

Aspects of QFT

We begin by discussing some familiar concepts from QFT I. In particular, chiral and gauge symmetries of fermion fields, and the theoretical formulation of Quantum Electrodynamics (QED).

This will be a bit of a review, which will serve to set our signs and notational conventions.

Natural Units

We will work almost exclusively with natural units,

where

$$\hbar = c = 1$$

In this system,

$$[\text{Length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$$

The mass, m , of a particle is equal to its rest energy, mc^2 , and also its inverse Compton wavelength, mc/\hbar .

For example,

$$\begin{aligned}m_{\text{electron}} &= 9.109 \times 10^{-28} \text{ g} \\&= 0.511 \text{ MeV} \\&= (3.862 \times 10^{-11} \text{ cm})^{-1}\end{aligned}$$

useful conversion

$$\begin{aligned}1 = \hbar c &= 197.3 \text{ MeV} \cdot \text{fm} \\&\approx 200 \text{ MeV} \cdot \text{fm}\end{aligned}$$

Relativity

We follow the conventions of Jackson and Peskin & Schroeder.

The metric tensor $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

where Greek indices

$$\mu, \nu = 0, 1, 2, 3$$

$$= t, x, y, z$$

and Roman indices

$$i, j = 1, 2, 3$$

Four vectors are $x^m = (x^0, \vec{x})$ contravariant

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x}) \quad \text{covariant}$$

↑ Einstein summation convention

scalar products

$$x_\mu = \sum_{\nu=0}^3 g_{\mu\nu} x^\nu$$

$$\begin{aligned} P \cdot x &= g_{\mu\nu} P^\mu x^\nu \\ &= P^0 x^0 - \vec{P} \cdot \vec{x} \end{aligned}$$

The relativistic dispersion relation is

$$P^2 = P^\mu P_\mu = E^2 - \vec{P}^2 = m^2 \geq 0$$

Derivatives are covariant, $\partial_m = \frac{\partial}{\partial x^m} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$

↑
↑
notice

The Levi-Civita tensor, $\epsilon^{\mu\nu\rho\sigma}$ totally antisymmetric

$$\epsilon^{0123} = +1 \Rightarrow \begin{cases} \epsilon_{0123} = -1 \\ \epsilon^{1230} = -1 \end{cases}$$

Fourier transforms

Recall the n -dimensional Dirac delta function: $\delta^{(n)}(x)$

which satisfies, $\int d^n x \delta^{(n)}(x) = 1$

The Fourier transform pairs are

$$\left\{ \begin{array}{l} f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \tilde{f}(k) \\ \tilde{f}(k) = \int d^4 x e^{ik \cdot x} f(x) \end{array} \right.$$

$k \cdot x = k \cdot x' - \vec{k} \cdot \vec{x}$

so that $\int d^4 x e^{ik \cdot x} = (2\pi)^4 \delta^{(4)}(k)$

Scalar Fields

Consider a real scalar field $\varphi(x)$ in 4D spacetime

Lagrange density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - V(\varphi)$$

↑ ↑ ↑
 kinetic term mass term interactions
 (e.g., $\frac{1}{4!} \lambda \varphi^4, \dots$)

The equations of motion for a real scalar field can be given by the Euler-Lagrange eqns.

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \right) - \frac{\delta \mathcal{L}}{\delta \varphi} = 0$$

which gives

$$(\square + m^2) \varphi = - \underbrace{\frac{\partial V}{\partial \varphi}}_{\text{Klein-Gordon operator}}$$

Note: $\square \equiv \partial_\mu \partial^\mu$

The action is given by

$$S = \int d^4x \mathcal{L}$$

Functional Quantization of Scalar Field

There are two primary formalisms to construct quantum theories of fields. One is canonical quantization, which was discussed in QFT I. The other is known as functional quantization or Path Integral quantization. The Path Integral (PI) has many advantages over canonical quantization:

- (1) Simple covariant way to quantize gauge theories
- (2) Suitable for perturbative methods
- (3) Based on commuting numbers (c-numbers) instead of operators.

Details on PI will be given in QFT II. Here, we will summarize main results. We will assume you are familiar with a cursory intro to PI & NRQM.

