

1. Consider a general *binary* reaction  $ab \rightarrow cd$ , where the masses of the particles are  $m_j$  and their four-momenta are  $p_j = (E_j, \mathbf{p}_j)$  with  $E_j^2 = m_j^2 + \mathbf{p}_j^2$  for each  $j = \{a, b, c, d\}$ . Prove the following results.

- (a) The *Mandelstam invariants* are defined as

$$s = (p_a + p_b)^2, \quad t = (p_a - p_c)^2, \quad u = (p_a - p_d)^2.$$

Show that  $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$ . **Hint:** Consider conservation of four-momentum.

**Solution:** Taking the sum  $s + t + u$ , we find

$$\begin{aligned} s + t + u &= (p_a + p_b)^2 + (p_a - p_c)^2 + (p_a - p_d)^2, \\ &= \sum_j p_j^2 + 2p_a^2 + 2p_a \cdot p_b - 2p_a \cdot p_c - 2p_a \cdot p_d, \\ &= \sum_j m_j^2 + 2p_a \cdot (p_a + p_b - p_c - p_d), \\ &= \sum_j m_j^2, \end{aligned}$$

where in the third line we used conservation of four-momentum states  $p_a + p_b = p_c + p_d$ .

- (b) Show in the *center-of-momentum* (CM) frame, the frame where  $\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}$ , that

$$s = (E_a + E_b)^2 = (E_c + E_d)^2.$$

Show that  $s \geq \max((m_a + m_b)^2, (m_c + m_d)^2)$ .

**Solution:** In the CM frame,  $\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}$ . Therefore,

$$s = (p_a + p_b)^2 = (E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2 = (E_a + E_b)^2. \quad (1)$$

Since  $\mathbf{p}_a + \mathbf{p}_b = \mathbf{p}_c + \mathbf{p}_d$  by momentum conservation, in the CM frame we also have  $\mathbf{p}_c + \mathbf{p}_d = \mathbf{0}$ . Therefore, we also find

$$s = (p_c + p_d)^2 = (E_c + E_d)^2 - (\mathbf{p}_c + \mathbf{p}_d)^2 = (E_c + E_d)^2. \quad (2)$$

Let  $\mathbf{p} \equiv \mathbf{p}_a = -\mathbf{p}_b$  and  $\mathbf{p}' \equiv \mathbf{p}_c = -\mathbf{p}_d$ . The energies of each particle are  $E_a = \sqrt{m_a^2 + \mathbf{p}^2}$ ,  $E_b = \sqrt{m_b^2 + \mathbf{p}^2}$ ,  $E_c = \sqrt{m_c^2 + \mathbf{p}'^2}$ , and  $E_d = \sqrt{m_d^2 + \mathbf{p}'^2}$ . The minimum energy for each particle is when  $\mathbf{p} = \mathbf{p}' = \mathbf{0}$ . So, the minimum  $s_{\min.}$  is given by  $s_{\min.} = (m_a + m_b)^2$  or  $s_{\min.} = (m_c + m_d)^2$ . Thus, the minimum  $s$  is given by  $\max((m_a + m_b)^2, (m_c + m_d)^2)$  since the physical scattering occurs only when the total energy can produce the pair of particles. Note since  $|\mathbf{p}|, |\mathbf{p}'| \in [0, \infty)$ ,  $s$  has no maximum bound.

- (c) Show in the CM frame that the energy of the particles are

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \quad E_b = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}, \quad E_c = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}}, \quad E_d = \frac{s - m_c^2 + m_d^2}{2\sqrt{s}},$$

and the momenta are

$$|\mathbf{p}_a| = |\mathbf{p}_b| = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_a^2, m_b^2), \quad |\mathbf{p}_c| = |\mathbf{p}_d| = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_c^2, m_d^2),$$

where  $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$  is the Källén triangle function.

