A differential equation is an equation involving independent variable, dependent variable and its one or more derivatives.

This taylor series can be used to evaluate $y(x)$ at $x = x_{i+1} = x_i + h$ giving ...

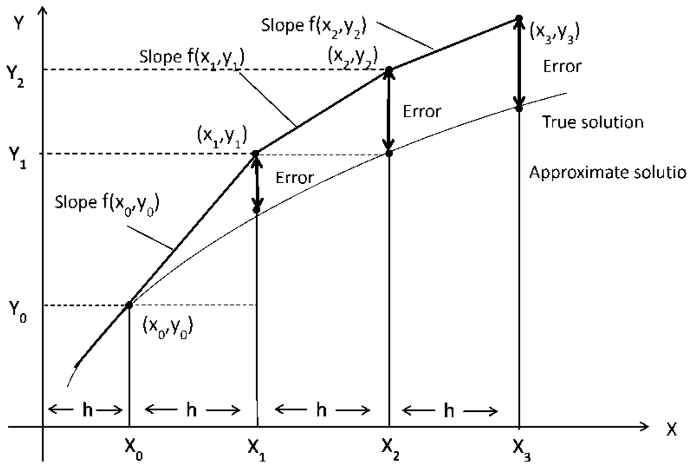
$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y_i^{(3)} + ...+ \frac{h^N}{N!}y_i^{(N)} + \frac{h^{N+1}}{(N+1)!}y^{N+1}(\xi)$$

12. Euler Formula

It is Taylor Series method of order 1, where N=1, second and higher powers of h are ignored; The Taylor formula for solving the IVP $\frac{dy}{dx} = f(x, y(x))$, $y(x_0) = y_0$,$a \le x \le b$; with step size h can be expressed as

$$y_{i+1} = y_i + h\,y'_i + O(h^2)$$

Ignoring second and higher powers of h , gives $y_{i+1} \approx y_i + h\,y'_i$; That is , same as , $y_{i+1} \approx y_i + h\,f(xi, yi)$, which has Simplest among all single step formulas.



All one-step methods can be expressed in this general form, with the only difference being the manner in which the slope is estimated. The simplest approach is to use the differential equation to estimate the slope in the form of the first derivative at x_i. In other words, the slope at the beginning of the interval is taken as an approximation of the average slope over the whole interval.

Example 4.2 *Solve the initial value problem* $yy' = x$, $y(0) = 1$, *using the Euler method in* $0 \le x \le 0.8$, *with* $h = 0.2$ *and* $h = 0.1$. *Compare the results with the exact solution at* $x = 0.8$. *Extrapolate the result.*

Solution We have $\quad y' = f(x, y) = (x/y).$

Euler method gives $\quad y_{i+1} = y_i + h\,f(x_i, y_i) = y_i + \dfrac{hx_i}{y_i}.$

Initial condition gives $\quad x_0 = 0,\ y_0 = 1.$

When $h = 0.2$, we get $\quad y_{i+1} = y_i + \dfrac{0.2\,x_i}{y_i}.$

We have the following results.

$$y(x_1) = y(0.2) \approx y_1 = y_0 + \frac{0.2\,x_0}{y_0} = 1.0.$$

$$y(x_2) = y(0.4) \approx y_2 = y_1 + \frac{0.2\,x_1}{y_1} = 1.0 + \frac{0.2(0.2)}{1.0} = 1.04.$$

$$y(x_3) = y(0.6) \approx y_3 = y_2 + \frac{0.2\,x_2}{y_2} = 1.04 + \frac{0.2(0.4)}{1.04} = 1.11692$$

$$y(x_4) = y(0.8) \approx y_4 = y_3 + \frac{0.2\,x_3}{y_3} = 1.11692 + \frac{0.2(0.6)}{1.11692} = 1.22436.$$

When $h = 0.1$, we get $y_{i+1} = y_i + \dfrac{0.1\,x_i}{y_i}$.

