Quadrotor with a Cable-Suspended Load
July 18, 2013

$m_Q, m_L \in \mathbb{R}$	Mass of quadrotor, load
$J_Q \in \mathbb{R}$	Inertia of quadrotor
$f \in \mathbb{R}$	Magnitude of thrust for quadrotor
$M \in \mathbb{R}$	Magnitude of moment for quadrotor in body frame
$l \in \mathbb{R}$	Length of suspension cable
$T \in \mathbb{R}$	Magnitude of tension in cable
$\mathbf{x}_Q, \mathbf{x}_L \in \mathbb{R}^2$	Position vector of center of mass of quadrotor, load in inertial frame
$\mathbf{v}_Q, \mathbf{v}_L \in \mathbb{R}^2$	Velocity vector of center of mass of quadrotor, load in inertial frame
$\mathbf{p} \in S^1$	Unit vector from quadrotor to load, $\mathbf{p} = [\sin(\phi_L) \quad -\cos(\phi_L)]^T$
$\mathbf{R} \in SO(2)$	Rotation matrix of quadrotor from body to inertial frame, $\mathbf{R} = [\cos(\phi_Q) \quad -\sin(\phi_Q); \sin(\phi_Q) \quad \cos(\phi_Q)]$
$\mathbf{e}_2, \mathbf{e}_3$	Axes of the inertial frame, $\mathbf{e}_2 = [1 \quad 0]^T$, $\mathbf{e}_3 = [0 \quad 1]^T$
$\mathbf{b}_2, \mathbf{b}_3$	Axes of quadrotor body frame,$\mathbf{b}_2 = \mathbf{R}\mathbf{e}_2$, $\mathbf{b}_3 = \mathbf{R}\mathbf{e}_3$
$\phi_Q \in (-\pi, \pi]$	Angle of quadrotor counter-clockwise from horizontal
$\phi_L \in (-\pi, \pi]$	Angle of load counter-clockwise from vertical
$\dot{\phi}_Q, \dot{\phi}_L \in \mathbb{R}$	Angular velocity of the quadrotor, load

1 Equations of Motion

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1) + g_1(\mathbf{x}_1)\mathbf{u}_1, & \mathbf{x}_1 \notin \mathcal{S}_1 \\ \mathbf{x}_2^+ = \Delta_1(\mathbf{x}_1^-), & \mathbf{x}_1^- \in \mathcal{S}_1 \\ \dot{\mathbf{x}}_2 = f_2(\mathbf{x}_2) + g_2(\mathbf{x}_2)\mathbf{u}_2, & \mathbf{x}_2 \notin \mathcal{S}_2 \\ \mathbf{x}_1^+ = \Delta_2(\mathbf{x}_2^-), & \mathbf{x}_2^- \in \mathcal{S}_2 \end{cases}$$

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1 = \begin{bmatrix} \dot{\mathbf{x}}_L \\ \dot{\mathbf{v}}_L \\ \dot{\phi}_L \\ \ddot{\phi}_L \\ \dot{\phi}_Q \\ \ddot{\phi}_Q \end{bmatrix} = \begin{bmatrix} \mathbf{v}_L \\ \frac{-m_Q l \dot{\phi}_L^2}{(m_Q+m_L)}\mathbf{p} - g\mathbf{e}_3 \\ \dot{\phi}_L \\ 0 \\ \dot{\phi}_Q \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{\cos(\phi_Q-\phi_L)}{(m_Q+m_L)}\mathbf{p} & 0 \\ 0 & 0 \\ \frac{\sin(\phi_Q-\phi_L)}{m_Q l} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_Q} \end{bmatrix} \begin{bmatrix} f \\ M \end{bmatrix}, & \mathbf{x}_1 \notin \{\mathbf{x}_1 \mid T \equiv \|m_L(\dot{\mathbf{v}}_L + g\mathbf{e}_3)\| = 0\} \\ \mathbf{x}_2^+ = \begin{bmatrix} \mathbf{x}_L^+ \\ \mathbf{v}_L^+ \\ \mathbf{x}_Q^+ \\ \mathbf{v}_Q^+ \\ \phi_Q^+ \\ \dot{\phi}_Q^+ \end{bmatrix} = \begin{bmatrix} \mathbf{x}_L^- \\ \mathbf{v}_L^- \\ \mathbf{x}_L^- - l\mathbf{p}^- \\ \mathbf{v}_L^- - l\dot{\mathbf{p}}^- \\ \phi_Q^- \\ \dot{\phi}_Q^- \end{bmatrix}, & \mathbf{x}_1^- \in \{\mathbf{x}_1 \mid T = 0\} \\ \dot{\mathbf{x}}_2 = \begin{bmatrix} \dot{\mathbf{x}}_L \\ \dot{\mathbf{v}}_L \\ \dot{\mathbf{x}}_Q \\ \dot{\mathbf{v}}_Q \\ \dot{\phi}_Q \\ \ddot{\phi}_Q \end{bmatrix} = \begin{bmatrix} \mathbf{v}_L \\ -g\mathbf{e}_3 \\ \mathbf{v}_Q \\ -g\mathbf{e}_3 \\ \dot{\phi}_Q \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{m_Q}\mathbf{b}_3 & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_Q} \end{bmatrix} \begin{bmatrix} f \\ M \end{bmatrix}, & \mathbf{x}_2 \notin \{\mathbf{x}_2 \mid \|\mathbf{x}_Q - \mathbf{x}_L\| = L\} \\ \mathbf{x}_1^+ = \begin{bmatrix} \mathbf{x}_L^+ \\ \mathbf{v}_L^+ \\ \phi_L^+ \\ \dot{\phi}_L^+ \\ \phi_Q^+ \\ \dot{\phi}_Q^+ \end{bmatrix} = \begin{bmatrix} \mathbf{x}_L^- \\ \frac{m_L\mathbf{v}_L^- + m_Q\mathbf{v}_Q^-}{(m_L+m_Q)} \\ \cos^{-1}(-\frac{\mathbf{x}_L^- - \mathbf{x}_Q^-}{l} \cdot \mathbf{e}_3) \\ 0 \\ \phi_Q^- \\ \dot{\phi}_Q^- \end{bmatrix}, & \mathbf{x}_2^- \in \{\mathbf{x}_2 \mid \|\mathbf{x}_Q - \mathbf{x}_L\| = L\} \end{cases}$$

1.1 When cable is taut

Let $\mathbf{e}_2 = [1 \;\; 0]^T$ and $\mathbf{e}_3 = [0 \;\; 1]^T$ be unit vectors in the plane; let $\mathbf{e}_1$ be a unit vector out of the plane
Constraint: $\mathbf{x}_Q = \mathbf{x}_L - l\mathbf{p}$
Applied forces: $\mathbf{f}_1 = f\mathbf{b}_3$ at $\mathbf{r}_1 = \mathbf{x}_Q = \mathbf{x}_L - l\mathbf{p}$, $\mathbf{f}_2 = M\mathbf{e}_1$ at $\mathbf{r}_2 = \phi_Q\mathbf{e}_1$

