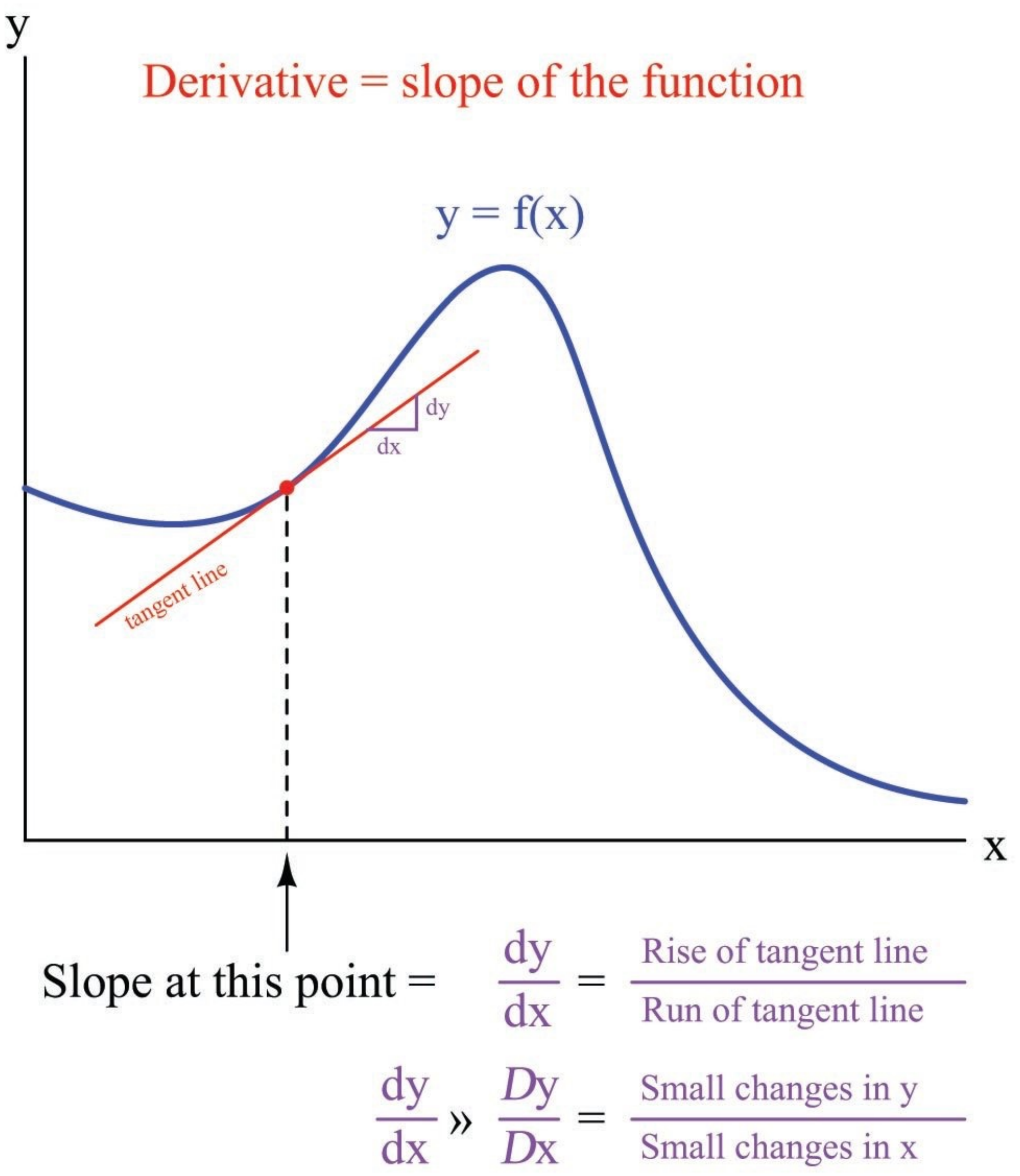
Newton Raphson Method

Derivative

- The derivative of a function at a chosen input value describes the rate of change of the function near that input value. The process of finding a derivative is called differentiation.