**Hint:** The following equivalent forms of the Källén function may be useful

$$\begin{aligned} \lambda(x, y, z) &= x^2 + y^2 + z^2 - 2(xy + yz + zx), \\ &= x^2 - 2(y + z)x + (y - z)^2, \\ &= [x - (\sqrt{y} + \sqrt{z})^2][x - (\sqrt{y} - \sqrt{z})^2], \\ &= (x - y - z)^2 - 4yz. \end{aligned}$$

**Solution:** In the CM frame,  $\sqrt{s} = E_a + E_b$ . Take the square of  $(\sqrt{s} - E_a)^2 = E_b^2$  to find

$$\begin{aligned} E_b^2 &= (\sqrt{s} - E_a)^2, \\ &= s + E_a^2 - 2\sqrt{s}E_a. \end{aligned}$$

Now, the on-shell condition gives  $E_a^2 = m_a^2 + \mathbf{p}_a^2$  and  $E_b^2 = m_b^2 + \mathbf{p}_b^2$ , with  $\mathbf{p}_a^2 = \mathbf{p}_b^2$  in the CM frame. So, solving for  $E_a$ ,

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}.$$

From  $\sqrt{s} = E_a + E_b$ , we then find

$$\begin{aligned} E_b &= \sqrt{s} - E_a, \\ &= \sqrt{s} - \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \\ &= \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}. \end{aligned}$$

Following similar arguments, we find  $E_c$  and  $E_d$ .

To find the momentum, use

$$\begin{aligned} |\mathbf{p}_a|^2 &= E_a^2 - m_a^2, \\ &= \left( \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \right)^2 - m_a^2, \\ &= \frac{1}{4s} \left( (s + m_a^2 - m_b^2)^2 - 4sm_a^2 \right), \\ &= \frac{1}{4s} \left( (m_b^2 - s - m_a^2)^2 - 4sm_a^2 \right). \end{aligned}$$

By the definition of the Källén function, we find

$$|\mathbf{p}_a|^2 = \frac{1}{4s} \lambda(s, m_a^2, m_b^2) \quad (3)$$

which holds since the  $\lambda$  function is symmetric in all arguments. In the CM frame,  $|\mathbf{p}_a| = |\mathbf{p}_b|$ , therefore

$$|\mathbf{p}_a| = |\mathbf{p}_b| = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_a^2, m_b^2).$$

We repeat the above arguments for both  $|\mathbf{p}_c|$  and  $|\mathbf{p}_d|$ , finding the desired results.

(d) Show in the CM frame that

$$t = t_0 - 2|\mathbf{p}_a||\mathbf{p}_c|(1 - \cos \theta),$$

where  $t_0 \equiv \Delta^2/4s - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2$  is the maximum value  $t$  can take with  $\Delta = (m_a^2 - m_b^2) - (m_c^2 - m_d^2)$ , and  $\theta$  is the *scattering angle* defined by

$$\cos \theta \equiv \frac{\mathbf{p}_a \cdot \mathbf{p}_c}{|\mathbf{p}_a||\mathbf{p}_c|}.$$

Show that  $t_1 \leq t \leq t_0$  where  $t_1 = t_0 - 4|\mathbf{p}_a||\mathbf{p}_c|$  is the minimum value  $t$  can take.

**Solution:** From the definition,  $t = (p_a - p_c)^2$ , we find

$$\begin{aligned} t &= (p_a - p_c)^2, \\ &= (E_a - E_c)^2 - (\mathbf{p}_a - \mathbf{p}_c)^2, \\ &= (E_a - E_c)^2 - \mathbf{p}_a^2 - \mathbf{p}_c^2 + 2|\mathbf{p}_a||\mathbf{p}_c| \cos \theta, \\ &= (E_a - E_c)^2 - \mathbf{p}_a^2 - \mathbf{p}_c^2 + (2|\mathbf{p}_a||\mathbf{p}_c| - 2|\mathbf{p}_a||\mathbf{p}_c|) + 2|\mathbf{p}_a||\mathbf{p}_c| \cos \theta, \\ &= (E_a - E_c)^2 - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2 - 2|\mathbf{p}_a||\mathbf{p}_c|(1 - \cos \theta). \end{aligned}$$

The maximum value of  $t$  occurs when  $\cos \theta = 1$ , so we define

$$\begin{aligned} t_0 &\equiv t|_{\cos \theta=1}, \\ &= (E_a - E_c)^2 - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2. \end{aligned}$$

so that

$$t = t_0 - 2|\mathbf{p}_a||\mathbf{p}_c|(1 - \cos \theta).$$

Now, we can manipulate  $t_0$  using the expressions for energies,

$$\begin{aligned} E_a - E_c &= \frac{1}{2\sqrt{s}}((m_a^2 - m_b^2) - (m_c^2 - m_d^2)), \\ &\equiv \frac{\Delta}{2\sqrt{s}}, \end{aligned}$$

so  $t_0 = \Delta^2/4s - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2$ . Note that when  $\mathbf{p}_a = \mathbf{p}_c = \mathbf{0}$ , then  $\Delta^2 \geq 0$ . For non-zero  $\mathbf{p}_a$  and  $\mathbf{p}_c$ ,  $t_0$  can be either positive or negative depending on the scattering process (e.g. if  $m_a = m_b = m_c = m_d$ , then  $t_0 \leq 0$ .) The minimum value of  $t$  coincides with  $\cos \theta = -1$ , or

$$\begin{aligned} t_1 &\equiv t|_{\cos \theta = -1}, \\ &= t_0 - 4|\mathbf{p}_a||\mathbf{p}_c|. \end{aligned}$$

So,  $t_1 \leq t \leq t_0$ .