$$\begin{aligned}
\text{Coordinates: } \mathbf{q} &= [\mathbf{x}_L \;\; \phi_L \;\; \phi_Q]^T \\
\text{Forces: } Q_j &= \sum_{i=0}^{n} \mathbf{f}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\
Q_{\mathbf{x}_L} &= f\mathbf{b}_3 \cdot \frac{\partial(\mathbf{x}_L - l\mathbf{p})}{\partial \mathbf{x}_L} + M\mathbf{e}_1 \cdot \frac{\partial \phi_Q \mathbf{e}_1}{\partial \mathbf{x}_L} \\
&= f\mathbf{b}_3 \\
Q_{\phi_L} &= f\mathbf{b}_3 \cdot \frac{\partial(\mathbf{x}_L - l\mathbf{p})}{\partial \phi_L} + M\mathbf{e}_1 \cdot \frac{\partial \phi_Q \mathbf{e}_1}{\partial \phi_L} \\
&= -f\mathbf{b}_3 \cdot L\left[\begin{smallmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{smallmatrix}\right] \\
&= Lf\sin(\phi_Q - \phi_L) \\
Q_{\phi_Q} &= f\mathbf{b}_3 \cdot \frac{\partial(\mathbf{x}_L - l\mathbf{p})}{\partial \phi_Q} + M\mathbf{e}_1 \cdot \frac{\partial \phi_Q \mathbf{e}_1}{\partial \phi_Q} \\
&= M
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} &= \mathcal{T} - \mathcal{U} \\
\mathcal{T} &= \frac{1}{2}m_Q\mathbf{v}_Q \cdot \mathbf{v}_Q + \frac{1}{2}m_L\mathbf{v}_L \cdot \mathbf{v}_L + \frac{1}{2}J_Q\dot{\phi}_Q^2 \\
&= \frac{1}{2}m_Q(\mathbf{v}_L - l\dot{\mathbf{p}}) \cdot (\mathbf{v}_L - l\dot{\mathbf{p}}) + \frac{1}{2}m_L\mathbf{v}_L \cdot \mathbf{v}_L + \frac{1}{2}J_Q\dot{\phi}_Q^2 \\
&= \frac{1}{2}(m_Q + m_L)(\mathbf{v}_L \cdot \mathbf{v}_L) - m_Q(\mathbf{v}_L \cdot l\dot{\mathbf{p}}) + \frac{1}{2}m_Q l^2(\dot{\mathbf{p}} \cdot \dot{\mathbf{p}}) + \frac{1}{2}J_Q\dot{\phi}_Q^2 \\
&= \frac{1}{2}(m_Q + m_L)(\mathbf{v}_L \cdot \mathbf{v}_L) - m_Q(\mathbf{v}_L \cdot l\dot{\phi}_L\left[\begin{smallmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{smallmatrix}\right]) + \frac{1}{2}m_Q l^2\dot{\phi}_L^2 + \frac{1}{2}J_Q\dot{\phi}_Q^2 \\
\mathcal{U} &= m_Q g\mathbf{e}_3 \cdot \mathbf{x}_Q + m_L g\mathbf{e}_3 \cdot \mathbf{x}_L \\
&= m_Q g\mathbf{e}_3 \cdot (\mathbf{x}_L - l\mathbf{p}) + m_L g\mathbf{e}_3 \cdot \mathbf{x}_L \\
&= (m_Q + m_L)g\mathbf{e}_3 \cdot \mathbf{x}_L + m_Q lg\mathbf{e}_3 \cdot \mathbf{p} \\
&= (m_Q + m_L)g\mathbf{e}_3 \cdot \mathbf{x}_L - m_Q lg\cos(\phi_L) \\
\mathcal{L} &= \frac{1}{2}(m_Q + m_L)(\mathbf{v}_L \cdot \mathbf{v}_L) - m_Q(\mathbf{v}_L \cdot l\dot{\phi}_L\left[\begin{smallmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{smallmatrix}\right]) + \frac{1}{2}m_Q l^2\dot{\phi}_L^2 + \frac{1}{2}J_Q\dot{\phi}_Q^2 \\
&\quad - (m_Q + m_L)g\mathbf{e}_3 \cdot \mathbf{x}_L + m_Q lg\cos(\phi_L)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{v}_L} - \frac{\partial \mathcal{L}}{\partial \mathbf{x}_L} &= f\mathbf{b}_3 \\
\frac{d}{dt}\left((m_Q + m_L)\mathbf{v}_L - m_Q l\dot{\phi}_L\left[\begin{smallmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{smallmatrix}\right]\right) - (m_Q + m_L)g\mathbf{e}_3 &= f\mathbf{b}_3 \\
(m_Q + m_L)\dot{\mathbf{v}}_L - m_Q l\ddot{\phi}_L\left[\begin{smallmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{smallmatrix}\right] + m_Q l\dot{\phi}_L^{\,2}\left[\begin{smallmatrix} \sin\phi_L \\ -\cos(\phi_L) \end{smallmatrix}\right] - (m_Q + m_L)g\mathbf{e}_3 &= f\mathbf{b}_3
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_L} - \frac{\partial \mathcal{L}}{\partial \phi_L} &= Lf\sin(\phi_Q - \phi_L) \\
\frac{d}{dt}\left(-m_Q l \mathbf{v}_L \cdot \left[\begin{smallmatrix}\cos(\phi_L)\\ \sin(\phi_L)\end{smallmatrix}\right] + m_Q l^2 \dot{\phi}_L\right) - \left(m_Q l \dot{\phi}_L \mathbf{v}_L \cdot \left[\begin{smallmatrix}\sin\phi_L\\ -\cos(\phi_L)\end{smallmatrix}\right] + m_Q l g \sin(\phi_L)\right) &= Lf\sin(\phi_Q - \phi_L) \\
-m_Q l \dot{\mathbf{v}}_L \cdot \left[\begin{smallmatrix}\cos(\phi_L)\\ \sin(\phi_L)\end{smallmatrix}\right] + m_Q l \dot{\phi}_L \mathbf{v}_L \cdot \left[\begin{smallmatrix}\sin\phi_L\\ -\cos(\phi_L)\end{smallmatrix}\right] + m_Q l^2 \ddot{\phi}_L & \\
-m_Q l \dot{\phi}_L \mathbf{v}_L \cdot \left[\begin{smallmatrix}\sin\phi_L\\ -\cos(\phi_L)\end{smallmatrix}\right] - m_Q l g \sin(\phi_L) &= Lf\sin(\phi_Q - \phi_L) \\
-m_Q l \dot{\mathbf{v}}_L \cdot \left[\begin{smallmatrix}\cos(\phi_L)\\ \sin(\phi_L)\end{smallmatrix}\right] + m_Q l^2 \ddot{\phi}_L - m_Q l g \sin(\phi_L) &= Lf\sin(\phi_Q - \phi_L)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_Q} - \frac{\partial \mathcal{L}}{\partial \phi_Q} &= M \\
\frac{d}{dt}\left(J_Q \dot{\phi}_Q\right) &= M \\
J_Q \ddot{\phi}_Q &= M
\end{aligned}
$$

Decoupling equations:

$$
\begin{aligned}
(m_Q + m_L)(\dot{\mathbf{v}}_L + g\mathbf{e}_3) &= (-f\cos(\phi_Q - \phi_L) - m_Q l \dot{\phi}_L^2)\mathbf{p} \\
m_Q l \ddot{\phi}_L &= f\sin(\phi_Q - \phi_L) \\
J_Q \ddot{\phi}_Q &= M
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= [\mathbf{x}_L \quad \mathbf{v}_L \quad \phi_L \quad \dot{\phi}_L \quad \phi_Q \quad \dot{\phi}_Q]^T \\
\dot{\mathbf{x}}_1 = \begin{bmatrix}\dot{\mathbf{x}}_L\\ \dot{\mathbf{v}}_L\\ \dot{\phi}_L\\ \ddot{\phi}_L\\ \dot{\phi}_Q\\ \ddot{\phi}_Q\end{bmatrix} &= \begin{bmatrix}\mathbf{v}_L\\ \frac{(f\cos(\phi_Q-\phi_L) - m_Q l \dot{\phi}_L^2)}{(m_Q+m_L)}\mathbf{p} - g\mathbf{e}_3\\ \dot{\phi}_L\\ \frac{f\sin(\phi_Q-\phi_L)}{m_Q l}\\ \dot{\phi}_Q\\ \frac{M}{J_Q}\end{bmatrix}
\end{aligned}
$$

