Local time subordinating Poisson processes to model semi-permeable membranes

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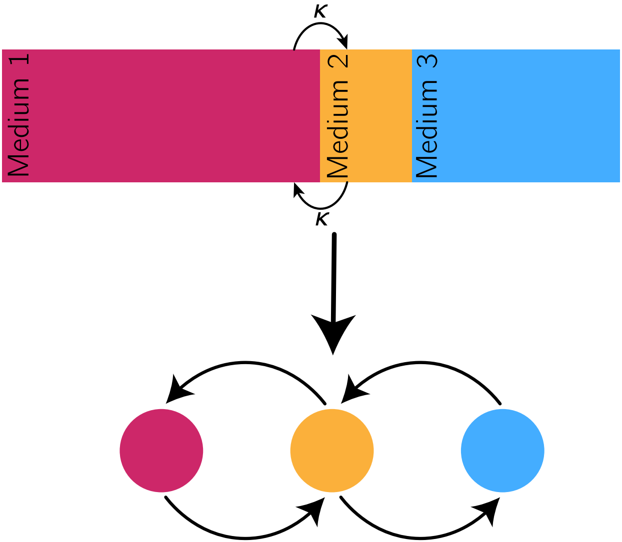
Coarse-graining



Local time

- Let X_t denote a stochastic process on a state space $S \subseteq \mathbb{R}^d$. For a point $y \in S$, we define the **local time** process as

$$L_t^y = \int_0^t \delta\left(X_s - y\right) \, d[X_s] \tag{1}$$

  - If X_t is a real-valued diffusion, $dX_t = \mu(X, t)\, dt + \sigma(X, t)\, dW_t$, then the quadratic variation is $[X_t] = \int_0^t \sigma^2(X_s, s)\, ds$ and so $d[X_t] = \sigma^2(X_t, t)\, dt$. Local time thus has units $[\text{length}]^{n+1}$ where $n < d$ is the dimension of the (local) manifold

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- Can reframe our problem as follows: A particle moves in a confining domain. When near the domain boundary, it is transported through to another domain, in a manner dependent upon the "number of attempts" to cross it (i.e., local time)[1]

Poisson point processes

- Choose Ω to be a domain with a smooth boundary $\partial\Omega$. The boundary local time is naturally defined as

$$L_t^{\partial S} = \int_{\partial\Omega} L_t^x \, d^{d-1}x \tag{2}$$

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- The crossing process of a semi-permeable membrane located at $\partial\Omega$ can be represented by a Poisson point process subordinated by the boundary local time $\ell_t := L_t^{\partial\Omega}$,[2]

$$\mathbb{P}(N(\ell) = n) = \frac{(q\ell)^n}{n!} e^{-q\ell}, \tag{3}$$

where $q = \kappa/D$ with κ the permeability of the membrane and D the (bulk) diffusivity[1]

The Skorokhod equation

- For a single domain $\Omega \subseteq \mathbb{R}^n$, whereby after reaction, the particle is absorbed by the boundary, the motion is described by the Skorokhod equation[3,1],

$$dX_t = \hat{n}(X)\, d\ell_t + \sqrt{2D}\, dW_t, \tag{4}$$

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- Naturally extended to include motion after surface reaction by inverting the direction of the boundary[2]

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  - A further extension is to promote the noise to be multiplicative with the boundary separating the media,

$$D(X) = D_1\mathbb{I}_{\Omega}(X) + D_2\mathbb{I}_{\mathbb{R}^d\setminus\Omega}(X)$$

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- Following KG 2025[2], we make use of

The forward Feynman-Kac equation

Let $L^* = \mu\partial + \frac{1}{2}\sigma\partial^2$ denote the (adjoint) diffusion generator and $dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t$ its corresponding diffusion process. Given the conditional expection of a test function ϕ can be written

$$\int_\Omega \phi(y)q(y, t \mid x_0)\, dy = \mathbb{E}_{x_0}\left[\exp\left(-\int_0^t V(X_s, s)ds\right)\phi(X_t)\right] \tag{6}$$

for a potential function V, then the (backwards-time) density q satisfies

$$\partial_t q(x, t \mid x_0) = (Lq)(x, t \mid x_0) - V(x, t)q(x, t \mid x_0) \tag{7}$$

with $q(x, t \mid x_0) = \delta(x - x_0)$ and appropriate boundary conditions on $\partial\Omega$.

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$$u(z, n, t \mid x_0) = \int_0^\infty \frac{(q\ell)^n}{n!} e^{-q\ell} p(z, \ell, t \mid x_0) \, d\ell \tag{8}$$

- Take the Laplace transform of the propagator $p(z, \ell, t \mid x_0)$,

$$\begin{aligned} P(z, \alpha, s \mid x_0) &= \int_0^\infty e^{-\alpha\ell} p(z, \ell, t \mid x_0) \, d\ell \\ &= \mathbb{E}_{x_0} \left[e^{-\alpha\ell_t} \delta(Z_t - z) \right] \end{aligned}$$

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- Identify $V = \alpha\delta(Z_t)$, amounting to a PDE (by FK)

$$\partial_t P(x, \alpha, t \mid x_0) = [L^* - \alpha\delta(x)]\, P(x, \alpha, t \mid x_0), \tag{9}$$

with the boundary condition $J(x, t) = 0$ for all $x \in \partial\Omega$

Stochastic to probabilistic

- Converting the PDE Eq. (9) back to in terms of the local time[2],

$$\partial_t p(x, \ell, t \mid x_0) = \left[L^* - \delta(x)\left(\partial_\ell - \delta(\ell^+)\right)\right] p(x, \ell, t \mid x_0) \quad (10)$$

