Physics 303

Classical Mechanics II

Mechanics in Noninertial Frames

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Noninertial Reference Frames

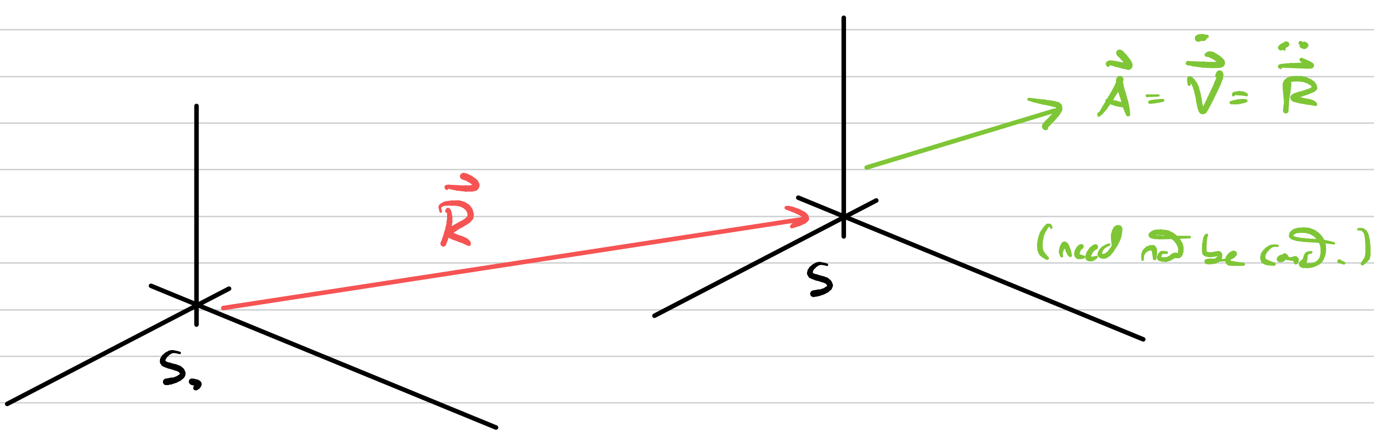
Newton's laws are valid only in inertial reference frames, that is frames which are not accelerating (translational or rotational).

However, many physically interesting systems involve accelerating frames, e.g., ballistic motion on Earth which is rotating about it's axis and revolving around the sun. Thus, it is useful to formulate mechanics of non-inertial reference frames.

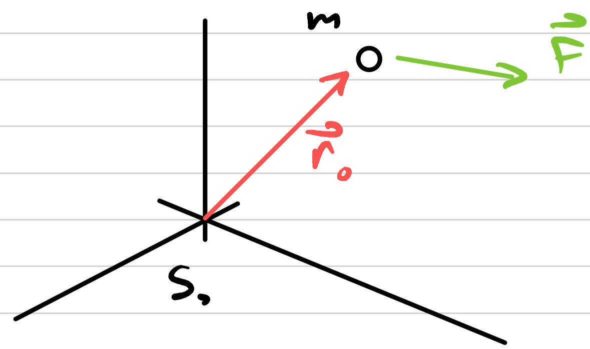
Accelerating Frames

Let's first consider the case of a frame with acceleration but no rotation.

Let S_0 be inertial frame, and S be accelerating frame wrt S_0 with acceleration $\vec{A}$



Consider the motion of a particle of mass m in both frames.



$\boxed{\text{Motion in } S_0}$

Since S_0 is inertial, NII holds

$$m\ddot{\vec{r}}_0 = \vec{F}$$

where $\vec{r}_0$ is position of particle in S_0

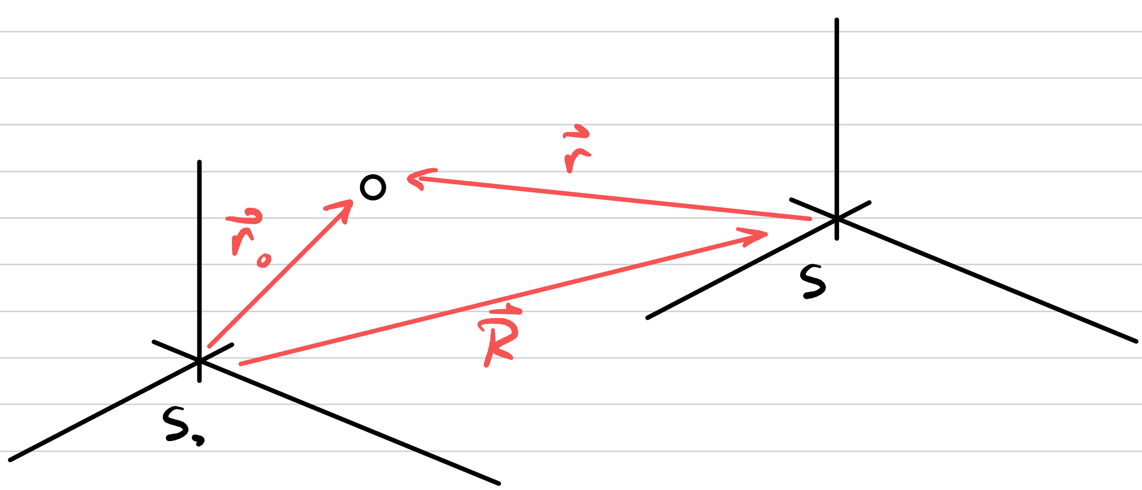
$\boxed{\text{Motion in } S}$

Let $\vec{r}$ be position of ball in S.

Velocity in S_0 relative to S

$$\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$$

velocity in S_0 — velocity in S — velocity of S relative to S_0



Differentiating in time,

$$\ddot{\vec{r}}_0 = \ddot{\vec{r}} + \vec{A}$$

where $\vec{A} = \dot{\vec{V}} \neq \vec{0}$ since S is accelerating.

From NII in S_0, $m\ddot{\vec{r}}_0 = \vec{F}$, so we find

$$\begin{aligned} m\ddot{\vec{r}} &= m\ddot{\vec{r}}_0 - m\vec{A} \\ &= \vec{F} - m\vec{A} \end{aligned}$$

This looks like NII in S except extra term.

We can continue to use NII provided we include an inertial force $\boxed{\vec{F}_{\text{inertial}} = -m\vec{A}}$ in S.

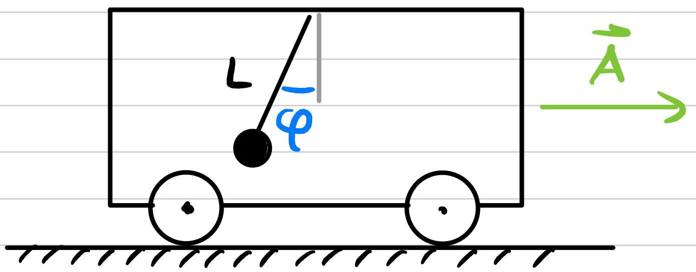
$\vec{F}_{\text{inertial}}$ is a force-like term or pseudo force

$\Rightarrow$ NII in S : $\boxed{m\ddot{\vec{r}} = \vec{F} + \vec{F}_{\text{inertial}}}$

forces   effect of accelerating frame.

Example

Consider a simple pendulum (mass m and length L) mounted inside a railroad cart accelerating to the right with a constant acceleration $\vec{A}$.

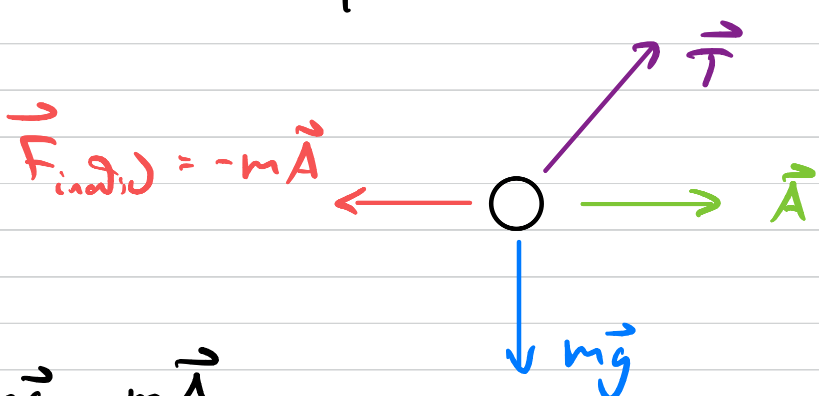


Find the equilibrium angle φ_{eq} which the pendulum will remain at rest wrt the cart.

Solution

Let S_0 be the frame of the ground, and S the frame of the cart.