- (e) Show that in the high-energy limit  $|\mathbf{p}_j| \approx E_j \approx \sqrt{s}/2$  for every  $j = \{a, b, c, d\}$ .

**Solution:** Consider first particle  $a$ . It's CM frame energy is

$$\begin{aligned} E_a &= \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \\ &= \frac{\sqrt{s}}{2} + \frac{m_a^2 - m_b^2}{2\sqrt{s}}, \\ &= \frac{\sqrt{s}}{2} + \mathcal{O}(s^{-1/2}), \end{aligned}$$

and for the momentum, using the second form of the Källén function, we find

$$\begin{aligned} |\mathbf{p}_a| &= \frac{\sqrt{s}}{2} \sqrt{1 - \frac{2(m_a^2 + m_b^2)}{s} + \frac{(m_a^2 - m_b^2)^2}{s^2}}, \\ &= \frac{\sqrt{s}}{2} - \frac{m_a^2 + m_b^2}{2\sqrt{s}} + \mathcal{O}(s^{-3/2}), \\ &= \frac{\sqrt{s}}{2} + \mathcal{O}(s^{-1/2}), \end{aligned}$$

where in the second line we performed a series expansion about  $1/s = 0$ . Therefore, both  $|\mathbf{p}_a|$  and  $E_a$  scale as

$$E_a = |\mathbf{p}_a| = \frac{\sqrt{s}}{2} + \mathcal{O}(s^{-1/2}),$$

as  $s \rightarrow \infty$ . Repeating this analysis for particles  $b$ ,  $c$ , and  $d$ , we find that at high-energy  $|\mathbf{p}_j| \approx E_j \approx \sqrt{s}/2$  for each  $j = \{a, b, c, d\}$ .

- (f) For the case where all masses are equal,  $m_a = m_b = m_c = m_d \equiv m$ , write expressions for kinematic quantities in parts (a) through (d).

**Solution:** By direct substitution into the general formulae we derived,

$$E \equiv E_a = E_b = E_c = E_d = \frac{\sqrt{s}}{2},$$

and

$$|\mathbf{p}| \equiv |\mathbf{p}_a| = |\mathbf{p}_b| = |\mathbf{p}_c| = |\mathbf{p}_d| = \frac{1}{2}\sqrt{s - 4m^2},$$

with  $s = 4E^2$ , and since  $E = \sqrt{m^2 + \mathbf{p}^2} \geq m$ , so  $s \geq 4m^2$ . For  $t, t_0 = 0$ , and

$$t = -2\mathbf{p}^2(1 - \cos \theta).$$

2. The two-body differential *Lorentz invariant phase space* for some initial total momentum  $P = (E, \mathbf{P})$  is defined as

$$d\Phi_2(P \rightarrow p_1 + p_2) = \frac{1}{\mathcal{S}} \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2),$$

where  $\mathcal{S}$  is a symmetry factor. Perform partial integrations to show that in the CM frame ( $\mathbf{P} = \mathbf{0}$ ) the differential phase space is given by

$$d\Phi_2(P \rightarrow p_1 + p_2) = \frac{1}{\mathcal{S}} \frac{|\mathbf{p}_1|}{4\pi\sqrt{s}} \frac{d\Omega}{4\pi} \Theta(\sqrt{s} - m_1 - m_2),$$

where  $d\Omega$  is the differential solid angle of  $\mathbf{p}_1$ ,  $s = P^2 = E^2$ , and  $\Theta(x)$  is the Heaviside step function.

Assume we are integrating against a test function  $f(\mathbf{p}_1, \mathbf{p}_2)$ . Since the phase space is Lorentz invariant, we can evaluate in any reference frame. We choose the CM frame. The four-dimensional Dirac delta can be written as

$$\delta^{(4)}(P - p_1 - p_2) = \delta^{(4)}(E - E_1 - E_2) \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2),$$

where we used  $\mathbf{P} = \mathbf{0}$ .

