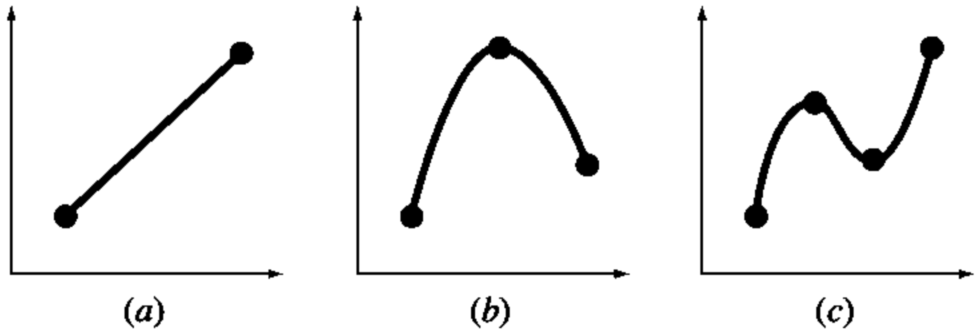
Interpolation

You will frequently have occasion to estimate intermediate values between precise data points. The most common method used for this purpose is polynomial interpolation. Recall that the general formula for an nth-order polynomial is

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

For n + 1 data points, there is one and only one polynomial of order n that passes through all the points. For example, there is only one straight line (that is, a first-order polynomial) that connects two points (Fig. 18.1a). Similarly, only one parabola connects a set of three points (Fig. 18.1b). Polynomial interpolation consists of determining the unique nth-order polynomial that fits n + 1 data points. This polynomial then provides a formula to compute intermediate values. Although there is one and only one nth-order polynomial that fits n + 1 points, there are a variety of mathematical formats in which this polynomial can be expressed. In this chapter, we will describe two alternatives that are well-suited for computer implementation: the Newton and the Lagrange polynomials.

FIGURE 18.1
Examples of interpolating polynomials: (*a*) first-order (linear) connecting two points, (*b*) second-order (quadratic or parabolic) connecting three points, and (*c*) third-order (cubic) connecting four points.



Let $f(x)$ be a continuous function defined on some interval $[a, b]$, and be prescribed at $n + 1$ distinct tabular points $x_0, x_1,..., x_n$ such that $a = x_0 < x_1 < x_2 < ... < x_n = b$. The distinct tabular points $x_0, x_1,..., x_n$ may be non-equispaced or equispaced, that is $x_{k+1} - x_k = h$, $k = 0, 1, 2,..., n-1$. The problem of polynomial approximation is to find a polynomial $P_n(x)$, of degree $\leq n$, which fits the given data exactly, that is,

$$P_n(x_i) = f(x_i), \quad i = 0, 1, 2,..., n. \tag{2.1}$$

The polynomial $P_n(x)$ is called the *interpolating polynomial*. The conditions given in (2.1) are called the *interpolating conditions*.

INTERPOLATION WITH UNEVENLY SPACED POINTS

1. **Divided differences**
2. **Lagrange Interpolation**

1. Newton's Divided Difference

Let the data, $(x_i, f(x_i))$, $i = 0, 1, 2,\ldots, n$, be given. We define the divided differences as follows.
First divided difference Consider any two consecutive data values $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$. Then, we define the first divided difference as

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad i = 0, 1, 2,\ldots, n-1. \tag{2.12}$$

Therefore, $$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \text{ etc.}$$

Second divided difference Consider any three consecutive data values $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+2}, f(x_{i+2}))$. Then, we define the second divided difference as

$$f[x_i\, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i} \quad i = 0, 1, 2,\ldots, n-2 \tag{2.13}$$

Therefore, $$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \text{ etc.}$$

The nth divided difference using all the data values in the table, is defined as

$$f[x_0,\ x_1,\ \ldots,\ x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0} \tag{2.14}$$

The nth divided difference can also be expressed in terms of the ordinates f_i. The denominators of the terms are same as the denominators of the Lagrange fundamental polynomials.

