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Group 2

Solution 1

1. The function $f(n)$ by definition of f gives the number of ways ~~th~~ to reach the n^{th} step on a stair.

We have to prove the relation

$f(n) = f(n-1) + f(n-2) \quad n \geq 3$

Now since to reach the n^{th} step only two ways are possible i.e. Either take a two step jump from $(n-2)^{th}$ step or take a one step jump from $(n-1)^{th}$ step. Ways to reach $(n-2)^{th}$ step $= f(n-2)$ and $(n-1)^{th}$ step $= f(n-1)$

Hence proved $f(n) = f(n-1) + f(n-2)$.

Note: I tried to use induction to prove this result see last page of this pdf. But it looked absurd since the induction hypo. was not even meaningfully used in induction step. Also this is a mathematical relation and not an algorithm so there is no question to prove an algorithm. (which could be done by PMI since mathematical relation would then have already been established)

Note: $f(1) = 1 \quad f(2) = 2$. else others satisfy the relation.

Solution Problem 1

2. (written in sml file also)

Algorithm.

$$\text{climbStair}(n) : \text{int} \to \text{int} = \begin{cases} 1 & n = 1 \\ 2 & n = 2 \\ \text{climbStair}(n-1) + \text{climbStair}(n-2) & n \geq 3 \end{cases}$$

3. To prove $f(n) = 2 + \sum_{i=1}^{n-2} f(i)$

we know that $f(n) = f(n-1) + f(n-2) \quad \forall\, n \geq 3$

so,

$$\sum_{k=3}^{n} f(k) = \sum_{k=3}^{n} f(k-1) + \sum_{k=3}^{n} f(k-2)$$

$$\sum_{k=3}^{n} f(k) = \sum_{j=2}^{n-1} f(j) + \sum_{i=1}^{n-2} f(i)$$ ([redacted] substitution of variables)

$$\sum_{k=3}^{n} f(k) - \sum_{j=2}^{n-1} f(j) = \sum_{i=1}^{n-2} f(i)$$

$$f(n) - f(2) = \sum_{i=1}^{n-2} f(i)$$ (obtained by opening the summation sign in previous step)

$$f(n) = 2 + \sum_{i=1}^{n-2} f(i)$$

[redacted]

Solution Problem 2

1. Let $n = d_k d_{k-1} \cdots d_1 d_0$ (d represents digits)

clearly $\left\lfloor \frac{n}{10} \right\rfloor = d_k d_{k-1} \cdots d_1$

also $n \bmod 10 = d_0$

$$f(n) = 2^k d_k + 2^{k-1} d_{k-1} + \cdots 2d_1 + d_0$$

$$= 2(2^{k-1} d_k + 2^{k-2} d_{k-1} + \cdots d_1) + n \bmod 10$$

$$= 2 f\left(\left\lfloor \frac{n}{10} \right\rfloor\right) + n \bmod 10$$

Hence prooved.

2. (Also written in SML code file)

Algorithm

$$\text{ModifiedDigitSum}(n) = \begin{cases} 2 \times \text{ModifiedDigitSum}\left(\left\lfloor \frac{n}{10} \right\rfloor\right) + n \bmod 10 & n \neq 0 \\ 0 & n = 0 \end{cases}$$

using $f(0) = 0$ and recurrence relation in f.

(int → int)

Solution Problem 3

1.

Helper function 1

$$\text{issquare}(a,b) = \begin{cases} \text{false} & b > a \\ \text{true} & a = b^2 \\ \text{issquare}(a, b+1) & \text{otherwise} \end{cases}$$

we will usually take argument $b=0$ while calling it in the definition of some other function to get to know whether a can be represented as a square.

Helper function 2

$$\text{canitself}(x,y) = \begin{cases} \text{false} & y > x \\ \text{true} & \text{when issquare}(x-y^2, 0) \text{ is true} \\ \text{canitself}(x, y+1) & \text{otherwise} \end{cases}$$

this function is to know whether x can be represented as sum of two squares. y will start from 0

Main function

$$\text{squaredcount}(n) = \begin{cases} 1 & n = 1 \\ \text{squaredcount}(n-1)+1 & \text{when canitself}(n,0) \text{ is true} \\ \text{squaredcount}(n-1) & \text{otherwise} \end{cases}$$

Proofs on next page.

First we start by proving the main function assumings the other two are correct.

The function squaredCount(n) is correct if it gives the number of Natural numbers before, including n which can be represented as sum of two squares.

Proof by PMI.

Base Case: $n=1$ answer should be 1 as $(1 = 1^2+0^2)$

squaredCount(1) = 1 so correct.

Induction Hypothesis: Let us assume that the function squaredCount (n) calculates correctly no. of Natural Numbers $\leq n$ which can be rep. as sum of 2 squares.

We now have to proove that this holds for $(n+1)$ also.

Induction Step:

Case 1: $n+1$ can be expressed as x^2+y^2. $x, y \geq 0$

So number of integers would be $=$ 1 + squaredCount (n) (by IH)

$=$ squaredCount (n+1)

Since canItself (n, 0) will evaluate as true.

Case 2: $n+1$ cannot be expressed as x^2+y^2. $x, y \geq 0$

then canItself (n, 0) is false so

squaredCount (n+1) = squaredCount (n)

And since squaredCount (n) evaluates correctly, the correctness is established.

Helper function Proof on next page

Proof for issquare (a, b) function.

Note: This is always called as issquare (a, 0) in starting.

First we show that this algorithm does terminate.

Since issquare (a, b) always calls issquare (a, b+1), b argument is increasing and so in the worst case it must exceed any finite a since set of Natural Numbers is not bounded.

Now we have to prove that it returns true whenever a is a perfect square and false when it is not.