So, we can integrate over the measure  $d^3\mathbf{p}_2$ , eliminating the spatial momentum Dirac delta functions,

$$d\Phi_2(P \rightarrow p_1 + p_2) = \frac{1}{(4\pi)^2} \frac{1}{\mathcal{S}} \frac{d^3\mathbf{p}_1}{E_1 E_2} \delta^{(4)}(E - E_1 - E_2).$$

Note that since  $\mathbf{p}_1 = -\mathbf{p}_2$ ,  $E_1 = \sqrt{m_1^2 + \mathbf{p}_1^2}$  and  $E_2 = \sqrt{m_2^2 + \mathbf{p}_1^2}$ . The remaining delta function can be evaluated by a change of variables to  $|\mathbf{p}_1|$ ,

$$\begin{aligned} \delta(E - E_1 - E_2) &= \left| \frac{\partial(E - E_1 - E_2)}{\partial|\mathbf{p}_1|} \right|^{-1} \delta(|\mathbf{p}_1| - |\mathbf{p}_1^*|), \\ &= \frac{E_1 E_2}{|\mathbf{p}_1| \sqrt{s}} \delta(|\mathbf{p}_1| - |\mathbf{p}_1^*|) \end{aligned}$$

where  $|\mathbf{p}_1^*|$  is the solution to  $E - E_1 - E_2 = 0$ .  
So, we find

$$\begin{aligned} d\Phi_2(P \rightarrow p_1 + p_2) &= \frac{1}{(4\pi)^2} \frac{1}{\mathcal{S}} \frac{d^3\mathbf{p}_1}{E_1 E_2} \frac{E_1 E_2}{|\mathbf{p}_1| \sqrt{s}} \delta(|\mathbf{p}_1| - |\mathbf{p}_1^*|), \\ &= \frac{1}{(4\pi)^2} \frac{1}{\mathcal{S}} \frac{d\Omega d|\mathbf{p}_1| |\mathbf{p}_1|^2}{E_1 E_2} \frac{E_1 E_2}{|\mathbf{p}_1| \sqrt{s}} \delta(|\mathbf{p}_1| - |\mathbf{p}_1^*|), \\ &= \frac{1}{\mathcal{S}} \frac{|\mathbf{p}_1^*|}{4\pi \sqrt{s}} \frac{d\Omega}{4\pi} \Theta(\sqrt{s} - m_1 - m_2). \end{aligned}$$

3. Consider the binary reaction  $ab \rightarrow cd$  where each particle is a scalar boson. The differential cross-section is defined as

$$d\sigma = \frac{1}{\mathcal{F}} |\mathcal{M}|^2 d\Phi_2(p_a + p_b \rightarrow p_c + p_d),$$

where  $\mathcal{F} = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$  is the flux factor. Show that the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_c|}{|\mathbf{p}_a|} \frac{1}{\mathcal{S}} |\mathcal{M}|^2,$$

where the solid angle is defined in the CM frame.

**Solution:** From Problem 2, we have an expression for the phase space in the CM frame,

$$d\Phi_2(P \rightarrow p_1 + p_2) = \frac{1}{\mathcal{S}} \frac{|\mathbf{p}_c|}{16\pi^2 \sqrt{s}} d\Omega,$$

where we leave the Heaviside function implicit. Therefore, we only need to express the flux factor in the CM frame. Note that  $s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2p_a \cdot p_b$ . So,  $(p_a \cdot p_b)^2 = (s - m_a^2 - m_b^2)^2/4$ , therefore

$$\begin{aligned} \mathcal{F} &= 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}, \\ &= 4\sqrt{\frac{(s - m_a^2 - m_b^2)^2}{4} - m_a^2 m_b^2}, \\ &= 2\lambda^{1/2}(s, m_a^2, m_b^2). \end{aligned}$$

Since  $2\sqrt{s}|\mathbf{p}_a| = \lambda^{1/2}(s, m_a^2, m_b^2)$ , we find  $\mathcal{F} = 4\sqrt{s}|\mathbf{p}_a|$ . Combining the pieces, we find the desired result

$$\begin{aligned} d\sigma &= \frac{1}{4\sqrt{s}|\mathbf{p}_a|} |\mathcal{M}|^2 \frac{1}{\mathcal{S}} \frac{|\mathbf{p}_c|}{16\pi^2 \sqrt{s}} d\Omega, \\ &= \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_c|}{|\mathbf{p}_a|} \frac{1}{\mathcal{S}} |\mathcal{M}|^2 d\Omega. \end{aligned}$$

4. Consider the elastic scattering of two scalar particles ( $\varphi\varphi \rightarrow \varphi\varphi$ ) of mass  $m$  described  $\lambda\varphi^4$  theory.