Note the tension force in the cable can be explicitly calculated by taking Newton's Law on the body of the load:

$$
\begin{aligned}
\mathbf{F} &= m_L \dot{\mathbf{v}}_L \\
-T\mathbf{p} - m_L g \mathbf{e}_3 &= m_L \dot{\mathbf{v}}_L \\
-T\mathbf{p} &= m_L \dot{\mathbf{v}}_L + m_L g \mathbf{e}_3 \\
\|T\| &= \|m_L \dot{\mathbf{v}}_L + m_L g \mathbf{e}_3\|
\end{aligned}
$$

1.2 When cable is slack

Let $\mathbf{e}_2 = [1 \;\; 0]^T$ and $\mathbf{e}_3 = [0 \;\; 1]^T$ be unit vectors in the plane; let $\mathbf{e}_1$ be a unit vector out of the plane
Applied forces: $\mathbf{f}_1 = f\mathbf{b}_3$ at $\mathbf{r}_1 = \mathbf{x}_Q$, $\mathbf{f}_2 = M\mathbf{e}_1$ at $\mathbf{r}_2 = \phi_Q \mathbf{e}_1$

$$\begin{aligned}
\text{Coordinates: } \mathbf{q} &= [\mathbf{x}_L \;\; \mathbf{x}_Q \;\; \phi_Q]^T \\
\text{Forces: } Q_j &= \sum_{i=0}^{n} \mathbf{f}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\
Q_{\mathbf{x}_L} &= f\mathbf{b}_3 \cdot \frac{\partial \mathbf{x}_Q}{\partial \mathbf{x}_L} + M\mathbf{e}_1 \cdot \frac{\partial \phi_Q \mathbf{e}_1}{\partial \mathbf{x}_L} \\
&= 0 \\
Q_{\mathbf{x}_Q} &= f\mathbf{b}_3 \cdot \frac{\partial \mathbf{x}_Q}{\partial \mathbf{x}_Q} + M\mathbf{e}_1 \cdot \frac{\partial \phi_Q \mathbf{e}_1}{\partial \mathbf{x}_Q} \\
&= f\mathbf{b}_3 \\
Q_{\phi_Q} &= f\mathbf{b}_3 \cdot \frac{\partial (\mathbf{x}_L - l\mathbf{p})}{\partial \phi_Q} + M\mathbf{e}_1 \cdot \frac{\partial \phi_Q \mathbf{e}_1}{\partial \phi_Q} \\
&= M
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} &= \mathcal{T} - \mathcal{U} \\
\mathcal{T} &= \frac{1}{2} m_Q \mathbf{v}_Q \cdot \mathbf{v}_Q + \frac{1}{2} m_L \mathbf{v}_L \cdot \mathbf{v}_L + \frac{1}{2} J_Q \dot{\phi}_Q^2 \\
\mathcal{U} &= m_Q g \mathbf{e}_3 \cdot \mathbf{x}_Q + m_L g \mathbf{e}_3 \cdot \mathbf{x}_L \\
\mathcal{L} &= \frac{1}{2} m_Q \mathbf{v}_Q \cdot \mathbf{v}_Q + \frac{1}{2} m_L \mathbf{v}_L \cdot \mathbf{v}_L + \frac{1}{2} J_Q \dot{\phi}_Q^2 - m_Q g \mathbf{e}_3 \cdot \mathbf{x}_Q - m_L g \mathbf{e}_3 \cdot \mathbf{x}_L
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_L} - \frac{\partial \mathcal{L}}{\partial \mathbf{x}_L} &= 0 \\
\frac{d}{dt} (m_L \mathbf{v}_L) + m_L g \mathbf{e}_3 &= 0 \\
m_L \dot{\mathbf{v}}_L + m_L g \mathbf{e}_3 &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_Q} - \frac{\partial \mathcal{L}}{\partial \mathbf{x}_Q} &= f\mathbf{b}_3 \\
\frac{d}{dt} (m_Q \mathbf{v}_Q) + m_Q g \mathbf{e}_3 &= f\mathbf{b}_3 \\
m_Q \dot{\mathbf{v}}_Q + m_Q g \mathbf{e}_3 &= f\mathbf{b}_3
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_Q} - \frac{\partial \mathcal{L}}{\partial \phi_Q} &= M \\
\frac{d}{dt} \left(J_Q \dot{\phi}_Q \right) &= M \\
J_Q \ddot{\phi}_Q &= M
\end{aligned}$$

$$\mathbf{x}_2 = [\mathbf{x}_L \quad \mathbf{v}_L \quad \mathbf{x}_Q \quad \mathbf{v}_Q \quad \phi_Q \quad \dot{\phi}_Q]^T$$

$$\dot{\mathbf{x}}_2 = \begin{bmatrix} \dot{\mathbf{x}}_L \\ \dot{\mathbf{v}}_L \\ \dot{\mathbf{x}}_Q \\ \dot{\mathbf{v}}_Q \\ \dot{\phi}_Q \\ \ddot{\phi}_Q \end{bmatrix} = \begin{bmatrix} \mathbf{v}_L \\ -g\mathbf{e}_3 \\ \mathbf{v}_Q \\ \frac{f}{m_Q}\mathbf{b}_3 - g\mathbf{e}_3 \\ \dot{\phi}_Q \\ \frac{M}{J_Q} \end{bmatrix}$$

2 Differential Flatness

2.1 $\mathbf{x}_1$ system:

2.1.1 Differential Flatness

Recall the equations of motion:

$$
\begin{aligned}
(m_Q + m_L)(\dot{\mathbf{v}}_L + g\mathbf{e}_3) &= (-f\cos(\phi_Q - \phi_L) - m_Q l \dot{\phi}_L^2)\mathbf{p} \\
m_Q l \ddot{\phi}_L &= f\sin(\phi_Q - \phi_L) \\
J_Q \ddot{\phi}_Q &= M
\end{aligned}
$$

Note the tension force in the cable can be explicitly calculated by taking Newton's Law on the body of the load:

$$
\begin{aligned}
\mathbf{F} &= m_L \dot{\mathbf{v}}_L \\
-T\mathbf{p} - m_L g \mathbf{e}_3 &= m_L \dot{\mathbf{v}}_L \\
-T\mathbf{p} &= m_L \dot{\mathbf{v}}_L + m_L g \mathbf{e}_3 \\
\|T\| &= \|m_L \dot{\mathbf{v}}_L + m_L g \mathbf{e}_3\|
\end{aligned}
$$

Choose flat outputs $\mathbf{y} = [\mathbf{x}_L]^T = [y_L \quad z_L]^T$

Derive $\dot{y}_L = v_{yL}$, $\dot{z}_L = v_{zL}$, and all higher derivatives from differentiation of y_L, z_L

