Newton-Cotes Integration Formulas

The Newton-Cotes formulas are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx \tag{21.1}$$

where $f_n(x)$ = a polynomial of the form

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

where n is the order of the polynomial. For example, in Fig. 21.1a, a first-order polynomial (a straight line) is used as an approximation. In Fig. 21.1b, a parabola is employed for the same purpose.

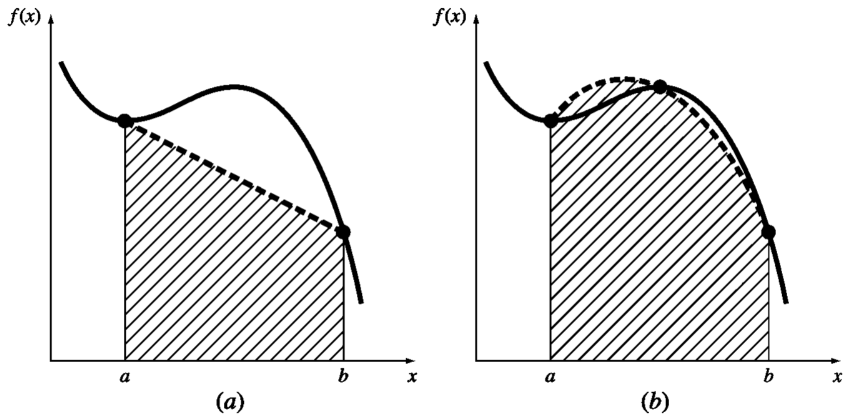


FIGURE 21.1 The approximation of an integral by the area under (a) a single straight line and (b) a single parabola.

For example, in Fig. 21.2, three straight-line segments are used to approximate the integral. Higher-order polynomials can be utilized for the same purpose. With this background, we now recognize that the "strip method" in Fig 21.2

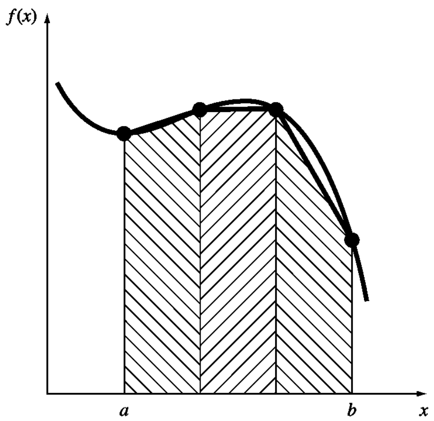


FIGURE 21.2 The approximation of an integral by the area under three straight-line segments.

THE TRAPEZOIDAL RULE

The *trapezoidal rule* is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial in Eq. (21.1) is first order:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

Geometrically, the trapezoidal rule is equivalent to approximating the area of the trapezoid under the straight line connecting $f(a)$ and $f(b)$ in Fig. 21.4. Recall from geometry that the formula for computing the area of a trapezoid is the height times the average of the bases (Fig. 21.5*a*). In our case, the concept is the same but the trapezoid is on its side (Fig. 21.5*b*). Therefore, the integral estimate can be represented as

$$I \cong \text{width} \times \text{average height} \tag{21.4}$$

FIGURE 21.4
Graphical depiction of the trapezoidal rule.

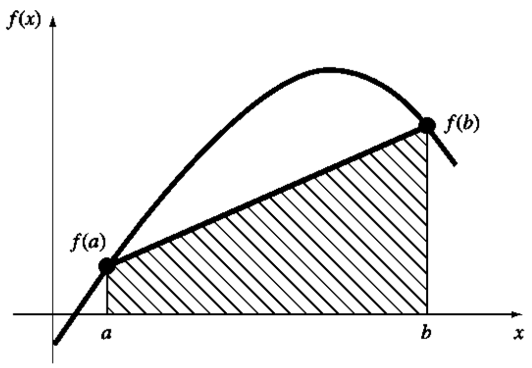
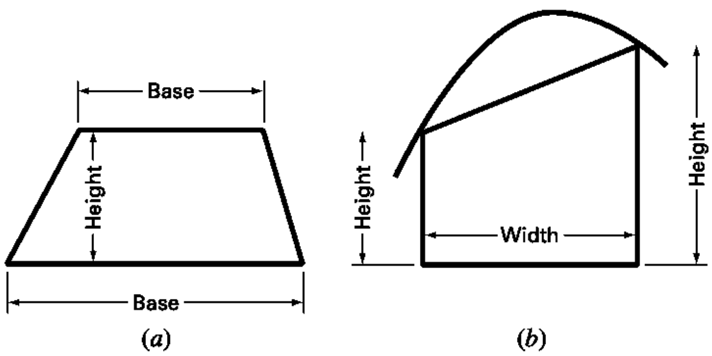


FIGURE 21.5
(*a*) The formula for computing the area of a trapezoid: height times the average of the bases.
(*b*) For the trapezoidal rule, the concept is the same but the trapezoid is on its side.



$$I = (b - a)\frac{f(a) + f(b)}{2}$$

which is called the *trapezoidal rule.*

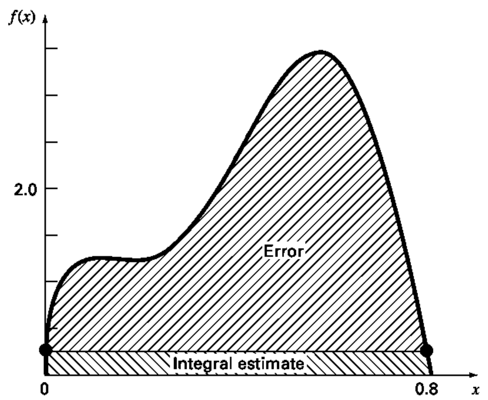
21.1.1 Error of the Trapezoidal Rule

When we employ the integral under a straight-line segment to approximate the integral under a curve, we obviously can incur an error that may be substantial (Fig. 21.6). An estimate for the local truncation error of a single application of the trapezoidal rule is (Box. 21.2)

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3 \tag{21.6}$$

where ξ lies somewhere in the interval from a to b. Equation (21.6) indicates that if the function being integrated is linear, the trapezoidal rule will be exact. Otherwise, for functions with second- and higher-order derivatives (that is, with curvature), some error can occur.

