Classification of Finite Simple Groups

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Simple Groups

Definition

A simple group is a nontrivial group with no normal subgroups aside from itself and the trivial group.

Galois Theory sparked the study of finite simple groups

Definition

A group G is solvable if there is a chain of subgroups $\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_k = G$ such that each G_i/G_{i-1} is abelian [4].

Examples

- If G is abelian, then $\{e\} \triangleleft G$, $G/\{e\}$ is abelian, so G is solvable.
- $\{e\} \triangleleft A_3 \triangleleft S_3$, $A_3/\{e\} \cong A_3 \cong \mathbb{Z}_3$, $S_3/A_3 \cong \mathbb{Z}_2$ so S_3 is solvable.
- If G is a nonabelian simple group, then G is not solvable.

Theorem

A polynomial is solvable by radicals if and only if its "Galois group" is solvable.

Classification of Abelian Finite Simple Groups

Theorem

A finite abelian group G is simple if and only if $G \cong \mathbb{Z}_p$ for some prime p.

Proof.

By LaGrange's Theorem, $\mathbb{Z}_p$ is simple. By the Fundamental Theorem of Finite Abelian Groups, $G \cong \mathbb{Z}_{{p_1}^{e_1}} \times \ldots \times \mathbb{Z}_{{p_k}^{e_k}}$. If $k > 1$, G has a normal subgroup isomorphic to $\mathbb{Z}_{{p_1}^{e_1}}$. If $k = 1$ and $e_1 > 1$, G has a normal subgroup isomorphic to $\mathbb{Z}_{p_1}$. □

Conjugate Subgroups

Definition

For subgroups H_1 and H_2 of G, H_1 is conjugate to H_2 if there exists a $g \in G$ such that $gH_1g^{-1} = H_2$.

Theorem

If $H \leq G$, H is normal if and only if H has no conjugates in G.

Proof.

- $\Rightarrow$ Let $H \triangleleft G$. Then $\forall g \in G$, $gHg^{-1} = H$, so H has no conjugates.
- $\Leftarrow$ Let $H_1 \ntriangleleft G$. Then $\exists g \in G$ such that $gH_1g^{-1} \neq H_1$. Let $H_2 = \{ghg^{-1} | h \in H_1\}$. Consider $gh_1g^{-1}, gh_2g^{-1} \in H_2$. Then $gh_1g^{-1}(gh_2g^{-1})^{-1} = g(h_1h_2^{-1})g^{-1} \in H_2$. So $H_2 \leq G$ by the 1 Step Subgroup Test. So H_1 is conjugate to H_2 and $H_1 \neq H_2$.

$\square$

Sylow Theorems

Theorem (Sylow Theorems [1], 1872)

If G is a finite group of order $p^k n$, p prime and $p \nmid n$, then

1. *G has $s_p \geq 1$ subgroups of order p^k, called Sylow p-subgroups*
2. *The Sylow p-subgroups are all conjugate*
3. $s_p \mid m$ *and* $s_p \equiv 1 \mod p$

Example

If $s_p = 1$ then the Sylow p-subgroup is normal so G is not simple.

Feit-Thompson Theorem

Theorem (Feit-Thompson Theorem [1], 1962)

Every finite group of odd order is solvable.

Corollary

Every nonabelian finite simple group has even order.

The Classification Theorem for Finite Simple Groups [1]

Theorem (The Classification Theorem for Finite Simple Groups [1], 2004)

Every finite simple group is isomorphic to one of the following:

- *The cyclic groups* $\mathbb{Z}_p$ *for prime* p
- *The alternating groups* A_n *for* $n \geq 5$
- *The classical groups of Lie type*
- *The exceptional groups of Lie type or the Tits group* ${}^2F_4(2)'$
- *26 sporadic finite simple groups*

Alternating Groups

Lemma

A_n is exactly the set of products of 3-cycles on n letters.

Lemma

All 3-cycles are conjugate in A_n.

Corollary

If $N \triangleleft A_n$ and N contains a 3-cycle, then $N = A_n$.

Classical Groups of Lie Type

For a prime-power q,

- Linear: $PSL_n(q)$, $n \geq 2$, except $PSL_2(2)$ and $PSL_2(3)$
- Unitary: $PSU_n(q)$, $n \geq 3$, except $PSU_3(2)$
- Symplectic: $PSp_{2n}(q)$, $n \geq 2$, except $PSp_4(2)$
- Orthogonal:
  - $P\Omega_{2n+1}(q)$, $n \geq 3$, q odd
  - $P\Omega_{2n}^{+}(q)$, $n \geq 4$
  - $P\Omega_{2n}^{-}(q)$, $n \geq 4$

Example

$PSL_n(q) = SL_n(q)/Z$ where $SL_n(q)$ is the group of $n \times n$ matrices with entries from the finite field $\mathbb{F}_q$ with determinant 1 and Z is the subgroup of $SL_n(q)$ that are scalar multiples of the identity [1].

Exceptional Groups of Lie Type

For a prime-power q,

- $G_2(q)$, $q \geq 3$;
- $F_4(q)$;
- $E_6(q)$;
- ${}^2E_6(q)$;
- ${}^3D_4(q)$;
- $E_7(q)$;
- $E_8(q)$
- ${}^2B_2(2^{2n+1})$, $n \geq 1$;
- ${}^2G_2(3^{2n+1})$, $n \geq 1$;
- ${}^2F_4(2^{2n+1})$, $n \geq 1$

Sporadic Finite Simple Groups

- The Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
- The Leech lattice groups $Co_1, Co_2, Co_3, McL, HS, Suz, J_2$
- The Fischer groups $Fi_{22}, Fi_{23}, Fi_{24}'$
- The monstrous groups $\mathbb{M}, \mathbb{B}, Th, HB, He$
- The pariahs $J_1, J_3, J_4, O'N, Ly, Ru$.

Sporadic Finite Simple Groups

Definition

If G is a group and $N \triangleleft H \leq G$, then H/N is called a subquotient of G [5].

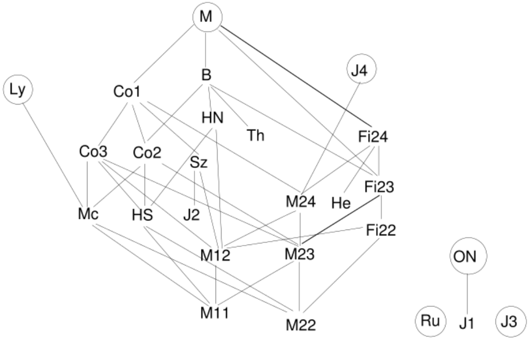

Figure: Subquotient diagram of sporadic finite simple groups [2]

The Monster

- Predicted by Bernd Fischer, 1973, and Robert Griess, 1976
- Existence proven in 220 pages in 1980 by Griess
- Order $2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ $\approx 8 \cdot 10^{53}$
- Wilson constructed two 196882×196882 matrices over $\mathbb{F}_2$ that generate the Monster

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