilon)}) W^{(\varepsilon)}, \tag{6}$$

$X^{(\varepsilon)}$, converges in probability, as $\varepsilon \to 0$, to the solution to the SDE given in Eq. (1), interpreted in the Stratonovich sense[2]

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- Implication: If the noise of our system is derived from microscopic degrees of freedom with finite correlation times, the SDE should be interpreted in the Stratonovich sense
- All forms give the same FPE[1] but translation between forms gives an additional drift

Analytical forms

- Following the notation of Eq. (1), let $\mu(X_t) = 0$ and $\sigma(X_t) = \sqrt{2D(X_t)}$ for a diffusivity field D,

$$dX_t = \sqrt{2D(X_t)} \circ dW_t = \frac{1}{2}\nabla D\, dt + \sqrt{2D(X_t)}\, dW_t \qquad (7)$$

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  - This latter form of SDE is much easier to numerically integrate due to the interpretation of the multiplication
- The FPE for the system, likewise, picks up an additional drift term (i.e., $\propto \nabla p$) when the diffusivity is heterogeneous,

$$\frac{\partial p(t,x)}{\partial t} = \nabla \cdot (D(x)\nabla p(t,x)) \qquad (8)$$

Formulating the problem in analysis (PDE)

Let (M, g) be a complete Riemannian manifold. Given a scalar field $\kappa : M \to \mathbb{R}$ which satisfies $0 < \lambda \leq \kappa(x) \leq \Lambda < \infty \; \forall x \in M$, what, if anything, can be said about the heterogeneous-conductivity heat equation

$$\partial_t u(t, x) = \nabla_g \cdot (\kappa(x) \nabla_g u(t, x)) \,, \tag{9}$$

for a smooth function u of $(t, x) \in \mathbb{R}^+ \times M$?

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Some interesting questions:

1. Well-posedness: Does the Cauchy problem admit a unit solution for a given κ and $u(0, x)$?
2. Stability: If $\kappa_n \to \kappa$, do the corresponding solutions $u_n \to u$?
3. Kernel: How does the the kernel compare with the heat kernel?
4. Phenomenological: What is the interplay between curvature and the diffusive behaviour?

Treating a simple case in $\mathbb{R}$

The equation now takes the much less daunting form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right)$$

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- For the sake of simplicity, let's choose the system

$$\kappa(x) = A\left[1 - \sin^{2n}\left(\frac{\pi x}{L}\right)\right] + \varepsilon, \tag{10}$$

- Reduces the PDE to – $\tau = A\pi^2 t/L^2$, $y = \pi x/L$, $b = \varepsilon/A$ –

$$\frac{\partial u}{\partial \tau} = -2n\sin^{2n-1}(y)\cos y\frac{\partial u}{\partial y} + \left(1 - \sin^{2n} y + b\right)\frac{\partial^2 u}{\partial y^2} \tag{11}$$

Treating a simple case in $\mathbb{R}$

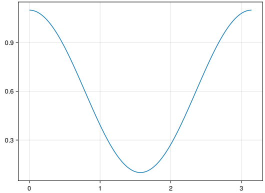

Figure: n=1

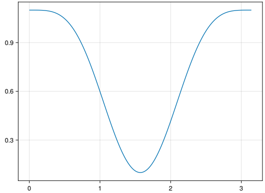

Figure: n=2

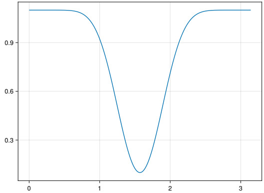

Figure: n=5

Numerical results in $\mathbb{R}$ (PDE)

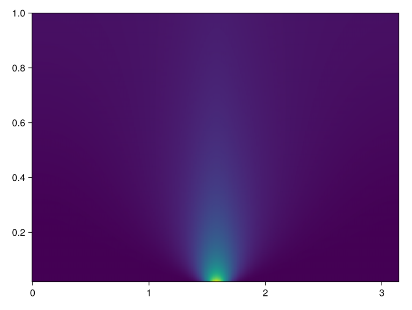

Figure: $n = 2,\ b = 0.1,$
$f(x) = \delta(x - \pi/2)$

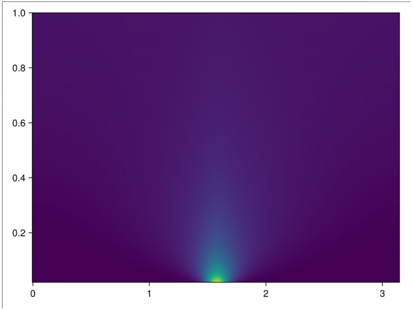

Figure: $n = 5,\ b = 0.1,$
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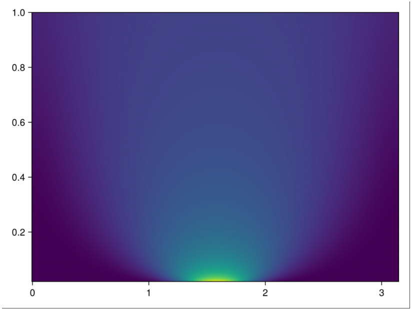