FIGURE 21.6
Graphical depiction of the use of a single application of the trapezoidal rule to approximate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $x = 0$ to 0.8.



Composite Trapezoidal Rule

One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment (Fig. 21.7). The areas of individual segments can then be added to yield the integral for the entire interval. The resulting equations are called *multiple-application,* or *composite, integration formulas*.

Figure 21.8 shows the general format and nomenclature we will use to characterize multiple-application integrals. There are $n + 1$ equally spaced base points $(x_0, x_1, x_2, \ldots, x_n)$. Consequently, there are n segments of equal width:

$$h = \frac{b-a}{n} \tag{21.7}$$

If a and b are designated as x_0 and x_n, respectively, the total integral can be represented as

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$\boxed{I = \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}_{\text{Average height}}}$$

An error for the multiple-application trapezoidal rule can be obtained by summing the individual errors for each segment to give

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

The error decreases as the number of segments increases. This is because the error is inversely related to the square of n. Therefore, doubling the number of segments quarters the error.

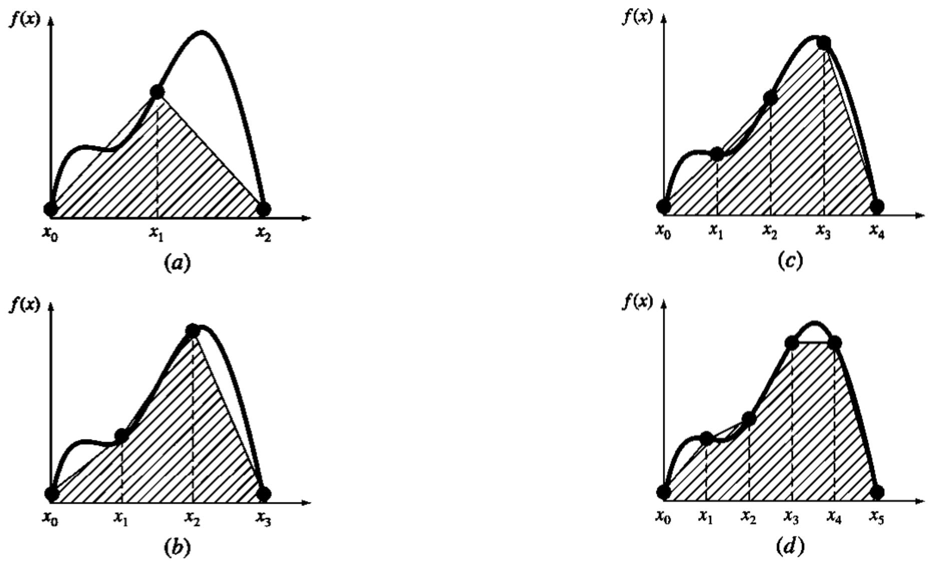


FIGURE 21.7 Illustration of the multiple-application trapezoidal rule. (a) Two segments, (b) three segments, (c) four segments, and (d) five segments.

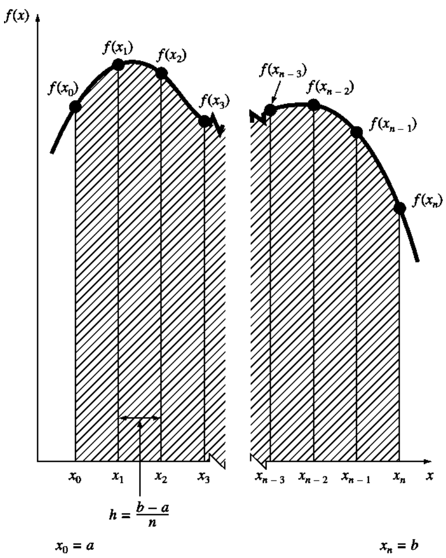


FIGURE 21.8 The general format and nomenclature for multiple-application integrals.

Example 3.12 *Find the approximate value of* $I = \int_0^1 \frac{dx}{1+x}$, *using the trapezium rule with* 2, 4 *and* 8 *equal subintervals. Using the exact solution, find the absolute errors.*

Solution With $N = 2$, 4 and 8, we have the following step lengths and nodal points.

$N = 2$: $h = \frac{b-a}{N} = \frac{1}{2}$. The nodes are 0, 0.5, 1.0.

$N = 4$: $h = \frac{b-a}{N} = \frac{1}{4}$. The nodes are 0, 0.25, 0.5, 0.75, 1.0.

$N = 8$: $h = \frac{b-a}{N} = \frac{1}{8}$. The nodes are 0, 0.125, 0.25, 0.375, 0.5, 0.675, 0.75, 0.875, 1.0.

We have the following tables of values.

$N = 2$:	x	0	0.5	1.0
	$f(x)$	1.0	0.666667	0.5

$N = 4$: We require the above values. The additional values required are the following:

x	0.25	0.75
$f(x)$	0.8	0.571429

$N = 8$: We require the above values. The additional values required are the following:

x	0.125	0.375	0.625	0.875
$f(x)$	0.888889	0.727273	0.615385	0.533333

Now, we compute the value of the integral.

$N = 2$:
$$I_1 = \frac{h}{2}[f(0) + 2f(0.5) + f(1.0)]$$
$$= 0.25 [1.0 + 2(0.666667) + 0.5] = 0.708334.$$

$N = 4$:
$$I_2 = \frac{h}{2}[f(0) + 2\{f(0.25) + f(0.5) + f(0.75)\} + f(1.0)]$$
$$= 0.125 [1.0 + 2 \{0.8 + 0.666667 + 0.571429\} + 0.5] = 0.697024.$$

$N = 8$:
$$I_3 = \frac{h}{2}[f(0) + 2\{f(0.125) + f(0.25) + f(0.375) + f(0.5)$$
$$+ f(0.625) + f(0.75) + f(0.875)\} + f(1.0)]$$
$$= 0.0625[1.0 + 2\{0.888889 + 0.8 + 0.727273 + 0.666667 + 0.615385$$
$$+ 0.571429 + 0.533333\} + 0.5] = 0.694122.$$

The exact value of the integral is $I = \ln 2 = 0.693147$.

