Linearization of Nonlinear Laws

The given data may not always follow a linear relationship. This can be ascertained from a plot of the given data. If a nonlinear model is to be fitted, it can be conveniently transformed to a linear relationship. Some nonlinear laws and their transformations are given as follows.

(a) $y = ax + \frac{b}{x}$

This can be written as

$$xy = ax^2 + b$$

Put $xy = Y$, $x^2 = X$. With these transformations, it becomes a linear model.

(b) $xy^a = b$

Taking logarithms of both sides, we get

$$\log_{10}x + a\log_{10}y = \log_{10}b.$$

In this case, we put

$$\log_{10}y = Y,\ \log_{10}x = X,$$

$$\frac{1}{a}\log_{10}b = A_0 \text{ and } -\frac{1}{a} = A_1,$$

so that

$$Y = A_0 + A_1X.$$

(c) $y = ab^x$

Taking logarithms of both sides, we obtain

$$\log_{10}y = \log_{10}a + x\log_{10}b$$
$$\Rightarrow Y = A_0 + A_1X,$$

where

$$Y = \log_{10}y,\ A_0 = \log_{10}a,$$
$$X = x, \text{ and } A_1 = \log_{10}b$$

(d) $y = ax^b$

We have

$$\log_{10}y = \log_{10}a + b\log_{10}x$$
$$\Rightarrow Y = A_0 + A_1X,$$

where

$$Y = \log_{10}y,\ A_0 = \log_{10}a,\ A_1 = b$$

and

$$X = \log_{10}x.$$

(e) $y = ae^{bx}$

In this case, we write

$$\ln y = \ln a + bx$$
$$\Rightarrow Y = A_0 + A_1X,$$

where

$$Y = \ln y,\ A_0 = \ln a,\ A_1 = b$$

and

$$X = x.$$

Using the method of least squares, find constants a and b such that the function $y = ae^{x}$ fits the following data:
(1.0, 2.473), (3.0, 6.722), (5.0, 18.274), (7.0, 49.673), (9.0, 135.026).
Solution

We have

$$y = ae^{bx}$$

Therefore,

$$\ln y = \ln a + bx$$

$$\Rightarrow Y = A_0 + A_1 X,$$

where

$$Y = \ln y,\ A_0 = \ln a,\ A_1 = b \text{ and } X = x.$$

The table of values is given below

X	$Y = \ln y$	X^2	XY
1	0.905	1	0.905
3	1.905	9	5.715
5	2.905	25	14.525
7	3.905	49	27.335
9	4.905	81	44.145
25	14.525	165	92.625

We obtain

$$\bar{X} = 5,\ \bar{Y} = 2.905$$

$$A_1 = \frac{5(92.625) - 25(14.525)}{5(165) - 625} = 0.5 = b.$$

Then

$$A_0 = \bar{Y} - A_1 \bar{X} = 2.905 - 0.5(5) = 0.405.$$

Hence,

$$a = e^{A_0} = e^{0.405} = 1.499.$$

It follows that the required curve is of the form

$$y = 1.499e^{0.5x}$$

Using the method of least squares, fit a curve of the form $y = \frac{x}{a+bx}$ to the following data:

(3, 7.148), (5, 10.231), (8, 13.509), (12, 16.434).

Solution

We have

$$y = \frac{x}{a + bx}$$

$$\Rightarrow \frac{1}{y} = \frac{a + bx}{x} = b + \frac{a}{x}$$

$$\Rightarrow Y = A_0 + A_1 X,$$

where

$$A_0 = b,\ A_1 = a,\ X = \frac{1}{x} \text{ and } Y = \frac{1}{y}.$$

The table of values is

X	Y	X^2	XY
0.333	0.140	0.111	0.047
0.200	0.098	0.040	0.020
0.125	0.074	0.016	0.009
0.083	0.061	0.007	0.005
0.741	0.373	0.174	0.081

We obtain

$$A_1 = a = \frac{4(0.081) - 0.741(0.373)}{4(0.174) - (0.741)^2} = 0.324,\ \bar{X} = 0.185,\ \bar{Y} = 0.093$$

and $A_0 = b = \bar{Y} - a\bar{X} = 0.0331.$

Hence the required fit is $Y = 0.0331 + 0.324(X)$, which simplifies to

$$y = \frac{x}{0.324 + 0.0331(x)}.$$