Figure: $n = 1$, $b = 0.1$ with $\sin \to \cos$

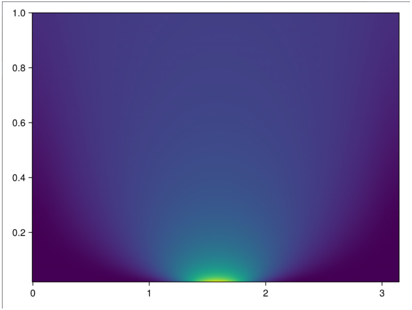

Figure: $n = 2$, $b = 0.1$ with $\sin \to \cos$

Numerical results in $\mathbb{R}$ (SDE)

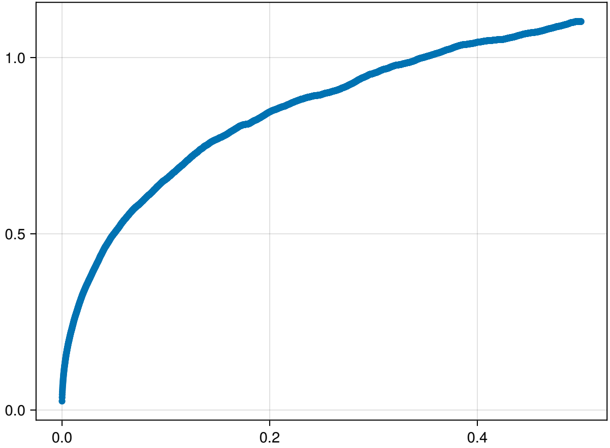

Figure: MSD with averaging performed over the entire trajectory; $n = 5$, $b = 0.1$

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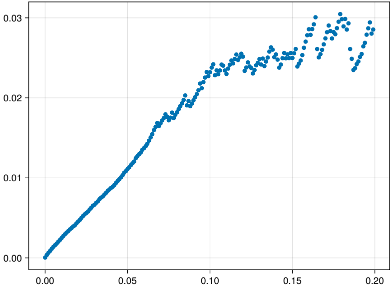
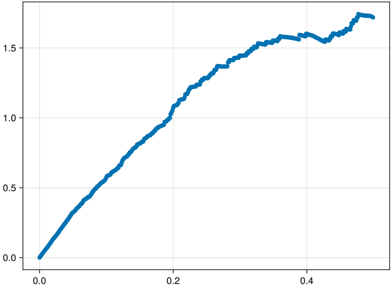


Figure: Slow phase MSD; $n = 5$, $b = 0.1$



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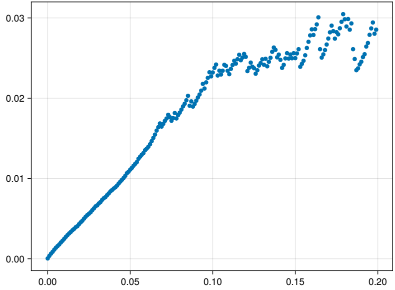

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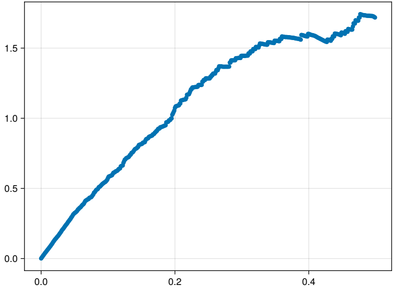

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- Phases distinguished by having diffusivities above and below $\kappa = b + 1/4$, respectively

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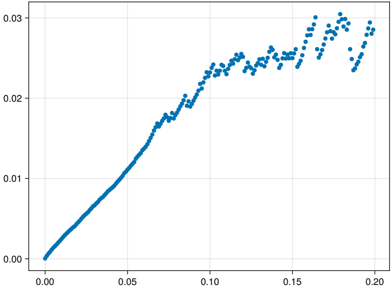

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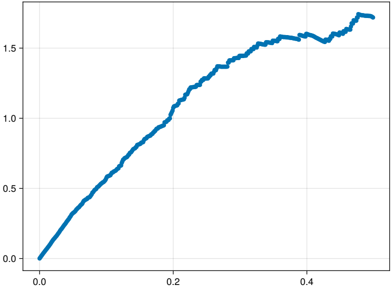

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- Observe anomalous diffusion in both domains on different timescales!

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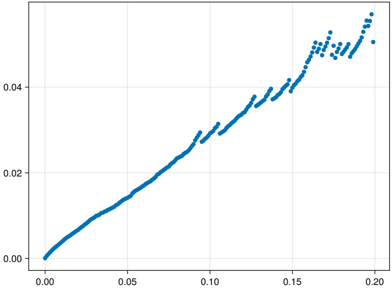

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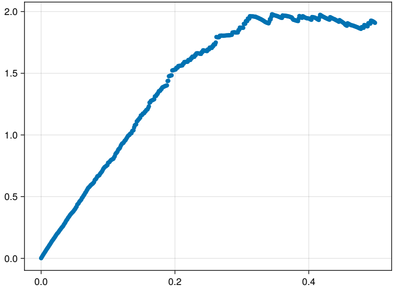

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References

[1] A. W. C. Lau and T. C. Lubensky, Phys. Rev. E **76**, 011123 (2007).

[2] E. Wong and M. Zakai, Ann. Math. Statist. **36**, 1560 (2005).