The errors in the solutions are the following:

$$| \text{Exact} - I_1 | = | 0.693147 - 0.708334 | = 0.015187$$
$$| \text{Exact} - I_2 | = | 0.693147 - 0.697024 | = 0.003877$$
$$| \text{Exact} - I_3 | = | 0.693147 - 0.694122 | = 0.000975.$$

Example 3.13 *Evaluate* $I = \int_1^2 \frac{dx}{5+3x}$ *with* 4 *and* 8 *subintervals using the trapezium rule. Compare with the exact solution and find the absolute errors in the solutions. Comment on the magnitudes of the errors obtained. Find the bound on the errors.*

Solution With $N = 4$ and 8, we have the following step lengths and nodal points.

$N = 4$: $h = \frac{b-a}{N} = \frac{1}{4}$. The nodes are 1, 1.25, 1.5, 1.75, 2.0.

$N = 8$: $\quad h = \dfrac{b-a}{N} = \dfrac{1}{8}$. The nodes are 1, 1.125, 1.25, 1.375, 1.5, 1.675, 1.75, 1.875, 2.0.

We have the following tables of values.

$N = 4$:

x	1.0	1.25	1.5	1.75	2.0
$f(x)$	0.125	0.11429	0.10526	0.09756	0.09091

$N = 8$: We require the above values. The additional values required are the following.

x	1.125	1.375	1.625	1.875
$f(x)$	0.11940	0.10959	0.10127	0.09412

Now, we compute the value of the integral.

$N = 4$:
$$I_1 = \frac{h}{2}\,[f(1) + 2\,\{f(1.25) + f(1.5) + f(1.75)\} + f(2.0)]$$
$$= 0.125\,[0.125 + 2\,\{0.11429 + 0.10526 + 0.09756\} + 0.09091]$$
$$= 0.10627.$$

$N = 8$:
$$I_2 = \frac{h}{2}\,[f(1) + 2\{f(1.125) + f(1.25) + f(1.375) + f(1.5)$$
$$+ f(1.625) + f(1.75) + f(1.875)\} + f(2.0)]$$
$$= 0.0625\,[0.125 + 2\{0.11940 + 0.11429 + 0.10959 + 0.10526 + 0.10127$$
$$+ 0.09756 + 0.09412\} + 0.09091]$$
$$= 0.10618.$$

The exact value of the integral is

$$I = \frac{1}{3}\Big[\ln(5+3x)\Big]_1^2 = \frac{1}{3}[\ln 11 - \ln 8] = 0.10615.$$

The errors in the solutions are the following:

$$|\text{ Exact} - I_1\,| = |\ 0.10615 - 0.10627\ | = 0.00012.$$
$$|\text{ Exact} - I_2\,| = |\ 0.10615 - 0.10618\ | = 0.00003.$$

Example 3.14 *Using the trapezium rule, evaluate the integral* $I = \int_0^1 \dfrac{dx}{x^2 + 6x + 10}$, *with* 2 *and* 4 *subintervals. Compare with the exact solution. Comment on the magnitudes of the errors obtained.*

Solution With $N = 2$ and 4, we have the following step lengths and nodal points.

$N = 2$: $\quad h = 0.5$. The nodes are 0.0, 0.5, 1.0.

$N = 4$: $\quad h = 0.25$. The nodes are 0.0, 0.25, 0.5, 0.75, 1.0.

We have the following tables of values.

$N = 2$:

x	0.0	0.5	1.0
$f(x)$	0.1	0.07547	0.05882

$N = 4$: We require the above values. The additional values required are the following.

x	0.25	0.75
$f(x)$	0.08649	0.06639

Now, we compute the value of the integral.

$N = 2$:
$$I_1 = \frac{h}{2}\,[f(0.0) + 2\,f(0.5) + f(1.0)]$$
$$= 0.25\,[0.1 + 2(0.07547) + 0.05882] = 0.07744.$$

$N = 4$:
$$I_2 = \frac{h}{2}\,[f(0.0) + 2\{f(0.25) + f(0.5) + f(0.75)\} + f(1.0)]$$
$$= 0.125[0.1 + 2(0.08649 + 0.07547 + 0.06639) + 0.05882] = 0.07694.$$

The exact value of the integral is

$$I = \int_0^1 \frac{dx}{(x+3)^2 + 1} = \Big[\tan^{-1}(x+3)\Big]_0^1 = \tan^{-1}(4) - \tan^{-1}(3) = 0.07677.$$

The errors in the solutions are the following:

$$|\text{ Exact} - I_1\,| = |\ 0.07677 - 0.07744\ | = 0.00067$$
$$|\text{ Exact} - I_2\,| = |\ 0.07677 - 0.07694\ | = 0.00017.$$

Example 3.15 *The velocity of a particle which starts from rest is given by the following table.*

t (sec)	0	2	4	6	8	10	12	14	16	18	20
v (ft/sec)	0	16	29	40	46	51	32	18	8	3	0

Evaluate using trapezium rule, the total distance travelled in 20 *seconds.*

Solution From the definition, we have

$$v = \frac{ds}{dt}, \text{ or } s = \int v\, dt.$$

Starting from rest, the distance travelled in 20 seconds is

$$s = \int_0^{20} v\, dt.$$

The step length is $h = 2$. Using the trapezium rule, we obtain

$$s = \frac{h}{2}\,[f(0) + 2\{f(2) + f(4) + f(6) + f(8) + f(10) + f(12) + f(14) + f(16) + f(18)\} + f(20)]$$

$$= 0 + 2\{16 + 29 + 40 + 46 + 51 + 32 + 18 + 8 + 3\} + 0 = 486 \text{ feet.}$$

Questions

1. What is the order of the trapezium rule for integrating $\int_a^b f(x)\,dx$? What is the expression for the error term?