Case 1. a is a perfect square

so $a = m^2$ where $m \geq 0$

Now since b starts from 0 and keeps increasing by 1 it must be equal to m at some point (can be prooved by contradiction). So when $b = m$, $b^2 = a$ and function will return true. The desired output.

Case 2, a is not a perfect square

so, $a \neq m^2$ for any $m \geq 0$ where $m \in$ (set of integers)

Now since b can only take integer values it will never be $b^2 = a$ by above argument. And by proof of termination, It will eventually return false which is desired output.

Hence, Prooved.

Having Prooved this the proof of canitself (x, y) if starts from zero is almost exactly same as the above proof

Solution Problem 4

1.

$$\pi = 3 + \frac{4}{2\cdot3\cdot4} - \frac{4}{4\cdot5\cdot6}$$

↳ 1st term is considered from here.

Helper function 1 (real → real)

$$abs(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

this is to give absolute value of x

Helper function 2 ((int → real) × int × int → real)

$$sum(f,a,b) = \begin{cases} 0 & a > b \\ f(a) + sum(f,a+1,b) & \text{otherwise} \end{cases}$$

this is to give $f(a) + f(a+1) ---- f(b-1) + f(b)$

Helper function 3 (int → real)

$$nterm(x) = \begin{cases} \dfrac{4}{2x(2x+1)(2x+2)} & x \text{ is odd } (x \bmod 2 = 1) \\ \dfrac{-4}{2x(2x+1)(2x+2)} & x \text{ is even } (x \bmod 2 = 0) \end{cases}$$

this given the n^{th} term of the sum thus will be used in sum(f,a,b) as f.

Helper function 4 (real × int → int)

$$greatest(t,x) = \begin{cases} x & abs(nterm(x)) < t \\ greatest(t,x+1) & \text{otherwise} \end{cases}$$

this in a way finds b which is used in sum(f,a,b). i.e upper limit.

(Note x will start from 1 when we call it)

final function on next page

$$\text{nilkanthasum}(t) = \begin{cases} 3 + \text{sum}(\text{nterm}, 1, \text{greatest}(t, 1)) & t \leq 3 \\ 3 & t > 3 \end{cases}$$

(this is also written in SML)

Here the function "greatest" helps to find the number of term to which sum is to be evaluated. "nterm" gives the n^{th} term and "sum" sums all the n^{th} terms till upper limit.

2. Proof for "abs", "nterm", "nilkanthasum" are fairly elementary.

– Proof for $\text{sum}(f, a, b) = \sum_{i=a}^{b} f(i)$

let $b - a + 1 = n$.

Base case: $n = 0$ so $b = a - 1$ makes no sense hence must evaluate to 0

$\text{sum}(f, a, b) = 0$ so true.

Induction Hypothesis: suppose $\text{sum}(f, a, b) = \sum_{i=a}^{b} f(i)$ is true for some $n \geq 0$. We have to prove the Hypothesis also applies to $n+1$.

Induction Step:

$\text{sum}(f, a-1, b) = f(a-1) + \text{sum}(f, a, b)$

(Here n+1 case is being seen where n was b − a + 1)

But since $\text{sum}(f, a, b) = \sum_{i=a}^{b} f(i)$ so,

$$\text{sum}(f, a-1, b) = f(a-1) + \sum_{i=a}^{b} f(i)$$

$$= \sum_{i=a-1}^{b} f(i)$$

Hence by induction "$\text{sum}(f, a, b)$" behaves as we wanted.

- Proof for greatest (t, n).

First we show that this function does terminate.
Since function is recursively calling greatest $(t, n+1)$
n is increasing with each functional call.
And since $|Interm(n+1)| < |Interm(n)|$ and
$\lim_{n \to \infty} |Interm(n)| = 0$ Hence we are bound to get
a n such that $|Interm(n)| < t \;\; \forall\, t > 0$

Now we will show that greatest (t, n) evalutes
the ~~largest~~ smallest n such that $|Interm(n)| < t$.

Proof by contradiction.

Suppose n is not the smallest integer that satisfies
above condition then let $y < n$ be such that
$|Interm(y)| < t$. Then the function greatest(t, y)
must have been evaluated to y and not
Greatest $(t, y+1)$ and Hence it is not possible that
the output be $n > y$. Hence it is a contradiction.

~~So, greatest since~~ Note: recursion will be started ~~in~~ from
~~subsequent~~ $n = 1$ always.

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~~Solution Problem 1~~

This is the seemingly incorrect proof that I had tried.

1. To Prove $f(n) = f(n-1) + f(n-2)$ for all $n \geq 3$

given: $f(1) = 1$

$f(2) = 2$

Base case: $f(3) = f(2) + f(1) = 3$

(This can also be verified by logic i.e. there are 3 ways $1+1+1$, $2+1$, $1+2$)

Induction Hypothesis – let $f(m) = f(m-1) + f(m-2)$ be true for all $3 \leq m < n$.

Induction step: we need to show that

$$f(n) = f(n-1) + f(n-2).$$

- Now logically to get on the n^{th} stair one has to either get a 2 step jump or 1 step jump.
- If we take a 2 step jump we were on $(n-2)^{th}$ stair and by IH ways to get there were $f(n-2)$.
- If we take 1 step jump we were on $n-1^{th}$ step and by IH ways to get there are $f(n-1)$.
- Hence $f(n) = f(n-1) + f(n-2)$.

This is attached here to get any recommendation on whether this was correct or fault in my thought process

~~[illegible]~~

Respected Sir,

~~Also~~ this is a request to provide feedback on the Proofs which were used in this assignment and were not taught in class. Also please tell if any of those functions could be meaningfully prooved by induction. Please also tell if anything could be improved while submitting future assignments.

Thank You
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You may also provide me the valuable feedback through MS teams.