In S, the forces on the pendulum are



So, NII in S

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A}$$
$$= \vec{T} + m(\vec{g} - \vec{A})$$

Let $\vec{g}_{eff} = \vec{g} - \vec{A}$, so $m\ddot{\vec{r}} = \vec{T} + m\vec{g}_{eff}$

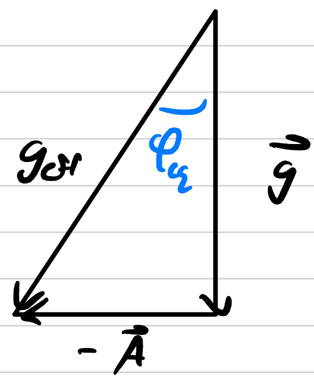
So, forces on pendulum are same as in S_0, except have effective gravity $\vec{g}_{eff}$

Equilibrium occurs when $\ddot{\vec{r}} = \vec{0} \Rightarrow \vec{T} = -m\vec{g}_{eff}$

so, φ_{eq} is defined $\vec{g}_{eff} = \vec{g} - \vec{A}$

$$\tan\varphi_{eq} = \frac{A}{g}$$

$$\Rightarrow \boxed{\varphi_{eq} = \tan^{-1}\left(\frac{A}{g}\right)}$$



For small oscillations about equilibrium, the EOM is

$$\ddot{\varphi} = -\omega^2\varphi \quad \text{with} \quad \omega = \sqrt{\frac{g_{eff}}{L}}$$

Now, $g_{eff} = \sqrt{A^2 + g^2}$, so the frequency is

$$\omega = \sqrt{\frac{\sqrt{g^2 + A^2}}{L}}$$ $\blacksquare$

The Tides

An example of accelerating systems is tidal motion.
Assume Earth is spherical, & that the oceans cover the entire surface.

We observe 2 tides per day (NOT one) so the motion is a little more complicated than the just due to the moon's gravitational attraction.

There are two effects occurring: The moon gives the Earth (oceans, too) an acceleration $\vec{A}$ toward the moon.

$\Rightarrow$ This is centripetal acceleration of Earth as two-body system orbit CM.

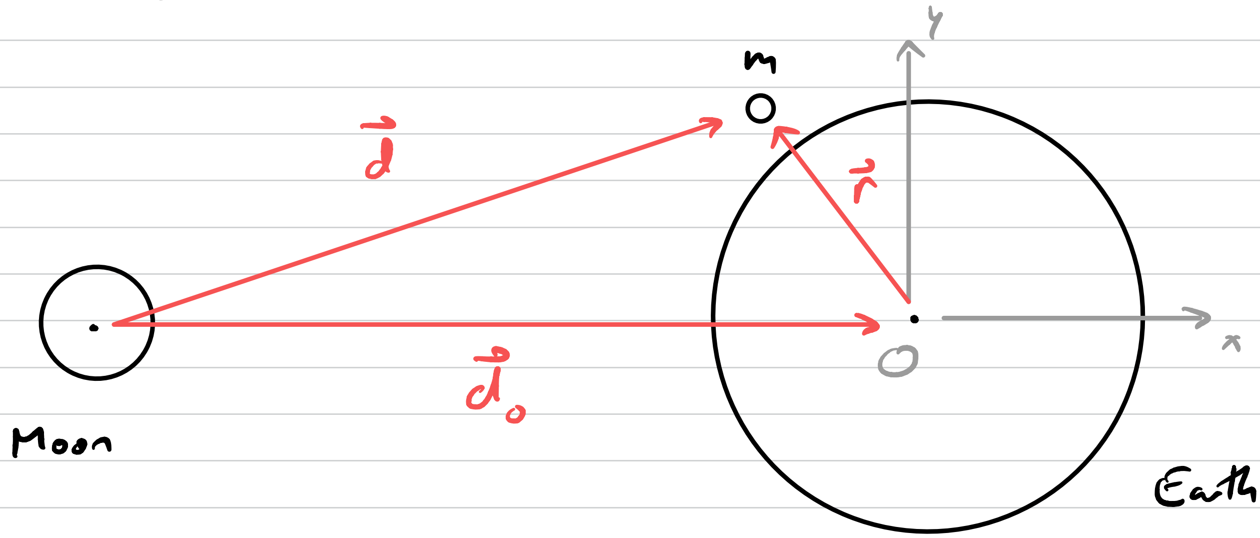
$\Rightarrow$ This acceleration is as if mass is at center of Earth

- If mass closer to moon, feels greater force
  - $\Rightarrow$ Ocean on moon side buldges toward moon.
- If mass on far side, feels weaker force
  - $\Rightarrow$ Ocean on far side buldges outward relative to Earth!

Let's look at the motion of a test mass near Earth.

Let S = frame of Earth (accelerating)

S_0 = frame of Moon (inertial)



The forces on m are

Now, $\vec{r}$ is position of m wrt Earth, but Earth is accelerating due to moon's gravity!

$$\Rightarrow \quad \vec{A} = -GM_m \frac{\hat{d}_0}{d_0^2}$$

Note sign wrt. $\hat{d}_0$

So, NII in S frame is

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A}$$

$$= \left(m\vec{g} - GM_m m \frac{\hat{d}}{d^2} + \vec{F}_{ng} \right) + GM_m m \frac{\hat{d}_0}{d_0^2}$$

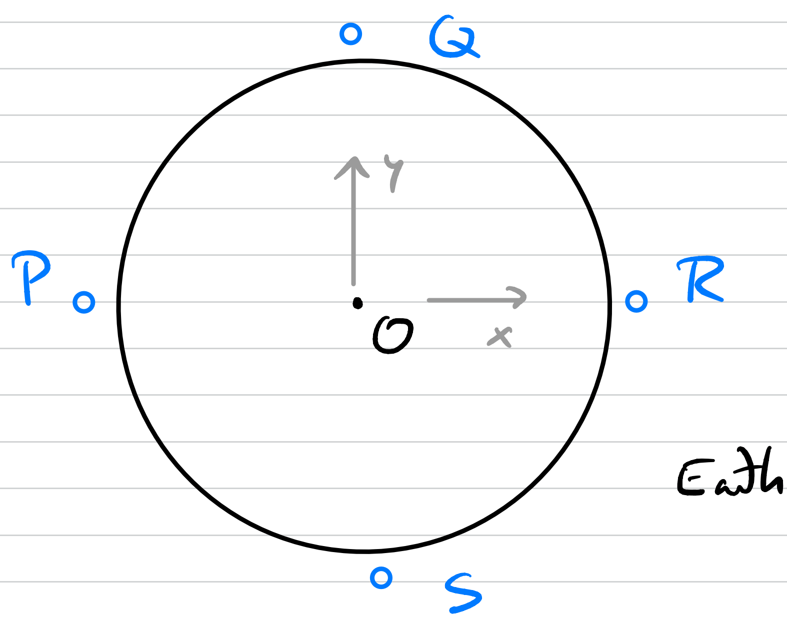
$$\Rightarrow \quad m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{tid} + \vec{F}_{ng}$$

where tidal force is

$$\boxed{\vec{F}_{tid} = -GM_m m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right)}$$

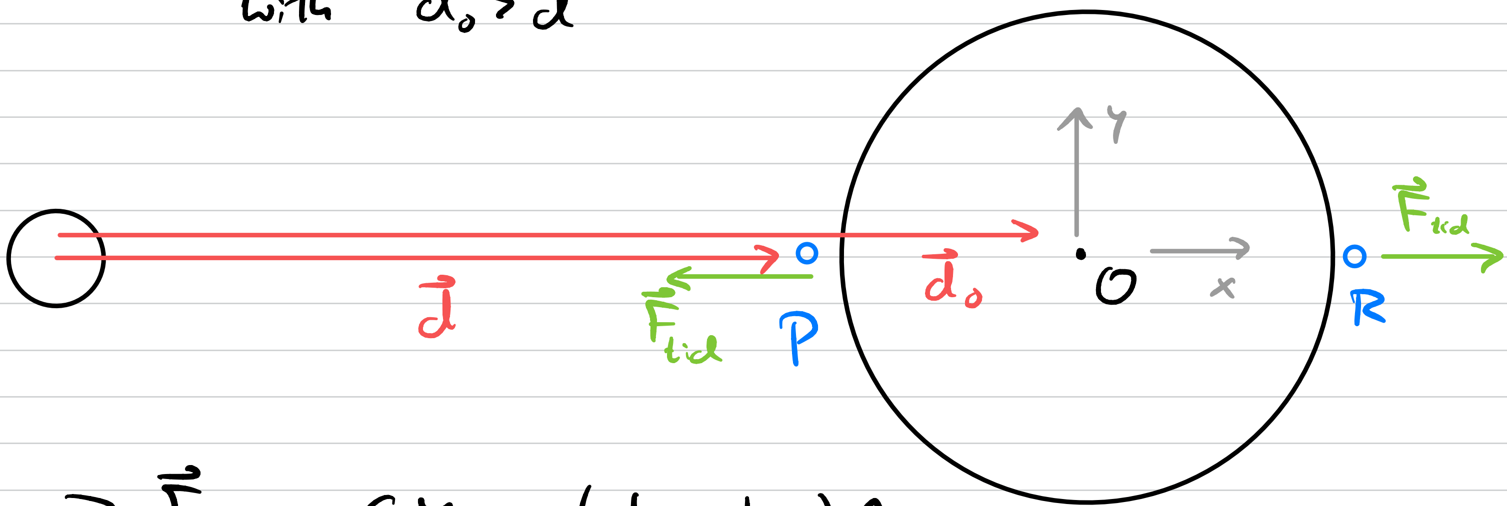
This force is difference of actual force on m and the force on m if it were at the centre.

