In the previous sections, *n*th-order polynomials were used to interpolate between *n* 1 l data points. For example, for eight points, we can derive a perfect seventh-order polynomial. This curve would capture all the meanderings (at least up to and including seventh derivatives) suggested by the points. However, there are cases where these functions can lead to erroneous results because of round-off error and overshoot. An alternative approach is to apply lower-order polynomials to subsets of data points. Such connecting polynomials are called *spline functions*.

For example, third-order curves employed to connect each pair of data points are called *cubic splines*. These functions can be constructed so that the connections between adjacent cubic equations are visually smooth. On the surface, it would seem that the third-order approximation of the splines would be inferior to the seventh-order expression. You might wonder why a spline would ever be preferable.

Figure 18.14 illustrates a situation where a spline performs better than a higher- order polynomial. This is the case where a function is generally smooth but undergoes an abrupt change somewhere along the region of interest. The step increase depicted in Fig. 18.14 is an extreme example of such a change and serves to illustrate the point.

Figure 18.14 *a* through *c* illustrates how higher-order polynomials tend to swing through wild oscillations in the vicinity of an abrupt change. In contrast, the spline also connects the points, but because it is limited to lower-order changes, the oscillations are kept to a minimum. As such, the spline usually provides a superior approximation of the behavior of functions that have local, abrupt changes.

The concept of the spline originated from the drafting technique of using a thin, flexible strip (called a *spline*) to draw smooth curves through a set of points. The process is depicted in Fig. 18.15 for a series of five pins (data points). In this technique, the drafter places paper over a wooden board and hammers nails or pins into the paper (and board) at the location of the data points. A smooth cubic curve results from interweaving the strip between the pins. Hence, the name “cubic spline” has been adopted for polynomials of this type.

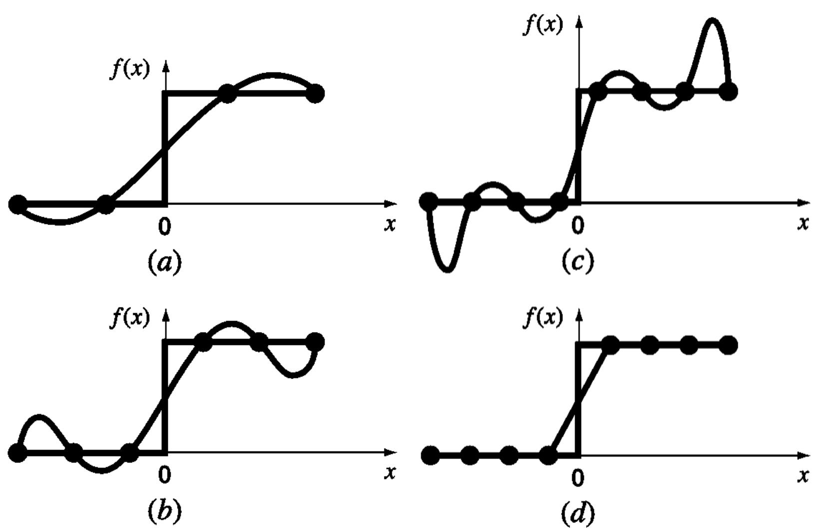
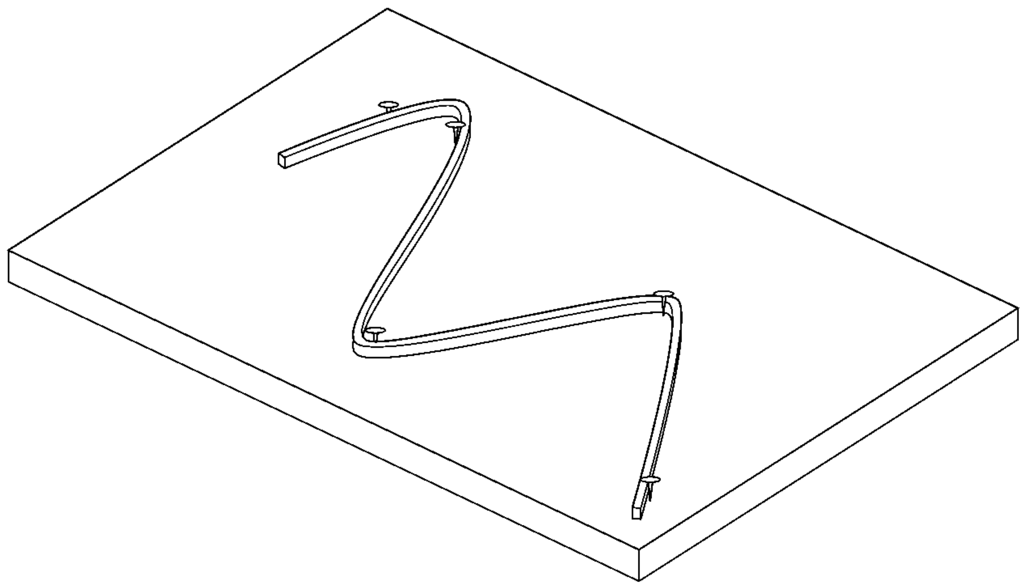


