Trajectory Generation with Tension constraints

August 22, 2013

r	derivative to minimize in cost function
	implies r initial conditions at each keyframe (position and $r-1$ derivatives, indexed from 0 (constant term)
n	order of desired trajectory, minimum order is $2r-1$
	implies $n+1$ coefficients, indexed from 0 (constant term)
d	number of dimensions to optimize, indexed from 1
	for example, $d=3$ when optimizing $[xyz]$ position of a trajectory,
m	number of pieces in trajectory
	implies $m+1$ keyframes in trajectory, indexed from 1
t_{des}	vertical vector of desired arrival times at keyframes, indexed from 0
p_{des}	matrix of desired positions, each row represents a derivative, each column represents a keyframe
	Inf represents unconstrained

1 Spline Interpolation

To analytically solve for the piecewise cubic spline $X(t)$ of m pieces going through positions $p_{des} = [X(t_0) \;\; X(t_1) \;\; X(t_2) \;\; ... \;\; X(t_m)]^T$:

$$X(t) = \begin{cases} X_1(t) = c_{1,3}t^3 + c_{1,2}t^2 + c_{1,1}t + c_{1,0}, & t_0 \le t < t_1 \\ X_2(t) = c_{2,3}t^3 + c_{2,2}t^2 + c_{2,1}t + ... + c_{2,0}, & t_1 \le t < t_2 \\ ... \\ X_m(t) = c_{m,3}t^3 + c_{m,2}t^2 + c_{m,1}t + ... + c_{m,0}, & t_{m-1} \le t < t_m \end{cases}$$

We can set up a system of $4m$ equations to solve for each of the $4m$ coefficients.

$2m$ Position Constraints:

$$\begin{gathered} X_1(t_0) = X(t_0) \\ X_1(t_1) = X(t_1) \\ X_2(t_1) = X(t_1) \\ X_2(t_2) = X(t_2) \\ ... \\ X_m(t_{m-1}) = X(t_{m-1}) \\ X_m(t_m) = X(t_m) \end{gathered}$$

$m-1$ Velocity Constraints:

$$\begin{gathered} \dot{X}_1(t_1) = \dot{X}_2(t_1) \\ \dot{X}_2(t_2) = \dot{X}_3(t_2) \\ ... \\ \dot{X}_{m-1}(t_{m-1}) = \dot{X}_m(t_{m-1}) \end{gathered}$$

$m-1$ Acceleration Constraints:

$$\begin{gathered} \ddot{X}_1(t_1) = \ddot{X}_2(t_1) \\ \ddot{X}_2(t_2) = \ddot{X}_3(t_2) \\ ... \\ \ddot{X}_{m-1}(t_{m-1}) = \ddot{X}_m(t_{m-1}) \end{gathered}$$

2 Endpoint Constraints (for example, velocity):

$$\begin{gathered} \dot{X}_1(t_0) = \dot{X}(t_0) \\ \dot{X}_m(t_m) = \dot{X}(t_m) \end{gathered}$$

The resulting $X(t)$ corresponds to the solution to the optimization problem of finding $X = [c_{1,3} \;\; c_{1,2} \;\; c_{1,1} \;\; c_{1,0} \;\; ... \;\; c_{m,0} \;\;]^T$ that minimizes the cost functional $J = \int_{t_0}^{t_1} \| \frac{d^2X(t)}{dt} \|^2 dt$ subject to $3m+1$ equality constraints $Ax = b$, where the equality constraints come from position constraints, endpoint constraints, and velocity continuity constraints.

In the general case, the minimum-order of the piece-wise polynomial to minimize the cost functional $J = \int_{t_0}^{t_1} \| \frac{d^{(r)}X(t)}{dt} \|^2 dt$ is $n = 2r - 1$. To analytically solve for the coefficients $X = [c_{1,n} \;\; c_{1,n-1} \;\; c_{1,n-2} \;\; ... \;\; c_{1,0} \;\;]^T$, we need $(n+1)m$ constraints. These constraints come from:

$2m$ Position Constraints
$(m-1)(r-1)$ Constraints for continuity of derivatives 1 to $r-1$
$2(r-1)$ Endpoint Constraints, for derivatives 1 to $r-1$

This gives a total of $2m+(m-1)(r-1)+2(r-1) = 2m+(m-1)(\frac{n+1}{2}-1)+2(\frac{n+1}{2}-1) = \frac{mn}{2}+\frac{m}{2}+m+\frac{n}{2}-\frac{1}{2}$ constraints. We thus need $((n+1)m) - \left(\frac{mn}{2}+\frac{m}{2}+m+\frac{n}{2}-\frac{1}{2}\right) = (\frac{n-1}{2})(m-1) = (r-1)(m-1)$ more constraints. This corresponds to constraining derivatives at intermediate points to be continuous up until the $2(r-1)$ derivative.

To solve for the minimum-order piecewise polynomial that of minimizes the cost functional of the rth derivative, we solve for coefficients using the constraints:

$2m$ Position Constraints
$2(m-1)(r-1)$ Constraints for continuity of derivatives 1 to $2(r-1)$
$2(r-1)$ Endpoint Constraints, for derivatives 1 to $r-1$