The vacuum-to-vacuum matrix element is defined as

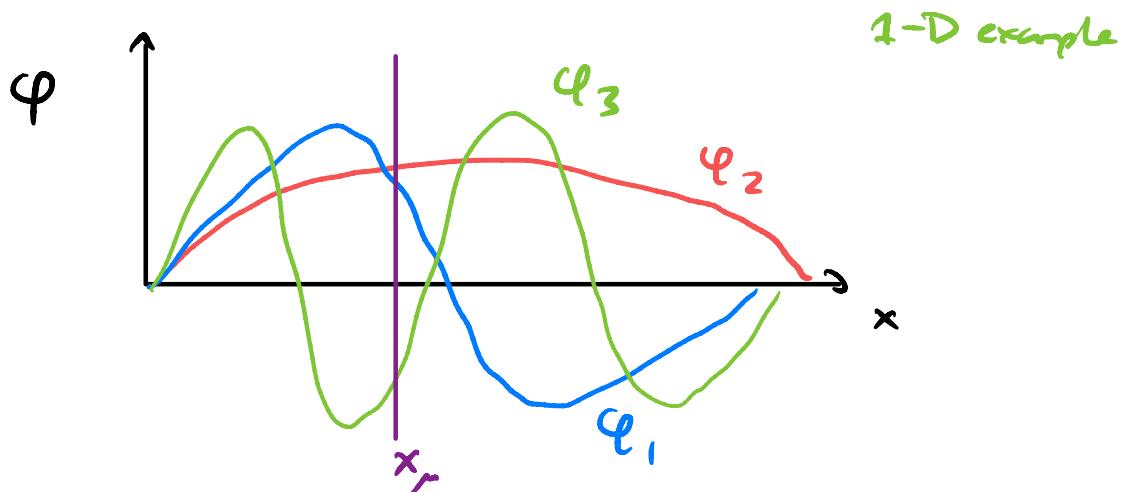
$$\langle 0|0 \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \int D\varphi e^{iS[\varphi]}$$

where,

- $S[\varphi]$ is the action \rightarrow weight for diagram
- The $T \rightarrow \infty(1-i\epsilon)$ ensures convergence
(often we will drop the limit)
- $D\varphi$ is defined on a lattice

$$D\varphi = \lim_{N \rightarrow \infty} \prod_{x_r} d\varphi(x_r)$$

Essentially, we are "summing" over field configurations at some spacetime point x_r



$$\begin{aligned} \int d\varphi(x) &\approx \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \dots \\ &\sim \int d\varphi(x) \end{aligned}$$

Generating functions

The central objects of all QFTs are time-ordered correlation functions, or Green's functions. The P.I. allows us to easily compute Green's functions using generating functions.

Define the generating function $Z[J]$ as

$$Z[J] = \frac{1}{N} \int D\varphi e^{i \int d^4x (L + J\varphi)}$$

↑
Normalization

such that $Z[0] = 1$
 $\Rightarrow \langle 0|0 \rangle \rightarrow 1$



Source term

Correlation functions are then given by

$$\langle 0|T\{\mathcal{O}[\hat{\varphi}]\}|0\rangle = \frac{1}{N} \int D\varphi \mathcal{O}[\varphi] e^{i \int d^4x (L + J\varphi)}$$

↑
operators

↑
c-number

↳ $\mathcal{O}[\hat{\varphi}]$ generic operator

e.g., $\mathcal{O}[\hat{\varphi}] = \varphi(x_1) \cdots \varphi(x_n)$

So, the generating function gives

$$G_n(x_1, \dots, x_n) = \frac{1}{n!} \int D\varphi \varphi(x_1) \cdots \varphi(x_n) e^{i S[\varphi]}$$

$$= \frac{(-i)^n S^n}{\delta J(x_1) \cdots \delta J(x_n)} Z[J] \Big|_{J=0}$$

Functional Derivatives

Definition :

$$\frac{\delta}{\delta J(x)} J(y) = S^{(4)}(x-y)$$

Example

$$\frac{\delta}{\delta J(x)} \int d^4y J(y) \varphi(y) = \varphi(x)$$

Example

$$\begin{aligned} & \frac{\delta}{\delta J(x)} \exp \left[i \int d^4y J(y) \varphi(y) \right] \\ &= \exp \left[i \int d^4y J(y) \varphi(y) \right] \frac{\delta}{\delta J(x)} i \int d^4y J(y) \varphi(y) \\ &= i \varphi(x) \exp \left[i \int d^4y J(y) \varphi(y) \right] \end{aligned}$$

Example

$$\frac{\delta}{\delta J(x)} \int d^4y \partial_\mu J(y) V^\mu(y)$$

$$= \frac{\delta}{\delta J(x)} \int d^4y J(y) (-\partial_\mu V^\mu(y)) + \underbrace{\frac{\delta}{\delta J(x)}}_{\text{(Boundary terms)}}$$

$$= -\partial_\mu V^\mu(x)$$

↑ gives zero except
 for topologically
 interesting terms.