From equation of motion:

$$
\begin{aligned}
T\mathbf{p} &= -(m_L \ddot{\mathbf{x}}_L + m_L g \mathbf{e}_3) \\
T &= \|m_L \ddot{\mathbf{x}}_L + m_L g \mathbf{e}_3\| \\
&= m_L(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{\frac{1}{2}} \\
\mathbf{p} &= \frac{-(m_L \ddot{\mathbf{x}}_L + m_L g \mathbf{e}_3)}{T} \\
&= -\frac{m_L}{T}\begin{bmatrix} \ddot{y}_L \\ \ddot{z}_L + g \end{bmatrix} = \begin{bmatrix} \sin(\phi_L) \\ -\cos(\phi_L) \end{bmatrix} \\
\phi_L &= \tan^{-1}\left(\frac{\mathbf{p}\cdot\mathbf{e}_2}{-\mathbf{p}\cdot\mathbf{e}_3}\right)
\end{aligned}
$$

Differentiating this equation of motion:

$$
\begin{aligned}
T\dot{\mathbf{p}} + \dot{T}\mathbf{p} &= -m_L \dddot{\mathbf{x}}_{\mathbf{L}} \\
\dot{\mathbf{p}} &= -\frac{1}{T}(m_L \dddot{\mathbf{x}}_{\mathbf{L}} + \dot{T}\mathbf{p})
\end{aligned}
$$

Where:

$$
\dot{T} = m_L \frac{\ddot{y}_L \dddot{y}_L + (\ddot{z}_L + g)\dddot{z}_L}{(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{\frac{1}{2}}}
$$

From $\dot{\mathbf{p}}$, we can find the state $\dot{\phi}_L$:

$$
\begin{aligned}
\dot{\mathbf{p}} &= \dot{\phi}_L \begin{bmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{bmatrix} \\
\dot{\phi}_L &= -\frac{1}{T}(m_L \dddot{\mathbf{x}}_{\mathbf{L}} + \dot{T}\mathbf{p}) \cdot \begin{bmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{bmatrix} \\
&= -\frac{m_L}{T}(\dddot{y}_L \cos(\phi_L) + \dddot{z}_L \sin(\phi_L)) \\
&= -\frac{m_L}{T}(\frac{m_L \dddot{y}_L(\ddot{z}_L + g)}{T} - \frac{m_L \dddot{z}_L \ddot{y}_L}{T}) \\
&= -\frac{m_L^2}{T^2}(\dddot{y}_L(\ddot{z}_L + g) - \dddot{z}_L \ddot{y}_L) \\
&= \frac{\dddot{z}_L \ddot{y}_L - \ddot{y}_L(\ddot{z}_L + g)}{\dddot{y}_L^2 + (\ddot{z}_L + g)^2}
\end{aligned}
$$

Differentiating again to find higher derivatives of $\mathbf{p}$ and ϕ_L:

$$
\begin{aligned}
2\dot{T}\dot{\mathbf{p}} + T\ddot{\mathbf{p}} + \ddot{T}\mathbf{p} &= -m_L \mathbf{x}_L^{(4)} \\
\ddot{\mathbf{p}} &= -\frac{1}{T}(m_L \mathbf{x}^{(\mathbf{4})}{}_L + \ddot{T}\mathbf{p} + 2\dot{T}\dot{\mathbf{p}})
\end{aligned}
$$

Where:

$$
\begin{aligned}
\ddot{T} &= m_L \left(\frac{(\dddot{y}_L^2 + \ddot{y}_L y_L^{(4)} + z_L^{(4)}(\ddot{z}_L + g) + \dddot{z}_L^2)}{(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{\frac{1}{2}}} - \frac{(\ddot{y}_L \dddot{y}_L + (\ddot{z}_L + g)\dddot{z}_L)^2}{(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{\frac{3}{2}}} \right) \\
&= m_L((\dddot{y}_L^2 + \ddot{y}_L y_L^{(4)} + z_L^{(4)}(\ddot{z}_L + g) + \dddot{z}_L^2)(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{1}{2}} \\
&- (\ddot{y}_L \dddot{y}_L + (\ddot{z}_L + g)\dddot{z}_L)^2(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{3}{2}})
\end{aligned}
$$

From $\ddot{\mathbf{p}}$, we can find the state $\ddot{\phi}_L$:

$$
\begin{aligned}
\ddot{\mathbf{p}} &= \ddot{\phi}_L \begin{bmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{bmatrix} + \dot{\phi}_L^2 \begin{bmatrix} -\sin(\phi_L) \\ \cos(\phi_L) \end{bmatrix} \\
\ddot{\phi}_L &= \ddot{\mathbf{p}} \cdot \begin{bmatrix} \cos(\phi_L) \\ \sin(\phi_L) \end{bmatrix}
\end{aligned}
$$

$$3\ddot{T}\dot{\mathbf{p}} + 3\dot{T}\ddot{\mathbf{p}} + T\dddot{\mathbf{p}} + \dddot{T}\mathbf{p} = -m_L\mathbf{x}_L^{(5)}$$

$$\dddot{\mathbf{p}} = -\frac{1}{T}\left(m_L\mathbf{x}_L^{(5)} + 3\ddot{T}\dot{\mathbf{p}} + 3\dot{T}\ddot{\mathbf{p}} + \dddot{T}\mathbf{p}\right)$$

Where:

$$\begin{aligned}\dddot{T} = m_L(&\\ &(3\dddot{y}_L y_L^{(4)} + \ddot{y}_L y_L^{(5)} + (\ddot{z}_L + g)z_L^{(5)} + 3\dddot{z}_L z_L^{(4)})(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{1}{2}}\\ &+ 3(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{1}{2}}(\ddot{y}_L\dddot{y}_L + (\ddot{z}_L + g)\dddot{z}_L)(\dddot{y}_L^2 + \ddot{y}_L y_L^{(4)} + (\ddot{z}_L + g)z_L^{(4)} + \dddot{z}_L^2)\\ &+ 3(\ddot{y}_L\dddot{y}_L + (\ddot{z}_L + g)\dddot{z}_L)^3(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{5}{2}})\end{aligned}$$

From $\dddot{\mathbf{p}}$, we can find the state $\dddot{\phi}_L$:

$$\dddot{\mathbf{p}} = (\dddot{\phi}_L - \dot{\phi}_L^3)\begin{bmatrix}\cos(\phi_L)\\ \sin(\phi_L)\end{bmatrix} + 3\dot{\phi}_L\ddot{\phi}_L\begin{bmatrix}-\sin(\phi_L)\\ \cos(\phi_L)\end{bmatrix}$$

$$\dddot{\phi}_L = \dddot{\mathbf{p}} \cdot \begin{bmatrix}\cos(\phi_L)\\ \sin(\phi_L)\end{bmatrix} + \dot{\phi}_L^3$$

$$4\dddot{T}\dot{\mathbf{p}} + 6\ddot{T}\ddot{\mathbf{p}} + 4\dot{T}\dddot{\mathbf{p}} + T\mathbf{p}^{(4)} + T^{(4)}\mathbf{p} = -m_L\mathbf{x}_L^{(6)}$$

$$\mathbf{p}^{(4)} = -\frac{1}{T}(m_L\mathbf{x}_L^{(6)} + 4\dddot{T}\dot{\mathbf{p}} + 6\ddot{T}\ddot{\mathbf{p}} + 4\dot{T}\dddot{\mathbf{p}} + T^{(4)}\mathbf{p})$$