FIGURE 18.14

A visual representation of a situation where the splines are superior to higher-order interpolating polynomials. The function to be fit undergoes an abrupt increase at *x* 5 0. Parts (*a*) through (*c*) indicate that the abrupt change induces oscillations in interpolating polynomials. In contrast, because it is limited to third-order curves with smooth transitions, a linear spline (*d*) provides a much more acceptable approximation.



The drafting technique of using a spline to draw smooth curves through a series of points. Notice how, at the end points, the spline straightens out. This is called a "natural" spline.

18.6.1 Linear Splines

The simplest connection between two points is a straight line. The first-order splines for a group of ordered data points can be defined as a set of linear functions,

$$
\begin{aligned}
f(x) &= f(x_0) + m_0(x - x_0) && x_0 \le x \le x_1 \\
f(x) &= f(x_1) + m_1(x - x_1) && x_1 \le x \le x_2 \\
&\ \ \vdots \\
f(x) &= f(x_{n-1}) + m_{n-1}(x - x_{n-1}) && x_{n-1} \le x \le x_n
\end{aligned}
$$

where m_i is the slope of the straight line connecting the points:

$$m_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \tag{18.27}$$

These equations can be used to evaluate the function at any point between x_0 and x_n by first locating the interval within which the point lies. Then the appropriate equation is used to determine the function value within the interval. The method is obviously identical to linear interpolation.

18.6.2 Quadratic Splines

To ensure that the mth derivatives are continuous at the knots, a spline of at least $m + 1$ order must be used. Third-order polynomials or cubic splines that ensure continuous first and second derivatives are most frequently used in practice. Although third and higher derivatives could be discontinuous when using cubic splines, they usually cannot be detected visually and consequently are ignored.