(a) At leading order in the coupling  $\lambda$ , the scattering amplitude is given by

$$i\mathcal{M} = -i\lambda + \mathcal{O}(\lambda^2).$$

Compute the total cross-section  $\sigma$  as a function of  $s$ .

**Solution:** For equal mass scattering,  $|\mathbf{p}_a| = |\mathbf{p}_c|$ . So, the differential cross section is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s} \frac{1}{2} |\mathcal{M}|^2, \\ &= \frac{\lambda^2}{128\pi^2 s} + \mathcal{O}(\lambda^3). \end{aligned}$$

Integrating, we have

$$\begin{aligned} \sigma &= \int d\Omega \frac{d\sigma}{d\Omega}, \\ &= \frac{\lambda^2}{32\pi s} + \mathcal{O}(\lambda^3). \end{aligned}$$

(b) As the energy approaches threshold,  $s \rightarrow 4m^2$ , the total cross-section can be written in terms of the *scattering length*  $a$ ,  $\sigma \rightarrow 4\pi a_0^2/\mathcal{S}$ . Determine  $a_0$  in terms of the coupling  $\lambda$ .

**Solution:** As  $s \rightarrow m^2$ , then

$$\sigma \rightarrow \frac{1}{\mathcal{S}} \frac{\lambda^2}{16\pi(4m^2)} + \mathcal{O}(\lambda^3) = \frac{4\pi a_0^2}{\mathcal{S}}.$$

So, we find

$$a_0 = \frac{\lambda}{16\pi m} + \mathcal{O}(\lambda^3).$$

(c) The *partial wave expansion* is defined as

$$\mathcal{M}(s, \theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \mathcal{M}_{\ell}(s) P_{\ell}(\cos \theta),$$

where  $\ell$  is the angular momentum,  $\theta$  is the scattering angle defined in the CM frame, and  $P_{\ell}(z)$  are the Legendre polynomials. Given the scattering amplitude at leading order in  $\lambda$ , calculate the *partial wave amplitudes*  $\mathcal{M}_{\ell}$  for every  $\ell$ .

**Hint:** The following properties of the Legendre polynomials may be useful. Given the first two polynomials,  $P_0(z) = 1$  and  $P_1(z) = z$ , all remaining  $P_{\ell}$  can be generated through the Bonnet recursion relation for  $\ell > 1$ ,

$$\ell P_{\ell}(z) = z(2\ell-1) P_{\ell-1}(z) - (\ell-1) P_{\ell-2}(z).$$

The polynomial are orthogonal over  $-1 \leq z \leq +1$ ,

$$\int_{-1}^{+1} dz P_{\ell'}(z) P_{\ell}(z) = \frac{2}{2\ell + 1} \delta_{\ell' \ell}.$$

**Solution:** From the orthogonality of  $P_{\ell}$ , we find

$$\mathcal{M}_{\ell}(s) = \frac{1}{2} \int_{-1}^{+1} d \cos \theta P_{\ell}(\cos \theta) \mathcal{M}(s, \theta).$$

Now,  $\mathcal{M} = -\lambda + \mathcal{O}(\lambda^2)$ , which is a constant at leading order. Recognizing that  $1 = P_0(\cos \theta)$ , we find

$$\begin{aligned} \mathcal{M}_{\ell}(s) &= -\frac{\lambda}{2} \int_{-1}^{+1} d \cos \theta P_{\ell}(\cos \theta) + \mathcal{O}(\lambda^2), \\ &= -\frac{\lambda}{2} \int_{-1}^{+1} d \cos \theta P_{\ell}(\cos \theta) P_0(\cos \theta) + \mathcal{O}(\lambda^2), \\ &= -\frac{\lambda}{2} \frac{2}{2\ell + 1} \delta_{\ell 0} + \mathcal{O}(\lambda^2), \\ &= -\lambda \delta_{\ell 0} + \mathcal{O}(\lambda^2) \end{aligned}$$

So, the scattering amplitude is the  $S$  wave ( $\ell = 0$ ) amplitude, all other  $\ell \neq 0$  partial wave amplitudes are identically zero at this order.