$$\begin{aligned}T^{(4)} = m_L(&\\ &-\frac{15}{8}((\ddot{z}_L + g)\dddot{z}_L + \ddot{y}_L\dddot{y}_L)^4(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{7}{2}}\\ &+ 9(\ddot{y}_L\dddot{y}_L + (\ddot{z}_L + g)\dddot{z}_L)^2((\ddot{z}_L + g)z_L^{(4)} + \ddot{y}_L y_L^{(4)} + \dddot{y}_L^2 + \dddot{z}_L^2)(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{5}{2}}\\ &- 3((\ddot{z}_L + g)z_L^{(4)} + \ddot{y}_L y_L^{(4)} + \dddot{y}_L^2 + \dddot{z}_L^2)^2(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{3}{2}}\\ &- 4((\ddot{z}_L + g)\dddot{z}_L + \ddot{y}_L\dddot{y}_L)(z_L^{(5)}(\ddot{z} + g) + \ddot{y}_L y_L^{(5)} + 3y_L^{(4)}\dddot{y}_L + 3\dddot{z}_L z_L^{(4)})(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{3}{2}}\\ &+ (z_L^{(6)}(\ddot{z}_L + g) + \ddot{y}_L y_L^{(6)} + 3{y_L^{(4)}}^2 + 4y_L^{(5)}\dddot{y}_L + 3{z_L^{(4)}}^2 + 4z_L^{(5)}\dddot{z}_L)(\ddot{y}_L^2 + (\ddot{z}_L + g)^2)^{-\frac{1}{2}})\end{aligned}$$

From $\mathbf{p}^{(4)}$, we can find $\phi_L^{(4)}$:

$$\mathbf{p}^{(4)} = (\phi_L^{(4)} - 6\dot{\phi}_L^2\ddot{\phi}_L)\begin{bmatrix}\cos(\phi_L)\\ \sin(\phi_L)\end{bmatrix} + (3\ddot{\phi}_L^2 - \dot{\phi}_L^4)\begin{bmatrix}-\sin(\phi_L)\\ \cos(\phi_L)\end{bmatrix}$$

$$\phi_L^{(4)} = \mathbf{p}^{(4)} \cdot \begin{bmatrix}\cos(\phi_L)\\ \sin(\phi_L)\end{bmatrix} + 6\dot{\phi}_L^2\ddot{\phi}_L$$

Using Newton's Equations on the quadrotor and the load to eliminate $T\mathbf{p}$ and solve for $f\mathbf{b}_3$:

$$\begin{aligned}
m_Q\ddot{\mathbf{x}}_Q &= f\mathbf{b}_3 - m_Q g\mathbf{e}_3 + T\mathbf{p} \\
m_L\ddot{\mathbf{x}}_L &= -T\mathbf{p} - m_L g\mathbf{e}_3 \\
f\mathbf{b}_3 &= m_Q\ddot{\mathbf{x}}_Q + m_L\ddot{\mathbf{x}}_L + m_Q g\mathbf{e}_3 + m_L g\mathbf{e}_3
\end{aligned}$$

Using the constraint $\ddot{\mathbf{x}}_Q = \ddot{\mathbf{x}}_L - l\ddot{\mathbf{p}}$:

$$\begin{aligned}
f\mathbf{b}_3 &= m_Q(\ddot{\mathbf{x}}_L - l\ddot{\mathbf{p}}) + m_L\ddot{\mathbf{x}}_L + m_Q g\mathbf{e}_3 + m_L g\mathbf{e}_3 \\
&= (m_Q + m_L)(\ddot{\mathbf{x}}_L + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b}_3 &= \frac{(m_Q + m_L)(\ddot{\mathbf{x}}_L + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}}{\|(m_Q + m_L)(\ddot{\mathbf{x}}_L + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}\|} = \begin{bmatrix} -\sin(\phi_Q) \\ \cos(\phi_Q) \end{bmatrix} \\
f &= ((m_Q + m_L)(\ddot{\mathbf{x}}_L + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}) \cdot \mathbf{b}_3 \\
\phi_Q &= \tan^{-1}\left(\frac{f\mathbf{b}_3 \cdot \mathbf{e}_2}{f\mathbf{b}_3 \cdot \mathbf{e}_3}\right) \\
&= \tan^{-1}\left(\frac{-(m_Q + m_L)\ddot{y}_L + m_Q l\ddot{\mathbf{p}} \cdot \mathbf{e}_2}{(m_Q + m_L)(\ddot{z}_L + g) - m_Q l\ddot{\mathbf{p}} \cdot \mathbf{e}_3}\right) \\
\dot{\phi}_Q &= ((lm_Q\dddot{p}_1 - (m_Q + m_L)\dddot{y}_L)((m_Q + m_L)(\ddot{z} + g) - lm_Q\ddot{p}_2) + (lm_Q\dddot{p}_2 - (m_Q + m_L)\dddot{z}_L)(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)) \\
&\quad \left((lm_Q\ddot{p}_2 - (m_Q + m_L)(\ddot{z}_L + g))^2(\frac{(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)^2}{(lm_Q\ddot{p}_2 + (m_Q + m_L)(\ddot{z}_L + g))^2} + 1)\right)^{-1}, \text{ where } \mathbf{p}^{(k)} = \begin{bmatrix} p_1^{(k)} \\ p_2^{(k)} \end{bmatrix} \\
\ddot{\phi} &= [(\frac{lm_Q p_1^{(4)} - (m_Q + m_L)y_L^{(4)}}{(m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2} - \frac{2(lm_Q\dddot{p}_1 - (m_Q + m_L)\dddot{y}_3)((m_Q + m_L)\dddot{z}_L - lm_Q\dddot{p}_2)}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^2} \\
&\quad - \frac{((m_Q + m_L)z_L^{(4)} - lm_Q p_2^{(4)})(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^2} + \frac{2((m_Q + m_L)\dddot{z}_L - lm_Q\dddot{p}_1)^2(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^3}) \\
&\quad (\frac{(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)^2}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^2} + 1)^{-1}] \\
&\quad - [(\frac{2(lm_Q\dddot{p}_1 - (m_Q + m_L)\dddot{y}_L)(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^2} - \frac{2((m_Q + m_L)\dddot{z}_L - lm_Q\dddot{p}_2)(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)^2}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^3}) \\
&\quad (\frac{lm_Q\dddot{p}_1 - (m_Q + m_L)\dddot{y}_L}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^2} - \frac{((m_Q + m_L)\dddot{z}_L - lm_Q\dddot{p}_2)(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^2}) \\
&\quad (\frac{(lm_Q\ddot{p}_1 - (m_Q + m_L)\ddot{y}_L)^2}{((m_Q + m_L)(\ddot{z}_L + g) - lm_Q\ddot{p}_2)^2} + 1)^{-2}]
\end{aligned}$$

2.1.2 Control Laws

Using the control law from the paper:
Load Position Control

$$
\begin{aligned}
f &= -k_p(e_{\mathbf{x}}) - k_d(\dot{e}_{\mathbf{x}}) + m_L\ddot{\mathbf{x}}_L^d + m_L g\mathbf{e}_3 + m_Q\ddot{\mathbf{x}}_Q^d + m_Q g\mathbf{e}_3 \\
&= -k_p(\mathbf{x}_L - \mathbf{x}_T) - k_d(\dot{\mathbf{x}}_L - \dot{\mathbf{x}}_T) + m_L\ddot{\mathbf{x}}_T + m_L g\mathbf{e}_3 + m_Q\ddot{\mathbf{x}}_Q^d + m_Q g\mathbf{e}_3 \\
&= -k_p(\mathbf{x}_L - \mathbf{x}_T) - k_d(\dot{\mathbf{x}}_L - \dot{\mathbf{x}}_T) + m_L\ddot{\mathbf{x}}_T + m_L g\mathbf{e}_3 + m_Q(\ddot{\mathbf{x}}_T - l\ddot{\mathbf{p}}) + m_Q g\mathbf{e}_3 \\
\mathbf{p}^d &= -\frac{(-k_p(\mathbf{x}_L - \mathbf{x}_T) - k_d(\dot{\mathbf{x}}_L - \dot{\mathbf{x}}_T) + m_L\ddot{\mathbf{x}}_T + m_L g\mathbf{e}_3)}{\| - k_p(\mathbf{x}_L - \mathbf{x}_T) - k_d(\dot{\mathbf{x}}_L - \dot{\mathbf{x}}_T) + m_L\ddot{\mathbf{x}}_T + m_L g\mathbf{e}_3\|} \\
\phi_L^d &= \tan^{-1}\left(-\frac{\mathbf{p}^d \cdot \mathbf{e}_2}{\mathbf{p}^d \cdot \mathbf{e}_3}\right)
\end{aligned}
$$