   Solution The order of the trapezium rule is 1. The expression for the error term is

$$\text{Error} = -\frac{(b-a)^3}{12} f''(\xi) = -\frac{h^3}{12} f''(\xi), \quad \text{where } a \le \xi \le b.$$

2. When does the trapezium rule for integrating $\int_a^b f(x)dx$ gives exact results?

   Solution Trapezium rule gives exact results when $f(x)$ is a polynomial of degree ≤ 1.

3. What is the restriction in the number of nodal points, required for using the trapezium rule for integrating $\int_a^b f(x)dx$?

   Solution There is no restriction in the number of nodal points, required for using the trapezium rule.

4. What is the geometric representation of the trapezium rule for integrating $\int_a^b f(x)dx$?

   Solution Geometrically, the right hand side of the trapezium rule is the area of the trapezoid with width $b - a$, and ordinates $f(a)$ and $f(b)$, which is an approximation to the area under the curve $y = f(x)$ above the x-axis and the ordinates $x = a$, and $x = b$.

5. State the composite trapezium rule for integrating $\int_a^b f(x)dx$, and give the bound on the error.

   Solution The composite trapezium rule is given by

$$\int_a^b f(x)dx = \frac{h}{2}\,[f(x_0) + 2\{f(x_1) + f(x_2) + \ldots + f(x_{n-1})\} + f(x_n)]$$

   where $nh = (b - a)$. The bound on the error is given by

$$|\text{ Error }| \le \frac{nh^3}{12} M_2 = \frac{(b-a)h^2}{12} M_2$$

   where $\quad M_2 = \max_{a \le x \le b} |f''(x)|$ and $nh = b - a$.

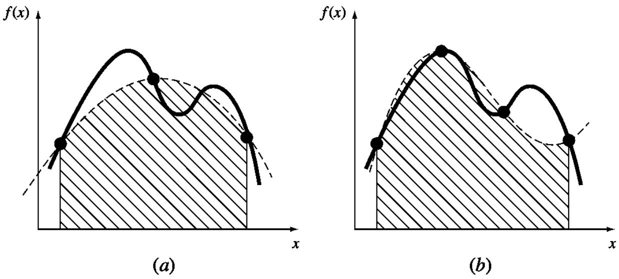
6. What is the geometric representation of the composite trapezium rule for integrating $\int_a^b f(x)dx$?

   Solution Geometrically, the right hand side of the composite trapezium rule is the sum of areas of the n trapezoids with width h, and ordinates $f(x_{i-1})$ and $f(x_i)$ $i = 1, 2, \ldots, n$. This sum is an approximation to the area under the curve $y = f(x)$ above the x-axis and the ordinates $x = a$ and $x = b$.

SIMPSON'S RULES

Aside from applying the trapezoidal rule with finer segmentation, another way to obtain a more accurate estimate of an integral is to use higher-order polynomials to connect the points. For example, if there is an extra point midway between f(a) and f(b), the three points can be connected with a parabola (Fig. 21.10a). If there are two points equally spaced between f(a) and f(b), the four points can be connected with a third-order polynomial (Fig. 21.10b). The formulas that result from taking the integrals under these polynomials are called Simpson's rules.

FIGURE 21.10
(*a*) Graphical depiction of Simpson's 1/3 rule: It consists of taking the area under a parabola connecting three points. (*b*) Graphical depiction of Simpson's 3/8 rule: It consists of taking the area under a cubic equation connecting four points.



$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

where, for this case, $h = (b - a)/2$.

Simpson's 1/3 rule can also be expressed using the format of Eq. (21.5):

$$I \cong \underbrace{(b-a)}_{\text{Width}}\underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{Average height}} \tag{21.15}$$

This equation is known as Simpson's 1/3 rule. It is the second Newton-Cotes closed integration formula. The label "1/3" stems from the fact that h is divided by 3

Composite Simpson's 1/3 Rule

We note that the Simpson's rule derived earlier uses three nodal points. Hence, we subdivide the given interval $[a, b]$ into even number of subintervals of equal length h. That is, we obtain an *odd number* of nodal points. The step length is given by $h = (b - a)/(n)$. The nodal points are given by
$a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$.

Just as with the trapezoidal rule, Simpson's rule can be improved by dividing the integration interval into a number of segments of equal width (Fig. 21.11):

$$h = \frac{b-a}{n} \tag{21.17}$$

The total integral can be represented as

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

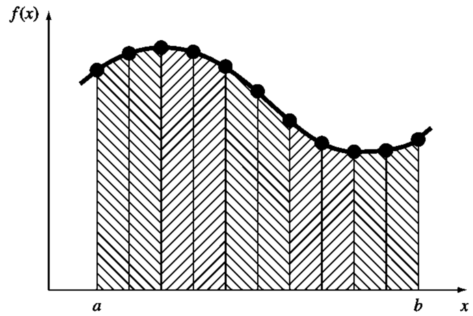
Substituting Simpson's 1/3 rule for the individual integral yields

$$I \cong 2h\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \cdots + 2h\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

or, combining terms and using Eq. (21.17),

$$I \cong \underbrace{(b-a)}_{\text{Width}}\underbrace{\frac{f(x_0) + 4\sum\limits_{i=1,3,5}^{n-1} f(x_i) + 2\sum\limits_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}} \tag{21.18}$$

FIGURE 21.11
Graphical representation of the multiple application of Simpson's 1/3 rule. Note that the method can be employed only if the number of segments is even.



Notice that, as illustrated in Fig. 21.11, an even number of segments must be utilized to implement the method. In addition, the coefficients "4" and "2" in Eq. (21.18) might seem peculiar at first glance. However, they follow naturally from Simpson's 1/3 rule. The odd points represent the middle term for each application and hence carry the weight of 4 from Eq. (21.15). The even points are common to adjacent applications and hence are counted twice.

An error estimate for the multiple-application Simpson's rule is obtained in the same fashion as for the trapezoidal rule by summing the individual errors for the segments and averaging the derivative to yield

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)} \tag{21.19}$$

where $\bar{f}^{(4)}$ is the average fourth derivative for the interval.