Let's look at this force at 4 special points



$\boxed{P}$ At point P, $\hat{d} = \hat{x}$, $\hat{d}_o = \hat{x}$

with $d_o > d$



$$\Rightarrow \vec{F}_{tid} = -GM_m m \left(\frac{1}{d^2} - \frac{1}{d_o^2} \right) \hat{x}$$

$$= -GM_m m \underbrace{\left(\frac{d_o^2 - d^2}{d_o^2 d^2} \right)}_{\text{term} > 0} \hat{x} \equiv -F_{tid}\, \hat{x}$$

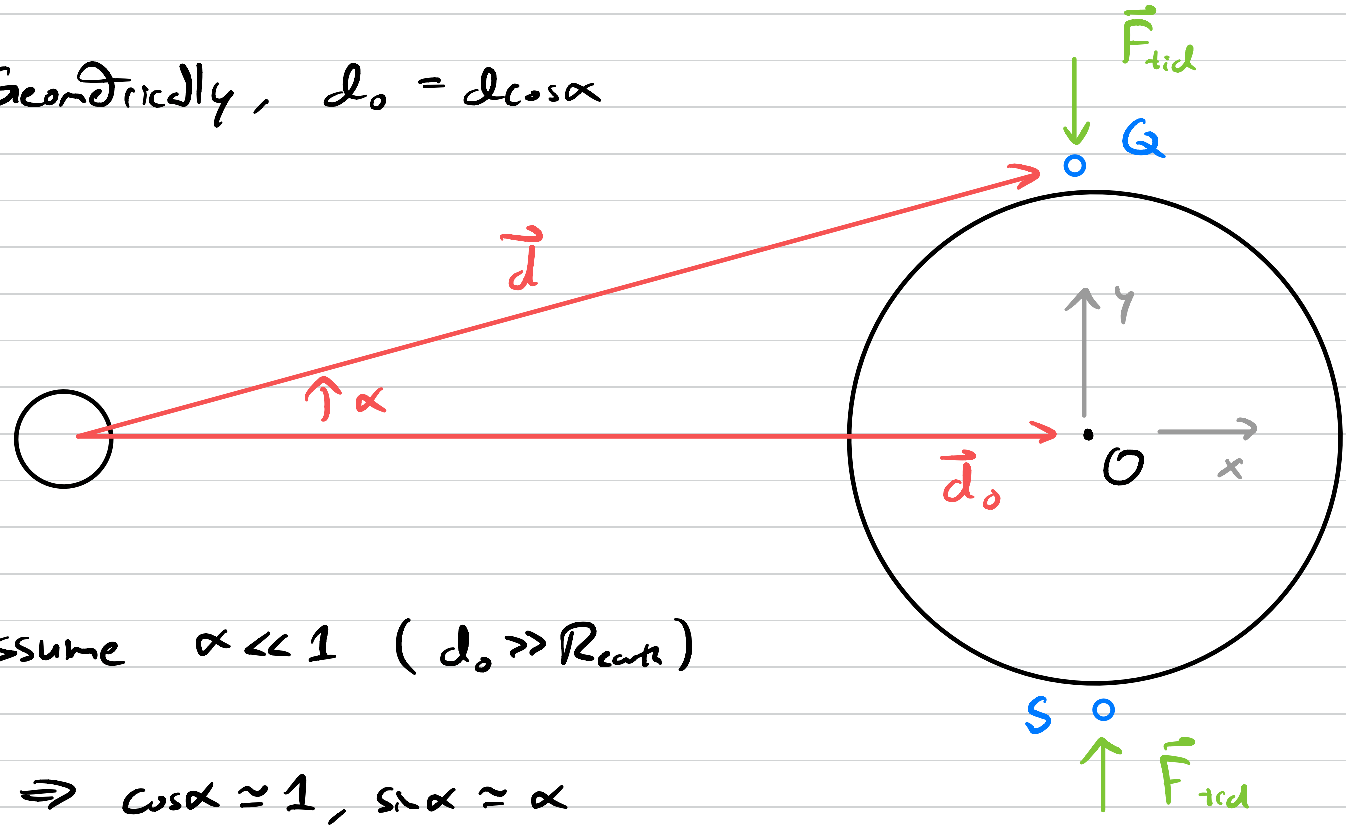
$\boxed{R}$ At R, now have $\vec{d} = d\hat{x}$, $\vec{d}_o = d_o \hat{x}$

but $\underline{d > d_o}$

$$\Rightarrow \vec{F}_{tid} = -GM_m m \underbrace{\left(\frac{d_o^2 - d^2}{d_o^2 d^2} \right)}_{\text{term} < 0} \hat{x} = +F_{tid}\, \hat{x}$$

$\boxed{Q}$ At point Q, Now have $\vec{d}_0 = d_0\,\hat{x}$
but $\vec{d} = d\cos\alpha\,\hat{x} + d\sin\alpha\,\hat{y}$

Geometrically, $d_0 = d\cos\alpha$



Assume $\alpha \ll 1$ ($d_0 \gg R_{earth}$)

$\Rightarrow \cos\alpha \simeq 1, \quad \sin\alpha \simeq \alpha$

So, $d_0 \simeq d \quad \Rightarrow \quad \vec{d} \simeq d_0\,\hat{x} + d_0\,\alpha\,\hat{y} = d_0\,\hat{d}$

$\Rightarrow \quad \hat{d} \simeq \hat{x} + \alpha\,\hat{y}$

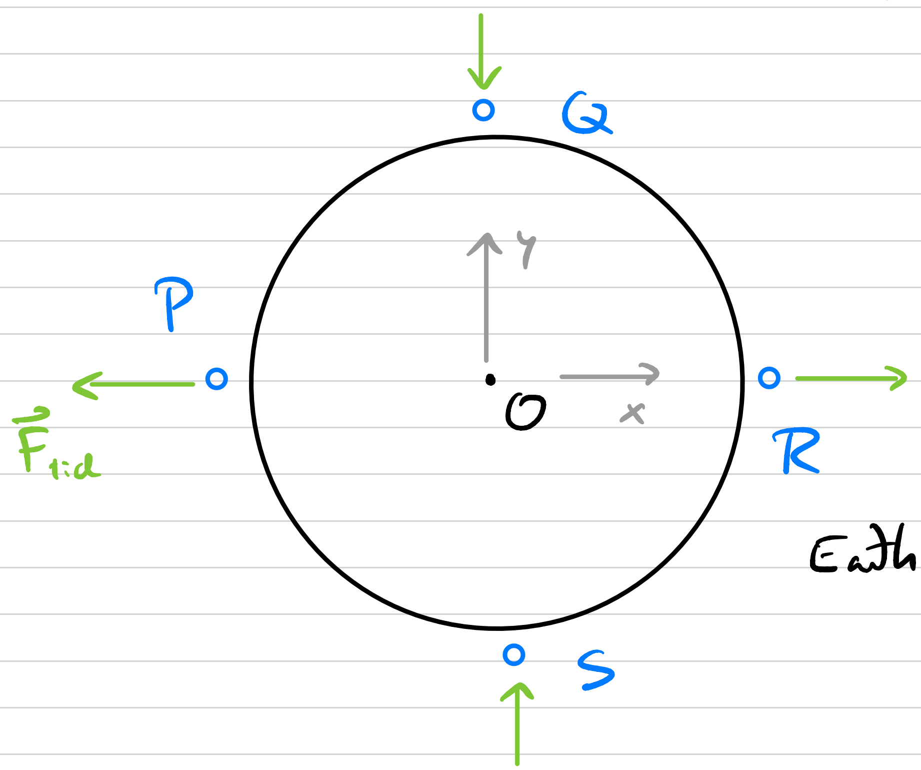
$$\therefore \quad \vec{F}_{tid} = -GM_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right)$$

$$\simeq -\frac{GM_m m}{d_0^2}\left(\hat{x} + \alpha\,\hat{y} - \hat{x}\right)$$

$$\simeq -\frac{GM_m m}{d_0^2}\,\alpha\,\hat{y} \equiv -F_{tid}\,\hat{y}$$

$\boxed{S}$ Similar to Q, $\vec{F}_{tid} \simeq +\vec{F}_{tid}\,\hat{y}$.

So, for the oceans, we get a bulging effect



How do we find the magnitude of the tides, i.e., the height difference between high and low tides.

$\Rightarrow$ look at equipotential surface.

Consider drop of water in ocean. Drop is in equilibrium, in Earth's reference frame, under the influence of three forces

- Earth's gravity $m\vec{g}$
- Tidal force
- Pressure force (Buoyancy)

Since fluid is static, $\vec{F}_p$ is normal to surface of water (Archimedes' principle)

So, NII on water drop $\vec{F}_p + m\vec{g} + \vec{F}_{tid} = \vec{0}$

Since $\hat{n} \cdot \vec{F}_p = F_p$

normal vector to surface

$\Rightarrow m\vec{g} + \vec{F}_{tid}$ is normal to surface as well.