Load Attitude Control

$$
\begin{aligned}
T_{nom}\mathbf{p}_{nom} &= -(m_L\ddot{\mathbf{x}}_T + m_L g\mathbf{e}_3) \\
\mathbf{p}_{nom} &= \frac{-(m_L\ddot{\mathbf{x}}_T + m_L g\mathbf{e}_3)}{\| - (m_L\ddot{\mathbf{x}}_T + m_L g\mathbf{e}_3)\|} \\
T_{nom} &= T_{nom} \cdot \mathbf{p}_{nom} \\
\phi_{L_{nom}} &= \tan^{-1}\left(\frac{T_{nom}\mathbf{p}_{nom} \cdot \mathbf{e}_2}{-T_{nom}\mathbf{p}_{nom} \cdot \mathbf{e}_3}\right) \\
\dot{\mathbf{p}}_{nom} &= -\frac{1}{T_{nom}}(m_L\dddot{\mathbf{x}}_{\mathbf{T}} + \dot{T}_{nom}\mathbf{p}_{nom}), \dot{T}_{nom} = m_L\frac{\ddot{y}_T\dddot{y}_T + (\ddot{z}_T + g)\dddot{z}_T}{(\ddot{y}_T^2 + (\ddot{z}_T + g)^2)^{\frac{1}{2}}} \\
\dot{\phi}_L^d &= \dot{\mathbf{p}}_{nom} \cdot \begin{bmatrix} \cos(\phi_{L_{nom}}) \\ \sin(\phi_{L_{nom}}) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\mathbf{p}}_{nom} &= -\frac{1}{T_{nom}}(m_L\mathbf{x}^{(\mathbf{4})}{}_T + \ddot{T}_{nom}\mathbf{p}_{nom} + 2\dot{T}_{nom}\dot{\mathbf{p}}_{nom}), \\
\ddot{T}_{nom} &= m_L\left(\frac{(\dddot{y}_T^2 + \ddot{y}_T y_T^{(4)} + z_T^{(4)}(\ddot{z}_T + g) + \dddot{z}_T^2)}{(\ddot{y}_T^2 + (\ddot{z}_T + g)^2)^{\frac{1}{2}}} - \frac{(\ddot{y}_T\dddot{y}_T + (\ddot{z}_T + g)\dddot{z}_T)^2}{(\ddot{y}_T^2 + (\ddot{z}_T + g)^2)^{\frac{3}{2}}}\right) \\
\ddot{\phi}_L^d &= \ddot{\mathbf{p}}_{nom} \cdot \begin{bmatrix} \cos(\phi_{L_{nom}}) \\ \sin(\phi_{L_{nom}}) \end{bmatrix} \\
f_{nom}\mathbf{b}_{3_{nom}} &= (m_Q + m_L)(\ddot{\mathbf{x}}_T + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}_{nom} \\
\mathbf{b}_{3_{nom}} &= \frac{(m_Q + m_L)(\ddot{\mathbf{x}}_T + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}_{nom}}{\|m_Q + m_L)(\ddot{\mathbf{x}}_T + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}_{nom}\|} \\
\phi_{Q_{nom}} &= \tan^{-1}\left(\frac{-f_{nom}\mathbf{b}_{3_{nom}} \cdot \mathbf{e}_2}{f_{nom}\mathbf{b}_{3_{nom}} \cdot \mathbf{e}_3}\right) \\
f_{nom} &= f_{nom}\mathbf{b}_{3_{nom}} \cdot \mathbf{b}_{3_{nom}} \\
\phi_Q^d &= \phi_{L_{nom}} + \sin^{-1}\left(-k_p^L e_L - k_d^L\dot{e}_L + \frac{\ddot{\phi}_L^d m_Q l}{f_{nom}}\right) \\
&= \phi_{L_{nom}} + \sin^{-1}\left(-k_p^L(\phi_L - \phi_L^d) - k_d^L(\dot{\phi}_L - \dot{\phi}_L^d) + \frac{\ddot{\phi}_L^d m_Q l}{f_{nom}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\dddot{\mathbf{p}}_{nom} &= -\frac{1}{T_{nom}}\left(m_L\mathbf{x}_T^{(5)} + 3\ddot{T}_{nom}\dot{\mathbf{p}}_{nom} + 3\dot{T}_{nom}\ddot{\mathbf{p}}_{nom} + \dddot{T}_{nom}\mathbf{p}_{nom}\right),\\
&\dddot{T}_{nom} = m_L(\\
&(3\dddot{y}_T y_T^{(4)} + \ddot{y}_T y_T^{(5)} + (\ddot{z}_T+g)z_T^{(5)} + 3\dddot{z}_T z_T^{(4)})(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{1}{2}}\\
&+3(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{1}{2}}(\ddot{y}_T\dddot{y}_T + (\ddot{z}_T+g)\dddot{z}_T)(\dddot{y}_T^2 + \ddot{y}_T y_T^{(4)} + (\ddot{z}_T+g)z_T^{(4)} + \dddot{z}_T^2)\\
&+3(\ddot{y}_T\dddot{y}_T + (\ddot{z}_T+g)\dddot{z}_T)^3(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{5}{2}})\\
\dddot{\phi}_{L_{nom}} &= \dddot{\mathbf{p}}_{nom}\cdot\begin{bmatrix}\cos(\phi_{L_{nom}})\\ \sin(\phi_{L_{nom}})\end{bmatrix} + \dot{\phi}_{L_{nom}}^3\\
\mathbf{p}_{nom}^{(4)} &= -\frac{1}{T_{nom}}(m_L\mathbf{x}_T^{(6)} + 4\dddot{T}_{nom}\dot{\mathbf{p}}_{nom} + 6\ddot{T}_{nom}\ddot{\mathbf{p}}_{nom} + 4\dot{T}_{nom}\dddot{\mathbf{p}}_{nom} + T^{(4)}\mathbf{p}_{nom})\\
T_{nom}^{(4)} &= m_L(\\
&-\frac{15}{8}((\ddot{z}_T+g)\dddot{z}_T + \ddot{y}_T\dddot{y}_T)^4(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{7}{2}}\\
&+9(\ddot{y}_T\dddot{y}_T + (\ddot{z}_T+g)\dddot{z}_T)^2((\ddot{z}_T+g)z_T^{(4)} + \ddot{y}_T y_T^{(4)} + \dddot{y}_T^2 + \dddot{z}_T^2)(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{5}{2}}\\
&-3((\ddot{z}_T+g)z_T^{(4)} + \ddot{y}_T y_T^{(4)} + \dddot{y}_T^2 + \dddot{z}_T^2)^2(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{3}{2}}\\
&-4((\ddot{z}_T+g)\dddot{z}_T + \ddot{y}_T\dddot{y}_T)(z_T^{(5)}(\ddot{z}_T+g) + \ddot{y}_T y_T^{(5)} + 3y_T^{(4)}\dddot{y}_T + 3\dddot{z}_T z_T^{(4)})(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{3}{2}}\\
&+(z_T^{(6)}(\ddot{z}_T+g) + \ddot{y}_T y_T^{(6)} + 3{y_T^{(4)}}^2 + 4y_T^{(5)}\dddot{y}_T + 3{z_T^{(4)}}^2 + 4z_T^{(5)}\dddot{z}_T)(\ddot{y}_T^2 + (\ddot{z}_T+g)^2)^{-\frac{1}{2}})\\
\phi_{L_{nom}}^{(4)} &= \mathbf{p}_{nom}^{(4)}\cdot\begin{bmatrix}\cos(\phi_{L_{nom}})\\ \sin(\phi_{L_{nom}})\end{bmatrix} + 3\dot{\phi}_{L_{nom}}^2\ddot{\phi}_{L_{nom}} + \ddot{\phi}_{L_{nom}}\dot{\phi}_{L_{nom}}^2 + 2\dot{\phi}_{L_{nom}}^2\\
\dot{f}_{nom} &= ((m_Q+m_L)\dddot{\mathbf{x}}_T - m_Q l\dddot{\mathbf{p}}_{nom})\cdot\mathbf{b}_{3_{nom}} + \dot{\phi}_{L_{nom}}((m_Q+m_L)(\ddot{\mathbf{x}}_T + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}_{nom})\cdot\mathbf{b}_{2_{nom}}\\
\ddot{f}_{nom} &= ((m_Q+m_L)\mathbf{x}_T^{(4)} - m_Q l\mathbf{p}_{\mathbf{nom}}^{(\mathbf{4})} + \dot{\phi}_{L_{nom}}^2((m_Q+m_L)(\ddot{\mathbf{x}}_T + g\mathbf{b}_{3_{nom}}) - m_Q l\ddot{\mathbf{p}}_{nom}))\cdot\mathbf{b}_{3_{nom}}\\
&+\ddot{\phi}_{L_{nom}}((m_Q+m_L)(\ddot{\mathbf{x}}_T + g\mathbf{e}_3) - m_Q l\ddot{\mathbf{p}}_{nom})\cdot\mathbf{b}_{2_{nom}}\\
\dot{\phi}_Q^d &= \frac{m_Q l\dddot{\phi}_{L_{nom}} - \dot{f}_{nom}\sin(\phi_{Q_{nom}} - \phi_{L_{nom}})}{f_{nom}\cos(\phi_{Q_{nom}} - \phi_{L_{nom}})} + \dot{\phi}_{L_{nom}}\\
\ddot{\phi}_Q^d &= \frac{m_Q l\phi_{L_{nom}}^{(4)} - \ddot{f}_{nom}\sin(\phi_{Q_{nom}} - \phi_{L_{nom}}) - 2\dot{f}_{nom}\cos(\phi_{Q_{nom}} - \phi_{L_{nom}})(\dot{\phi}_{Q_{nom}} - \dot{\phi}_{L_{nom}})}{\cos(\phi_{Q_{nom}} - \phi_{L_{nom}})}\\
&+\frac{\dot{f}_{nom}\sin(\phi_{Q_{nom}} - \phi_{L_{nom}})(\dot{\phi}_{Q_{nom}} - \dot{\phi}_{L_{nom}})^2}{\cos(\phi_{Q_{nom}} - \phi_{L_{nom}})}\\
&+\ddot{\phi}_{L_{nom}}\\
M &= J_Q(-k_p^Q e_Q - k_d^Q\dot{e}_Q + \ddot{\phi}_Q^d)\\
&= J_Q(-k_p^Q(\phi_Q - \phi_Q^d) - k_d^Q(\dot{\phi}_Q - \dot{\phi}_Q^d) + \ddot{\phi}_Q^d)
\end{aligned}
$$