Free Klein-Gordon theory

The generating function can be evaluated in closed form for the Free KG theory.

$$Z_{KG}[J] = \frac{1}{N_{KG}} \int D\varphi e^{i \int d^4x \left(\frac{1}{2} (\partial_\mu \varphi)^2 - \frac{m^2}{2} \varphi^2 + J_\mu \varphi_\mu \right)}$$

↴ $= \partial_\mu(\varphi \partial^\mu \varphi) - \varphi \square \varphi$
 ↴ surface term

$$\hookrightarrow N_{KG} = Z[0]$$

$$= \frac{1}{N_{KG}} \int D\varphi e^{i \int_{x,y} \varphi_x i \tilde{D}_{x,y}^{-1} \varphi_y + i \int_x J_x \varphi_x}$$

where $\int_x = \int d^4x$ and $\int_y i \tilde{D}_{x,y}^{-1} \varphi_y = -(\partial_x^2 + m^2) \varphi_x$

Now,

$$\int_y i D_{y,x}^{-1} \varphi_y = -(\partial_x^2 + m^2) \varphi_x$$

$$\Rightarrow i D_{y,x}^{-1} = -(\partial_x^2 + m^2) \delta^{(4)}(y-x) \quad \xrightarrow{(y-x)_0 \rightarrow T(1-\epsilon\epsilon)}$$
$$= -(\partial_x^2 + m^2) \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)}$$
$$= \int \frac{d^4 p}{(2\pi)^4} (\rho^2 - m^2 + i\epsilon) e^{-ip \cdot (y-x)}$$

Then, we get Formal propagator

$$D_{y,x} = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{\rho^2 - m^2 + i\epsilon} e^{-ip \cdot (y-x)} = i \Delta(y-x)$$

such that $\int \int_y D_{x,y} D_{y,z}^{-1} = \delta^{(4)}(x-z)$

The integrals are now Gaussians

Recall

$$\int_{i=1}^n dx_i e^{-x_i A_{ij} x_j} = (2\pi)^{n/2} \frac{1}{\sqrt{\det A}}$$

Let us complete the square

$$\begin{aligned}
 & \frac{i}{2} \int_{y,x} \varphi_y (iD_{y,x}^{-1}) \varphi_x + i \int_x J_x \varphi_x \\
 &= \frac{i}{2} \int_{y,x} (\varphi_y - \int_z J_z i D_{z,y}) (i D_{y,x}^{-1}) (\varphi_x - \int_{x'} i D_{x,z'} J_{z'}) \\
 &\quad + \frac{i}{2} \int_{y,x} J_y i D_{y,x} J_x \\
 &= \frac{i}{2} \int_{y,x} \varphi_y' i D_{y,x}^{-1} \varphi_x' + \frac{i}{2} \int_{y,x} J_y i D_{y,x} J_x
 \end{aligned}$$

so, integrate over φ' , find

$$Z_{k_0}[\gamma] = e^{\frac{i}{2} \int x^\alpha \int d^\alpha y J(x) i \Delta(x-y) J(y)}$$

Simple application.

$$\begin{aligned}
 G_2(x_1, x_2) &= \langle 0 | T\{\varphi(x_1) \varphi(x_2)\} | 0 \rangle \\
 &= \frac{1}{N_{k_0}} \int D\varphi \varphi(x_1) \varphi(x_2) e^{i L_{k_0}} \\
 &= \left. \frac{(-i)^2 \delta^2}{\delta J(x_1) \delta J(x_2)} Z_{k_0}[\gamma] \right|_{\gamma=0} \\
 &= i \Delta(x-y)
 \end{aligned}$$