Derivative

- Geometrically, the derivative at a point is the slope of the tangent line to the graph of the function at that point, provided that the derivative exists and is defined at that point. For a real valued function of a single real variable, the derivative of a function at a point generally determines the best linear approximation to the function at that point.



Derivatives

$f'(c) = 0$, c is a constant term

$f'(x^n) = nx^{n-1}$

$f'(e^x) = e^x$

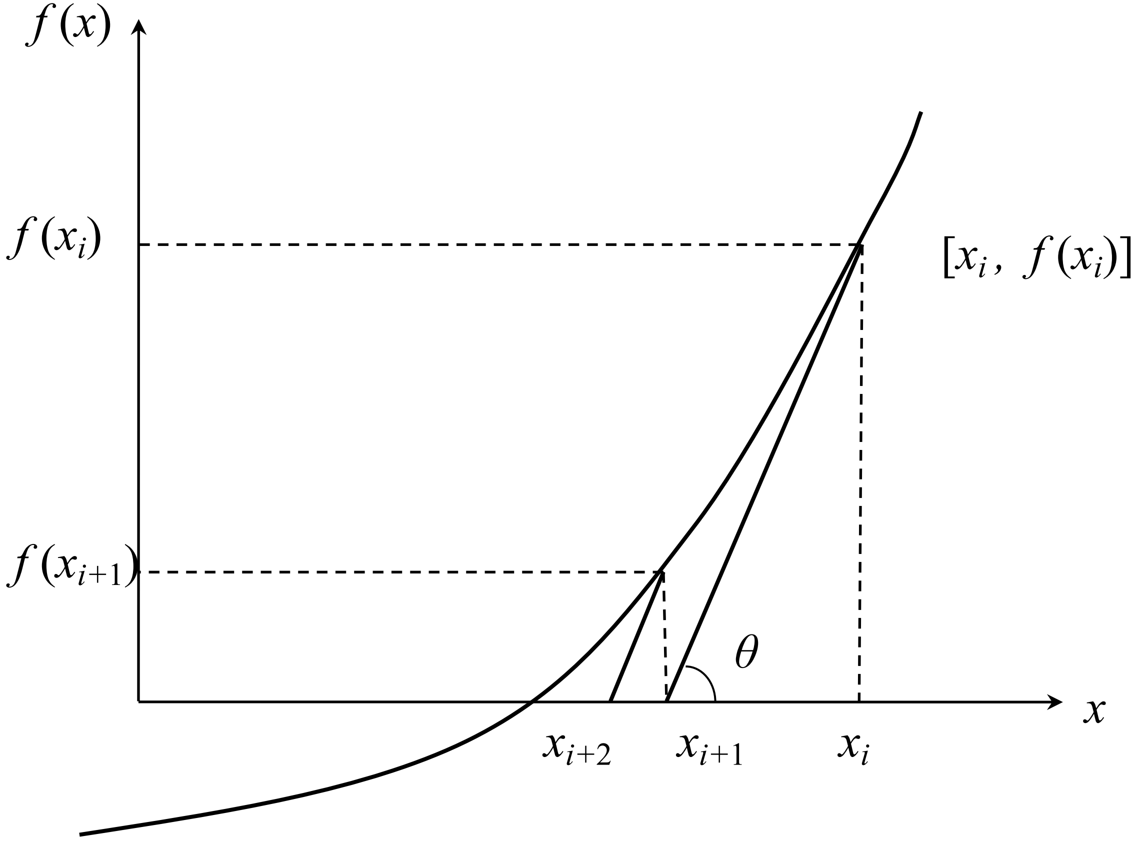
Newton Raphson Method

- Methods such as the bisection method and the false position method of finding roots of a nonlinear equation $f(x) = 0$ require bracketing of the root by two guesses. Such methods are called bracketing methods. These methods are always convergent since they are based on reducing the interval between the two guesses so as to zero in on the root of the equation.

