Infinite Powers

How Calculus Reveals the Secrets of the Universe

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possible to calculate whatever we want about a curved shape by adding up all the straight little pieces. Figuring out exactly how to do this—no easy feat—took the efforts of the world's greatest mathematicians over many centuries. Collectively, however, and sometimes through bitter rivalries, they eventually began to make headway on the riddle of curves. Spinoffs today, as we'll see in chapter 2, include the math needed to draw realistic-looking hair, clothing, and faces of characters in computer-animated movies and the calculations required for doctors to perform facial surgery on a virtual patient before they operate on the real one.

The quest to solve the mystery of curves reached a fever pitch when it became clear that curves were much more than geometric diversions. They were a key to unlocking the secrets of nature. They arose naturally in the parabolic arc of a ball in flight, in the elliptical orbit of Mars as it moved around the sun, and in the convex shape of a lens that could bend and focus light where it was needed, as was required for the burgeoning development of microscopes and telescopes in late Renaissance Europe.

And so began the second great obsession: a fascination with the mysteries of motion on Earth and in the solar system. Through observation and ingenious experiments, scientists discovered tantalizing numerical patterns in the simplest moving things. They measured the swinging of a pendulum, clocked the accelerating descent of a ball rolling down a ramp, and charted the stately procession of planets across the sky. The patterns they found enraptured them—indeed, Johannes Kepler fell into a state of self-described "sacred frenzy" when he found his laws of planetary motion—because those patterns seemed to be signs of God's handiwork. From a more secular perspective, the patterns reinforced the claim that nature was deeply mathematical, just as the Pythagoreans had maintained. The only catch was that nobody could explain the marvelous new patterns, at least not with the existing forms of math. Arithmetic and geometry were not up to the task, even in the hands of the greatest mathematicians.

The trouble was that the motions weren't steady. A ball rolling down a ramp kept changing its speed, and a planet revolving around the sun kept changing its direction of travel. Worse yet, the planets moved faster when they got close to the sun and slowed down as they receded from it. There was no known way to deal with motion that kept changing in ever-changing ways. Earlier mathematicians had worked out the mathematics of the most trivial kind of motion, namely, motion at a constant speed where distance equals rate times time. But when speed changed and kept on changing continuously, all bets were off. Motion was proving to be as much of a conceptual Mount Everest as curves were.

As we'll see in the middle chapters of this book, the next great advances in calculus grew out of the quest to solve the mystery of motion. The Infinity Principle came to the rescue, just as it had for curves. This time the act of wishful fantasy was to pretend that motion at a changing speed was made up of infinitely many, infinitesimally brief motions at a *constant* speed. To visualize what this would mean, imagine being in a car with a jerky driver at the wheel. As you anxiously watch the speedometer, it moves up and down with every jerk. But over a millisecond, even the jerkiest driver can't make the speedometer needle move by much. And over an interval much shorter than that—an infinitesimal time interval—the needle won't move at all. Nobody can tap the gas pedal that fast.

These ideas coalesced in the younger half of calculus, differential calculus. It was precisely what was needed to work with the infinitesimally small changes of time and distance that arose in the study of ever-changing motion as well as with the infinitesimal straight pieces of curves that arose in analytic geometry, the newfangled study of curves defined by algebraic equations that was all the rage in the first half of the 1600s. Yes, at one time, algebra was a craze, as we'll see. Its popularity was a boon for all fields of mathematics, including geometry, but it also created an unruly jungle of new curves to explore. Thus, the mysteries of curves and motion collided. They were now both at the center stage of calculus in the mid-1600s, banging into each other, creating mathematical mayhem and confusion. Out of the tumult, differential calculus began to flower, but not without controversy. Some mathematicians were criticized for playing fast and loose with infinity. Others derided algebra as a scab of symbols. With all the bickering, progress was fitful and slow.

And then a child was born on Christmas Day. This young messiah of calculus was an unlikely hero. Born premature and fatherless and abandoned by his mother at age three, he was a lonesome boy with dark thoughts who grew into a secretive, suspicious young man. Yet Isaac Newton would make a mark on the world like no one before or since.

First, he solved the holy grail of calculus: he discovered how to put the pieces of a curve back together again—and how to do it easily, quickly, and systematically. By combining the symbols of algebra with the power of infinity, he found a way to represent any curve as a sum of infinitely many simpler curves described by powers of a variable x, like x^2, x^3, x^4, and so on. With these ingredients alone, he could cook up any curve he wanted by putting in a pinch of x and a dash of x^2 and a heaping tablespoon of x^3. It was like a master recipe and a universal spice rack, butcher shop, and vegetable garden, all rolled into one. With it he could solve any problem about shapes or motions that had ever been considered.

Then he cracked the code of the universe. Newton discovered that motion of any kind always unfolds one infinitesimal step at a time, steered from moment to moment by mathematical laws written in the language of calculus. With just a handful of differential equations (his laws of motion and gravity), he could explain everything from the arc of a cannonball to the orbits of the planets. His astonishing "system of the world" unified heaven and earth, launched the Enlightenment, and changed Western culture. Its impact on the philosophers and poets of Europe was immense. He even influenced Thomas Jefferson and the writing of the Declaration of Independence, as we'll see. In our own time, Newton's ideas underpinned the space program by providing the mathematics necessary for trajectory design, the work done at NASA by African-American mathematician Katherine Johnson and her colleagues (the heroines of the book and hit movie *Hidden Figures*).