Interacting φ^4 -theory - Perturbation Theory

Consider $\lambda\varphi^4$ theory,

↗ Coupling

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

$$= \mathcal{L}_{KG} + \mathcal{L}_\lambda$$

where

$$\mathcal{L}_\lambda = -\frac{\lambda}{4!}\varphi^4$$

In general, can integrate $Z[J]$. But, if λ is small,
can expand perturbation series

$$\begin{aligned} Z[J] &= \frac{1}{N_{\text{d.f.}}} \int D\varphi e^{i \int d^4x (\mathcal{L}_{\text{KG}} + J\varphi)} \\ &= \frac{1}{N_{\text{d.f.}}} \int D\varphi \underbrace{e^{i \int d^4x \mathcal{L}_{\text{KG}}}}_{= N_\lambda} e^{i \int d^4x (\mathcal{L}_{\text{KG}} + J\varphi)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \int d^4x \mathcal{L}_{\text{KG}} \right)^n \end{aligned}$$

Note:

$$N_\lambda = Z[0]$$

$$= N_{KG} \langle 0 | T \{ e^{i \int d^4x \mathcal{L}_{\text{KG}}} \} | 0 \rangle$$

- vacuum-to-vacuum
diagrams

So,

$$Z[J] = \frac{1}{N_{\text{vac}}} \int D\varphi e^{i \int d^4x L_{\text{eff}}} \frac{e^{i \int d^4x (L_{\text{vac}} + J\varphi)}}{= N_{kG} Z_{kG}[J]}$$

↓

$$= N_{kG} \langle 0|T(e^{i \int d^4x L_{\text{eff}}})|0\rangle$$
$$= \frac{1}{N_{\text{vac}}} e^{i \int d^4x L_{\text{eff}} [-i \frac{\delta}{\delta J}]} Z_{kG}[J]$$

where

$$N_{\text{vac}} = e^{i \int d^4x L_{\text{eff}} [-i \frac{\delta}{\delta J}]} Z_{kG}[J] \Big|_{J=0}$$

and $f(\varphi) e^{i \int J\varphi} = f\left(-i \frac{\delta}{\delta J}\right) e^{i \int J\varphi}$

We know $Z_{kG}[J]$, so from $e^{i \int d^4x L_{\text{eff}}} = \sum_{n=0}^{\infty} \frac{1}{n!} (i \int d^4x L_{\text{eff}})^n$

we can solve for correlation functions

as a series in coupling!

$$Z[J] = \frac{1}{N_{\text{vac}}} e^{i \int d^4x L_{\text{eff}} [-i \frac{\delta}{\delta J}]} Z_{kG}[J]$$

Scattering amplitudes and LSZ theorem

Our primary goal in SM phenomenology is to compute scattering amplitudes of particles, from which we can predict/compare to experiment.

Scattering amplitudes are defined through the S-matrix

$$S_{\beta\alpha} = \langle \beta, \text{out} | \alpha, \text{in} \rangle$$

where

$$|\alpha, \text{in}\rangle = |p_1, p_2, \dots, \text{in}\rangle \quad \text{incoming particles}$$

$$\langle \beta, \text{out} | = \langle h_1, h_2, \dots, \text{out} | \quad \text{outgoing particles}$$

We use the standard normalization for a relativistic single particle S-matrix

$$\langle p' | p \rangle = (2\pi)^3 2E \delta^{(3)}(\vec{p}' - \vec{p})$$

$E = \sqrt{h^2 + \vec{p}^2}$

The S-matrix operator converts in-states to out-states,

$$\begin{aligned}S_{\rho\alpha} &= \langle \rho, \text{out} | \alpha, \text{in} \rangle \\&= \langle \rho, \text{in} | S | \alpha, \text{in} \rangle\end{aligned}$$

The S-matrix is a unitary operator

$$S^\dagger S = SS^\dagger = \mathbb{1}$$

Proof

$$\begin{aligned}S_{\rho\alpha} &= \langle \rho, \text{in} | \alpha, \text{in} \rangle \\&= \sum_y \underbrace{\langle \rho, \text{in} | \gamma, \text{out} \rangle}_{S_{\gamma\rho}^* = S_{\rho\gamma}^+} \underbrace{\langle \gamma, \text{out} | \alpha, \text{in} \rangle}_{S_{\gamma\alpha}} \\&= (S^\dagger S)_{\rho\alpha} = (\mathbb{1})_{\rho\alpha} \quad \blacksquare\end{aligned}$$

The main task now is to connect the S-matrix to the QFT correlation functions. We can do this by the LSD theorem

$$\begin{aligned}
 S_{\mu\nu} = & \text{D.T.} + (iz)^{\frac{n}{2}} (\bar{z})^{-\frac{n}{2}} \int d^4x_1 \dots \int d^4y_1 \dots \\
 & \times \exp \left[-i \left(\sum_{i=1}^n \alpha_i \cdot x_i - \sum_{j=1}^n p_j \cdot y_j \right) \right] \\
 & \times (\square_{x_1} + m^2) \dots \langle 0 | T \{ \phi_{x_1} \dots \phi_{y_1} \dots \} | 0 \rangle (\square_{y_1} + m^2) \dots
 \end{aligned}$$

Here D.T. are disconnected terms, e.g., \equiv , $\cancel{\otimes}$, etc. and z is the wavefunction renormalization of the scalar field, $z = \langle 0 | \phi | 0 \rangle / \langle 0 | \rho | 0 \rangle$.