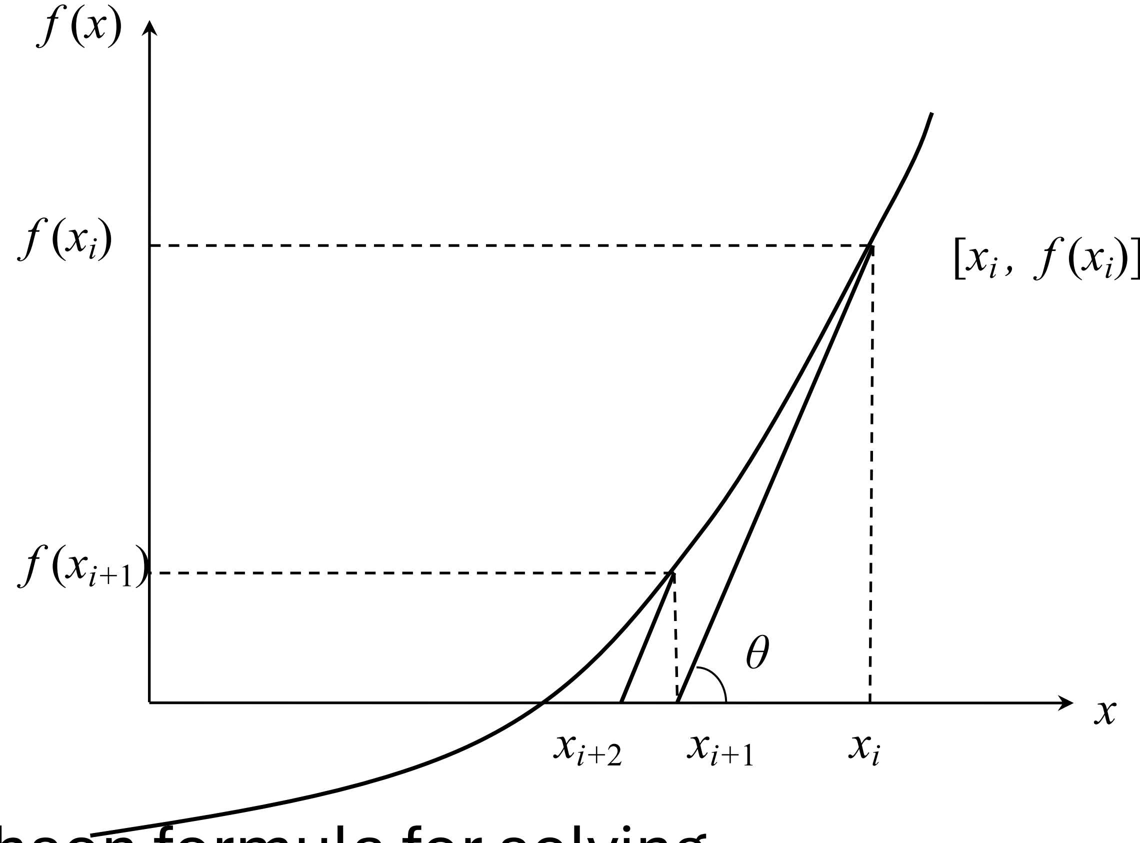
- In the Newton-Raphson method, the root is not bracketed. In fact, only one initial guess of the root is needed to get the iterative process started to find the root of an equation. The method hence falls in the category of open methods. Convergence in open methods is not guaranteed but if the method does converge, it does so much faster than the bracketing methods

Newton Raphson Method

- The Newton-Raphson method is based on the principle that if the initial guess of the root of f(x) =0 is at x_i , then if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root



Newton Raphson Method



- Using the definition of the slope of

a function, at $x = x_i$

gives

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- This equation is called the Newton-Raphson formula for solving nonlinear equations of the form f(x) = 0. So starting with an initial guess, x_i , one can find the next guess, x_{i+1} , by using the above equation. One can repeat this process until one finds the root within a desirable tolerance.