The divided differences can be written in a tabular form as in Table 2.1.

Table 2.1. Divided differences (d.d).

x	$f(x)$	*First d.d*	*Second d.d*	*Third d.d*
x_0	f_0			
		$f[x_0, x_1]$		
x_1	f_1		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	f_2		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	f_3			

Hence, we can write the interpolating polynomial as

$$\begin{aligned} f(x) &= P_n(x) \\ &= f(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2] + \ldots \\ &\quad + (x - x_0)(x - x_1)\ldots(x - x_{n-1})\, f[x_0, x_1, \ldots,\ x_n] \end{aligned} \tag{2.16}$$

This polynomial is called the *Newton's divided difference interpolating polynomial*.

Note:

1. From the divided difference table, we can determine the degree of the interpolating polynomial. Suppose that all the kth divided differences in the kth column are equal (same). Then, all the $(k + 1)$th divided differences in the $(k + 1)$th column are zeros. Therefore, from (2.16), we conclude that the data represents a kth degree polynomial. Otherwise, the data represents an nth degree polynomial.
2. Newton's divided difference interpolating polynomial possesses the permanence property. Suppose that we add a new data value $(x_{n+1}, f(x_{n+1}))$ at the distinct point x_{n+1}, at the end of the given table of values. This new data of values can be represented by a $(n + 1)$th degree polynomial. Now, the $(n + 1)$th column of the divided difference table contains the $(n + 1)$th divided difference. Therefore, we require to add the term $(x - x_0)(x - x_1)...(x - x_{n-1})(x - x_n) f[x_0, x_1, ..., x_n, x_{n+1}]$ to the previously obtained nth degree interpolating polynomial given in (2.16).

Example 2.7 *Obtain the divided difference table for the data*

x	-1	0	2	3
$f(x)$	-8	3	1	12

Solution We have the following divided difference table for the data.

Divided difference table. Example 2.7.

x	$f(x)$	*First d.d*	*Second d.d*	*Third d.d*
-1	-8			
		$\frac{3+8}{0+1} = 11$		
0	3		$\frac{-1-11}{2+1} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4+4}{3+1} = 2$
2	1		$\frac{11+1}{3-0} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

Example 2.8 *Find f(x) as a polynomial in x for the following data by Newton's divided difference formula*

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Solution We form the divided difference table for the data.

The Newton's divided difference formula gives

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2 f[x_0, x_1, x_2, x_3] \\ &+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f[x_0, x_1, x_2, x_3, x_4] \end{aligned}$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)\, x\, (-14)$$
$$+ (x + 4)(x + 1)\, x\, (x - 2)(3)$$
$$= 1245 - 404x - 1616 + (x^2 + 5x + 4)(94) + (x^3 + 5x^2 + 4x)(-14)$$
$$+ (x^4 + 3x^3 - 6x^2 - 8x)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Divided difference table. Example 2.8.

x	$f(x)$	*First d.d*	*Second d.d*	*Third d.d*	*Fourth d.d*
– 4	1245				
		– 404			
– 1	33		94		
		– 28		– 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Example 2.9 *Find f(x) as a polynomial in x for the following data by Newton's divided difference formula*

x	– 2	– 1	0	1	3	4
$f(x)$	9	16	17	18	44	81

Hence, interpolate at $x = 0.5$ *and* $x = 3.1$.

Solution We form the divided difference table for the given data.