We have the following results.

$$y(x_1) = y(0.1) \approx y_1 = y_0 + \frac{0.1x_0}{y_0} = 1.0.$$

$$y(x_2) = y(0.2) \approx y_2 = y_1 + \frac{0.1x_1}{y_1} = 1.0 + \frac{0.1(0.1)}{1.0} = 1.01.$$

$$y(x_3) = y(0.3) \approx y_3 = y_2 + \frac{0.1x_2}{y_2} = 1.01 + \frac{0.1(0.2)}{1.01} = 1.02980.$$

$$y(x_4) = y(0.4) \approx y_4 = y_3 + \frac{0.1x_3}{y_3} = 1.0298 + \frac{0.1(0.3)}{1.0298} = 1.05893.$$

$$y(x_5) = y(0.5) \approx y_5 = y_4 + \frac{0.1x_4}{y_4} = 1.05893 + \frac{0.1(0.4)}{1.05893} = 1.09670.$$

$$y(x_6) = y(0.6) \approx y_6 = y_5 + \frac{0.1x_5}{y_5} = 1.0967 + \frac{0.1(0.5)}{1.0967} = 1.14229.$$

$$y(x_7) = y(0.7) \approx y_7 = y_6 + \frac{0.1x_6}{y_6} = 1.14229 + \frac{0.1(0.6)}{1.14229} = 1.19482.$$

$$y(x_8) = y(0.8) \approx y_8 = y_7 + \frac{0.1x_7}{y_7} = 1.19482 + \frac{0.1(0.7)}{1.19482} = 1.25341.$$

Example 4.3 *Consider the initial value problem* $y' = x(y + 1)$, $y(0) = 1$. *Compute* $y(0.2)$ *with* $h = 0.1$ *using (i) Euler method (ii) Taylor series method of order two, and (iii) fourth order Taylor series method. If the exact solution is* $y = -1 + 2e^{x^2/2}$, *find the magnitudes of the actual errors for* $y(0.2)$. *In the solutions obtained by the Euler method, find the estimate of the errors.*

Solution We have $f(x, y) = x(y + 1)$, $x_0 = 0$, $y_0 = 1$.

(*i*) Euler's method: $y_{i+1} = y_i + h\,f(x_i, y_i) = y_i + 0.1[x_i\,(y_i + 1)]$.

With $x_0 = 0$, $y_0 = 1$, we get

$$y(0.1) \approx y_1 = y_0 + 0.1[x_0\,(y_0 + 1)] = 1 + 0.1[0] = 1.0.$$

With $x_1 = 0.1$, $y_1 = 1.0$, we get

$$y(0.2) \approx y_2 = y_1 + 0.1\,[x_1(y_1 + 1)]$$
$$= 1.0 + 0.1[(0.1)(2)] = 1.02.$$

(*ii*) Taylor series second order method.

$$y_{i+1} = y_i + h\,y_i' + \frac{h^2}{2!}\,y_i'' = y_i + 0.1\,y_i' + 0.005y_i''.$$

We have $\quad y'' = xy' + y + 1.$

With $x_0 = 0$, $y_0 = 1$, we get

$$y_0' = 0,\ y_0'' = x_0y_0' + y_0\ + 1 = 0 + 1 + 1 = 2.$$
$$y(0.1) \approx y_1 = y_0 + 0.1y_0' + 0.005y_0''$$
$$= 1 + 0 + 0.005\,[\,2] = 1.01.$$

With $x_1 = 0.1$, $y_1 = 1.01$, we get

$$y_1' = 0.1(1.01 + 1) = 0.201.$$
$$y_1'' = x_1y_1' + y_1 + 1 = (0.1)(0.201) + 1.01 + 1 = 2.0301.$$
$$y(0.2) \approx y_2 = y_1 + 0.1\,y_1' + 0.005y_1'$$
$$= 1.01 + 0.1\,(0.201) + 0.005(2.0301)] = 1.04025.$$

Example 8.5 To illustrate Euler's method, we consider the differential equation $y' = -y$ with the condition $y(0) = 1$.

