Successive Approximation
or
Fixed Point Iteration

Fixed Point Iteration

Fixed-point iteration (or, as it is also called successive approximation) is obtained by rearranging the function $f(x) = 0$ so that x is on the left-hand side of the equation:

$$x = g(x)$$

For example,

$$x^2 - 2x + 3 = 0$$

can be simply manipulated to yield

$$x = \frac{x^2 + 3}{2}$$

Fixed Point Iteration Example

Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

$\epsilon = 0.001$

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Starting with an initial guess of $x_0 = 0$,

The function can be separated directly and expressed in the form

$x_{i+1} = e^{-xi}$

Fixed Point Iteration Example

Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

Sr No	x_i	$\lvert x_k - x_{k-1} \rvert$
0	0	-
1	1	1
2	0.367879	0.632121
3	0.692201	0.324322
4	0.500473	0.191728
5	0.606244	0.105771
6	0.545396	0.060848
7	0.579612	0.034216
8	0.560115	0.019497
9	0.571143	0.011028
10	0.564879	0.006264

Fixed Point Iteration Example 2

The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in [1,2],

Find the root with $\epsilon = 0.001$

Fixed Point Iteration Example 2

The equation can be written in the forms:

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

(e) $x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$

Fixed Point Iteration Example

Sr No	x_i
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Fixed Point Iteration Example 2

With $x_0 = 1.5$, Table lists the results of the fixed-point iteration for all five choices of g.

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	

Fixed-Point Theorem

Let $g \in C[a.b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \text{ for all } x \in (a, b)$$

converge to the unique fixed point p *in* $[a, b]$.

Descartes' Rule of Signs

A polynomial of degree n has the form

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

The polynomial equation $P_n(x) = 0$ has exactly n roots, which may be real or complex. If the coefficients are real, the complex roots always occur in conjugate pairs $(x_r + i x_i,\ x_r - i x_i)$, where x_r and x_i are the real and imaginary parts, respectively. For real coefficients, the number of real roots can be estimated from the *rule of Descartes*:

- The number of positive, real roots equals the number of sign changes in the expression for $P_n(x)$, or less by an even number.
- The number of negative, real roots is equal to the number of sign changes in $P_n(-x)$, or less by an even number.

Descartes' Rule of Signs

As an example, consider $P_3(x) = x^3 - 2x^2 - 8x + 27$. Since the sign changes twice, $P_3(x) = 0$ has either two or zero positive real roots. In contrast, $P_3(-x) = -x^3 - 2x^2 + 8x + 27$ contains a single sign change; hence $P_3(x)$ possesses one negative real zero.

Descartes' Rule of Signs

1. $x^6 - 3x^5 + 2x^4 - 6x^3 - x^2 + 4x - 1 = 0$
2. $3x^5 + 2x^4 + x^3 - 2x^2 + x - 2 = 0$

Descartes' Rule of Signs solution 1

For example, the polynomial equation $p(x) = x^6 - 3x^5 + 2x^4 - 6x^3 - x^2 + 4x - 1 = 0$ will have 5, 3, or 1 positive and 1 negative real root. We can assume then that the number of real roots are at least 2 but can be as many as 6! (There are actually 3 positive, 1 negative, and 2 complex roots.)

Descartes' Rule of Signs solution 2

Clearly there are three changes of sign and hence the number of positive real roots is three or one. Thus, it must have a real root. In fact, every polynomial equation of odd degree has a real root.

We can also use Descarte's rule to determine the number of negative roots by finding the number of changes of signs in $p_n(-x)$. For the above equation, $p_n(-x) = -3x^5 + 2x^4 - x^3 - 2x^2 - x - 2 = 0$ and it has two changes of sign. Thus, it has either two negative real roots or none.

Budan’s Theorem

Let $p^k(a)$ be the value of the k^{th} derivative of $p(x)$ at $x = a$. Let v_a be the number of changes in sign in the sequence of numbers $p(a), p^1(a), \ldots, \mathrm{p}^{\mathrm{n}}(a)$ then number of roots of $p(x)$ in the open interval (a,b) is either equal to $|v_a - v_b|$ or is less than that by a multiple of 2.

Budan’s Theorem

$$p(x) = -6 + 10x - 6x^2 + 4x^4$$

In order to apply Budan’s theorem, we need to determine derivatives.

$$p^1(x) = 10 - 12x + 16x^3$$

$$p^2(x) = -12 + 48x^2$$

$$p^3(x) = 86x$$

$$p^4(x) = 86$$

Budan's Theorem

Let us try to obtain information about possibility of roots in $(-2,-1)$, $(-1,0)$, $(0,1)$ and $(1,2)$

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$$p^3(x) = 86x$$

$$p^4(x) = 86$$

x	$p(x)$	$p^1(x)$	$p^2(x)$	$p^3(x)$	$p^4(x)$	v_a
-2						
-1						
0						
1						
2						

Budan's Theorem

Let us try to obtain information about possibility of roots in $(-2,-1), (-1,0), (0,1)$ and $(1,2)$

x	$p(x)$	$p^1(x)$	$p^2(x)$	$p^3(x)$	$p^4(x)$	V_a
-2	$+$	$-$	$+$	$-$	$+$	4
-1	$-$	$+$	$+$	$-$	$+$	3
0	$-$	$+$	$-$	$+$	$+$	3
1	$+$	$+$	$+$	$+$	$+$	0
2	$+$	$+$	$+$	$+$	$+$	0

Budan's Theorem

$$|v_{-2} - v_{-1}| = 4 - 3 = 1$$

$$|v_{-1} - v_0| = 0$$

$$|v_0 - v_1| = 3$$

$$|v_1 - v_2| = 0$$

By Budan's Theorem, there is one root between -1 and -2 and 1 or 3 roots between 0 and 1.

So Budan's theorem can be used to estimate an upperbound on number of real roots of a polynomial $p(x)$ with real coefficients, within any interval.