Birge Vieta Method

Let us take an example first. Suppose we wish to evaluate, polynomial

$$p(x) = 5x^4 + 7x^3 + 4x^2 + 3x + 8 \qquad \text{... (9)}$$

If evaluated directly, there are $4+3+2+1=10$ number of multiplications and 4 additions.

In general, if $p(x)$ is given by equation (1)

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \; a_n \neq 0$$

Its evaluation would require $1+2+3+\ldots+n = \frac{n}{2}(n+1)$ multiplications and n additions.

Birge Vieta Method

Now, let us express $p(x)$ in (9) as $p(x) = 8 + x(3 + x(4 + x(7 + 5x)))$ ... (10)

In direct evaluation, we are wasting our efforts, x^2 evaluated in $4x^2$ term is not used in calculation of x^3 and similarly value of x^3 is not used in calculating value of x^4.

If $p(x)$ is evaluated using equation (10), there are 4 additions (no saving in additions) but only 4 multiplications.

In general, $p(x)$ can be represented by $p(x) = a_0 + x(a_1 + x(a_2 + \ldots + x(a_{n-1} + xa_n))\ldots)$ and this expression will need n additions and n multiplications. It leads to a considerable saving of multiplication operations from $\dfrac{n^2 + n}{2}$ to only n operations.

Birge Vieta Method

This method of using nested parenthesis is very well known as Horner's method of polynomial evaluation. There is no more efficient way with respect to requiring fewer arithmetic operations.

Now, let us develop algorithm of Newton Raphson method for finding roots of a polynomial which uses Horner's method for evaluation of polynomial and its derivative. This method is known as **Birge Vieta Method**.

Birge Vieta Method

If we devide $p(x)$ by $\left(x-\bar{x}\right)$, quotient shall be a polynomial of *degree (n-1)* and remainder would be constant.

$\therefore\ p(x)=\left(x-\bar{x}\right)\left(b_1+b_2x+\ldots+b_nx^{n-1}\right)+b_0$ ... (11)

$\therefore\ p\left(\bar{x}\right)=b_0$ ... (12)

So, we need to determine the value of b_0.

Rewrite (19) as $a_0+a_1x+a_2x^2+\ldots+a_nx^n=\left(x-\bar{x}\right)\left(b_1+b_2x+\ldots+b_nx^{n-1}\right)+b_0$

To compute b_0, we equate coefficients of equal powers of x on both sides of (11).

Birge Vieta Method

Equating coefficients of x^n, we get, $a_n = b_n$

Computing coefficients of x^{n-1}, gives $a_{n-1} = b_{n-1} - \bar{x} b_n$

In general, equating of x^i, we obtain $a_j = b_j - \bar{x} b_{j+1}$, $\mathrm{j} = \mathrm{n} - 1, \mathrm{n} - 2, \ldots, 0$

So b_j's can be obtained as follows:

$b_n = a_n$ and $b_j = a_j + \bar{x} b_{j+1}$, $\mathrm{j} = \mathrm{n} - 1, \mathrm{n} - 2, \ldots, 1, 0$ (b_j's are calculated in decreasing order)

Birge Vieta Method

Now, let us tackle the issue of evaluating derivative of $p(x)$.

Let us express equation (19) as $p(x) = (x - \bar{x})g(x) + b_0$

Differentiation of $p(x)$ gives us $p'(x) = g(x) + (x - \bar{x})g'(x)$ ... (12)

Where $g(x) = b_1 + b_2 x + \ldots + b_n x^{n-1}$

From (12), $p'(\bar{x}) = g(\bar{x})$

So $p'(\bar{x})$ can be easily calculated using the same above technique.

So $c_n = b_n$ and $c_j = b_j + \bar{x} c_{j+1}$, $\mathrm{j} = \mathrm{n} - 1, \mathrm{n} - 2, \ldots, 1$

Birge Vieta Method

So by Newton's formula

$$x_{k+1} = x_k - \frac{b_0}{c_1}; \quad \mathrm{k} = 0, 1, 2, \ldots$$

Where b_0 and c_1 are calculated as follows:

$$c_n = b_n = a_n$$

$$b_j = a_j + x_k b_{j+1}, \text{ for } \mathrm{j} = \mathrm{n} - 1, \ldots, 0$$

$$c_j = b_j + x_k c_{j+1}, \text{ for } \mathrm{j} = \mathrm{n} - 1, \ldots, 1$$

Step 1 Input: $p(x)$ (the given polynomial coefficients $a_0, a_1 ..., a_n$)

x_0 the initial guess for the root

ϵ: the error tolerance (X – TOL)

N : the maximum number of iterations

Step 2 $k = 0$, $c_n = b_n = a_n$

Step 3 $\mathrm{j = n - 1}$

Step 4 $b_j = a_j + x_k b_{j+1}$

$c_j = b_j + x_k c_{j+1}$

Step 5 $\mathrm{j = j - 1}$, if $j \geq 1$ go to step 4 else

Step 6 $b_0 = a_0 + x_k b_1, \quad x_{k+1} = x_k - \dfrac{b_0}{c_1}$

Step 7 If $|x_{k+1} - x_k| \langle \in$ exit

Output: estimate of the root is x_{k+1} else

Step 8 $k = k + 1$, if $k \leq n$ go to step 4

Step 9 Else output: does not converge in N iterations.

Birge Vieta Example

- Find the root of $f(x) = x^4 - 9x^3 - 2x^2 + 120x - 130$

Initial guess = -3

Birge Vieta Example

- Find the root of $f(x) = x^4 - 9x^3 - 2x^2 + 120x - 130$
- Iteration 1

i	ai	bi	ci
4	1.000000	1.000000	1.000000
3	-9.000000	-12.000000	-15.000000
2	-2.000000	34.000000	79.000000
1	120.000000	18.000000	-219.000000
0	-130.000000	-184.000000	473.000000

Value of x_1 = -3.840183

Birge Vieta Example

- Find the root of $f(x) = x^4 - 9x^3 - 2x^2 + 120x - 130$
- Iteration 2

i	ai	bi	ci
4	1.000000	1.000000	1.000000
3	-9.000000	-12.840182	-16.680365
2	-2.000000	47.308643	111.364288
1	120.000000	-61.673828	-489.333038
0	-130.000000	106.838760	1985.966919

Value of x_1 = -3.621847

Birge Vieta Example

- Find the root of $f(x) = x^3 + x - 1$

Initial guess = 0

Birge Vieta Example

- Find the root of $f(x) = x^3 + x - 1$
- Iteration 1

i	ai	bi	ci
3	1.000000	1.000000	1.000000
2	0.000000	0.000000	0.000000
1	1.000000	1.000000	1.000000
0	-1.000000	-1.000000	-1.000000

Value of x_1 = 1.000000

Birge Vieta Example

- Find the root of $f(x) = x^3 + x - 1$
- Iteration 2

i	ai	bi	ci
3	1.000000	1.000000	1.000000
2	0.000000	1.000000	2.000000
1	1.000000	2.000000	4.000000
0	-1.000000	1.000000	5.000000

Value of x_1 = 0.750000

Birge Vieta Example

- Find the root of $f(x) = x^3 + x - 1$
- Iteration 3

i	ai	bi	ci
3	1.000000	1.000000	1.000000
2	0.000000	0.750000	1.500000
1	1.000000	1.562500	2.687500
0	-1.000000	0.171875	2.187500

Value of x_1 = 0.686047