We have noted that the Simpson 1/3 rule and the composite Simpson's 1/3 rule are of order 3. This can be verified from the error expressions. If $f(x)$ is a polynomial of degree ≤ 3, then $f^{(4)}(x) = 0$. This result implies that error is zero and the composite Simpson's 1/3 rule produces exact results for polynomials of degree ≤ 3.

Multiple-application version of Simpson's 1/3 rule yields very accurate results. For this reason, it is considered superior to the trapezoidal rule for most applications. However, as mentioned previously, it is limited to cases where the values are equi-spaced. Further, it is limited to situations where there are an even number of segments and an odd number of points.

Example 3.16 *Find the approximate value of* $I = \int_0^1 \frac{dx}{1+x}$, *using the Simpson's* 1/3 *rule with* 2, 4 *and* 8 *equal subintervals. Using the exact solution, find the absolute errors.*

Solution With $n = 2N = 2$, 4 and 8, or $N = 1, 2, 4$ we have the following step lengths and nodal points.

$N = 1$: $h = \frac{b-a}{2N} = \frac{1}{2}$. The nodes are 0, 0.5, 1.0.

$N = 2$: $h = \frac{b-a}{2N} = \frac{1}{4}$. The nodes are 0, 0.25, 0.5, 0.75, 1.0.

$N = 4$: $h = \frac{b-a}{2N} = \frac{1}{8}$. The nodes are 0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1.0.

We have the following tables of values.

$n = 2N = 2$:

x	0	0.5	1.0
$f(x)$	1.0	0.666667	0.5

$n = 2N = 4$: We require the above values. The additional values required are the following.

x	0.25	0.75
$f(x)$	0.8	0.571429

$n = 2N = 8$: We require the above values. The additional values required are the following.

x	0.125	0.375	0.625	0.875
$f(x)$	0.888889	0.727273	0.615385	0.533333

Now, we compute the value of the integral.

$$n = 2N = 2: \quad I_1 = \frac{h}{3}\,[f(0) + 4f(0.5) + f(1.0)]$$

$$= \frac{1}{6}\,[1.0 + 4(0.666667) + 0.5] = 0.674444.$$

$$n = 2N = 4: \quad I_2 = \frac{h}{3}\,[f(0) + 4\{f(0.25) + f(0.75)\} + 2f(0.5) + f(1.0)]$$

$$= \frac{1}{12}\,[1.0 + 4\,\{0.8 + 0.571429\} + 2(0.666667) + 0.5] = 0.693254.$$

$$n = 2N = 8: \quad I_3 = \frac{h}{3}\,[f(0) + 4\{f(0.125) + f(0.375) + f(0.625) + f(0.875)\}$$

$$+ 2\{f(0.25) + f(0.5) + f(0.75)\} + f(1.0)]$$

$$= \frac{1}{24}\,[1.0 + 4\,\{0.888889 + 0.727273 + 0.615385 + 0.533333\}$$

$$+ 2\,\{0.8 + 0.666667 + 0.571429\} + 0.5]$$

$$= 0.693155.$$

The exact value of the integral is $I = \ln 2 = 0.693147$.

The errors in the solutions are the following:

$$|\text{ Exact} - I_1\,| = |\,0.693147 - 0.694444\,| = 0.001297.$$
$$|\text{ Exact} - I_2\,| = |\,0.693147 - 0.693254\,| = 0.000107.$$
$$|\text{ Exact} - I_3\,| = |\,0.693147 - 0.693155\,| = 0.000008.$$

Example 3.17 *Evaluate* $I = \int_1^2 \frac{dx}{5+3x}$, *using the Simpson's* 1/3 *rule with* 4 *and* 8 *subintervals. Compare with the exact solution and find the absolute errors in the solutions.*

Solution With $N = 2N = 4$, 8 or $N = 2$, 4, we have the following step lengths and nodal points.

$N = 2$: $h = \frac{b-a}{2N} = \frac{1}{4}$. The nodes are 1, 1.25, 1.5, 1.75, 2.0.

$N = 4$: $h = \frac{b-a}{2N} = \frac{1}{8}$. The nodes are 1, 1.125, 1.25, 1.375, 1.5, 1.675, 1.75, 1.875, 2.0.

We have the following tables of values.

$n = 2N = 4$:

x	1.0	1.25	1.5	1.75	2.0
$f(x)$	0.125	0.11429	0.10526	0.09756	0.09091

$n = 2N = 8$: We require the above values. The additional values required are the following.

x	1.125	1.375	1.625	1.875
$f(x)$	0.11940	0.10959	0.10127	0.09412

Now, we compute the value of the integral.

$$n = 2N = 4: \quad I_1 = \frac{h}{3}\,[f(1) + 4\{(1.25) + f(1.75)\} + 2f(1.5) + f(2.0)]$$

$$= \frac{0.25}{3}\,[0.125 + 4\{0.11429 + 0.09756\} + 2(0.10526) + 0.09091]$$

$$= 0.10615.$$

$$n = 2N = 8: \quad I_2 = \frac{h}{3}\,[f(1) + 4\{f(1.125) + f(1.375) + f(1.625) + f(1.875)\}$$

$$+ 2\{f(1.25) + f(1.5) + f(1.75)\} + f(2.0)]$$

$$= \frac{0.125}{3}\,[0.125 + 4\{0.11940 + 0.10959 + 0.10127 + 0.09412\}$$

$$+ 2\{0.11429 + 0.10526 + 0.09756\} + 0.09091]$$

$$= 0.10615.$$

The exact value of the integral is $I = \frac{1}{3}\,[\ln 11 - \ln 8] = 0.10615.$

The results obtained with $n = 2N = 4$ and $n = 2N = 8$ are accurate to all the places.

Example 3.18 *Using Simpson's 1/3 rule, evaluate the integral* $I = \int_0^1 \frac{dx}{x^2 + 6x + 10}$, *with 2 and 4 subintervals. Compare with the exact solution.*

Solution With $n = 2N = 2$ and 4, or $N = 1, 2$, we have the following step lengths and nodal points.