Algorithm

The steps of the Newton-Raphson method to find the root of an equation f(x) = 0 are

1. Let k =1, Evaluate $f'(x_i)$
2. Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3. Compare the absolute approximate error $|c_k - c_{k-1}|$ or absolute relative approximate error $\frac{|c_k - c_{k-1}|}{|c_k|}$ with the pre-specified relative error tolerance ϵ.

where

c_k = estimated root from present iteration

c_{k-1}= estimated root from previous iteration

$If\ |c_k - c_{k-1}| < \epsilon\ or\ \frac{|c_k - c_{k-1}|}{|c_k|} < \epsilon$ then exit

Else

k = k + 1

go to Step 2

Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Advantages and Disadvantages

Advantages:
1. Only one initial guess is needed.
2. Very rapid convergence. It has quadratic convergence which implies, number of correct figures in the estimate is nearly doubled at each successive step.

Disadvantages:
1. If initial guess is not sufficiently near the root, it may diverge.
2. Function must be differentiable.
3. In case of multiple roots, pace of convergence slows down,
4. There can be many instances, when Newton Raphson may fail to converge,

Example 1

Eg 1. *Find the real root of the equation* $x^3 - 4x - 9 = 0$ *by* Newton Raphson method *correct to two decimal places. Take* x_0 *= 2.5,*

$\epsilon = 0.001$

$f'(x) = 3x^2 - 4$

x	0	1	2	3
f(x)	-9	-12	-9	6

$$f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375$$

- $f'(2.5) = 3(2.5)^2 - 4 = 14.75$
- $x_{i+1} = 2.5 - \frac{-3.375}{14.75} = 2.7288$

Solution 1

x_i	f(x_i)	f '(x_i)	x_{i+1}	$\lvert c_k - c_{k-1} \rvert$
2.5	-3.375	14.75	2.7288	-
2.7288	0.4046	18.3393	2.7067	0.0221

$f(2.7288) = 0.4046$

$f'(2.7288) = 18.3393$

$x_{i+1} = 2.7288 - \frac{0.4046}{18.3393} = 2.7067$

Solution 1

x_i	f(x_i)	f ‘(x_i)	x_{i+1}	$\lvert c_k - c_{k-1} \rvert$
2.5	-3.375	14.75	2.7288	-
2.7288	0.4046	18.3393	2.7067	0.0221
2.7067	0.004	17.9795	2.7065	0.0002

$f(2.7067) = 0.004$

$f'(2.7067) = 17.9795$

$x_{i+1} = 2.7067 - \frac{0.004}{17.9795}$ = 2.7065

ϵ = 0.001 $\lvert c_k - c_{k-1} \rvert$ = 0.0002 < ϵ

The approximate real root is 2.7065

Solution 2

Eg 2. *Find the real root of the equation* $x^3 - x - 1$ *by* Newton Raphson method *correct to three decimal places. Take* x_0 *= 1.5,*

ϵ = 0.001

f'(x) = $3x^2 - 1$

$f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875$

- f'(1.5) = $3(1.5)^2 - 1 = 5.75$
- $x_{i+1} = 1.5 - \frac{0.875}{5.75}$ = 1.3478

Solution 2

x_i	f(x_i)	f ‘(x_i)	x_{i+1}	$\lvert c_k - c_{k-1} \rvert$
1.5	0.875	5.75	1.3478	-
1.3478	0.1007	4.4499	1.3252	0.0226

$f(1.3478) = 0.1007$

$f'(1.3478) = 4.4499$

$x_{i+1} = 1.3478 - \frac{0.1007}{4.4499}$ = 1.3252

Solution 2

x_i	f(x_i)	f '(x_i)	x_{i+1}	$\lvert c_k - c_{k-1} \rvert$
1.5	0.875	5.75	1.3478	-
1.3478	0.1007	4.4499	1.3252	0.0226
1.3252	0.0021	4.2646	1.3247	0.0005

$f(1.3252) = 0.0021$

$f'(1.3252) = 4.2646$

$x_{i+1} = 1.3252 - \frac{0.0021}{4.2646}$ = 1.3247

ϵ = 0.001 $\lvert c_k - c_{k-1} \rvert$ = 0.0005 < ϵ

The approximate real root is 1.3247

Example 3

This polynomial obviously has roots at x = 2 and at x = 4; one is a double root, the other a triple root:

$f(x) = (x-2)^3(x-4)^2$

$= x^5 - 14x^4 + 76x^3 - 200x^2 + 256x - 128$

a. Which root can you get with bisection? Which root can't you get?