Since, the fourth order differences are zeros, the data represents a third degree polynomial. Newton's divided difference formula gives the polynomial as

$$f(x) = f(x_0) + (x - x_0)\, f\,[x_0, x_1] + (x - x_0)(x - x_1)\, f\,[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)\,(x - x_2)\, f\,[x_0, x_1, x_2, x_3]$$
$$= 9 + (x + 2)(7) + (x + 2)(x + 1)(-3) + (x + 2)(x + 1)\, x(1)$$
$$= 9 + 7x + 14 - 3x^2 - 9x - 6 + x^3 + 3x^2 + 2x = x^3 + 17.$$

Hence, $f(0.5) = (0.5)^3 + 17 = 17.125.$

$f(3.1) = (3.1)^3 + 17 = 47.791.$

Divided difference table. Example 2.9.

x	$f(x)$	*First d.d*	*Second d.d*	*Third d.d*	*Fourth d.d*
– 2	9				
		7			
– 1	16		– 3		
		1		1	
0	17		0		0
		1		1	
1	18		4		0
		13		1	
3	44		8		
		37			
4	81				

Example 2.10 *Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula*

x	1	3	4	5	7	10
$f(x)$	3	31	69	131	351	1011

Hence, interpolate at $x = 3.5$ and $x = 8.0$.

Solution We form the divided difference table for the data.

Divided difference table. Example 2.10.

x	$f(x)$	*First d.d*	*Second d.d*	*Third d.d*	*Fourth d.d*
1	3				
		14			
3	31		8		
		38		1	
4	69		12		0
		62		1	
5	131		16		0
		110		1	
7	351		22		
		220			
10	1011				

Since, the fourth order differences are zeros, the data represents a third degree polynomial. Newton's divided difference formula gives the polynomial as

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &\quad + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \\ &= 3 + (x - 1)(14) + (x - 1)(x - 3)(8) + (x - 1)(x - 3)(x - 4)(1) \\ &= 3 + 14x - 14 + 8x^2 - 32x + 24 + x^3 - 8x^2 + 19x - 12 = x^3 + x + 1. \end{aligned}$$

Hence $\quad f(3.5) \approx P_3(3.5) = (3.5)^3 + 3.5 + 1 = 47.375,$

$$f(8.0) \approx P_3(8.0) = (8.0)^3 + 8.0 + 1 = 521.0.$$

Q. Give two uses of interpolating polynomials.

Solution The first use is to reconstruct the function $f(x)$ when it is not given explicitly and only values of $f(x)$ and/ or its certain order derivatives are given at a set of distinct points called *nodes* or *tabular points*. The second use is to perform the required operations which were intended for $f(x)$, like determination of roots, differentiation and integration etc. can be carried out using the approximating polynomial $P(x)$. The approximating polynomial $P(x)$ can be used to predict the value of $f(x)$ at a non-tabular point.

Q. Does the Newton's divided difference interpolating polynomial have the permanence property?

Solution Newton's divided difference interpolating polynomial has the permanence property. Suppose that to the given data $(x_i, f(x_i))$, $i = 0, 1, 2,..., n$, a new data value $(x_{n+1}, f(x_{n+1}))$ at the distinct point x_{n+1} is added at the end of the table. Then, the $(n + 1)$th column of the divided difference table has the $(n + 1)$th divided difference. Hence, the data represents a polynomial of degree $(n + 1)$. We need to add only one extra term $(x - x_0)(x - x_1)...(x - x_n) f[x_0, x_1, ... x_{n+1}]$ to the previously obtained nth degree divided difference polynomial.

Q. Distinguish between interpolation and extrapolation

Solution. Interpolation: It is the estimation for some such values which lie inside the given Values. Extrapolation: It is the estimation for some such values which lie outside the given Values.

2. Lagrange Interpolation

Let the data

x	x_0	x_1	x_2	...	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	...	$f(x_n)$

be given at distinct unevenly spaced points or non-uniform points $x_0, x_1,..., x_n$. This data may also be given at evenly spaced points.

For this data, we can fit a unique polynomial of degree $\le n$. Since the interpolating polynomial must use all the ordinates $f(x_0), f(x_1),... f(x_n)$, it can be written as a linear combination of these ordinates. That is, we can write the polynomial as

$$P_n(x) = l_0(x)\, f(x_0) + l_1(x)\, f(x_1) + ... + l_n(x)\, f(x_n)$$

$$= l_0(x)\, f_0 + l_1(x)\, f_1 + ... + l_n(x)\, f(x_n) \tag{2.2}$$

where $f(x_i) = f_i$ and $l_i(x)$, $i = 0, 1, 2, ...,n$ are polynomials of degree n. This polynomial fits the data given in (2.1) exactly.