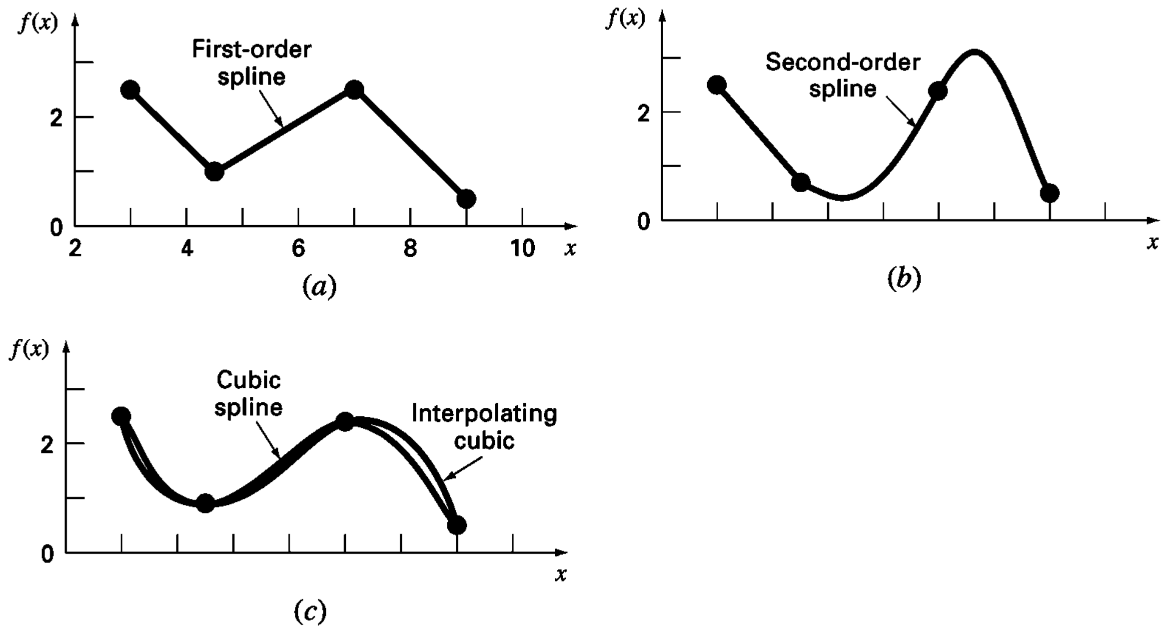


FIGURE 18.16
Spline fits of a set of four points. (*a*) Linear spline, (*b*) quadratic spline, and (*c*) cubic spline, with a cubic interpolating polynomial also plotted.

The objective in quadratic splines is to derive a second-order polynomial for each interval between data points. The polynomial for each interval can be represented generally as

$$f_i(x) = a_i x^2 + b_i x + c_i \qquad (18.28)$$

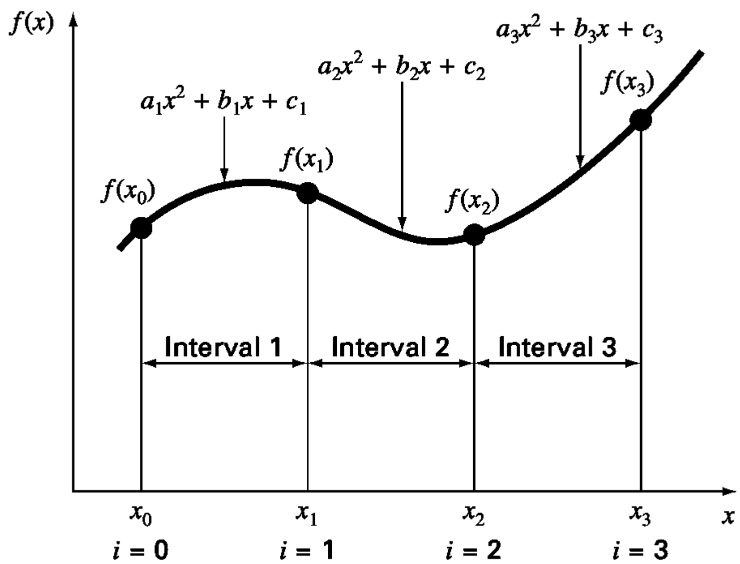


FIGURE 18.17
Notation used to derive quadratic splines. Notice that there are n intervals and $n + 1$ data points. The example shown is for $n = 3$.

Figure 18.17 has been included to help clarify the notation. For $n + 1$ data points ($i = 0, 1, 2, \ldots, n$), there are n intervals and, consequently, $3n$ unknown constants (the a's, b's, and c's) to evaluate. Therefore, $3n$ equations or conditions are required to evaluate the unknowns. These are:

1. *The function values of adjacent polynomials must be equal at the interior knots.* This condition can be represented as

$$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1}) \qquad (18.29)$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1}) \qquad (18.30)$$

for i = 2 to n. Because only interior knots are used, Eqs. (18.29) and (18.30) each provide n - 1 conditions for a total of 2n - 2 conditions.
2. *The first and last functions must pass through the end points.* This adds two additional equations:

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0) \quad (18.31)$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n) \quad (18.32)$$

for a total of 2n - 2 + 2 = 2n conditions.
3. *The first derivatives at the interior knots must be equal.* The first derivative of Eq. (18.28) is

$$f'(x) = 2ax + b$$

$$2a_{i-1}x_{i-1} + b_i = 2a_i x_{i-1} + b_i \quad (18.33)$$

for i = 2 to n. This provides another n - 1 conditions for a total of 2n + n - 1 = 3n - 1. Because we have 3n unknowns, we are one condition short. Unless we have some additional information regarding the functions or their derivatives, we must make an arbitrary choice to successfully compute the constants. Although there are a number of different choices that can be made, we select the following:
4. *Assume that the second derivative is zero at the first point.* Because the second derivative of Eq. (18.28) is $2a_i$ this condition can be expressed mathematically as $2a_i = 0$ (18.34) The visual interpretation of this condition is that the first two points will be connected by a straight line.

