Quadrotor with a Cable-Suspended Load in 1-Dimension

September 5, 2013

$m_Q, m_L \in \mathbb{R}$	Mass of quadrotor, load
$f \in \mathbb{R}$	Magnitude of thrust for quadrotor
$l \in \mathbb{R}$	Length of suspension cable
$T \in \mathbb{R}$	Magnitude of tension in cable
$x_Q, \mathbf{x}_L \in R^2$	Position of center of mass of quadrotor, load
$v_Q, \mathbf{v}_L \in R^2$	Velocity of center of mass of quadrotor, load

1 Equations of Motion

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1) + g_1(\mathbf{x}_1)\mathbf{u}_1, & \mathbf{x}_1 \notin \mathcal{S}_1 \\ \mathbf{x}_2^+ = \Delta_1(\mathbf{x}_1^-), & \mathbf{x}_1^- \in \mathcal{S}_1 \\ \dot{\mathbf{x}}_2 = f_2(\mathbf{x}_2) + g_2(\mathbf{x}_2)\mathbf{u}_2, & \mathbf{x}_2 \notin \mathcal{S}_2 \\ \mathbf{x}_1^+ = \Delta_2(\mathbf{x}_2^-), & \mathbf{x}_2^- \in \mathcal{S}_2 \end{cases}$$

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1 = \begin{bmatrix} \dot{x}_L \\ \dot{v}_L \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_L \\ v_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_L + m_Q} \end{bmatrix} f + \begin{bmatrix} 0 \\ -g \end{bmatrix}, & \mathbf{x}_1 \notin \{\mathbf{x}_1 \mid T \equiv \|m_L(\dot{v}_L + g)\| = 0\} \\ \mathbf{x}_2^+ = \begin{bmatrix} x_L^+ \\ v_L^+ \\ x_Q^+ \\ v_Q^+ \end{bmatrix} = \begin{bmatrix} x_L^- \\ v_L^- \\ x_L^- + l \\ v_L^- + l \end{bmatrix}, & \mathbf{x}_1^- \in \{\mathbf{x}_1 \mid T = 0\} \\ \dot{\mathbf{x}}_2 = \begin{bmatrix} \dot{x}_L \\ \dot{v}_L \\ \dot{x}_Q \\ \dot{v}_Q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_L \\ v_L \\ x_Q \\ v_Q \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_Q} \end{bmatrix} f + \begin{bmatrix} 0 \\ -g \\ 0 \\ -g \end{bmatrix}, & \mathbf{x}_2 \notin \{\mathbf{x}_2 \mid \|x_Q - x_L\| = L\} \\ \mathbf{x}_1^+ = \begin{bmatrix} x_L^+ \\ v_L^+ \end{bmatrix} = \begin{bmatrix} x_L^- \\ v_Q^- \end{bmatrix}, & \mathbf{x}_2^- \in \{\mathbf{x}_2 \mid \|x_Q - x_L\| = L\} \end{cases}$$

1.1 When cable is taut

Assume that the quadrotor is moving in the $y-z$ plane, where its only degree of freedom is in the z direction. $+z$ is pointing upwards.
Constraint: $x_Q = x_L + l$

$$\begin{aligned}\sum \mathbf{F} &= m\ddot{x} \\ -m_Q g - T + f &= m_Q \dot{v}_Q \\ -m_L g + T &= m_L \dot{v}_L\end{aligned}$$

Solving for $\dot{v}_L$ in terms of only f:

$$\begin{aligned}T &= -m_Q \dot{v}_Q - m_Q g + f \\ m_L \dot{v}_L &= -m_L g + (-m_Q \dot{v}_Q - m_Q g + f)\end{aligned}$$

Differentiating the constraint gives:

$$\begin{aligned}\dot{x}_Q = v_Q &= \dot{x}_L = v_L \\ \dot{v}_Q &= \dot{v}_L \\ (m_L + m_Q)\dot{v}_L &= -(m_L + m_Q)g + f \\ \dot{v}_L &= -g + \frac{f}{m_L + m_Q}\end{aligned}$$

This gives our equations of motion:

$$\begin{aligned}\dot{x}_L &= v_L \\ \dot{v}_L &= -g + \frac{f}{m_L + m_Q}\end{aligned}$$

Note that the system transitions to $\mathbf{x}_2$ when the tension is 0. The tension force in the cable can be explicitly from the equation of motion:

$$\begin{aligned}-m_L g + T &= m_L \dot{v}_L \\ T &= m_L(\dot{v}_L + g)\end{aligned}$$

Note that this force will always go from positive to negative at the transition. In order to exert a tension force on the load, there must be a net force upwards, as tension cannot act downwards on the load. Thus, when tension is non-zero, $\dot{v}_L > -g$ the load simply enters free fall at the point $\dot{v}_L = g$, and it's not feasible to have $\dot{v}_L < -g$.

1.2 When cable is slack

Assume that the quadrotor is moving in the $y-z$ plane, where its only degree of freedom is in the z direction.
No constraints. Load is in free fall while quad is controlled by its input force f.

$$
\begin{aligned}
\sum \mathbf{F} &= m\ddot{x} \\
-m_Q g + f &= m_Q \dot{v}_Q \\
-m_L g &= m_L \dot{v}_L
\end{aligned}
$$

Solving for $\dot{v}_L$ and $\dot{v}_Q$ in terms of only f:

$$
\begin{aligned}
\dot{v}_L &= -g \\
\dot{v}_Q &= -g + \frac{f}{m_Q}
\end{aligned}
$$

This makes our equations of motion:

$$
\begin{aligned}
\dot{x}_L &= v_L \\
\dot{v}_L &= -g \\
\dot{x}_Q &= v_Q \\
\dot{v}_Q &= -g + \frac{f}{m_Q}
\end{aligned}
$$

Note that the system transitions to $\mathbf{x}_2$ when the quad and load are at a distance l apart again, with the load falling away from the quad. In other words, we transition when:

$$(x_Q - x_L) - l = 0$$

When the load is falling away from the quad, we go from $(x_Q - x_L) < l$ to $(x_Q - x_L) = l$, or when $(x_Q - x_L) - l$ is increasing. This means that $\frac{dL(t)}{dt} = \frac{d((x_Q - x_L) - l)}{dt} = v_Q - v_L \geq 0$, or $v_Q \geq v_L$.

2 Differential Flatness

2.1 $\mathbf{x}_1$ system:

Recall the equation of motion:

$$\dot{x}_L = v_L$$
$$\dot{v}_L = -g + \frac{f}{m_L + m_Q}$$

Choose flat output $\mathbf{y} = [x_L]$

Derive $\dot{x}_L = v_L$, $\ddot{x}_L = \dot{v}_L$ from differentiation of x_L
We can simply find the nominal input force f from the equation of motion:

$$f = (m_L + m_Q)\dot{v}_L + g$$

2.2 $\mathbf{x}_2$ system:

Recall the equations of motion:

$$\dot{x}_L = v_L$$
$$\dot{v}_L = -g$$
$$\dot{x}_Q = v_Q$$
$$\dot{v}_Q = -g + \frac{f}{m_Q}$$

Flat output $\mathbf{y} = [x_Q]$

x_L and v_L are known from initial conditions because load is in free fall:

$$v_L(t) = -gt + v_L(t_0)$$
$$x_L(t) = -gt^2 + v_L(t_0)t + x_L(t_0)$$

Derive $\dot{x}_Q = v_Q$, $\ddot{x}_Q = \dot{v}_Q$, and all higher derivatives from differentiation of x_Q
We can simply derive the input f from the equation of motion:

$$f = m_Q(\dot{v}_Q + g)$$