Successive application of Eq. (8.8) with $h = 0.01$ gives

$$y(0.01) = 1 + 0.1(-1) = 0.99$$
$$y(0.02) = 0.99 + 0.01\,(-0.99) = 0.9801$$
$$y(0.03) = 0.9801 + 0.01\,(-0.9801) = 0.9703$$
$$y(0.04) = 0.9703 + 0.01\,(-0.9703) = 0.9606.$$

The exact solution is $y = e^{-x}$ and from this the value at $x = 0.04$ is 0.9608.

Runge-Kutta methods

Runge-Kutta (*RK*) methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives. The basic idea of Runge-Kutta methods is to approximate the integral by a weighted average of slopes and approximate slopes at a number of points in $[x_i, x_{i+1}]$.

Runge-Kutta method of fourth order The most commonly used Runge-Kutta method is a method which uses four slopes. The method is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{4.51}$$

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right),$$

$$k_4 = hf(x_i + h, y_i + k_3).$$

Example 4.8 *Solve the initial value problem*

$$y' = -2xy^2, \; y(0) = 1$$

with $h = 0.2$ *on the interval* $[0, 0.4]$. *Use* (*i*) *the Heun's method (second order Runge-Kutta method);* (*ii*) *the fourth order classical Runge-Kutta method. Compare with the exact solution* $y(x) = 1/(1 + x^2)$.

Solution

(*ii*) For $i = 0$, we have $x_0 = 0$, $y_0 = 1$.

$$k_1 = hf(x_0, y_0) = -2(0.2)(0)(1)^2 = 0,$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = -2(0.2)(0.1)(1)^2 = -0.04,$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = -2\,(0.2)(0.1)(0.98)^2 = -0.038416,$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = -2(0.2)(0.2)(0.961584)^2 = -0.0739715,$$

$$y(0.2) \approx y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.0 + \frac{1}{6}[0.0 - 0.08 - 0.076832 - 0.0739715] = 0.9615328.$$

For $i = 1$, we have $\quad x_1 = 0$, $y_1 = 0.9615328$.

$$k_1 = hf(x_1, y_1) = -2(0.2)(0.2)(0.9615328)^2 = -\,0.0739636,$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right) = -2(0.2)(0.3)\ (0.924551)^2 = -\ 0.1025753,$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_2\right) = -2(0.2)(0.3)(0.9102451)^2 = -\ 0.0994255,$$

$$k_4 = hf\ (x_1 + h,\ y_1 + k_3) = -2(0.2)(0.4)(0.86210734)^2 = -\ 0.1189166,$$

$$y(0.4) \approx y_2 = y_1 + \frac{1}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.9615328 + \frac{1}{6}\ [-0.0739636\ -0.2051506\ -0.1988510\ -$$

$$0.1189166]$$

$$= 0.8620525$$

Example 4.9 *Given $y' = x^3 + y$, $y(0) = 2$, compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ using the Runge-Kutta method of fourth order.*

Solution We have $x_0 = 0,\ y_0 = 2,\ f(x, y) = x^3 + y,\ h = 0.2.$

For $i = 0$, we have $x_0 = 0,\ y_0 = 2.$

$$k_1 = hf(x_0, y_0) = 0.2\ f\ (0, 2) = (0.2)(2) = 0.4,$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = 0.2\ f(0.1,\ 2.2)$$

$$= (0.2)(2.201) = 0.4402,$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = 0.2\ f(0.1,\ 2.2201)$$

$$= (0.2)(2.2211) = 0.44422,$$

$$k_4 = hf\ (x_0 + h,\ y_0 + k_3) = 0.2\ f\ (0.2,\ 2.44422)$$

$$= (0.2)(2.45222) = 0.490444,$$

$$y(0.2) \approx y_1 = y_0 + \frac{1}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.0 + \frac{1}{6}\,[0.4 + 2(0.4402) + 2(0.44422) + 0.490444]$$

$$= 2.443214.$$

For $i = 1$, we have $\quad x_1 = 0.2,\ y_1 = 2.443214.$

$$k_1 = h\,f\,(x_1, y_1) = 0.2\,f(0.2,\ 2.443214) = (0.2)(2.451214) = 0.490243,$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3,\ 2.443214 + 0.245122)$$