18.6.3 Cubic Splines

The objective in cubic splines is to derive a third-order polynomial for each interval between knots, as in

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i \quad (18.35)$$

Thus, for n + 1 data points (i = 0, 1, 2, . . . , n), there are n intervals and, consequently, 4n unknown constants to evaluate. Just as for quadratic splines, 4n conditions are required to evaluate the unknowns. These are:
1. The function values must be equal at the interior knots (2n - 2 conditions).
2. The first and last functions must pass through the end points (2 conditions).
3. The first derivatives at the interior knots must be equal (n - 1 conditions).
4. The second derivatives at the interior knots must be equal (n - 1 conditions).
5. The second derivatives at the end knots are zero (2 conditions).
The visual interpretation of condition 5 is that the function becomes a straight line at the end knots. Specification of such an end condition leads to what is termed a "natural" spline. It is given this name because the drafting spline naturally behaves in this fashion (Fig. 18.15). If the value of the second derivative at the end knots is nonzero (that is, there is some curvature), this information can be used alternatively to supply the two final conditions.
The above five types of conditions provide the total of 4n equations required to solve for the 4n coefficients.

Note that h is the length of the interval $[x_{i-1}, x_i]$ of equi-spaced data, $x_i - x_{i-1} = h$
Denote $f''(x_i) = M_i \; and \; f''(x_{i-1}) = M_{i-1}$
$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f_{i+1} - 2f_i + f_{i-1})$ for I = 1, 2, n-1
Then, the spline in the interval x_{i-1}, x_i can be derived by

$$F_i(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right]$$

$$+ \frac{(x_i - x)}{h}\left[f_{i-1} - \frac{h^2}{6} M_{i-1}\right] + \frac{(x - x_{i-1})}{h}\left[f_i - \frac{h^2}{6} M_i\right]$$

Example 2.22 *Obtain the cubic spline approximation for the following data.*

x	0	1	2
$f(x)$	– 1	3	29

with $M_0 = 0$, $M_2 = 0$. *Hence, interpolate at* $x = 0.5, 1.5$.

Solution We have equispaced data with $h = 1$. We obtain from (2.63),

$$M_{i-1} + 4M_i + M_{i+1} = 6(f_{i+1} - 2f_i + f_{i-1}),\ i = 1.$$

For $i = 1$, we get

$$M_0 + 4M_1 + M_2 = 6(f_2 - 2f_1 + f_0).$$

Since, $M_0 = 0$, $M_2 = 0$, we get

$$4M_1 = 6[29 - 2(3) - 1] = 132, \quad \text{or} \quad M_1 = 33.$$

The spline is given by

$$F_i(x) = \frac{1}{6}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + (x_i - x)\left[f_{i-1} - \frac{1}{6}M_{i-1}\right] + (x - x_{i-1})\left[f_i - \frac{1}{6}M_i\right]$$

We have the following splines.

On [0, 1]:

$$F(x) = \frac{1}{6}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + (x_1 - x)\left[f_0 - \frac{1}{6}M_0\right] + (x - x_0)\left[f_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}(33)x^3 + (1 - x)(-1) + x\left(3 - \frac{1}{6}(33)\right)$$

$$= \frac{1}{2}(11x^3 - 3x - 2).$$

On [1, 2]:

$$F(x) = \frac{1}{6}[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2] + (x_2 - x)\left[f_1 - \frac{1}{6}M_1\right] + (x - x_1)\left[f_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}[(2 - x)^3 (33)] + (2 - x)\left[3 - \frac{1}{6}(33)\right] + (x - 1)[29]$$

$$= \frac{11}{2}(2 - x)^3 + \frac{63}{2}x - 34.$$

Since, 0.5 lies in the interval (0, 1), we obtain

$$F(0.5) = \frac{1}{2}\left(\frac{11}{8} - \frac{3}{2} - 2\right) = -\frac{17}{16}.$$

