Forward Difference Interpolation for Evenly spaced points

Let the data $(x_i, f(x_i))$ be given with uniform spacing, that is, the nodal points are given by $x_i = x_0 + ih$, $i = 0, 1, 2, ..., n$.

Forward difference operator Δ When the operator Δ is applied on $f(x_i)$, we obtain

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f_{i+1} - f_i .$$

That is,

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0),$$

$$\Delta f(x_1) = f(x_1 + h) - f(x_1) = f(x_2) - f(x_1), \text{ etc.}$$

These differences are called the first forward differences.

The second forward difference is defined by

$$\begin{aligned}\Delta^2 f(x_i) &= \Delta[\Delta f(x_i)] = \Delta[f(x_i + h) - f(x_i)] = \Delta f(x_i + h) - \Delta f(x_i)\\ &= [f(x_i + 2h) - f(x_i + h)] - [f(x_i + h) - f(x_i)]\\ &= f(x_i + 2h) - 2f(x_i + h) + f(x_i) = f_{i+2} - 2f_{i+1} + f_i.\end{aligned}$$

The third forward difference is defined by

$$\begin{aligned}\Delta^3 f(x_i) &= \Delta[\Delta^2 f(x_i)] = \Delta f(x_i + 2h) - 2\Delta f(x_i + h) + \Delta f(x_i)\\ &= f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i.\end{aligned}$$

Table 2.2. Forward differences.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	$f(x_0)$			
		$\Delta f_0 = f_1 - f_0$		
x_1	$f(x_1)$		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	
		$\Delta f_1 = f_2 - f_1$		$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
x_2	$f(x_2)$		$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	
		$\Delta f_2 = f_3 - f_2$		
x_3	$f(x_3)$			

we get

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{1!\,h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \ldots + (x - x_0)(x - x_1)\ldots(x - x_{n-1})\frac{1}{n!\,h^n}\Delta^n f_0$$

as the **Newton's Forward Difference Interpolating polynomial**

Example 2.17 *For the data*

x	– 2	– 1	0	1	2	3
$f(x)$	15	5	1	3	11	25

construct the forward difference formula. Hence, find f (0.5).

Solution We have the following forward difference table.

Forward difference table. Example 2.17.

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
– 2	15			
		– 10		
– 1	5		6	
		– 4		0
0	1		6	
		2		0
1	3		6	
		8		0
2	11		6	
		14		
3	25			

From the table, we conclude that the data represents a quadratic polynomial. We have $h = 1$. The Newton's forward difference formula is given by

$$f(x) = f(x_0) + (x - x_0)\left(\frac{\Delta f_0}{h}\right) + (x - x_0)(x - x_1)\left(\frac{\Delta^2 f_0}{2h^2}\right)$$

$$= 15 + (x + 2)(-10) + (x + 2)(x + 1)\left(\frac{6}{2}\right)$$

$$= 15 - 10x - 20 + 3x^2 + 9x + 6 = 3x^2 - x + 1.$$

We obtain $f(0.5) = 3(0.5)^2 - 0.5 + 1 = 0.75 - 0.5 + 1 = 1.25.$

Backward Difference Interpolation for Evenly spaced points

Backward difference operator ∇ When the operator ∇ is applied on $f(x_i)$, we obtain

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - f_{i-1}.$$

That is,

$$\nabla f(x_1) = f(x_1) - f(x_0),$$

$$\nabla f(x_2) = f(x_2) - f(x_1), \text{ etc.}$$

These differences are called the first backward differences.

The second backward difference is defined by

$$\nabla^2 f(x_i) = \nabla[\nabla f(x_i)] = \nabla[f(x_i) - f(x_i - h)] = \nabla f(x_i) - \nabla f(x_i - h)$$

$$= f(x_i) - f(x_i - h)] - [f(x_i - h) - f(x_i - 2h)]$$

$$= f(x_i) - 2f(x_i - h) + f(x_i - 2h) = f_i - 2f_{i-1} + f_{i-2}$$

The third backward difference is defined by

$$\nabla^3 f(x_i) = \nabla[\nabla^2 f(x_i)] = \nabla f(x_i) - 2\nabla f(x_i - h) + \nabla f(x_i - 2h)$$

$$= f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}.$$

Table 2.3. Backward differences.

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	$f(x_0)$			
		$\nabla f_1 = f_1 - f_0$		
x_1	$f(x_1)$		$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	
		$\nabla f_2 = f_2 - f_1$		$\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$
x_2	$f(x_2)$		$\nabla^2 f_3 = \nabla f_3 - \nabla f_2$	
		$\nabla f_3 = f_3 - f_2$		
x_3	$f(x_3)$			

Substituting in (2.45), we obtain the *Newton's backward difference interpolation formula* as

$$f(x) = f(x_n) + (x - x_n)\frac{1}{1!\,h}\nabla f(x_n) + (x - x_n)(x - x_{n-1})\frac{1}{2!\,h^2}\nabla^2 f(x_n) + \ldots$$

$$+ (x - x_n)(x - x_{n-1}) \ldots (x - x_1)\frac{1}{n!\,h^n}\nabla^n f(x_n). \qquad (2.46)$$