Both $m\vec{g}$ & $\vec{F}_{tid}$ are conservative

$$\Rightarrow \quad m\vec{g} = -\vec{\nabla} U_{eg} \;, \quad \vec{F}_{tid} = -\vec{\nabla} U_{tid}$$

potential from Earth's gravity

potential from tides

To get potential energy, consider

$$\vec{F} = -GM_m m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_o}{d_o^2} \right)$$

But, $\vec{d} = \vec{d}_o + \vec{r}$

Again, $r/d_o \ll 1$ if $r \sim R_e$

$$\Rightarrow \frac{\hat{d}}{d^2} = \frac{\vec{d}}{d^3} = \frac{\vec{d}_o + \vec{r}}{d_o^3 \left(1 + \left(\frac{r}{d_o}\right)^2 + 2\frac{\vec{r}\cdot\vec{d}_o}{d_o^2} \right)^{3/2}} \simeq \frac{\vec{d}_o + \vec{r}}{d_o^3 \left(1 + 3\frac{\vec{d}_o\cdot\vec{r}}{d_o^2} \right)}$$

$$\simeq \frac{\vec{d}_o}{d_o^3} + \left[\frac{\vec{r}}{d_o^3} - \frac{3\vec{d}_o}{d_o^3} \frac{\vec{d}_o\cdot\vec{r}}{d_o^2} \right]$$

Therefore the tidal force is

$$\vec{F}_{tid} \simeq -\frac{GM_m m}{d_o^2}\left[\frac{\vec{r}}{d_o} - \frac{3\hat{d}_o(\hat{d}_o\cdot\vec{r})}{d_o}\right] \quad (*)$$

$$= -\frac{GM_m m}{d_o^2}\underbrace{\left[\hat{r} - 3\hat{d}_o(\hat{d}_o\cdot\hat{r})\right]}_{\text{angular form factor}}\frac{|\vec{r}|}{d_o}$$

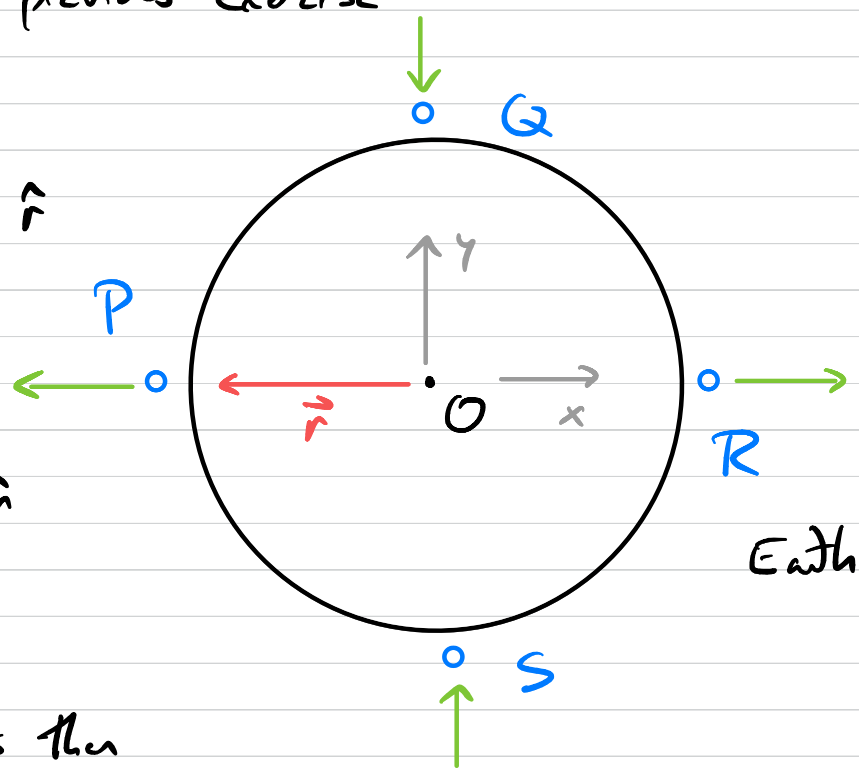
Notice, we recover our previous exercise

e.g., if $\hat{r} \parallel \hat{d}_o$

$$\Rightarrow \vec{F}_{tid} = \frac{GM_m m}{d_o^2}\cdot 2\frac{|\vec{r}|}{d_o}\hat{r}$$

or, if $\hat{r}\cdot\hat{d}_o = 0$

$$\Rightarrow \vec{F}_{tid} \simeq -\frac{GM_m m}{d_o^2}\frac{|\vec{r}|}{d_o}\hat{r}$$



So, the potential energy is then

$$\vec{F}_{tid} = -\vec{\nabla}_{\vec{r}}\, U_{tid}$$

Notice in (*), $\vec{r} = \frac{1}{2}\vec{\nabla}_{\vec{r}}\, \vec{r}^{\,2}$

$$\hat{d}_o(\hat{d}_o\cdot\vec{r}) = \frac{1}{2}\vec{\nabla}_{\vec{r}}(\hat{d}_o\cdot\vec{r})^2$$

$$\Rightarrow \vec{F}_{tid} \simeq -\frac{GM_m m}{d_0}\frac{1}{d_0^2}\left[\frac{1}{2}\vec{\nabla}_{\vec{r}}\,\vec{r}^{\,2} - \frac{3}{2}\vec{\nabla}_{\vec{r}}\left(\hat{d}_0\cdot\vec{r}\right)^2\right]$$

$$= -\vec{\nabla}_{\vec{r}}\left[\text{const.} - \frac{GM_m m}{d_0}\left(\frac{r}{d_0}\right)^2\left[\frac{3(\hat{d}_0\cdot\hat{r})^2 - 1}{2}\right]\right]$$

$$\equiv -\vec{\nabla}_{\vec{r}}\, U_{tid}$$

where

$$\boxed{U_{tid} = \text{const} - \frac{GM_m m}{d_0}\left(\frac{r}{d_0}\right)^2\left[\frac{3(\hat{d}_0\cdot\hat{r})^2 - 1}{2}\right]}$$

* Note $P_\ell(z)$ = Legendre polynomial

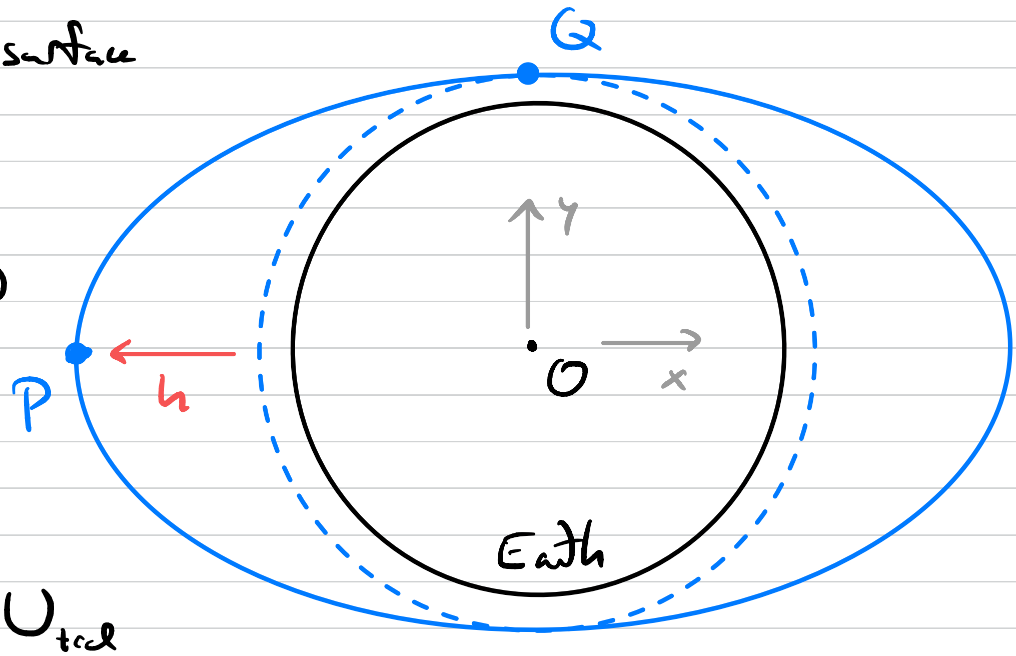
$$P_2(\hat{d}_0\cdot\hat{r}) = \frac{3}{2}(\hat{d}_0\cdot\hat{r})^2 - \frac{1}{2}$$

$$\Rightarrow U_{tid} = \text{const} - \frac{GM_m m}{d_0^2}\left(\frac{r}{d_0}\right)^2 P_2(\hat{d}_0\cdot\hat{r})$$

This is the additional PE term to Earth's gravity.