$N = 1$: $h = 0.5$. The nodes are 0.0, 0.5, 1.0.

$N = 2$: $h = 0.25$. The nodes are 0.0, 0.25, 0.5, 0.75, 1.0.

We have the following values of the integrand.

$n = 2N = 2$:

x	0.0	0.5	1.0
$f(x)$	0.1	0.07547	0.05882

$n = 2N = 4$: We require the above values. The additional values required are the following.

x	0.25	0.75
$f(x)$	0.08649	0.06639

Now, we compute the value of the integral.

$$n = 2N = 2: \quad I_1 = \frac{h}{3}[f(0.0) + 4\, f(0.5) + f(1.0)]$$

$$= \frac{0.5}{3}[0.1 + 4(0.07547) + 0.05882] = 0.07678.$$

$$n = 2N = 4: \quad I_2 = \frac{h}{3}[f(0.0) + 4\,\{f(0.25) + f(0.75)\} + 2\, f(0.5) + f(1.0)]$$

$$= \frac{0.25}{3}[0.1 + 4(0.08649 + 0.06639) + 2(0.07547) + 0.05882] = 0.07677.$$

Example 3.19 *The velocity of a particle which starts from rest is given by the following table.*

t (sec)	0	2	4	6	8	10	12	14	16	18	20
v (ft/sec)	0	16	29	40	46	51	32	18	8	3	0

Evaluate using Simpson's 1/3 rule, the total distance travelled in 20 seconds.

Solution From the definition, we have

$$v = \frac{ds}{dt}, \quad \text{or} \quad s = \int v\, dt.$$

Starting from rest, the distance travelled in 20 seconds is

$$s = \int_0^{20} v\, dt.$$

The step length is $h = 2$. Using the Simpson's rule, we obtain

$$s = \frac{h}{3}[f(0) + 4\{f(2) + f(6) + f(10) + f(14) + f(18)\} + 2\{f(4) + f(8) + f(12) + f(16)\} + f(20)]$$

$$= \frac{2}{3}[0 + 4\{16 + 40 + 51 + 18 + 3\} + 2\{29 + 46 + 32 + 8\} + 0]$$

$$= 494.667 \text{ feet.}$$

21.2.3 Simpson's 3/8 Rule

In a similar manner to the derivation of the trapezoidal and Simpson's 1/3 rule, a third-order Lagrange polynomial can be fit to four points and integrated:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_3(x)\,dx$$

to yield

$$I \cong \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

where $h = (b - a)/3$. This equation is called *Simpson's* 3/8 *rule* because h is multiplied by 3/8. It is the third Newton-Cotes closed integration formula. The 3/8 rule can also be expressed in the form of Eq. (21.5):

$$I \cong \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{Average height}} \tag{21.20}$$

Thus, the two interior points are given weights of three-eighths, whereas the end points are weighted with one-eighth. Simpson's 3/8 rule has an error of

$$E_t = -\frac{3}{80}h^5 f^{(4)}(\xi)$$

or, because $h = (b - a)/3$,

$$E_t = -\frac{(b - a)^5}{6480} f^{(4)}(\xi) \tag{21.21}$$

Because the denominator of Eq. (21.21) is larger than for Eq. (21.16), the 3/8 rule is somewhat more accurate than the 1/3 rule.

Simpson's 1/3 rule is usually the method of preference because it attains third-order accuracy with three points rather than the four points required for the 3/8 version. However, the 3/8 rule has utility when the number of segments is odd. For instance, in Example 21.5 we used Simpson's rule to integrate the function for four segments. Suppose that you desired an estimate for five segments. One option would be to use a multiple-application version of the trapezoidal rule as was done in Examples 21.2 and 21.3. This may not be advisable, however, because of the large truncation error associated with this method. An alternative would be to apply Simpson's 1/3 rule to the first two segments and Simpson's 3/8 rule to the last three (Fig. 21.12). In this way, we could obtain an estimate with third-order accuracy across the entire interval.

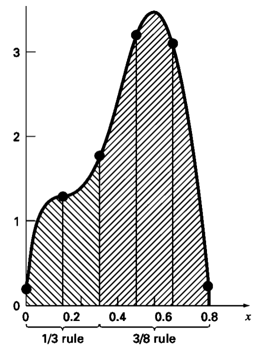


FIGURE 21.12 Illustration of how Simpson's 1/3 and 3/8 rules can be applied in tandem to handle multiple applications with odd numbers of intervals.

As in the case of the Simpson's 1/3 rule, if the length of the interval $[a, b]$ is large, then $b - a$ is also large and the error expression given becomes meaningless. In this case, we subdivide $[a, b]$ into a number of subintervals of equal length such that the number of subintervals is divisible by 3. That is, the number of intervals must be 6 or 9 or 12 etc., so that we get 7 or 10 or 13 nodal points etc. Then, we apply the Simpson's 3/8 rule to evaluate each integral. The rule is then called the *composite Simpson's* 3/8 *rule.* For example, if we divide $[a, b]$ into 6 parts, then we get the seven nodal points as

$x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, \ldots, x_6 = x_0 + 6h$

The Simpson's 3/8 rule becomes

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx$$

$$= \frac{3h}{8}\,[\{f(x_0) + 3\,f(x_1) + 3\,f(x_2) + f(x_3)\} + \{f(x_3) + 3f(x_4) + 3\,f(x_5) + f\,(x_6)\}]$$

$$= \frac{3h}{8}\,[f(x_0) + 3\,f(x_1) + 3\,f(x_2) + 2\,f(x_3) + 3\,f(x_4) + 3\,f(x_5) + f(x_6)]$$

.

Example 3.20 *Using the Simpson's* 3/8 *rule, evaluate* $I = \int_1^2 \frac{dx}{5+3x}$ *with* 3 *and* 6 *subintervals.*

Compare with the exact solution.

Solution With $n = 3N = 3$ and 6, we have the following step lengths and nodal points.