$$= (0.2)(2.715336) = 0.543067,$$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3,\ 2.443214 + 0.271534)$$

$$= (0.2)(2.741748) = 0.548350,$$

$$k_4 = hf\,(x_1 + h,\ y_1 + k_3) = 0.2\,f(0.4,\ 2.443214 + 0.548350)$$

$$= (0.2)(3.055564) = 0.611113,$$

$$y(0.4) \approx y_2 = y_1 + \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.443214 + \frac{1}{6}\,[0.490243 + 2(0.543067) + 2(0.548350) + 0.611113]$$

$$= 2.990579.$$

For $i = 2$, we have $\quad x_2 = 0.4,\ y_2 = 2.990579.$

$$k_1 = hf\,(x_2, y_2) = 0.2\,f\,(0.4,\ 2.990579) = (0.2)(3.054579) = 0.610916,$$

$$k_2 = hf\left(x_2 + \frac{h}{2},\ y_2 + \frac{1}{2}k_1\right) = 0.2\,f(0.5,\ 2.990579 + 0.305458)$$

$$= (0.2)(3.421037) = 0.684207,$$

$$k_3 = hf\left(x_2 + \frac{h}{2},\ y_2 + \frac{1}{2}k_2\right) = 0.2f\,(0.5,\ 2.990579 + 0.342104)$$

$$= (0.2)(3.457683) = 0.691537,$$

$$k_4 = hf\,(x_2 + h,\ y_2 + k_3) = 0.2\,f\,(0.6,\ 2.990579 + 0.691537)$$

$$= (0.2)\,(3.898116) = 0.779623.$$

$$y(0.6) \approx y_3 = y_2 + \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.990579 + \frac{1}{6}\,[0.610916 + 2(0.684207) + 2(0.691537) + 0.779623]$$

$$= 3.680917.$$

Predictor Corrector Methods:

Now, we define the predictor-corrector methods. We denote P for predictor and C for corrector.

P: Predict an approximation to the solution y_{i+1} at the current point, using an explicit method. Denote this approximation as $y_{i+1}^{(p)}$.

C: Correct the approximation $y_{i+1}^{(p)}$, using a corrector, that is, an implicit method. Denote this corrected value as $y_{i+1}^{(c)}$. The corrector is used 1 or 2 or 3 times, depending on the orders of explicit and implicit methods used.

Milne Simpson's Predictor Corrector Method

Predictor P is given by:

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}[2f_i - f_{i-1} + 2f_{i-2}]. \tag{4.97}$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}.

Corrector C: Milne-Simpson's method of fourth order.

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1})]. \tag{4.98}$$

The method requires the starting values y_i, y_{i-1}.

The combination requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

Adam Bashforth Moulton Predictor Corrector Method

Predictor P: Adams-Bashforth method of fourth order.

$$y_{i+1}^{(p)} = y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}]. \tag{4.95}$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}.

Corrector C: Adams-Moulton method of fourth order.

$$y_{i+1}^{(c)} = y_i + \frac{h}{24}[9f(x_{i+1}, y_{i+1}^{(p)}) + 19f_i - 5f_{i-1} + f_{i-2}]. \tag{4.96}$$

The method requires the starting values y_i, y_{i-1}, y_{i-2}.

The combination requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

Given $\dfrac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by the Adams-Bashforth method.

Solution:

Here $f(x, y) = x^2(1 + y)$

Starting values of the Adams-Bashforth method with $h = 0.1$ are

$x = 1.0,$	$y_{-3} = 1.000,$	$f_{-3} = (1.0)^2(1 + 1.000) = 2.000$
$x = 1.1,$	$y_{-2} = 1.233,$	$f_{-2} = 2.702$
$x = 1.2,$	$y_{-1} = 1.548,$	$f_{-1} = 3.669$
$x = 1.3,$	$y_{0} = 1.979,$	$f_{0} = 5.035$

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}\left(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}\right)$$

$$x_4 = 1.4, \quad y_1^{(p)} = 2.573 \quad f_1 = 7.004$$

Using the *corrector*

$$y_1^{(c)} = y_0 + \frac{h}{24}\left(9f_1 + 19f_0 - 5f_{-1} + f_{-2}\right)$$

$$y_1^{(c)} = 1.979 + \frac{0.1}{24}\left(9\times 7.004 + 19\times 5.035 - 5\times 3.669 + 2.702\right) = 2.575$$

Hence $y(1.4) = 2.575$

Example 4.14 *Using the Adams-Bashforth predictor-corrector equations, evaluate $y(1.4)$, if y satisfies*

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$.

