

Forward Kinematics

1 TRANSFORMATION MATRICES

The DH table for the QArm manipulator is presented in Table 1 (See 1. Frame Assignments for more information).

| i | a_i | α_i | d_i | θ_i |
|-----|-------------|------------|-------------|------------|
| 1 | 0 | $-\pi/2$ | λ_1 | θ_1 |
| 2 | λ_2 | 0 | 0 | θ_2 |
| 3 | 0 | $-\pi/2$ | 0 | θ_3 |
| 4 | 0 | 0 | λ_3 | θ_4 |

Table 1. DH Table

Begin with the general form of the transformation matrix,

$${}^{i-1}T_i = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

Using Equation 1 and Table 1, we can find all the transformation matrices. For brevity, we can abbreviate cosines as c and sines as s. In addition, the revolute joint variables θ will be dropped. Thus, the term $\cos \theta_1$ can be abbreviated as c_1 . Similarly, a term such as $\sin(\theta_1 + \theta_2)$ can be abbreviated as s_{12} .

$$\begin{aligned} {}^0T_1 &= \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & \lambda_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^1T_2 &= \begin{bmatrix} c_2 & -s_2 & 0 & \lambda_2 c_2 \\ s_2 & c_2 & 0 & \lambda_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^2T_3 &= \begin{bmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^3T_4 &= \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & \lambda_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (2)$$

2 FORWARD KINEMATICS FORMULATION

The joint vector $\vec{\theta}$ represents the joint states of the manipulator,

$$\vec{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \quad (3)$$

and the position \vec{p} of the end-effector {4} (expressed in base frame {0})

$$\vec{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^0p_4 \quad (4)$$

The goal of the forward kinematics formulation is to provide a mapping from the joint space to the position space

$$\vec{p} = f_{FPK}(\vec{\theta}) \quad (5)$$

Using the matrices in equation 2, we can proceed to derive the matrices 0T_2 , 0T_3 and 0T_4 .

$${}^0T_2 = {}^0T_1 {}^1T_2 = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & \lambda_2c_1c_2 \\ s_1c_2 & -s_1s_2 & c_1 & \lambda_2s_1c_2 \\ -s_2 & -c_2 & 0 & \lambda_1 - \lambda_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$${}^0T_3 = {}^0T_2 {}^2T_3 = \begin{bmatrix} c_1c_{23} & s_1 & -c_1s_{23} & \lambda_2c_1c_2 \\ s_1c_{23} & -c_1 & -s_1s_{23} & \lambda_2s_1c_2 \\ -s_{23} & 0 & -c_{23} & \lambda_1 - \lambda_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

$${}^0T_4 = {}^0T_3 {}^3T_4 = \begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & -c_1c_{23}s_4 + s_1c_4 & -c_1s_{23} & \lambda_2c_1c_2 - \lambda_3c_1s_{23} \\ s_1c_{23}c_4 - c_1s_4 & -s_1c_{23}s_4 - c_1c_4 & -s_1s_{23} & \lambda_2s_1c_2 - \lambda_3s_1s_{23} \\ -s_{23}c_4 & s_{23}s_4 & -c_{23} & \lambda_1 - \lambda_2s_2 - \lambda_3c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

Consider the vector representing the position of the end-effector frame $\{4\}$ \vec{p} . From the matrix 0T_4 in equation 8, we can extract this vector expressed in frame $\{0\}$ as 0p_4 , and can also extract the orientation (via rotation matrix) 0R_4 of the end-effector frame $\{4\}$ with respect to the base frame $\{0\}$.

$$\begin{aligned} p_x &= \lambda_2c_1c_2 - \lambda_3c_1s_{23} \\ p_y &= \lambda_2s_1c_2 - \lambda_3s_1s_{23} \\ p_z &= \lambda_1 - \lambda_2s_2 - \lambda_3c_{23} \end{aligned} \quad (9)$$

Equation 9 represents the forward kinematics formulation f_{FPK} . Note that the wrist rotation angle θ_4 does not play a part in the position of the end-effector, and correspondingly does not appear in f_{FPK} . However, it does play a part in the end-effector orientation,

$${}^0R_4 = \begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & -c_1c_{23}s_4 + s_1c_4 & -c_1s_{23} \\ s_1c_{23}c_4 - c_1s_4 & -s_1c_{23}s_4 - c_1c_4 & -s_1s_{23} \\ -s_{23}c_4 & s_{23}s_4 & -c_{23} \end{bmatrix} \quad (10)$$