With the mysteries of curves and motion now settled, calculus moved on to its third lifelong obsession: the mystery of change. It's a cliché, but it's true all the same—nothing is constant but change. It's rainy one day and sunny the next. The stock market rises and falls. Emboldened by the Newtonian paradigm, the later practitioners of calculus asked: Are there laws of change similar to Newton's laws of motion? Are there laws for population growth, the spread of epidemics, and the flow of blood in an artery? Can calculus be used to describe how electrical signals propagate along nerves or to predict the flow of traffic on a highway?

By pursuing this ambitious agenda, always in cooperation with other parts of science and technology, calculus has helped make the world modern. Using observation and experiment, scientists worked out the laws of change and then used calculus to solve them and make predictions. For example, in 1917 Albert Einstein applied calculus to a simple model of atomic transitions to predict a remarkable effect called stimulated emission (which is what the *s* and *e* stand for in *laser*, an acronym for *light amplification by stimulated emission of radiation*). He theorized that under certain circumstances, light passing through matter could stimulate the production of more light at the same wavelength and moving in the same direction, creating a cascade of light through a kind of chain reaction that would result in an intense, coherent beam. A few decades later, the prediction proved to be accurate. The first working lasers were built in the early 1960s. Since then, they have been used in everything from compact-disc players and laser-guided weaponry to supermarket bar-code scanners and medical lasers.

The laws of change in medicine are not as well understood as those in physics. Yet even when applied to rudimentary models, calculus has been able to make lifesaving contributions. For example, in chapter 8 we'll see how a differential-equation model developed by an immunologist and an AIDS researcher played a part in shaping the modern three-drug combination therapy for patients infected with HIV. The insights provided by the model overturned the prevailing view that the virus was lying dormant in the body; in fact, it was in a raging battle with the immune system every minute of every day. With the new understanding that calculus helped provide, HIV infection has been transformed from a near-certain death sentence to a manageable chronic disease—at least for those with access to combination-drug therapy.

Admittedly, some aspects of our ever-changing world lie beyond the approximations and wishful thinking inherent in the Infinity Principle. In the subatomic realm, for example, physicists can no longer think of an electron as a classical particle following a smooth path in the same way that a planet or a cannonball does. According to quantum mechanics, trajectories become jittery, blurry, and poorly defined at the microscopic scale, so we need to describe the behavior of electrons as probability waves instead of Newtonian trajectories. As soon as we do that, however, calculus returns triumphantly. It governs the evolution of probability waves through something called the Schrödinger equation.

It's incredible but true: Even in the subatomic realm where Newtonian physics breaks down, Newtonian calculus still works. In fact, it works spectacularly well. As we'll see in the pages ahead, it has teamed up with quantum mechanics to predict the remarkable effects that underlie medical imaging, from MRI and CT scans to the more exotic positron emission tomography.

It's time for us to take a closer look at the language of the universe. Naturally, the place to start is at infinity.

1

Infinity

THE BEGINNINGS OF mathematics were grounded in everyday concerns. Shepherds needed to keep track of their flocks. Farmers needed to weigh the grain reaped in the harvest. Tax collectors had to decide how many cows or chickens each peasant owed the king. Out of such practical demands came the invention of numbers. At first they were tallied on fingers and toes. Later they were scratched on animal bones. As their representation evolved from scratches to symbols, numbers facilitated everything from taxation and trade to accounting and census taking. We see evidence of all this in Mesopotamian clay tablets written more than five thousand years ago: row after row of entries recorded with the wedge-shaped symbols called cuneiform.

Along with numbers, shapes mattered too. In ancient Egypt, the measurement of lines and angles was of paramount importance. Each year surveyors had to redraw the boundaries of farmers' fields after the summer flooding of the Nile washed the borderlines away. That activity later gave its name to the study of shape in general: *geometry*, from the Greek *gē*, "earth," and *metrēs*, "measurer."

At the start, geometry was hard-edged and sharp-cornered. Its predilection for straight lines, planes, and angles reflected its utilitarian origins—triangles were useful as ramps, pyramids as monuments and tombs, and rectangles as tabletops, altars, and plots of land. Builders and carpenters used right angles for plumb lines. For sailors, architects, and priests, knowledge of straight-line geometry was essential for surveying, navigating, keeping the calendar, predicting eclipses, and erecting temples and shrines.

Yet even when geometry was fixated on straightness, one curve always stood out, the most perfect of all: the circle. We see circles in tree rings, in the ripples on a pond, in the shape of the sun and the moon. Circles surround us in nature. And as we gaze at circles, they gaze back at us, literally. There they are in the eyes of our loved ones, in the circular outlines of their pupils and irises. Circles span the practical and the emotional, as wheels and wedding rings, and they are mystical too. Their eternal return suggests the cycle of the seasons, reincarnation, eternal life, and never-ending love. No wonder circles have commanded attention for as long as humanity has studied shapes.

Mathematically, circles embody change without change. A point moving around the circumference of a circle changes direction without ever changing its distance from a center. It's a minimal form of change, a way to change and curve in the slightest way possible. And, of course, circles are symmetrical. If you rotate a circle about its center, it looks unchanged. That rotational symmetry may be why circles are so ubiquitous. Whenever some aspect of nature doesn't care about direction, circles are bound to appear. Consider what happens when a raindrop hits a puddle: tiny ripples expand outward from the point of impact. Because they spread equally fast in all directions and because they started at a single point, the ripples *have* to be circles. Symmetry demands it.

Circles can also give birth to other curved shapes. If we imagine skewering a circle on its diameter and spinning it around that axis in three-dimensional space, the rotating circle makes a sphere, the shape of a globe or a ball. When a circle is moved vertically into the third dimension along a straight line at right angles to its plane, it makes a cylinder, the shape of a can or a hatbox. If it shrinks at the same time as it's moving vertically, it makes a cone; if it expands as it moves vertically, it makes a truncated cone (the shape of a lampshade).