Linear interpolation

For $n = 1$, we have the data

x	x_0	x_1
$f(x)$	$f(x_0)$	$f(x_1)$

The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)},\; l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}. \qquad (2.6$$

The Lagrange linear interpolation polynomial is given by

$$P_1(x) = l_0(x)\, f(x_0) + l_1(x)\, f(x_1). \qquad (2.7$$

Quadratic interpolation

For $n = 2$, we have the data

x	x_0	x_1	x_2
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$

The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)},\; l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)},\; l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

The Lagrange quadratic interpolation polynomial is given by

$$P_1(x) = l_0(x)\, f(x_0) + l_1(x)\, f(x_1) + l_2(x)\, f(x_2). \tag{2.8}$$

Example 2.1 *Using the data* $\sin(0.1) = 0.09983$ *and* $\sin(0.2) = 0.19867$, *find an approximate value of* $\sin(0.15)$ *by Lagrange interpolation.*

Solution We have two data values. The Lagrange linear polynomial is given by

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

$$= \frac{(x - 0.2)}{(0.1 - 0.2)} (0.09983) + \frac{(x - 0.1)}{(0.2 - 0.1)} (0.19867).$$

Hence,
$$f(0.15) = P_1(0.15) = \frac{(0.15 - 0.2)}{(0.1 - 0.2)} (0.09983) + \frac{(0.15 - 0.1)}{(0.2 - 0.1)} (0.19867)$$

$$= (0.5)(0.09983) + (0.5)(0.19867) = 0.14925.$$

Example 2.3 *Given that* $f(0) = 1$, $f(1) = 3$, $f(3) = 55$, *find the unique polynomial of degree* 2 *or less, which fits the given data.*

Solution We have $x_0 = 0$, $f_0 = 1$, $x_1 = 1$, $f_1 = 3$, $x_2 = 3$, $f_2 = 55$. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 1)(x - 3)}{(-1)(-3)} = \frac{1}{3}(x^2 - 4x + 3).$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x - 3)}{(1)(-2)} = \frac{1}{2}(3x - x^2).$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x - 1)}{(3)(2)} = \frac{1}{6}(x^2 - x).$$

Hence, the Lagrange quadratic polynomial is given by

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$= \frac{1}{3}(x^2 - 4x + 3) + \frac{1}{2}(3x - x^2)(3) + \frac{55}{6}(x^2 - x) = 8x^2 - 6x + 1.$$

Example 2.5 *Construct the Lagrange interpolation polynomial for the data*

x	-1	1	4	7
$f(x)$	-2	0	63	342

Hence, interpolate at $x = 5$.

Solution The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x - 1)(x - 4)(x - 7)}{(-1 - 1)(-1 - 4)(-1 - 7)}$$

$$= -\frac{1}{80}(x^3 - 12x^2 + 39x - 28).$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x+1)(x-4)(x-7)}{(1+1)(1-4)(1-7)}$$

$$= \frac{1}{36}(x^3 - 10x^2 + 17x + 28).$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x-1)(x-7)}{(4+1)(4-1)(4-7)}$$

$$= -\frac{1}{45}(x^3 - 7x^2 - x + 7).$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x-1)(x-4)}{(7+1)(7-1)(7-4)}$$

$$= \frac{1}{144}(x^3 - 4x^2 - x + 4).$$

Note that we need not compute $l_1(x)$ since $f(x_1) = 0$.