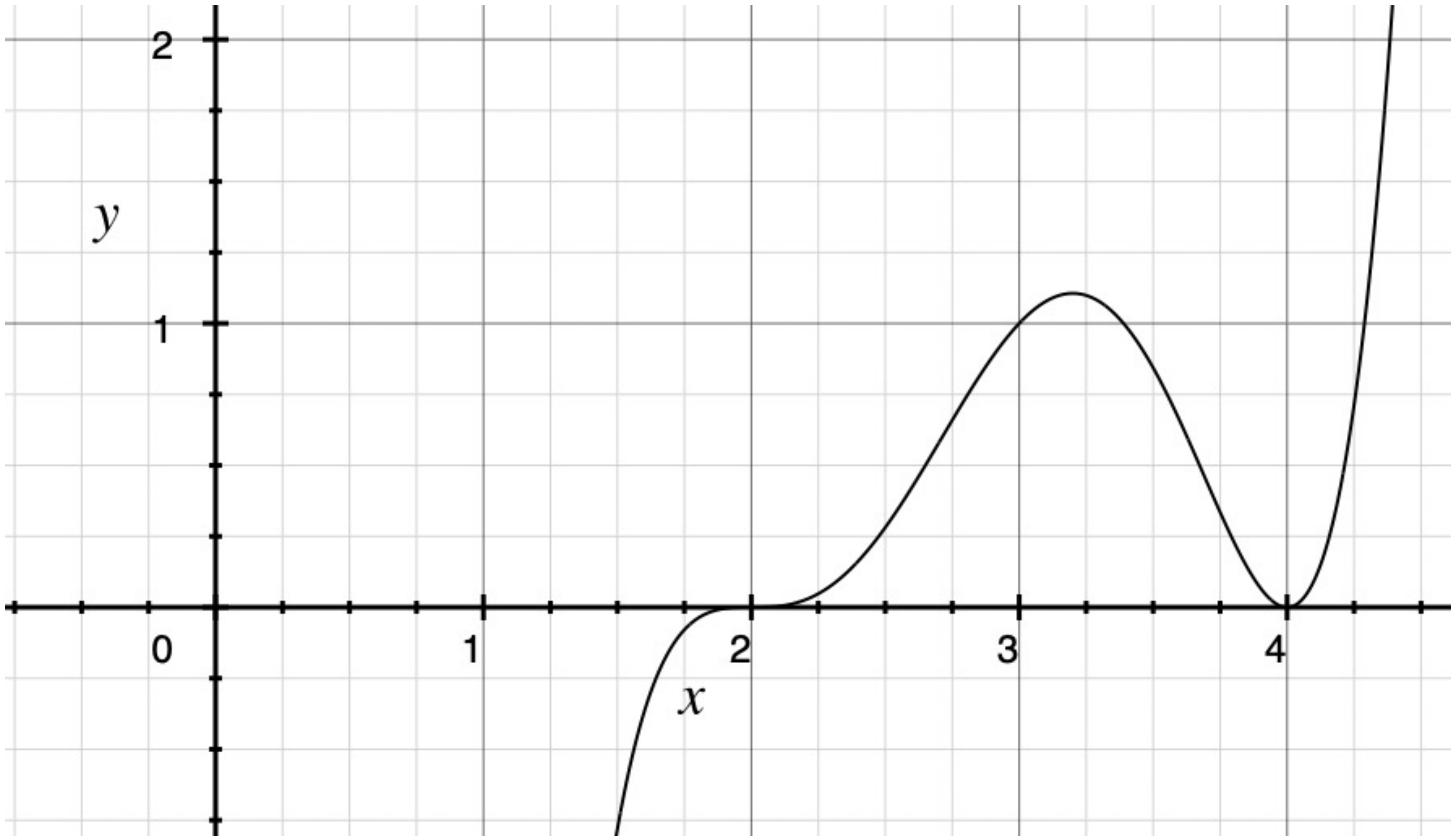
b. Repeat part (a) with the secant method.