2.2 $\mathbf{x}_2$ system:

$$\begin{aligned} m_L\dot{\mathbf{v}}_L + m_L g\mathbf{e}_3 &= 0 \\ m_Q\dot{\mathbf{v}}_Q + m_Q g\mathbf{e}_3 &= f\mathbf{b}_3 \\ J_Q\ddot{\phi}_Q &= M \end{aligned}$$

2.2.1 Differential Flatness

Flat outputs $\mathbf{y} = [\mathbf{x}_Q]^T = [y_Q \;\; z_Q]^T$

$\mathbf{x}_L$ and $\mathbf{v}_L$ are known from initial conditions because load is in free fall:

$$\begin{aligned} \dot{\mathbf{x}}_L &= \mathbf{v}_L \\ \dot{\mathbf{v}}_L &= -g\mathbf{e}_3 \end{aligned}$$

Derive $\dot{y}_Q = v_{yQ}$, $\dot{z}_Q = v_{zQ}$, and all higher derivatives from differentiation of y_Q, z_Q

From equation of motion:

$$\begin{aligned} f\mathbf{b}_3 &= m_Q\ddot{\mathbf{x}}_Q + m_Q g\mathbf{e}_3 \\ f &= \|m_Q\ddot{\mathbf{x}}_Q + m_Q g\mathbf{e}_3\| \\ &= m_Q\left(\ddot{y}_Q^2 + (\ddot{z}_Q + g)^2\right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{b}_3 = \begin{bmatrix} -\sin(\phi_Q) \\ \cos(\phi_Q) \end{bmatrix} &= \frac{m_Q\ddot{\mathbf{x}}_Q + m_Q g\mathbf{e}_3}{\|m_Q\ddot{\mathbf{x}}_Q + m_Q g\mathbf{e}_3\|} \\ \sin(\phi_Q) &= -\frac{m_Q}{f}\ddot{y}_Q \\ \cos(\phi_Q) &= \frac{m_Q}{f}(\ddot{z}_Q + g) \\ \phi_Q &= \tan^{-1}\left(\frac{-f\mathbf{b}_3\cdot\mathbf{e}_2}{f\mathbf{b}_3\cdot\mathbf{e}_3}\right) \\ &= \tan^{-1}\left(\frac{-\ddot{y}_Q}{\ddot{z}_Q + g}\right) \end{aligned}$$

Differentiating the equation of motion:

$$\begin{aligned} m_Q\dddot{\mathbf{x}}_Q &= \dot{f}\mathbf{b}_3 + f\left({}^{\mathcal{I}}\omega^{\mathcal{B}} \times \mathbf{b}_3\right) \\ &= \dot{f}\mathbf{b}_3 + f\left(\dot{\phi}_Q\mathbf{b}_1 \times \mathbf{b}_3\right) \\ &= \dot{f}\mathbf{b}_3 - f\dot{\phi}_Q\mathbf{b}_2 \\ \dot{\phi}_Q &= -\frac{m_Q}{f}\left(\dddot{\mathbf{x}}_Q\cdot\mathbf{b}_2\right) \\ &= -\frac{m_Q}{f}\left(\dddot{y}_Q\cos(\phi_Q) + \dddot{z}_Q\sin(\phi_Q)\right) \\ &= -\frac{m_Q^2}{f^2}\left(\dddot{y}_Q(\ddot{z}_Q + g) - \dddot{z}_Q\ddot{y}_Q\right) \\ &= \frac{\left(\dddot{z}_Q\ddot{y}_Q - \dddot{y}_Q(\ddot{z}_Q + g)\right)}{\left(\ddot{y}_Q^2 + (\ddot{z}_Q + g)^2\right)} \end{aligned}$$