$n = 3N = 3$: $\quad h = \frac{b-a}{3N} = \frac{1}{3}$. The nodes are 1, 4/3, 5/3, 2.0.

$n = 3N = 6$: $\quad h = \frac{b-a}{3N} = \frac{1}{6}$. The nodes are 1, 7/6, 8/6, 9/6, 10/6, 11/6, 2.0

We have the following tables of values.

$n = 3N = 3$:

x	1.0	4/3	5/3	2.0
$f(x)$	0.125	0.11111	0.10000	0.09091

$n = 3N = 6$: We require the above values. The additional values required are the following.

x	7/6	9/6	11/6
$f(x)$	0.11765	0.10526	0.09524

Now, we compute the value of the integral.

$n = 3N = 3$: $\quad I_1 = \frac{3h}{8}\,[f(1) + 3\,f(4/3) + 3\,f(5/3) + f(2.0)]$

$= 0.125[0.125 + 3\{0.11111 + 0.10000\} + 0.09091] = 0.10616.$

$n = 3N = 6$: $\quad I_2 = \frac{3h}{8}\,[f(1) + 3\{f(7/6) + f(8/6) + f(10/6) + f(11/6)\}$

$+ 2\,f(9/6)\} + f(2.0)]$

$$= \frac{1}{16}\,[0.125 + 3\,\{0.11765 + 0.11111 + 0.10000 + 0.09524\}$$

$$+ 2(0.10526) + 0.09091] = 0.10615.$$

The exact value of the integral is $I = \frac{1}{3}\,[\log 11 - \log 8] = 0.10615.$

The magnitude of the error for $n = 3$ is 0.00001 and for $n = 6$ the result is correct to all places.

GAUSS QUADRATURE or GAUSS-LEGENDRE formulas

A characteristic of Newton-Cotes formulas was that the integral estimate was based on evenly spaced function values. Consequently, the location of the base points used in these equations was predetermined or fixed.

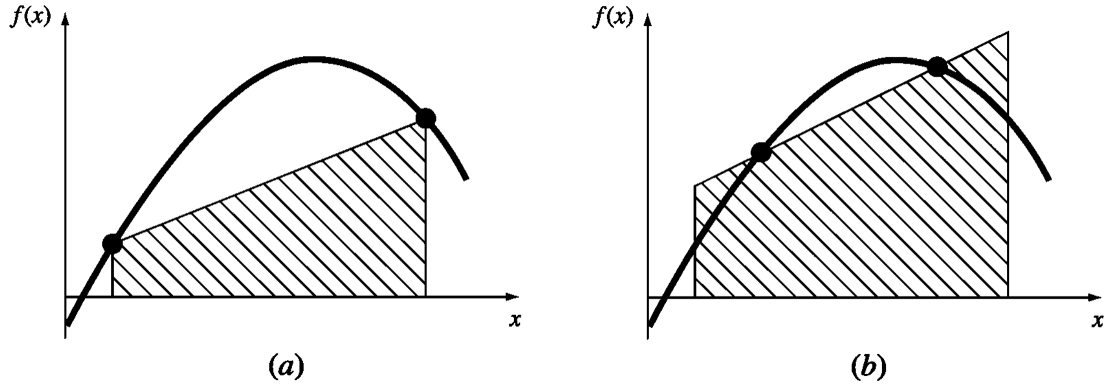


FIGURE 22.6

(*a*) Graphical depiction of the trapezoidal rule as the area under the straight line joining fixed end points. (*b*) An improved integral estimate obtained by taking the area under the straight line passing through two intermediate points. By positioning these points wisely, the positive and negative errors are balanced, and an improved integral estimate results.

For example, as depicted in Fig. 22.6*a*, the trapezoidal rule is based on taking the area under the straight line connecting the function values at the ends of the integration interval. The formula that is used to compute this area is

$$I \cong (b - a)\frac{f(a) + f(b)}{2} \tag{22.16}$$

where a and b = the limits of integration and $b - a$ = the width of the integration interval. Because the trapezoidal rule must pass through the end points, there are cases such as Fig. 22.6*a* where the formula results in a large error.

Now, suppose that the constraint of fixed base points was removed and we were free to evaluate the area under a straight line joining any two points on the curve. By positioning these points wisely, we could define a straight line that would balance the positive and negative errors. Hence, as in Fig. 22.6*b*, we would arrive at an improved estimate of the integral.

Gauss quadrature is the name for one class of techniques to implement such a strategy. The particular Gauss quadrature formulas described in this section are called *Gauss-Legendre* formulas. Before describing the approach, we will show how numerical integration formulas such as the trapezoidal rule can be derived using the method of undetermined coefficients. This method will then be employed to develop the Gauss-Legendre formulas.

Therefore, the two point Gauss rule (Gauss-Legendre rule) is given by

$$\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

Therefore, the three point Gauss rule (Gauss-Legendre rule) is given by

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right].$$

As mentioned earlier, the limits of integration for Gauss-Legendre integration rules are $[-1, 1]$. Therefore, we transform the limits $[a, b]$ to $[-1, 1]$, using a linear transformation.

Let the transformation be $x = pt + q$.

When $x = a$, we have $t = -1$: $a = -p + q$.

When $x = b$, we have $t = 1$: $b = p + q$.

Solving, we get $p(b - a)/2$, $q = (b + a)/2$.

The required transformation is $x = \frac{1}{2}[(b - a)t + (b + a)]$. (3.75)

Then, $f(x) = f\{[(b - a)t + (b + a)]/2\}$ and $dx = [(b - a)/2]dt$.

The integral becomes

$$I = \int_a^b f(x)\,dx = \int_{-1}^{1} f\left\{\frac{1}{2}[(b-a)t + (b+a)]\right\}\left\{\frac{1}{2}(b-a)\right\}dt = \int_{-1}^{1} g(t)\,dt \qquad (3.76)$$

where $g(t) = \left\{\frac{1}{2}(b-a)\right\} f\left\{\frac{1}{2}[(b-a)t + (b+a)]\right\}$.