Let's look at two specific points, P & Q

Since the ocean surface is equipotential



$\Rightarrow U(P) = U(Q)$

Since $U = U_{eg} + U_{tid}$

$$\Rightarrow \quad U_{eg}(P) - U_{eg}(Q) = U_{tid}(Q) - U_{tid}(P)$$

Now, $U_{eg}(P) - U_{eg}(Q) = mgh$

and $U_{tid}(Q) - U_{tid}(P)$

$$= -\frac{GM_m m}{d_0}\left(\frac{R_e}{d_0}\right)^2 \frac{3}{2}\left[(\hat{d}_0 \cdot \hat{r}_Q)^2 - (\hat{d}_0 \cdot \hat{r}_P)^2\right]$$

So, $\hat{r}_Q = \hat{y} \Rightarrow \hat{d}_0 \cdot \hat{r}_Q = 0$ since $\hat{d}_0 = \hat{x}$

$\hat{r}_P = -\hat{x} \Rightarrow \hat{d}_0 \cdot \hat{r}_P = -1$

$$\Rightarrow U_{tid}(Q) - U_{tid}(P) = \frac{3}{2}\frac{GM_m m}{d_0}\left(\frac{R_e}{d_0}\right)^2$$

Equating, $mgh = \frac{3}{2}\frac{GM_m m}{d_o}\left(\frac{R_e}{d_o}\right)^2$

Recall that $g = \frac{GM_e}{R_e^2}$

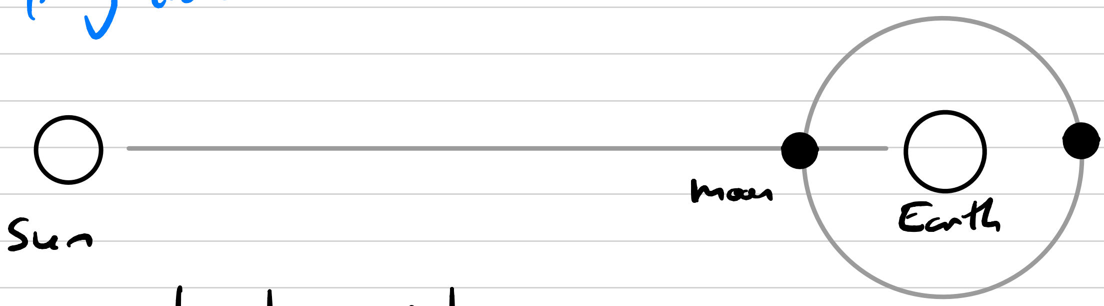
$$\Rightarrow \boxed{h = \frac{3}{2}\frac{M_m R_e^4}{M_e d_o^3}} \Rightarrow h \simeq 54 \text{ cm}$$

Similarly, the sun impacts the tides as $h = 25\text{cm}$

The combined effect is complicated, but two special cases

Spring tide

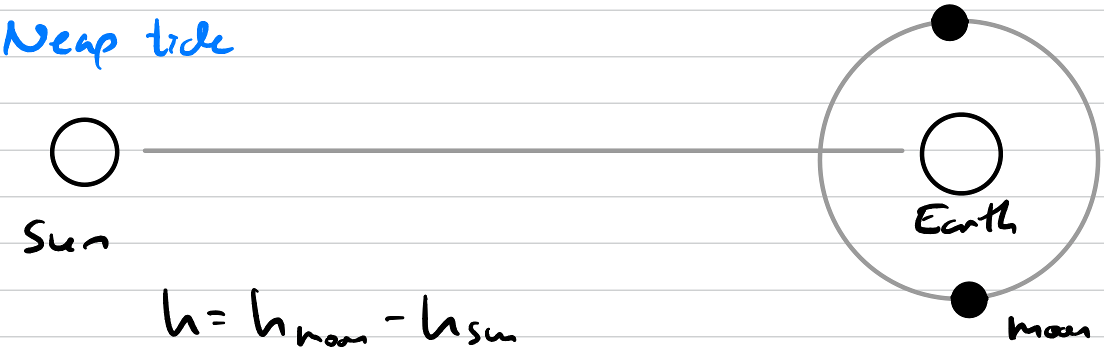
full moon



$$h = h_{moon} + h_{sun}$$
$$\simeq 79 \text{ cm}$$

Neap tide

half moon



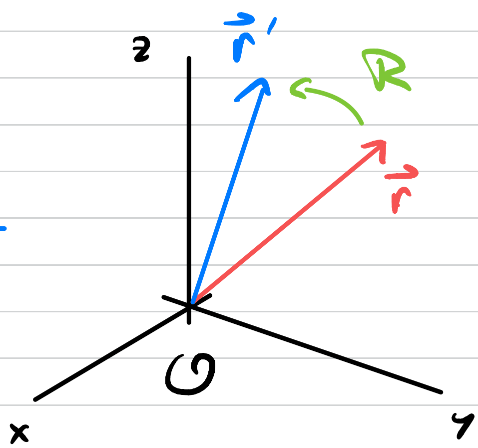
$$h = h_{moon} - h_{sun}$$
$$\simeq 29 \text{ cm}$$

Rotation Matrices

As we move toward rotating reference frames, we will use many concepts of linear algebra. It is therefore useful to review some essential tools of linear algebra, which is the mathematical language of rotational transformations.

Rotation Matrices

Let $\vec{r}$ be a vector with components $\vec{r} = (x, y, z)$ wrt some coordinate system $\mathcal{O}$.



A rotation is a linear transformation such that

$$\vec{r}\,' = \mathcal{R}\,\vec{r}$$

where $\vec{r}\,' = (x', y', z')$ in $\mathcal{O}$ & $|\vec{r}\,'| = |\vec{r}|$

Component-wise, $\boxed{r_i' = \sum_j \mathcal{R}_{ij}\, r_j}$ $\quad i,j = 1,2,3$ or x,y,z

Both $\vec{r}\,', \vec{r}$ are vectors in $\mathbb{R}^3$

$$\Rightarrow \boxed{\mathcal{R} \text{ is a } 3\times 3 \text{ matrix}}$$

9 components

or,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For a physical rotation, we require that the length of $\vec{r}$ is unchanged

$$\Rightarrow \quad \vec{r} \cdot \vec{r} = r^2 = r'^2 = \vec{r}' \cdot \vec{r}'$$

What does this impose on R?

careful w/ repeated indices

$$r'^2 = \sum_i r_i' r_i' = \sum_i \left(\sum_j R_{ij} r_j \right) \left(\sum_k R_{ik} r_k \right)$$

$$= \sum_{j,k} \left(\sum_i R_{ij} R_{ik} \right) r_j r_k$$

$$= \sum_j r_j r_j = r^2$$

require

$$\Rightarrow \sum_i R_{ij} R_{ik} = \delta_{jk}$$

Kronecker δ

$$\Rightarrow \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Recall matrix multiplication & transposition

- if A is $N \times N'$ matrix & B is $N' \times N''$ matrix then $C = A \cdot B$ is $N \times N''$ matrix with elements

$$C_{ij} = (AB)_{ij} = \sum_{k=1}^{N'} A_{ik} B_{kj}$$

transpose

- If A is $N \times N'$ matrix, then A^T is $N' \times N$ matrix with elements

$$(A^T)_{ij} = A_{ji}$$

So, for rotation matrices,

$$\begin{aligned}\delta_{jk} &= \sum_i R_{ij} R_{ik} \\ &= \sum_i (R^T)_{ji} R_{ik} \\ &= (R^T R)_{jk}\end{aligned}$$

$$\Rightarrow \boxed{R^T R = I}$$

identity matrix

Definition: An $N \times N$ square matrix M
which respects $M^T M = I$
is called an orthogonal matrix

Orthogonal matrices preserve vector norms.

$\Rightarrow$ Rotations in $\mathbb{R}^3$ are described by a 3×3 orthogonal matrix R.

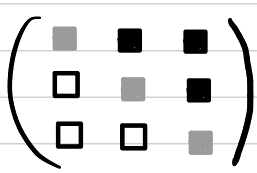
Note that inverse rotations are also rotations

$$R^{-1} R = I \quad \Rightarrow \quad \boxed{R^{-1} = R^T}$$

A 3×3 orthogonal matrix has only 3 real degrees of freedom

Why?

- R has 9 elements, all real
- $R^T R = I$ is symmetric 3×3 matrix

 $\Rightarrow$ 6 independent constraints

$\Rightarrow$ $9 - 6 = 3$ degrees of freedom

symmetric matrix
$M^T = M$

Various parametrizations exists

- axis angle, $R(\hat{n}, \theta)$

$\uparrow$ rotate an angle θ about $\hat{n}$ axis

Basis examples

$$R(\hat{x},\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}$$

$$R(\hat{y},\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

$$R(\hat{z},\theta_z) = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Infinitesimal Transformations

Consider now an infinitesimal rotation,

$$r_i' = \sum_j R_{ij} r_j$$

small

$$\simeq \sum_j \left(\delta_{ij} + M_{ij} + \cdots \right) r_j$$

$$= r_i + \underbrace{\sum_j M_{ij} r_j}_{\text{small correction}} + \cdots$$

Since $R^T R = I$

$$\Rightarrow (I + M + \cdots)^T (I + M + \cdots) = I$$

$$\Rightarrow I + M^T + M + \cdots = I \Rightarrow M^T = -M$$

So, the infinitesimal correction M_{ij} is antisymmetric

$$M^T = -M \Rightarrow M_{ji} = -M_{ij}$$

Consider $\mathcal{R}(\hat{x}, \theta_x)$ for small θ_x, $\cos\theta_x \simeq 1$, $\sin\theta_x \simeq \theta_x$

$$\mathcal{R}(\hat{x}, \theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}$$

$$\simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_x \\ 0 & \theta_x & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{identity}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta_x \\ 0 & \theta_x & 0 \end{pmatrix}}_{\text{antisymmetric}} \equiv I + M_x$$

A generic rotation can be parametrized

$$\mathcal{R}(\theta_i) \simeq I + M(\theta_i)$$

$$\hookrightarrow M = \begin{pmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{pmatrix}$$