The Lagrange interpolation polynomial is given by

$$P_3(x) = l_0(x)\, f(x_0) + l_1(x)\, f(x_1) + l_2(x)\, f(x_2) + l_3(x)\, f(x_3)$$

$$= -\frac{1}{80}(x^3 - 12x^2 + 39x - 28)(-2) - \frac{1}{45}(x^3 - 7x^2 - x + 7)(63)$$

$$+ \frac{1}{144}(x^3 - 4x^2 - x + 4)(342)$$

$$= \left(\frac{1}{40} - \frac{7}{5} + \frac{171}{72}\right)x^3 + \left(-\frac{3}{10} + \frac{49}{5} - \frac{171}{18}\right)x^2 + \left(\frac{39}{40} + \frac{7}{5} - \frac{171}{72}\right)x + \left(-\frac{7}{10} - \frac{49}{5} + \frac{171}{8}\right)$$

$$= x^3 - 1.$$

Hence, $f(5) = P_3(5) = 5^3 - 1 = 124.$

Note:

For a given data, it is possible to construct the Lagrange interpolation polynomial. However, it is very difficult and time consuming to collect and simplify the coefficients of x^i, $i = 0, 1, 2,..., n$. Now, assume that we have determined the Lagrange interpolation polynomial of degree n based on the data values $(x_i, f(x_i))$, $i = 0, 1, 2,..., n$ at the $(n + 1)$ distinct points. Suppose that to this given data, a new value $(x_{n+1}, f(x_{n+1}))$ at the distinct point x_{n+1} is added at the end of the table. If we require the Lagrange interpolating polynomial for this new data, then we need to compute all the Lagrange fundamental polynomials again. The nth degree Lagrange polynomial obtained earlier is of no use. This is the disadvantage of the Lagrange interpolation. However, Lagrange interpolation is a fundamental result and is used in proving many theoretical results of interpolation.

Suppose that the data $(x_i, f(x_i))$, $i = 0, 1, 2,..., n$, is given. Assume that a new value $(x_{n+1}, f(x_{n+1}))$ at the distinct point x_{n+1} is added at the end of the table. The data, $(x_i, f(x_i))$, $i = 0, 1, 2,..., n + 1$, represents a polynomial of degree $\leq (n + 1)$. If this polynomial of degree $(n + 1)$ can be obtained by adding an extra term to the previously obtained nth degree interpolating polynomial, then the interpolating polynomial is said to have the *permanence property*. We observe that the Lagrange interpolating polynomial does not have the permanence property.

Q. What is the disadvantage of Lagrange interpolation?

Solution Assume that we have determined the Lagrange interpolation polynomial of degree n based on the data values $(x_i, f(x_i))$, $i = 0, 1, 2,..., n$ given at the $(n + 1)$ distinct points. Suppose that to this given data, a new value $(x_{n+1}, f(x_{n+1}))$ at the distinct point x_{n+1} is added at the end of the table. If we require the Lagrange interpolating polynomial of degree $(n + 1)$ for this new data, then we need to compute all the Lagrange fundamental polynomials again. The nth degree Lagrange polynomial obtained earlier is of no use. This is the disadvantage of the Lagrange interpolation.

Q. Does the Lagrange interpolating polynomial have the permanence property?

Solution Lagrange interpolating polynomial does not have the permanence property. Suppose that to the given data $(x_i, f(x_i))$, $i = 0, 1, 2,..., n$, a new value $(x_{n+1}, f(x_{n+1}))$ at the distinct point x_{n+1} is added at the end of the table. If we require the Lagrange interpolating polynomial for this new data, then we need to compute all the Lagrange fundamental polynomials again. The nth degree Lagrange polynomial obtained earlier is of no use.

Q. Define inverse interpolation.

Solution Suppose that a data $(x_i, f(x_i))$, $i = 0, 1, 2,..., n$, is given. In interpolation, we predict the value of the ordinate $f(x')$ at a non-tabular point $x = x'$. In many applications, we require the value of the abscissa x' for a given value of the ordinate $f(x')$. For this problem, we consider the given data as $(f(x_i), x_i)$, $i = 0, 1, 2,..., n$ and construct the interpolation polynomial. That is, we consider $f(x)$ as the independent variable and x as the dependent variable. This procedure is called inverse interpolation