Differentiation

We assume that a function $f(x)$ is given in a tabular form at a set of $n + 1$ distinct points $x_0, x_1, ..., x_n$. From the given tabular data, we require approximations to the derivatives $f^{(r)}(x')$, $r \geq 1$, where x' may be a tabular or a non-tabular point. We consider the cases $r = 1, 2$.

3.2.1.1 Derivatives Using Newton's Forward Difference Formula

Consider the data $(x_i, f(x_i))$ given at equispaced points $x_i = x_0 + ih$, $i = 0, 1, 2, ..., n$ where h is the step length. The Newton's forward difference formula is given by

At $x = x_0$, that is, at $s = 0$, we obtain the approximation to the derivative $f'(x)$ as

$$f'(x_0) = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0 - ...\right] \tag{3.4}$$

Example 3.1 *Find dy/dx at $x = 1$ from the following table of values*

x	1	2	3	4
y	1	8	27	64

Solution We have the following forward difference table.

Forward difference table. Example 3.1.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		
4	64			

We have $h = 1$, $x_0 = 1$, and $x = x_0 + sh = 1 + s$. For $x = 1$, we get $s = 0$.

Therefore,

$$\frac{dy}{dx}(1) = \frac{1}{h}\left(\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0\right)$$

$$= 7 - \frac{1}{2}(12) + \frac{1}{3}(6) = 3.$$

Example 3.3 *Find $f'(3)$ and $f''(3)$ for the following data:*

x	3.0	3.2	3.4	3.6	3.8	4.0
$f(x)$	– 14	– 10.032	– 5.296	– 0.256	6.672	14

Solution We have $h = 0.2$ and $x = x_0 + sh = 3.0 + s(0.2)$. For $x = 3$, we get $s = 0$.
We have the following difference table.

Forward difference table. Example 3.3.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
3.0	– 14					
		3.968				
3.2	– 10.032		0.768			
		4.736		– 0.464		
3.4	– 5.296		0.304		2.048	
		5.040		1.584		– 5.120
3.6	– 0.256		1.888		– 3.072	
		6.928		–1.488		
3.8	6.672		0.400			
		7.328				
4.0	14					

We have the following results:

$$f'(x_0) = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0\right]$$

$$f'(3.0) = \frac{1}{0.2}\left[3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(-0.464) - \frac{1}{4}(2.048) + \frac{1}{5}(-5.120)\right] = 9.4667.$$

Questions

1. What are the drawbacks of numerical differentiation?

Solution Numerical differentiation has two main drawbacks. (*i*) On the right hand side of the approximation to $f'(x)$, we have the multiplying factor $1/h$, and on the right hand side of the approximation to $f''(x)$, we have the multiplying factor $1/h2$. Since h is small, this implies that we may be multiplying by a large number. For example, if $h = 0.01$, the multiplying factor on the right hand side of the approximation to $f'(x)$ is 100, while the multiplying factor on the right hand side of the approximation to $f''(x)$ is 10000. Therefore, the round-off errors in the values of $f(x)$ and hence in the forward differences, when multiplied by these multiplying factors may seriously effect the solu- tion and the numerical process may become unstable. (*ii*) When a data is given, we do not know whether it represents a continuous function or a piecewise continuous func- tion. It is possible that the function may not be differentiable at some points in its domain. If we try to find the derivatives at these points where the function is not differ- entiable, the result is unpredictable.