We will focus on tree-level calculations, thus $z=1$

We define the scattering amplitude $M(\omega, \Gamma)$ as

$$S_{\mu\nu} = \text{D.T.} + (2\pi)^4 \delta^{(4)}(p' - p) i M_{\mu\nu}$$

↑ ↑
convention

conserves
total momentum

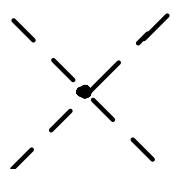
The scattering amplitude is the central observable we want to compute observables. Using the expansion for the correlation functions, we can generate an expansion for the scattering amplitude.

It is convenient to use a diagrammatic representation for the S-matrix called Feynman diagrams.

Feynman Rules - $\lambda \varphi^4$

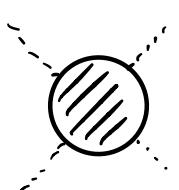
$$\overleftarrow{p} = i\Delta(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

$$= \int d^4x e^{ip \cdot x} i\Delta(x)$$



$$= i\Gamma = -i\lambda$$

Example - $\varphi\varphi \rightarrow \varphi\varphi$ amplitude at leading order



$$= \text{ (crossed lines)} + \mathcal{O}(\lambda^2)$$

$$\Rightarrow iM = -i\lambda + \mathcal{O}(\lambda^2)$$

Observables

Two observables we will concentrate on are cross-sections and decay rates.

Cross-Section (σ) - measures the transition rate of a beam of particles colliding with a target in some region \mathcal{S} final states per unit time & unit flux.

$$\sigma = \frac{\# \text{ events}}{\text{unit time} \times \text{unit flux}}$$

$\sqcap = \frac{\# \text{ particles}}{\text{cross-sectional area} \times \text{time}}$

For $2 \rightarrow n$ processes

$$d\sigma_{\alpha \rightarrow \rho} = \frac{1}{F} |M_{\rho\alpha}|^2 d\Omega_n$$

where $F = 4 \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$ is flux factor

and

$$d\Omega_n = (2\pi)^n \delta(P_\rho - P_\alpha) \prod_{j=1}^n \frac{d^3 k_j}{(2\pi)^3 2 E_j}$$

is n -body phase space with $P_\alpha = p_a + p_b$

$$P_\rho = \sum_{j=1}^n k_j$$

The total cross-section is then

$$\sigma_{\alpha \rightarrow p} = \frac{1}{F} \int d\Phi_n |M_{\alpha n}|^2$$

for particles with spin, can determine unpolarized cross-section by spin-averaging

$$\sigma_{\alpha \rightarrow p} = \frac{1}{F} \frac{1}{2s_u+1} \frac{1}{2s_s+1} \sum_{\text{up to } n} \sum_{\text{final spins}} \int d\Phi_n |M_{\alpha n}|^2$$

Decay Rate (Γ) - Measures the disintegration rate for a particle decaying to lighter particles.

$$\Gamma = \frac{\# \text{ decays}}{\text{unit time}}$$

For $1 \rightarrow n$, we have

$$\Gamma_{\alpha \rightarrow p} = \frac{1}{2E_\alpha} \int d\Phi_n |M_{\alpha n}|^2$$

Here $E_\alpha = \sqrt{M^2 + \vec{p}_\alpha^2}$ is energy of decaying particle.

Γ is not Lorentz invariant. It is common to quote the rest-frame decay rate, $E_\alpha = M$.

The partial width is the decay rate to some particular decay mode. The total decay rate is the sum of partial widths

$$\Gamma_{\alpha, \text{tot}} = \sum_p \Gamma_{\alpha \rightarrow p}$$

The branching ratio for some particular channel is

$$\text{BR}(\alpha \rightarrow p) = \frac{\Gamma_{\alpha \rightarrow p}}{\Gamma_{\text{tot}}}$$