$\theta_i = \{\theta_x, \theta_y, \theta_z\}$

Can define vector $\vec{\theta} = (\theta_x, \theta_y, \theta_z)$

$= \varphi\,\hat{n}$

Useful to separate $M(\theta)$ as

$$M(\theta) = \sum_i \theta_i J_i$$

↳ called generators of rotation

Where $(J_i)_{jk} = -\epsilon_{ijk}$

↳ Levi-Civita symbol

fully antisymmetric

vector of matrices

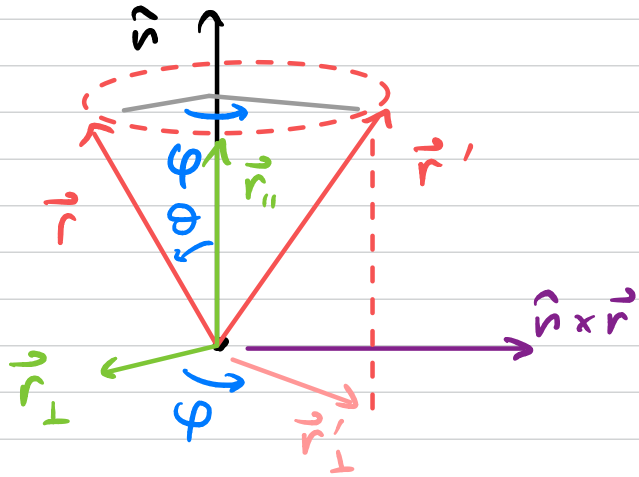
$$\epsilon_{ijk} = \begin{cases} +1 & \text{for } ijk = 123, 231, 312 \quad (\text{even}) \\ -1 & \text{for } ijk = 132, 213, 321 \quad (\text{odd}) \\ 0 & \text{otherwise} \end{cases}$$

So, for small rotations,

$$\boxed{\begin{aligned} R_{jk} &= \delta_{jk} + \sum_i \theta_i (J_i)_{jk} + \mathcal{O}(\theta^2) \\ &= \delta_{jk} - \sum_i \theta_i \epsilon_{ijk} + \mathcal{O}(\theta^2) \end{aligned}}$$

Axis-Angle Representation

Consider a rotation of $\vec{r}$ to $\vec{r}\,'$, about some axis $\hat{n}$ an angle φ such that $|\vec{r}| = |\vec{r}\,'| = r$



Define parallel & perpendicular components of $\vec{r}$ wrt $\hat{n}$

$$\vec{r}_{\parallel} = (\vec{r}\cdot\hat{n})\,\hat{n} \quad \& \quad \vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel}$$

Can use triple cross here

$$= r\cos\theta\,\hat{n} \qquad\qquad = \vec{r} - (\vec{r}\cdot\hat{n})\,\hat{n} = |\vec{r}_{\perp}|\,\hat{r}_{\perp}$$

Now $|\vec{r}_{\perp}| = r\sin\theta = |\hat{n}\times\vec{r}| \Rightarrow \vec{r}_{\perp} = \underline{r\sin\theta}\,\hat{r}_{\perp}$

Since we rotate about $\hat{n}$, $\vec{r}\,'_{\parallel} = (\vec{r}\,'\cdot\hat{n})\,\hat{n}$

$$= r\cos\theta\,\hat{n}$$

fixed

and $|\vec{r}\,'_{\perp}| = r\sin\theta$

So, can decompose $\vec{r}\,'$ along $\hat{n}$, $\hat{r}_{\perp}$, & $\hat{n}\times\vec{r}$ axis

$$\vec{r}\,' = r\cos\theta\,\hat{n} + \underline{r\sin\theta}\cos\varphi\,\underline{\hat{r}_{\perp}} + r\sin\theta\sin\varphi\,\frac{\hat{n}\times\vec{r}}{|\hat{n}\times\vec{r}|}$$

$$= (\vec{r}\cdot\hat{n})\hat{n} + \cos\varphi\left[\vec{r} - (\vec{r}\cdot\hat{n})\,\hat{n}\right] + \sin\varphi\,(\hat{n}\times\vec{r})$$

So, under a rotation $\vec{r} \to \vec{r}\,' = R \cdot \vec{r}$

In the axis-angle representation, the components of R can be read off

$$\vec{r}\,' = \cos\varphi\, \vec{r} + [1-\cos\varphi](\vec{r}\cdot\hat{n})\,\hat{n} + \sin\varphi\,(\hat{n}\times\vec{r})$$

In cartesian components, $\vec{r} = (r_1, r_2, r_3)$, $\hat{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$

and

$$\vec{r}\cdot\hat{n} = \sum_j r_j \hat{n}_j = \sum_{j,k} r_j \hat{n}_k\, \delta_{jk}$$

↳ Kronecker

$$(\hat{n}\times\vec{r})_i = \sum_{j,k} \epsilon_{ijk}\, \hat{n}_j\, \vec{r}_k$$

↳ Levi-Civita

$$\Rightarrow r'_i = \cos\varphi\, r_i + [1-\cos\varphi]\left(\sum_j \hat{n}_j r_j\right)\hat{n}_i$$

$$- \sin\varphi \sum_{j,k} \epsilon_{ijk}\, r_j\, \hat{n}_k \qquad \epsilon_{ijk} = -\epsilon_{ikj}$$

$$= \sum_j \left[\cos\varphi\, \delta_{ij} + (1-\cos\varphi)\hat{n}_i\hat{n}_j - \sum_k \epsilon_{ijk}\,\hat{n}_k \sin\varphi\right] r_j$$

$$\equiv \sum_j R_{ij}\, r_j$$

$$\Rightarrow \boxed{R_{ij}(\hat{n},\varphi) = \cos\varphi\, \delta_{ij} + (1-\cos\varphi)\,\hat{n}_i\hat{n}_j - \sum_k \epsilon_{ijk}\,\hat{n}_k \sin\varphi}$$

Rodriguez formula

Notice that for small angles, $\varphi \ll 1$

$$R_{ij}(\hat{n},\varphi) \simeq \delta_{ij} - \sum_k \epsilon_{ijk}\hat{n}_k \varphi + \mathcal{O}(\varphi^2)$$

$$\Rightarrow \vec{r}\,' \simeq \vec{r} + \varphi(\hat{n}\times\vec{r}) + \mathcal{O}(\varphi^2)$$

So, if rotating a vector by $\Delta\varphi$ in Δt

$$\vec{r}\,' \simeq \vec{r} + \Delta\varphi(\hat{n}\times\vec{r})$$

$$\Rightarrow \vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}\,'}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\varphi}{\Delta t}(\hat{n}\times\vec{r}) \equiv \omega(\hat{n}\times\vec{r})$$

↳ angular speed

Define angular velocity $\boxed{\vec{\omega} \equiv \omega\hat{n}}$

Notice that $R\cdot\hat{n} = \hat{n}$

$$\hat{n}_i' = \cos\varphi\,\hat{n}_i + [1-\cos\varphi]\left(\cancelto{1}{\sum_j \hat{n}_j\hat{n}_j}\right)\hat{n}_i - \sin\varphi\,\cancelto{0}{\sum_{j,k}\epsilon_{ijk}\hat{n}_j\hat{n}_k}$$

$$\Rightarrow \hat{n}_i' = \cos\varphi\,\hat{n}_i + [1-\cos\varphi]\,\hat{n}_i = \hat{n}_i$$

This is in fact a statement of Euler's theorem

Euler's Theorem

Euler's theorem states that, in 3D space, any motion of a rigid body relative to a fixed point O, such that a point of the rigid body is fixed at O, is equivalent to a rotation about some axis through O.

$\Rightarrow$ Any composition of two rotations is also a rotation.

A modern version of the theorem is

Theorem: If R is a proper 3×3 rotation matrix ($R^T R = R R^T = I$ and $\det R = +1$), then $\exists$ a non-zero vector $\vec{n}$ s.t. $R\vec{n} = \vec{n}$

Proof: $R\vec{n} = \vec{n} \Rightarrow (R - I)\vec{n} = 0$, that is $\vec{n}$ is an eigenvector of R with eigenvalue $\lambda = 1$.

$\Rightarrow \det(R - I) = 0$ to be a characteristic solution.