Solution The Adams-Bashforth predictor-corrector method is given by

Predictor P: Adams-Bashforth method of fourth order.

$$y_{i+1}^{(p)} = y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}.

Corrector C: Adams-Moulton method of fourth order.

$$y_{i+1}^{(c)} = y_i + \frac{h}{24}[9f(x_{i+1}, y_{i+1}^{(p)}) + 19f_i - 5f_{i-1} + f_{i-2}]$$

The method requires the starting values y_i, y_{i-1}, y_{i-2}.

The combination requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. With $h = 0.1$, we are given the values

$$y(1) = 1,\ y(1.1) = 0.996,\ y(1.2) = 0.986,\ y(1.3) = 0.972.$$

We have $\quad f(x, y) = \dfrac{1}{x^2} - \dfrac{y}{x}.$

Predictor application

For $i = 3$, we obtain

$$y_4^{(0)} = y_4^{(p)} = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

We have

$$f_0 = f(x_0, y_0) = f(1, 1) = 1 - 1 = 0,$$

$$f_1 = f(x_1, y_1) = f(1.1, 0.996) = -0.079008,$$

$$f_2 = f(x_2, y_2) = f(1.2, 0.986) = -0.127222,$$

$$f_3 = f(x_3, y_3) = f(1.3, 0.972) = -0.155976.$$

$$y_4^{(0)} = 0.972 + \frac{0.1}{24}[55(-0.155976) - 59(-0.127222) + 37(-0.079008) - 9(0)]$$

$$= 0.955351.$$

Corrector application

Now, $f(x_4, y_4^{(0)}) = f(1.4, 0.955351) = -\ 0.172189.$

First iteration

$$y_4^{(1)} = y_4^{(c)} = y_3 + \frac{h}{24}[9f(x_4, y_4^{(0)}) + 19f_3 - 5f_2 + f_1]$$

$$= 0.972 + \frac{0.1}{24}\ [9(-\ 0.172189) + 19(-\ 0.155976) - 5(-\ 0.127222)$$

$$+ (-\ 0.079008)] = 0.955516.$$

Second iteration

$$f(x_4, y_4^{(1)}) = f(1.4, 0.955516) = -\ 0.172307.$$

$$y_4^{(2)} = y_3 + \frac{h}{24}[9f(x_4, y_4^{(1)}) + 19f_3 - 5f_2 + f_1]$$

$$= 0.972 + \frac{0.1}{24}\ [9(-\ 0.172307) + 19(-\ 0.155976) - 5(-\ 0.127222)$$

$$+ (-\ 0.079008)] = 0.955512.$$

Now, $|\ y_4^{(2)} - y_4^{(1)}\ | = |\ 0.955512 - 0.955516\ | = 0.000004.$

Therefore, $y(1.4) = 0.955512$. The result is correct to five decimal places.

Example 4.15 *Given* $y' = x^3 + y$, $y(0) = 2$, *the values* $y(0.2) = 2.073$, $y(0.4) = 2.452$, *and* $y(0.6) = 3.023$ *are got by Runge-Kutta method of fourth order. Find* $y(0.8)$ *by Milne's predictor-corrector method taking* $h = 0.2$.

Solution Milne's predictor-corrector method is given by

Predictor P: Adams-Bashforth method of fourth order.

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}[2f_i - f_{i-1} + 2f_{i-2}].$$

Corrector C: Milne-Simpson's method of fourth order.