Since, 1.5 lies in the interval (1, 2), we obtain

$$F(1.5) = \frac{1}{2}[11(2 - 1.5)^3 + 63(1.5) - 68] = \frac{223}{16}.$$

Example 2.23 *Obtain the cubic spline approximation for the following data.*

x	0	1	2	3
$f(x)$	1	2	33	244

with $M_0 = 0$, $M_3 = 0$. *Hence, interpolate at* $x = 2.5$.

Solution We have equispaced data with $h = 1$. We obtain from (2.63),

$$M_{i-1} + 4M_i + M_{i+1} = 6(f_{i+1} - 2f_i + f_{i-1}) \quad i = 1, 2.$$

For $i = 1$, we get

$$M_0 + 4M_1 + M_2 = 6(f_2 - 2f_1 + f_0) = 6(33 - 4 + 1) = 180.$$

For $i = 2$, we get

$$M_1 + 4M_2 + M_3 = 6(f_3 - 2f_2 + f_1) = 6(244 - 66 + 2) = 1080.$$

Since, $M_0 = 0$, $M_3 = 0$, we get

$$4M_1 + M_2 = 180, \quad M_1 + 4M_2 = 1080.$$

The solution is $M_1 = -24$, $M_2 = 276$.

The cubic splines in the corresponding intervals are as follows.

On [0, 1]:

$$F(x) = \frac{1}{6}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + (x_1 - x)\left[f_0 - \frac{1}{6}M_0\right] + (x - x_0)\left[f_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}x^3(-24) + (1 - x) + x\left[2 - \frac{1}{6}(-24)\right] = -4x^3 + 5x + 1.$$

On [1, 2]:

$$F(x) = \frac{1}{6}[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2] + (x_2 - x)\left[f_1 - \frac{1}{6}M_1\right] + (x - x_1)\left[f_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}[(2 - x)^3(-24) + (x - 1)^3(276)] + (2 - x)\left[2 - \frac{1}{6}(-24)\right] + (x - 1)\left[33 - \frac{1}{6}(276)\right]$$

$$= \frac{1}{6}[(8 - 12x + 6x^2 - x^3)(-24) + (x^3 - 3x^2 + 3x - 1)(276)] + 6(2 - x) - 13(x - 1)$$

$$= 50x^3 - 162x^2 + 167x - 53.$$

On [2, 3]:

$$F(x) = \frac{1}{6}[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3] + (x_3 - x)\left[f_2 - \frac{1}{6}M_2\right] + (x - x_2)\left[f_3 - \frac{1}{6}M_3\right]$$

$$= \frac{1}{6}[(3 - x)^3(276)] + (3 - x)\left[33 - \frac{1}{6}(276)\right] + (x - 2)(244)$$

$$= \frac{1}{6}[(27 - 27x + 9x^2 - x^3)(276)] - 13(3 - x) + 244(x - 2)$$

$$= -46x^3 + 414x^2 - 985x + 715.$$

The estimate at $x = 2.5$ is

$$F(2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715 = 121.25.$$

Questions

1. What are the advantages of cubic spline fitting?

Solution Splines provide better approximation to the behaviour of functions that have abrupt local changes. Further, splines perform better than higher order polynomial approximations.

2. Write the relation between the second derivatives $M_i(x)$ in cubic splines with equal mesh spacing.

Solution The required relation is

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f_{i+1} - 2f_i + f_{i-1}), \quad i = 1, 2, ..., n - 1.$$

3. Write the end conditions on $M_i(x)$ in natural cubic splines.

Solution The required conditions are $M_0(x) = 0$, $M_n(x) = 0$.

Reference Books:

NUMERICAL METHODS - S.R.K. Iyengar •R.K. Jain

Numerical Methods for Engineers - Steven C. Chapra Raymond P .Canale