Assignment Sheet - I

Stylianos Sygletos
s.sygletos@aston.ac.uk

May 17, 2021

Problem 1 [2 points]

Two balls are placed in a box as follows: A fair coin is tossed and a white ball is placed in the box if a head occurs, otherwise a red ball is placed in the box. The coin is tossed again and a red ball is placed in the box if a tail occurs, otherwise a white ball is placed in the box. Balls are drawn from the box three times in succession (always with replacing the drawn ball back in the box). It is found that on all three occasions a red ball is drawn. What is the probability that both balls in the box are red?

Problem 2 [2 points]

Let the observations resulting from an experiment be x_n, $n = 1, 2, ..., N$. Assume that they are independent and that originate from a Gaussian PDF $\mathcal{N}(\mu, \sigma^2)$. Both the mean and the variance are unknown. Prove that the maximum likelihood (ML) estimates of these quantities are given by:

$$\hat{\mu}_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n, \qquad \hat{\sigma}^2_{ML} = \frac{1}{N}\sum_{n=1}^{N}(x_n - \hat{\mu}_{ML})^2 \tag{1}$$

Exercise 3 [2 points]

Using the below equations :

$$\mathrm{E}[x] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) x dx = \mu$$
$$\mathrm{E}[x^2] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) x^2 dx = \mu^2 + \sigma^2$$

show that

$$\mathrm{E}[x_n x_m] = \mu^2 + I_{nm}\sigma^2$$

where x_n and x_m denote data points sampled from a Gaussian distribution with mean μ and variance σ^2, and I_{nm} satisfies $I_{nm} = 1$ if $n = m$ and $I_{nm} = 0$ otherwise. Hence, prove the below results:

$$\mathrm{E}[\mu_{ML}] = \mu \quad \text{and} \quad \mathrm{E}[\sigma^2_{ML}] = \left(\frac{N-1}{N}\right)\sigma^2$$

Exercise 4 [2 points]

Obtain the maximum likelihood estimate (ML) of the parameter $\lambda > 0$ of the exponential distribution:

$$p(x) = \begin{cases} \lambda \exp(-\lambda x), & x \geqslant 0, \\ 0, & x < 0, \end{cases} \tag{2}$$

Exercise 5 [2 points]

Let $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{x} \in \mathcal{R}^2$ and $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ where ρ is the correlation coefficient. Show that the pdf is given by:

$$\begin{aligned} p(x_1, x_2) =& \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ & \exp\left[-\frac{1}{2(1-\rho^2)} \left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - 2\rho \frac{(x_1-\mu_1)}{\sigma_1} \frac{(x_2-\mu_2)}{\sigma_2} \right) \right] \end{aligned}$$

Exercise 6 [2 points]

Consider a bivariate Gaussian distribution $p(x_1, x_2) = \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \sigma_1\sigma_2 \begin{bmatrix} \frac{\sigma_1}{\sigma_2} & \rho \\ \rho & \frac{\sigma_2}{\sigma_1} \end{bmatrix} \tag{3}$$

where the correlation coefficient is given by

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$$

i. What is the $P(x_2|x_1)$? Simplify your answer by expressing it in terms of $\rho, \sigma_2, \sigma_1, \mu_1, \mu_2$ and x_1 [1pt]

ii. Assume $\sigma_1 = \sigma_2 = 1$. What is $P(x_2|x_1)$ now? [1pt]

Exercise 7 [3 points]

1. Derive the mean and the variance of the binomial and the uniform distribution [2pt]
2. Derive the mean and covariance matrix of the multivariate Gaussian [1pt].