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

We are given that

$$f(x, y) = x^3 + y,\ x_0 = 0,\ y_0 = 2,\ y(0.2) = y_1 = 2.073,$$

$$y(0.4) = y_2 = 2.452,\ y(0.6) = y_3 = 3.023.$$

Predictor application

For $i = 3$, we obtain

$$y_4^{(0)} = y_4^{(p)} = y_0 + \frac{4(0.2)}{3}[2f_3 - f_2 + 2f_1].$$

We have $f_0 = f(x_0, y_0) = f(0, 2) = 2,$

$f_1 = f(x_1, y_1) = f(0.2, 2.073) = 2.081,$

$f_2 = f(x_2, y_2) = f(0.4, 2.452) = 2.516,$

$f_3 = f(x_3, y_3) = f(0.6, 3.023) = 3.239.$

$$y_4^{(0)} = 2 + \frac{0.8}{3}[2(3.239) - 2.516 + 2(2.081)] = 4.1664.$$

Corrector application

First iteration For $i = 3$, we get

$$y_4^{(1)} = y_2 + \frac{0.2}{3}[f(x_4, y_4^{(0)}) + 4f_3 + f_2]$$

Now, $f(x_4, y_4^{(0)}) = f(0.8, 4.1664) = 4.6784.$

$$y_4^{(1)} = 2.452 + \frac{0.2}{3}[4.6784 + 4(3.239) + 2.516] = 3.79536.$$

Second iteration

$$y_4^{(2)} = y_2 + \frac{0.2}{3}[f(x_4, y_4^{(1)}) + 4f_3 + f_2]$$

Now, $f(x_4, y_4^{(1)}) = f(0.8, 4.6784) = 4.30736.$

$$y_4^{(2)} = 2.452 + \frac{0.2}{3}[4.30736 + 4(3.239) + 2.516] = 3.770624.$$

We have $| y_4^{(2)} - y_4^{(1)} | = | 3.770624 - 3.79536 | = 0.024736.$

The result is accurate to one decimal place.

Third iteration

$$y_4^{(3)} = y_2 + \frac{0.2}{3}[f(x_4, y_4^{(2)}) + 4f_3 + f_2]$$

Now, $f(x_4, y_4^{(2)}) = f(0.8, 3.770624) = 4.282624.$

$$y_4^{(3)} = 2.452 + \frac{0.2}{3}[4.282624 + 4(3.239) + 2.516] = 3.768975.$$

We have $| y_4^{(3)} - y_4^{(2)} | = | 3.768975 - 3.770624 | = 0.001649.$

The result is accurate to two decimal places.

Fourth iteration

$$y_4^{(4)} = y_2 + \frac{0.2}{3}[f(x_4, y_4^{(3)}) + 4f_3 + f_2]$$

Now, $\quad f(x_4, y_4^{(3)}) = f(0.8, 3.76897) = 4.280975.$

$$y_4^{(4)} = 2.452 + \frac{0.2}{3}\ [4.280975 + 4(3.239) + 2.516] = 3.768865.$$

We have $\mid y_4^{(4)} - y_4^{(3)} \mid = \mid 3.768865 - 3.768975 \mid = 0.000100.$

The result is accurate to three decimal places.

The required result can be taken as $y(0.8) = 3.7689$.

Example 4.16 *Using Milne's predictor-corrector method, find* $y(0.4)$ *for the initial value problem*

$$y' = x^2 + y^2,\ y(0) = 1, \ \textit{with } h = 0.1.$$

Calculate all the required initial values by Euler's method. The result is to be accurate to three decimal places.

Solution Milne's predictor-corrector method is given by

Predictor P: Adams-Bashforth method of fourth order.

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}\ [2f_i - f_{i-1} + 2f_{i-2}]\ .$$

Corrector C: Milne-Simpson's method of fourth order.