Note for a 3×3 matrix A, $\det(-A) = (-1)^3 \det A = -\det A$

Also, if $\det R = 1$, then $\det(R^{-1}) = \dfrac{1}{\det R} = 1$

Therefore, compute

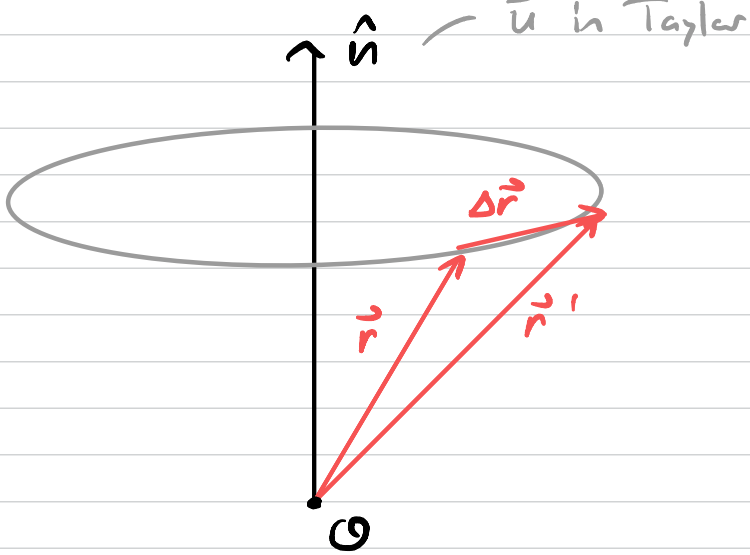
$$\det(R-I) = \det((R-I)^T)$$
$$= \det(R^T - I) \qquad R^T = R^{-1},\ I = R^{-1}R$$
$$= \det(R^{-1} - R^{-1}R)$$
$$= \det(R^{-1})\det(I-R)$$
$$= \det(-(R-I))$$
$$= -\det(R-I)$$

$$\Rightarrow \det(R-I) = -\det(R-I) \Rightarrow \det(R-I) = 0$$

$$\Rightarrow R\vec{n} = \vec{n} \qquad \blacksquare$$

Angular Velocity

Consider the rotation of a vector $\vec{r}$ to $\vec{r}\,'$ in a time Δt about some axis $\vec{n}$



for small rotations,

$$r_j' = \sum_k R_{jk} r_k$$

$$\simeq \sum_k \left(\delta_{jk} - \sum_i \Delta\theta_i \, \epsilon_{ijk} \right) r_k$$

$$= r_j - \sum_i \sum_k \Delta\theta_i \, \epsilon_{ijk} r_k$$

So,

$$\Delta r_j = r_j' - r_j = -\sum_{i,k} \epsilon_{ijk} \Delta\theta_i r_k$$

$$= \sum_{i,k} \epsilon_{jik} \Delta\theta_i r_k$$

$\epsilon_{ijk} = -\epsilon_{jik}$

In a time Δt,

$$\frac{\Delta r_j}{\Delta t} = \sum_{i,k} \epsilon_{jik} \frac{\Delta\theta_i}{\Delta t} r_k$$

Define the angular velocity vector as

$$\boxed{\omega_i = \lim_{\Delta t \to 0} \frac{\Delta\theta_i}{\Delta t} = \frac{d\theta_i}{dt}}$$

So,

$$v_j = \sum_{i,k} \epsilon_{jik}\, \omega_i\, r_k$$

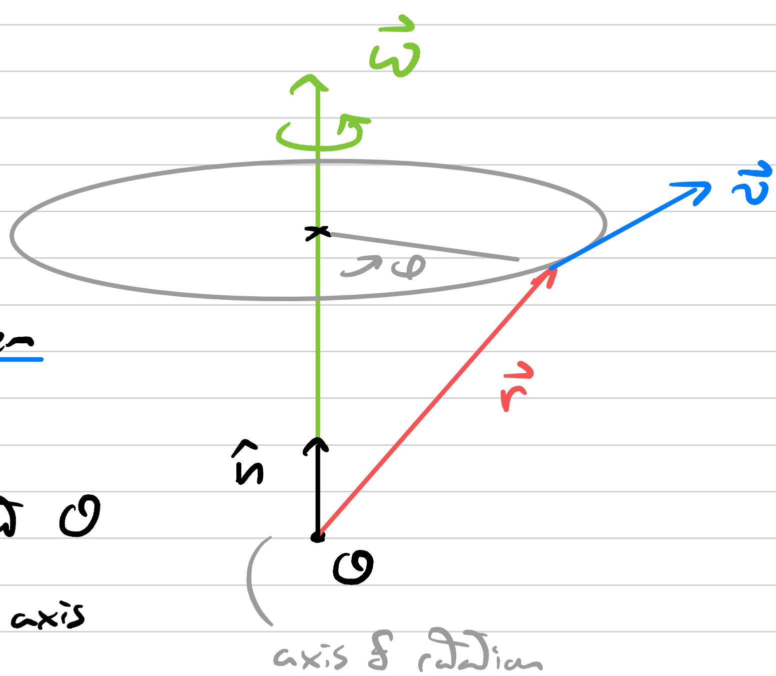
Recall the cross-product – $(\vec{c})_j = (\vec{a} \times \vec{b})_j$

$$= \sum_{i,k} \epsilon_{jik}\, a_i b_k$$

$$\Rightarrow \boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

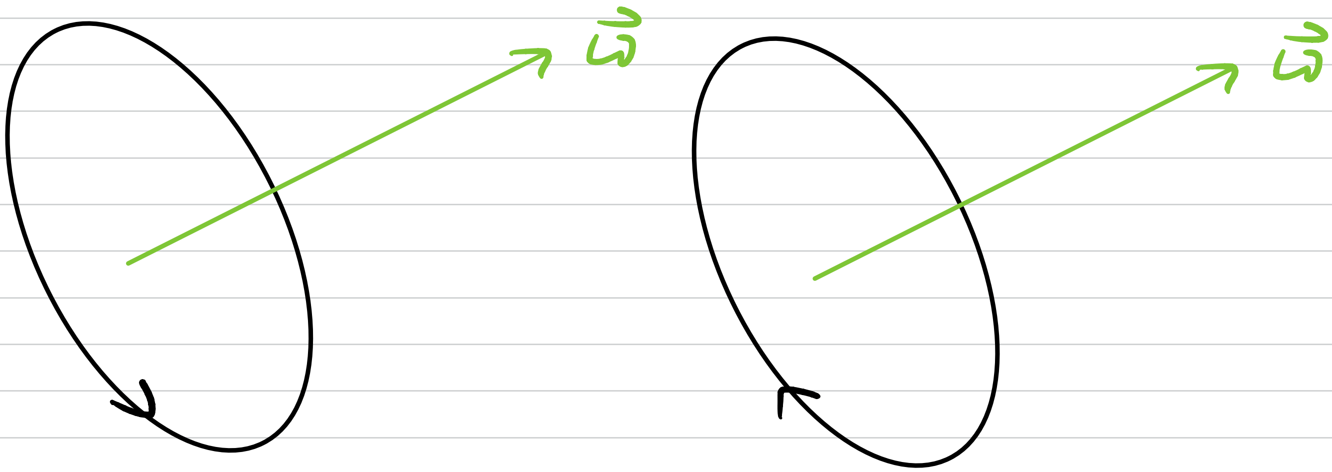
Direction of $\vec{\omega}$ is $\hat{n}$, can write $\vec{\omega} = \omega \hat{n}$

with $\omega = \dot{\varphi}$



This description of $\vec{\omega}$ is a result of Euler's theorem

– Most general motion of any body relative to fixed point O is a rotation about some axis through O.



positive rotation   negative rotation

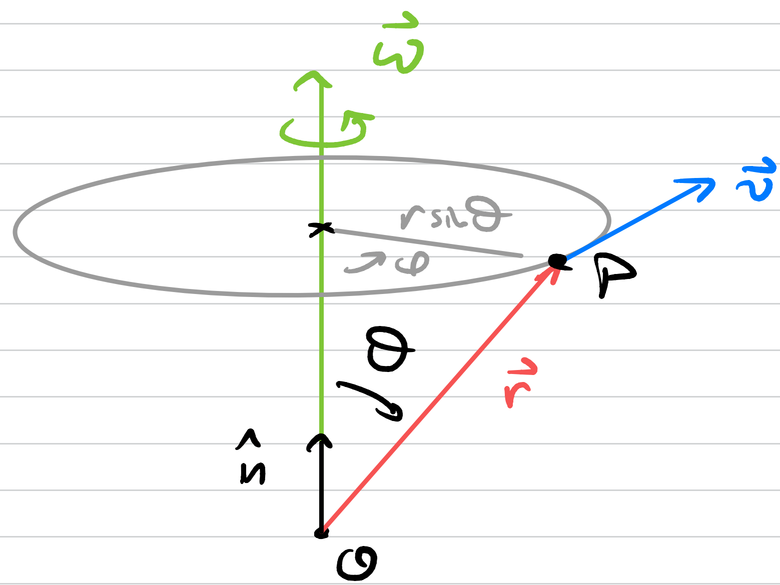
"right-hand rule"

We can also see $\vec{v} = \vec{\omega} \times \vec{r}$ geometrically

- position of P wrt O

  (r, θ, φ)

  where θ = colatitude



- P moves with speed

  $$v = r \sin\theta \, \dot{\varphi}$$
  $$= r \omega \sin\theta$$

- Geometrically, $\vec{C} = \vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$