Example 2.20 *Using Newton's backward difference interpolation, interpolate at* $x = 1.0$ *from the following data.*

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	– 1.699	– 1.073	– 0.375	0.443	1.429	2.631

Solution The step length is $h = 0.2$. We have the difference table as given below.

$$= 2.631 + (x - 1.1)\left(\frac{1.202}{0.2}\right) + (x - 1.1)(x - 0.9)\left(\frac{0.216}{2(0.04)}\right)$$

$$+ (x - 1.1)(x - 0.9)(x - 0.7)\left(\frac{0.048}{6(0.008)}\right)$$

$$= 2.631 + 6.01(x - 1.1) + 2.7(x - 1.1)(x - 0.9) + (x - 1.1)(x - 0.9)(x - 0.7)$$

Since, we have not been asked to find the interpolation polynomial, we may not simplify this expression. At $x = 1.0$, we obtain

$$f(1.0) = 2.631 + 6.01(1.0 - 1.1) + 2.7(1.0 - 1.1)(1.0 - 0.9) + (1.0 - 1.1)(1.0 - 0.9)(1.0 - 0.7)$$
$$= 2.631 + 6.01(- 0.1) + 2.7(- 0.1)(0.1) + (- 0.1)(0.1)(- 0.3) = 2.004.$$

Difference table. Example 2.20.

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
0.1	– 1.699					
		0.626				
0.3	– 1.073		0.072			
		0.698		0.048		
0.5	– 0.375		0.120		0.0	
		0.818		0.048		0.0
0.7	0.443		0.168		0.0	
		0.986		0.048		
0.9	1.429		0.216			
		1.202				
1.1	2.631					

Example 2.19 *For the following data, calculate the differences and obtain the Newton's forward and backward difference interpolation polynomials. Are these polynomials different? Interpolate at* $x = 0.25$ *and* $x = 0.35$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

Solution The step length is $h = 0.1$. We have the following difference table.

Since, the third and higher order differences are zero, the data represents a quadratic polynomial. The third column represents the first forward/ backward differences and the fourth column represents the second forward/ backward differences.

The forward difference polynomial is given by

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!h^2}$$

$$= 1.4 + (x - 0.1)\left(\frac{0.16}{0.1}\right) + (x - 0.1)(x - 0.2)\left(\frac{0.04}{0.02}\right)$$

$$= 2x^2 + x + 1.28.$$

The backward difference polynomial is given by

$$f(x) = f(x_n) + (x - x_n)\frac{\nabla f_n}{h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 f_n}{2!h^2}$$

$$= 2.28 + (x - 0.5)\left(\frac{0.28}{0.1}\right) + (x - 0.5)(x - 0.4)\left(\frac{0.04}{0.02}\right)$$

$$= 2x^2 + x + 1.28.$$

Both the polynomials are identical, since the interpolation polynomial is unique. We obtain

$$f(0.25) = 2(0.25)^2 + 0.25 + 1.28 = 1.655$$

$$f(0.35) = 2(0.35)^2 + (0.35) + 1.28 = 1.875.$$

Difference table. Example 2.19.

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

Questions

Give the relation between the divided differences and forward or backward differences.

Solution The required relation is

$$f[x_0, x_1, ..., x_n] = \frac{1}{n!\,h^n} \Delta^n f_0 = \frac{1}{n!\,h^n} \nabla^n f_n .$$

Does the Newton's forward difference formula has permanence property ?

Solution Yes. The Newton's forward difference formula has permanence property. Suppose we add a new data value $(x_{n+1}, f(x_{n+1}))$ at the end of the given table of values. Then, the $(n + 1)$th column of the forward difference table has the $(n + 1)$th forward difference. Then, the Newton's forward difference formula becomes

$$f(x) = f(x_0) + (x - x_0) \frac{\Delta f_0}{1!\,h} + (x - x_0)(x - x_1) \frac{\Delta^2 f_0}{2!\,h^2} + ...$$

$$+ (x - x_0)(x - x_1) \ldots (x - x_n) \frac{\Delta^{n+1} f_0}{(n + 1)!\,h^{n+1}}$$

For performing interpolation for a given data, when do we use the Newton's forward and backward difference formulas?

Solution We use the forward difference interpolation when we want to interpolate near the top of the table and backward difference interpolation when we want to interpolate near the bottom of the table.

Can we decide the degree of the polynomial that a data represents by writing the forward or backward difference tables?

Solution Given a table of values, we can determine the degree of the forward/ backward difference polynomial using the difference table. The kth column of the difference table contains the kth forward/ backward differences. If the values of these differences are same, then the $(k + 1)$th and higher order differences are zero. Hence, the given data represents a kth degree polynomial.

What are the advantages of Lagrange' interpolation over Forward or Backward difference?

Solution The forward and backward interpolation formulae of Newton can be used only when the values of the independent variable x are equally spaced and can also be used when the differences of the dependent variable f(x) become smaller ultimately. But Lagrange's interpolation be used whether the values of x, the independent variable are equally spaced or not and whether the difference of f(x) become smaller or not