4.1 Example 1

Let us take some examples. Suppose we are interested in approximating the integral $\int_1^2 \frac{1}{x}dx$

Here, $a = 1$, $b = 2$, $f(x) = \frac{1}{x}$,

Substitute $x = \frac{2-1}{2}t + \frac{2+1}{2}$, that is,

x = t/2 + 3/2

$$\therefore \int_1^2 \frac{1}{x}dx = \frac{1}{2}\int_{-1}^{1} f\left(\frac{t}{2} + \frac{3}{2}\right)dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{2}{t+3}dt$$

$$= \frac{1}{2}\left[\frac{2}{\left(3 - \frac{1}{\sqrt{3}}\right)} + \frac{2}{\left(3 + \frac{1}{\sqrt{3}}\right)}\right]$$

$$= \frac{2}{2}\left[\frac{1}{\left(3-\frac{1}{\sqrt{3}}\right)}+\frac{1}{\left(3+\frac{1}{\sqrt{3}}\right)}\right]$$

$$= \frac{3+\frac{1}{\sqrt{3}}+3-\frac{1}{\sqrt{3}}}{\left(3-\frac{1}{\sqrt{3}}\right)\left(3+\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{6}{\left(9-\frac{1}{3}\right)} = \frac{6\times 3}{(27-1)} = \frac{18}{26} = \frac{9}{13}$$

=0.6923076923

Example 3.23 *Evaluate the integral* $I = \int_1^2 \frac{2x}{1+x^4}\,dx$, *using Gauss one point, two point and three point rules. Compare with the exact solution* $I = \tan^{-1}(4) - (\pi/4)$.

Solution We reduce the interval [1, 2] to [– 1, 1] to apply the Gauss rules.

Writing $x = a\,t + b$, we get

$1 = -a + b,\ 2 = a + b.$

Solving, we get $a = 1/2$, $b = 3/2$. Therefore, $x = (t + 3)/2$, $dx = dt/2$.

The integral becomes

$$I = \int_{-1}^{1} \frac{8(t+3)}{[16+(t+3)^4]}\,dt = \int_{-1}^{1} f(t)\,dt$$

where $f(t) = 8(t + 3)/[16 + (t + 3)^4]$.

Using the two point Gauss rule, we obtain

$$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = f(-0.577350) + f(0.577350)$$

$$= 0.384183 + 0.159193 = 0.543376.$$

Using the three point Gauss rule, we obtain

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\ [5\ f(-0.774597) + 8\ f(0) + 5f(0.774597)]$$

$$= \frac{1}{9}\ [5(0.439299) + 8(0.247423) + 5(0.137889) = 0.540592.$$

Example 3.24 *Evaluate the integral* $I = \int_0^1 \frac{dx}{1+x}$, *using the Gauss three point formula. Compare with the exact solution.*

Solution We reduce the interval [0, 1] to [– 1, 1] to apply the Gauss three point rule.

Writing $x = a\,t + b$, we get

$$0 = -a + b,\ 1 = a + b$$

Solving, we get $a = 1/2$, $b = 1/2$. Therefore, $x = (t + 1)/2$, $dx = dt/2$.

The integral becomes

$$I = \int_{-1}^{1} \frac{dt}{t+3} = \int_{-1}^{1} f(t)\,dt$$

where $f(t) = 1/(t + 3)$.

Using the three point Gauss rule, we obtain

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\ [5(0.449357) + 8(0.333333) + 5(0.264929)] = 0.693122.$$

Example 3.25 *Evaluate the integral* $I = \int_0^2 \frac{(x^2 + 2x + 1)}{1 + (x+1)^4}\,dx$, *by Gauss three point formula.*

Solution We reduce the interval [0, 2] to [– 1, 1] to apply the Gauss three point rule.

Writing $x = a\,t + b$, we get

$$0 = -a + b,\ 2 = a + b.$$

Solving, we get $a = 1$, $b = 1$. Therefore, $x = t + 1$, and $dx = dt$.

The integral becomes

$$I = \int_{-1}^{1} \frac{(t+2)^2}{1 + (t+2)^4}\,dt = \int_{-1}^{1} f(t)\,dt$$

where $f(t) = (t + 2)^2/[1 + (t + 2)^4]$.

Using the three point Gauss rule, we obtain

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\ [5(0.461347) + 8(0.235194) + 5(0.127742)] = 0.536422.$$

Questions

Why is Trapezoidal rule so called ?
Solution The trapezoidal rule is so called, because it approximates the integral by the sum of n trapezoids.
When does Simpson's rule give exact result?
Solution Simpson's rule will give exact result, if the entire curve y =f(x) is itself a parabola.
When do you apply Simpson's 1/3 rule?
Solution The interval of integration must be divided into an even number of subintervals of width h.
When does the Simpson's 1/3 rule for integration gives exact results?
Solution Simpson's 1/3 rule gives exact results when f(x) is a polynomial of degree ≤ 3.
What is the restriction in number of nodal points, required for using the Simpson's 1/3 rule for integrating?
Solution The number of nodal points must be odd for using the Simpson's 1/3 rule or the number of subintervals must be even.
What is the restriction in the number of nodal points, required for using the Simpson's 3/8 rule for integrating f(x) ?
Solution The number of subintervals must be divisible by 3.
What are the disadvantages of the Simpson's 3/8 rule compared with the Simpson's 1/3 rule?
Solution The disadvantages are the following: (i) The number of subintervals must be divisible by 3. (ii) It is of the same order as the Simpson's 1/3 rule, which only requires that the number of nodal points must be odd. (iii) The error constant c in the case of Simpson's 3/8 rule is c = 3/80, which is much larger than the error constant c = 1/90, in the case of Simpson's 1/3 rule. Therefore, the error in the case of the Simpson's 3/8 rule is larger than the error in the case Simpson 1/3 rule.
In which circumstances Gauss quadrature cannot be used?
Solution Because Gauss quadrature requires function evaluations at nonuniformly spaced points within the integration interval, it is not appropriate for cases where the function is unknown. Thus, it is not suited for engineering problems that deal with tabulated data. However, where the function is known, its efficiency can be a decided advantage.

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