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

We are given that

$$f(x, y) = x^2 + y^2,\ x_0 = 0,\ y_0 = 1.$$

Euler's method gives

$$y_{i+1} = y_i + hf(x_i, y_i) = y_i + 0.1(x_i^2 + y_i^2).$$

With $x_0 = 0$, $y_0 = 1$, we get

$$y_1 = y_0 + 0.1\ (x_0^2 + y_0^2) = 1.0 + 0.1\ (0 + 1.0) = 1.1.$$

$$y_2 = y_1 + 0.1\ (x_1^2 + y_1^2) = 1.1 + 0.1\ (0.01 + 1.21) = 1.222.$$

$$y_3 = y_2 + 0.1\ (x_2^2 + y_2^2) = 1.222 + 0.1[0.04 + (1.222)^2] = 1.375328.$$

Predictor application

For $i = 3$, we obtain

$$y_4^{(0)} = y_4^{(p)} = y_0 + \frac{4(0.1)}{3}[2f_3 - f_2 + 2f_1]$$

We have $f_1 = f(x_1, y_1) = f(0.1, 1.1) = 1.22,$

$f_2 = f(x_2, y_2) = f(0.1, 1.222) = 1.533284,$

$f_3 = f(x_3, y_3) = f(0.3, 1.375328) = 1.981527.$

$$y_4^{(0)} = 1.0 + \frac{0.4}{3}\,[2(1.981527) - 1.533284 + 2(1.22)] = 1.649303.$$

Corrector application

First iteration For $i = 3$, we get

$$y_4^{(1)} = y_2 + \frac{0.1}{3}[f(x_4, y_4^{(0)}) + 4f_3 + f_2]$$

Now, $f(x_4, y_4^{(0)}) = f(0.4, 1.649303) = 2.880200.$

$$y_4^{(1)} = 1.222 + \frac{0.1}{3}\,[2.880200 + 4(1.981527) + 1.533284] = 1.633320.$$

Second iteration

$$y_4^{(2)} = y_2 + \frac{0.1}{3}[f(x_4, y_4^{(1)}) + 4f_3 + f_2]$$

Now, $f(x_4, y_4^{(1)}) = f(0.4, 1.633320) = 2.827734.$

$$y_4^{(2)} = 1.222 + \frac{0.1}{3}\,[2.827734 + 4(1.981527) + 1.533284] = 1.631571.$$

We have $|\ y_4^{(2)} - y_4^{(1)}\ | = |\ 1.631571 - 1.633320\ | = 0.001749.$

The result is accurate to two decimal places.

Third iteration

$$y_4^{(3)} = y_2 + \frac{0.1}{3}[f(x_4, y_4^{(2)}) + 4f_3 + f_2]$$

Now, $f(x_4, y_4^{(2)}) = f(0.4, 1.631571) = 2.822024.$

$$y_4^{(3)} = 1.222 + \frac{0.1}{3}\,[2.822024 + 4(1.981527) + 1.533284] = 1.631381.$$

We have $|\ y_4^{(3)} - y_4^{(2)}\ | = |\ 1.631381 - 1.631571\ | = 0.00019.$

The result is accurate to three decimal places.

The required result can be taken as $y(0.4) \approx 1.63138.$

Questions

1. What is the disadvantage of the Taylor series method?
   Solution Taylor series method requires the computation of higher order derivatives.
2. Why the Runge-Kutta method is the most commonly used method?
3. **Solution** Using two slopes in the method, we have obtained methods of second order, which we have called as second order Runge-Kutta methods. The method has one arbitrary parameter, whose value is suitably chosen. The methods using four evaluations of slopes have two arbitrary parameters. The values of these parameters are chosen such that the method becomes simple for computations.

1. Are the multi step methods self starting ?

Solution Multi step methods are not self starting, since a k-step multi step method requires the k previous values $y_i, y_{i-1}, ..., y_{i-k+1}$. The k values that are required for starting the application of the method are obtained using some single step method like Euler method, Taylor series method or Runge-Kutta method, which is of the same or lower order than the order of the multi step method.