Differentiating the equation of motion again:

$$
\begin{aligned}
m_Q \dddot{\mathbf{x}}_Q &= \ddot{f}\mathbf{b}_3 + \dot{f}\left({}^{\mathcal{I}}\omega^{\mathcal{B}} \times \mathbf{b}_3\right) - \Big((\dot{f}\dot{\phi}_Q + f\ddot{\phi}_Q)\mathbf{b}_2 + f\dot{\phi}_Q({}^{\mathcal{I}}\omega^{\mathcal{B}} \times \mathbf{b}_2)\Big) \\
&= \ddot{f}\mathbf{b}_3 + \dot{f}\left(\dot{\phi}_Q\mathbf{b}_1 \times \mathbf{b}_3\right) - \Big((\dot{f}\dot{\phi}_Q + f\ddot{\phi}_Q)\mathbf{b}_2 + f\dot{\phi}_Q(\dot{\phi}_Q\mathbf{b}_1 \times \mathbf{b}_2)\Big) \\
&= \ddot{f}\mathbf{b}_3 - \dot{f}\dot{\phi}_Q\mathbf{b}_2 - \Big((\dot{f}\dot{\phi}_Q + f\ddot{\phi}_Q)\mathbf{b}_2 + f\dot{\phi}_Q^2\mathbf{b}_3\Big) \\
&= -(2\dot{f}\dot{\phi}_Q + f\ddot{\phi}_Q)\mathbf{b}_2 + (\ddot{f} - f\dot{\phi}_Q^2)\mathbf{b}_3 \\
\ddot{\phi}_Q &= -\frac{m_Q}{f}(\dddot{\mathbf{x}}_Q \cdot \mathbf{b}_2) - 2\frac{\dot{f}\dot{\phi}_Q}{f}
\end{aligned}
$$

Where:

$$
\begin{aligned}
\dot{f} &= m_Q^2 \frac{(\ddot{y}_Q\dddot{y}_Q + (\ddot{z}_Q + g)\dddot{z}_Q)}{f} \\
\ddot{\phi}_Q &= -\frac{m_Q}{f}(\dddot{y}_Q\cos(\phi_Q) + \dddot{z}_Q\sin(\phi_Q)) \\
&- \frac{2m_Q^2(\ddot{y}_Q\dddot{y}_Q + (\ddot{z}_Q + g)\dddot{z}_Q)}{f^2}\dot{\phi}_Q \\
&= -\frac{m_Q^2}{f^2}\left(\dddot{y}_Q(\ddot{z}_Q + g) - \dddot{z}_Q\ddot{y}_Q\right) \\
&+ \frac{2m_Q^4}{f^4}(\ddot{y}_Q\dddot{y}_Q + (\ddot{z}_Q + g)\dddot{z}_Q)(\dddot{y}_Q(\ddot{z}_Q + g) - \dddot{z}_Q\ddot{y}_Q) \\
&= \frac{(\dddot{z}_Q\ddot{y}_Q - \dddot{y}_Q(\ddot{z}_Q + g))}{\left(\ddot{y}_Q^2 + (\ddot{z}_Q + g)^2\right)} \\
&+ \frac{2(\ddot{y}_Q\dddot{y}_Q + (\ddot{z}_Q + g)\dddot{z}_Q)(\dddot{y}_Q(\ddot{z}_Q + g) - \dddot{z}_Q\ddot{y}_Q)}{\left(\ddot{y}_Q^2 + (\ddot{z}_Q + g)^2\right)^2}
\end{aligned}
$$

The moment input be found from:

$$
M = J_Q\ddot{\phi}_Q
$$

2.2.2 Control Laws

Use control laws from paper with desired trajectory: $\sigma_T(t) = [\mathbf{x}_T(t)] = [y_T(t) \;\; z_T(t)]^T$:

$$\begin{aligned}
\mathbf{F1} &= m_Q g\mathbf{e}_3 + m_Q\ddot{\mathbf{x}}_T \\
\mathbf{F} &= -K_p\left(\mathbf{x} - \mathbf{x}_T\right) - K_d\left(\dot{\mathbf{x}} - \dot{\mathbf{x}}_T\right) + \mathbf{F1} \\
f &= \mathbf{F}\cdot\mathbf{b}_3 \\
\phi_Q^d &= \tan^{-1}\left(\frac{-\mathbf{F}\cdot\mathbf{e}_2}{\mathbf{F}\cdot\mathbf{e}_3}\right)
\end{aligned}$$

$$\begin{aligned}
f_{nom}\mathbf{b}_{3_{nom}} &= m_Q\ddot{\mathbf{x}}_T + m_Q g\mathbf{e}_3 \\
\mathbf{b}_{3_{nom}} &= \frac{m_Q\ddot{\mathbf{x}}_T + m_Q g\mathbf{e}_3}{\|m_Q\ddot{\mathbf{x}}_T + m_Q g\mathbf{e}_3\|} \\
\phi_{Q_{nom}} &= \tan^{-1}\left(\frac{f_{nom}\mathbf{b}_{3_{nom}}\cdot\mathbf{e}_2}{f_{nom}\mathbf{b}_{3_{nom}}\cdot\mathbf{e}_3}\right) \\
f_{nom} &= f_{nom}\mathbf{b}_{3_{nom}}\cdot\mathbf{b}_{3_{nom}} \\
\dot{\phi}_Q^d &= -\frac{m_Q}{f_{nom}}\left(\dddot{\mathbf{x}}_T\cdot\mathbf{b}_{2_{nom}}\right), \text{ where } \mathbf{b}_{2_{nom}} = \begin{bmatrix}\cos(\phi_{Q_{nom}}) \\ \sin(\phi_{Q_{nom}})\end{bmatrix} \\
\ddot{\phi}_Q^d &= -\frac{m_Q}{f_{nom}}(\ddddot{\mathbf{x}}_T\cdot\mathbf{b}_{2_{nom}}) - 2\frac{\dot{f}\dot{\phi}_Q^d}{f_{nom}}, \dot{f} = \frac{(\ddot{y}_T\dddot{y}_T + (\ddot{z}_T + g)\dddot{z}_T)}{f_{nom}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}^d &= J_Q\ddot{\phi}_Q^d \\
\mathbf{M} &= J_Q(-K_{p_\phi}(\phi_Q - \phi_Q^d) - K_{d_\phi}(\dot{\phi}_Q - \dot{\phi}_Q^d)) + \mathbf{M}^d
\end{aligned}$$

2.3 Differential Flatness of Hybrid System

"A Differentially-Flat Hybrid System is a hybrid system where each subsystem is differentially-flat, the switching surfaces are functions of the flat outputs and their derivatives, and moreover the flat outputs map from one subsystem to a subsequent subsystem through the sufficiently smooth transition maps."

The switching surfaces are $\mathcal{S}_1 = \{\mathbf{x}_1 \mid T \equiv \|m_L(\dot{\mathbf{v}}_L + g\mathbf{e}_3)\| = 0\}$, which is in terms of the flat outputs $\mathbf{x}_L$. The second switching surface is $\mathcal{S}_2 = \{\mathbf{x}_2 \mid \|\mathbf{x}_Q - \mathbf{x}_L\| = l\}$, which is in terms of the flat output $\mathbf{x}_Q$ and $\mathbf{x}_L$ which can be determined from initial conditions.