- Compute the time derivative of the density u, Eq. (8),

$$\begin{aligned}
\partial_t u &= \int_0^\infty \frac{(q\ell)^n}{n!} e^{-q\ell} \partial_t p \, d\ell \\
&= \int_0^\infty \frac{(q\ell)^n}{n!} e^{-q\ell} \left[L^* - \delta(x)\left(\partial_\ell + \delta(\ell^+)\right)\right] p \, d\ell \\
&= L^* u - \delta(x) \int_0^\infty \frac{(q\ell)^n}{n!} e^{-q\ell} \partial_\ell p \, d\ell \\
&= L^* u + \delta(x) \int_0^\infty \frac{q e^{-q\ell}}{n!} \left[(q\ell)^{n-1} - (q\ell)^n\right] p \, d\ell
\end{aligned}$$

References

[1] D. S. Grebenkov, Phys. Rev. Lett. **125**, 078102 (2020).

[2] T. Kay and L. Giuggioli, Phys. Rev. Res. **7**, 013097 (2025).

[3] A. V. Skorokhod, Theory of Probability & Its Applications **6**, 264 (1961).

Extended Skorokhod equation I

- We treat the one dimensional case with a single barrier at the origin. Kay & Giuggioli (2025)[2] showed that the correction EoM is

$$X_t = (-1)^{N(L_t^0)} \left| x_0 + \sqrt{2D} W_t \right| \tag{11}$$

- Consider the reflected Brownian motion, $Z_t = |Y_t| = \left| x_0 + \sqrt{2D} W_t \right|$. By Tanaka's formula, this can be written

$$Z_t = \int_0^t \operatorname{sgn}\left(x_0 + \sqrt{2D} W_s \right) dY_s + L_t^0 \tag{12}$$

which can be written in differential form as

$$dZ_t = \sqrt{2D} d\tilde{W}_t + dL_t^0, \tag{13}$$

using that the stochastic integral is itself a Brownian motion and $dY_t = \sqrt{2D} dW_t$

Extended Skorokhod equation II

- We can now use the product rule to determine the differential form of X_t, with $\sigma_t = (-1)^{N(\ell_t)}$, so $dX_t = \sigma_t \, dZ_t + Z_t \, d\sigma_t$, but $d\sigma_t \propto dL_t^0$ is only non-zero when $Z_t = 0$ so the second term is identically zero
- Recognize that $\sigma_t d\tilde{W}_t$ remains white noise,

$$dX_t = \sigma_t \, dZ_t = \sqrt{2D} \, dW_t + (-1)^{N(L_t^0)} dL_t^0, \tag{14}$$

which we call the "extended" Skorokhod, equation, since it allows for motion on all of $\mathbb{R}$ instead of $\mathbb{R}^+$ as in the initial work of Skorokhod[3]

Feynman-Kac equation I

Let V be a function with sufficient regularity conditions, ϕ a test function and X_t a diffusion process with generator $L = \mu_i \partial^i + \frac{1}{2}\sigma^2 \partial^2$. We define the process

$$M_t = e^{-A_t}\phi(X_t), \qquad A_t = \int_0^t V(X_t)\, ds.$$

By the chain rule, we have

$$dM_t = \left[(L\phi)(X_t, t) - V(X_t, t)\phi(X_t)\right] e^{-A_t}\, dt + \partial_i \phi \sigma_i e^{-A_t}\, dW_t.$$

Taking the expectation value the noise process vanishes, leaving

$$\frac{d}{dt}\mathbb{E}_{x_0}[M_t] = \mathbb{E}_{x_0}\left[e^{-A_t}\left(L - V(X_t, t)\right)\phi(X_t)\right].$$

Feynman-Kac equation II

Since we define the FK-weighted, conditional backwards-time propagator q to act as

$$\int_\Omega q(y,t \mid x_0)\phi(y)\,dy = \mathbb{E}_{x_0}\left[e^{-A_t}\phi(X_t)\right].$$

Comparing with the time derivative of $\mathbb{E}_{x_0}[M_t]$, we write

$$\frac{d}{dt}\int_\Omega q(y,t \mid x_0)\phi(y)\,dy = \int_\Omega q(y,t \mid x_0)\left(L - V(y,t)\right)\phi(y)\,dy.$$

Formally, integration by parts leaves the regular (non-adjoint) generator L acting on q

Feynman-Kac equation III

For the sake of concreteness, in one spatial dimension,

$$\begin{aligned}\int_a^b q(L\phi)dy &= \int_a^b q(y,t \mid x_0)\left(\mu\frac{d\phi}{dy} + \frac{1}{2}\sigma^2\frac{d^2\phi}{dy^2}\right) dy \\ &= \int_a^b \phi L^* q\, dy + \left[J\phi + \frac{1}{2}q\sigma^2\phi'\right]_a^b,\end{aligned}$$

where J is the probability current $J(x,t) = \mu q - \frac{1}{2}(\sigma^2 q)'$. If we take a and b to be reflecting then we impose a Neumann-type BC $J(a,t) = J(b,t) = 0$ and likewise $\phi'(a) = \phi'(b) = 0$ (restrict space of test functions)