2. Why do we require predictor-corrector methods for solving the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$?

Solution If we perform analysis for numerical stability of single or multi step methods, we find that all explicit methods require very small step lengths to be used for convergence. If the solution of the problem is required over a large interval, we may need to use the method, thousands or even millions of steps, which is computationally very expensive. Most implicit methods have strong stability properties, that is, we can use sufficiently large step lengths for computations and we can obtain convergence. However, we need to solve a nonlinear algebraic equation for the solution at each nodal point. This procedure may also be computationally expensive as convergence is to be obtained for the solution of the nonlinear equation at each nodal point. Therefore, we combine the explicit methods (which have weak stability properties) and implicit methods (which have strong stability properties) to obtain new methods. Such methods are called *predictor-corrector methods.*

3. What are predictor-corrector methods for solving the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$? Comment on the order of the methods used as predictors and correctors.

Solution We combine the explicit methods (which have weak stability properties) and implicit methods (which have strong stability properties) to obtain new methods. Such methods are called *predictor-corrector methods.* We denote P for predictor and C for corrector.

P: Predict an approximation to the solution y_{i+1} at the current point, using an explicit method. Denote this approximation as $y_{i+1}^{(p)}$.

C: Correct the approximation $y_{i+1}^{(p)}$, using a corrector, that is, an implicit method. Denote this corrected value as $y_{i+1}^{(c)}$. The corrector is used 1 or 2 or 3 times, depending on the orders of explicit and implicit methods used.

The order of the predictor should be less than or equal to the order of the corrector. If the orders of the predictor and corrector are same, then we may require only one or two corrector iterations at each nodal point. For example, if the predictor and corrector are both of fourth order, then the combination (*P-C* method) is also of fourth order and we may require one or two corrector iterations at each point. If the order of the predictor is less than the order of the corrector, then we require more iterations of the corrector. For example, if we use a first order predictor and a second order corrector, then one application of the combination gives a result of first order. If corrector is iterated once more, then the order of the combination increases by one, that is, the result is now of second order. If we iterate a third time, then the truncation error of the combination reduces, that is, we may get a better result. Further iterations may not change the results.

4. Write Adams-Bashforth predictor-corrector method for solving the initial value problem $y' = f(x, y)$, $y(x_0) = b_0$. Comment on the order and the required starting values.

Solution The Adams-Bashforth predictor-corrector method is given by

Predictor P: Adams-Bashforth method of fourth order.

$$y_{i+1}^{(p)} = y_i + \frac{h}{24} [55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}].$$

$$\text{Error term} = \frac{251}{720} h^5 f^{(4)} (\xi_4) = \frac{251}{720} h^5 y^{(5)} (\xi_4).$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}.

Corrector C: Adams-Moulton method of fourth order.

$$y_{i+1}^{(c)} = y_i + \frac{h}{24} [9f(x_{i+1}, y_{i+1}^{(p)}) + 19f_i - 5f_{i-1} + f_{i-2}].$$

$$\text{Error term} = -\frac{19}{720} h^5 f^{(4)} (\xi_4) = -\frac{19}{720} h^5 y^{(5)} (\xi_4).$$

The method requires the starting values y_i, y_{i-1}, y_{i-2}.

The combination requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

5. Write the Milne's predictor-corrector method for solving the initial value problem $y' = f(x, y)$, $y(x_0) = b_0$. Comment on the order and the required starting values.

Solution Milne's predictor-corrector method is given by

Predictor P: Adams-Bashforth method of fourth order

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3} [2f_i - f_{i-1} + 2f_{i-2}].$$

$$\text{Error term} = \frac{14}{45} h^5 f^{(4)} (\xi) = \frac{14}{45} h^5 y^{(5)} (\xi).$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}.

Corrector C: Milne-Simpson's method of fourth order

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3} [f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}]$$

$$\text{Error term} = -\frac{1}{90} h^5 f^{(4)} (\xi) = -\frac{1}{90} h^5 y^{(5)} (\xi).$$

The method requires the starting values y_i, y_{i-1}.

The combination requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

6. How many prior values are required to predict the next value in Adams-Bashforth-Moulton method ?

Solution The Adams-Bashforth-Moulton predictor-corrector method requires four starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

7. How many prior values are required to predict the next value in Milne's predictor-corrector method ?

Solution The Milne's predictor-corrector method requires four starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.

Reference Books:

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