c. If you begin with the interval [1,5], which root will you get with

(1) bisection, (2) the secant method, (3) false position?

d. Use Newton's method with x_0 = 3.2 Does it converge? Can Newton method converge to root x = 4?

Solution 3



Solution 3

This polynomial obviously has roots at x = 2 and at x = 4; one is a double root, the other a triple root:

$f(x) = (x-2)^3(x-4)^2$

$= x^5 - 14x^4 + 76x^3 - 200x^2 + 256x - 128$

a.

b.

c.

d. Derivate is zero, so the method does not converge.

Solution 3

d.

Yes x_0 = 4.1, can converge to x = 4

x_i	f(x_i)	f ‘(x_i)	x_{i+1}	$\lvert c_k - c_{k-1} \rvert$
4.1	0.0926	1.9845	4.0533	
4.0533	0.0246	0.9594	4.0277	0.0256
4.0277	0.0064	0.4707	4.0141	0.0136
4.0141	0.0016	0.233	4.0071	0.007
4.0071	0.0004	0.1159	4.0036	0.0035

Example 4

This quadratic has two nearly equal roots:

P (x) is $x^2 - 4x + 3.9999$

a. Which root do you get with Newton's method starting at x = 2.1?
b. Repeat part (a) but starting with x = 1.9.

c. What happens with Newton's method starting from

x = 2.0?

Solution 4

a. Starting from x_0 = 2.1, convergence is to x = 2.0108

b. Starting from x_0 = 1.9, convergence is to x = 1.9892.

c. Starting from x_0 = 2.0 fails, f'(2.0) = zero.

Example 5

- f(x) = 3x - e^x

Starting from 0.5

Solution 5

- f(x) = 3x - e^x. f'(x) = $3 - e^x$

x_i	f(x_i)	f '(x_i)	x_{i+1}	$\lvert c_k - c_{k-1} \rvert$
0.5	-0.148	1.3513	0.6101	
0.6101	-0.0104	1.1595	0.619	0.0089
0.619	0	1.1429	0.6191	0.0001

Successive Approximation
or
Fixed Point Iteration

Fixed Point Iteration

Fixed-point iteration (or, as it is also called successive approximation) is obtained by rearranging the function $f(x) = 0$ so that x is on the left-hand side of the equation:

$$x = g(x)$$

For example,

$$x^2 - 2x + 3 = 0$$

can be simply manipulated to yield

$$x = \frac{x^2 + 3}{2}$$

Fixed Point Iteration Example

Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

$\epsilon = 0.001$

Fixed Point Iteration Example

Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

$\epsilon = 0.001$

Starting with an initial guess of $x_0 = 0$,

The function can be separated directly and expressed in the form

$x_{i+1} = e^{-xi}$

Fixed Point Iteration Example

Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

Sr No	x_i	$\lvert x_k - x_{k-1} \rvert$
0	0	-
1	1	1
2	0.367879	0.632121
3		
4		
5		
6		
7		
8		
9		
10		

Fixed Point Iteration Example

Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

Sr No	x_i	$\lvert x_k - x_{k-1} \rvert$
0	0	-
1	1	1
2	0.367879	0.632121
3	0.692201	0.324322
4	0.500473	0.191728
5	0.606244	0.105771
6	0.545396	0.060848
7	0.579612	0.034216
8	0.560115	0.019497
9	0.571143	0.011028
10	0.564879	0.006264

Fixed Point Iteration Example 2

The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in [1,2],

Find the root with $\epsilon = 0.001$

Fixed Point Iteration Example 2

The equation can be written in the forms:

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

(e) $x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$