  ↳ perpendicular to plane spanned by $\vec{A}$ & $\vec{B}$

  $\Rightarrow \vec{v} = \vec{\omega} \times \vec{r}$

  since $\vec{v} \parallel \vec{\omega} \times \vec{r}$

Note for any vector $\vec{e}$ fixed on the rotating body,

$$\boxed{\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}}$$

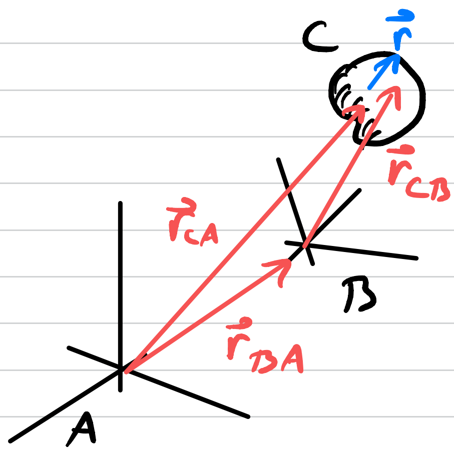
Angular velocities add linearly

Suppose frame B rotating with $\vec{\omega}_{BA}$ wrt. frame A

Body C is rotating with $\vec{\omega}_{CB}$ wrt frame B

$$\vec{r}_{CA} = \vec{r}_{CB} + \vec{r}_{BA}$$

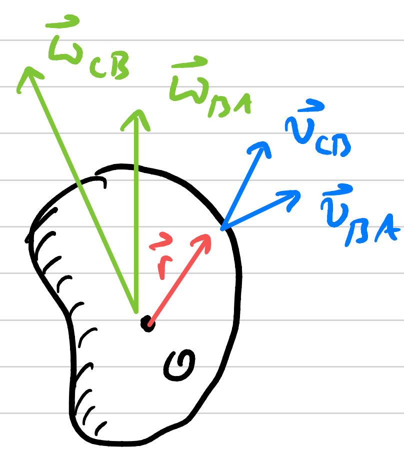
$$\Rightarrow \quad \vec{v}_{CA} = \vec{v}_{CB} + \vec{v}_{BA}$$



Let $\vec{r}$ be vector fixed in C

$$\begin{aligned}
\vec{\omega}_{CA} \times \vec{r} &= \vec{v}_{CA} \\
&= \vec{v}_{CB} + \vec{v}_{BA} \\
&= \vec{\omega}_{CB} \times \vec{r} + \vec{\omega}_{BA} \times \vec{r} \\
&= (\vec{\omega}_{CB} + \vec{\omega}_{BA}) \times \vec{r}
\end{aligned}$$

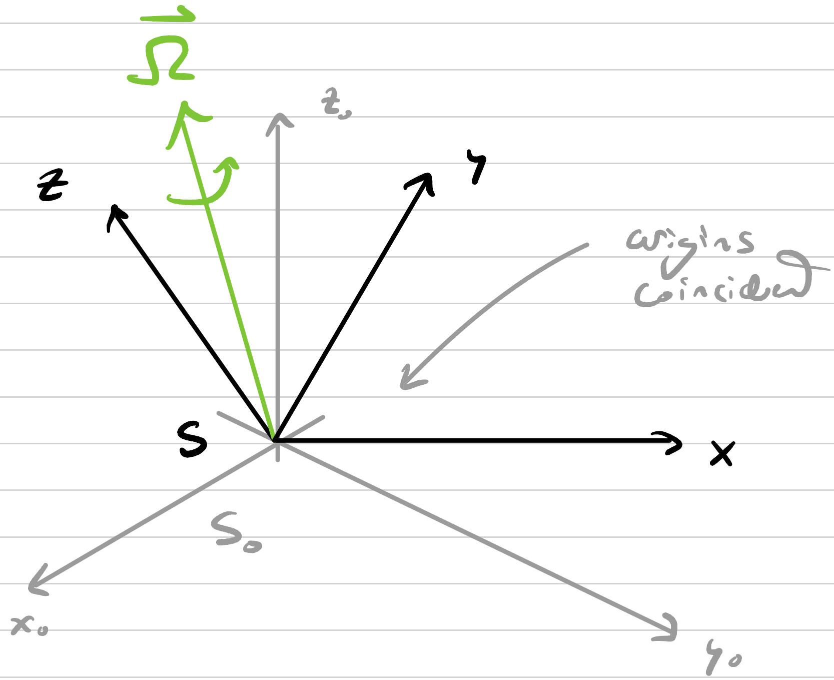
$$\Rightarrow \quad \boxed{\vec{\omega}_{CA} = \vec{\omega}_{CB} + \vec{\omega}_{BA}}$$



Time Derivatives in a Rotating Frame

Having discussed the mathematical description of rotations, we are in position to describe motion in rotating frames.

Frame S rotating w/ ang. velocity $\vec{\Omega}$ wrt S_0.

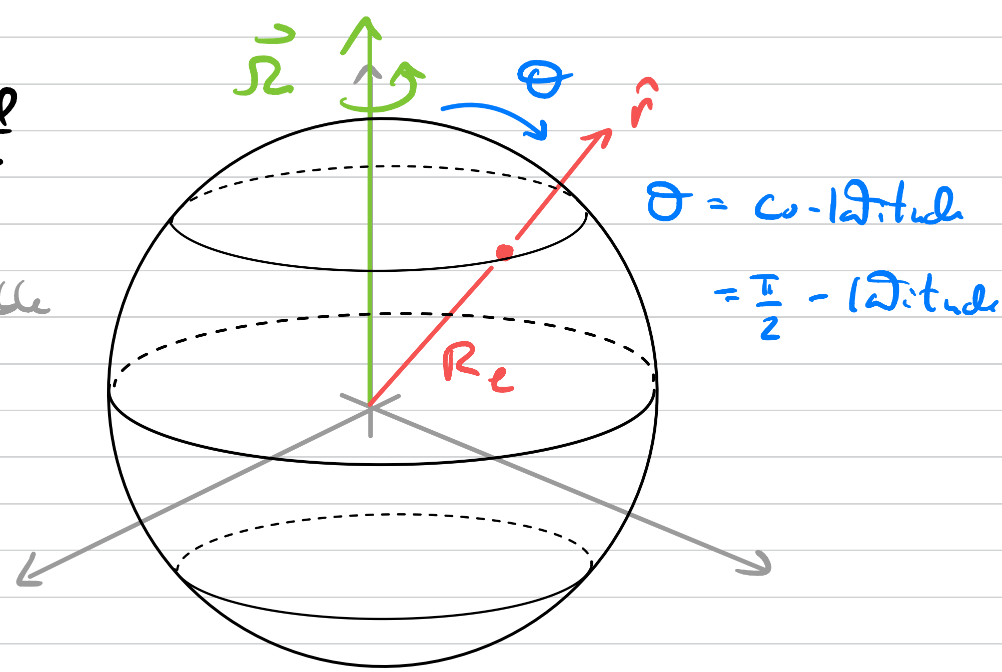


An example we will revisit often is the motion on Earth, which is rotating at a rate

$$\Omega = \frac{2\pi \text{ rad}}{24 \times 3600 \text{ s}}$$

$$\simeq 7.3 \times 10^{-5} \frac{\text{rad}}{\text{s}}$$

small, but not negligible



In the case of the Earth, S_0 is some inertial frame w/ axes fixed relative to distant stars. This frame, while inertial, is arbitrary and relatively inconvenient compared w/ Earth frame S.

$\Rightarrow$ Useful to analyze physics in non-inertial frame — closer connection to observable physics.

Consider a vector $\vec{Q}$ (e.g., velocity, position, ...)

We want to relate the rate of change between S_0 & S

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} \quad \text{vs.} \quad \left(\frac{d\vec{Q}}{dt}\right)_S$$

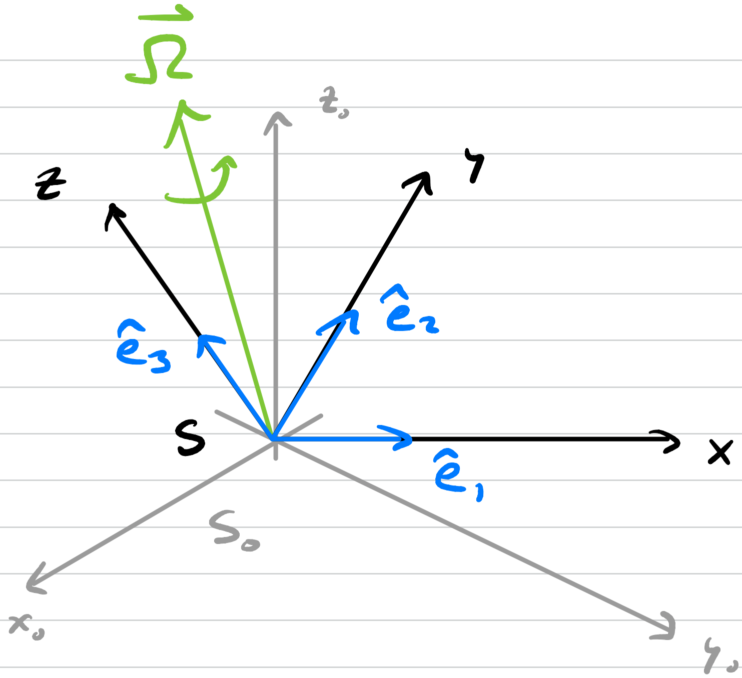
relative to inertial frame S_0 relative to rotating frame S

Let $\vec{Q} = \sum_{i=1}^{3} Q_i \hat{e}_i$

axes fixed to rotating frame

For example,

$\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$, $\hat{e}_3 = \hat{z}$



Since $\hat{e}_j$ fixed in S,

$$\left(\frac{d\vec{Q}}{dt}\right)_S = \sum_{j=1}^{3} \frac{dQ_j}{dt}\,\hat{e}_j$$

But, in frame S_0, $\hat{e}_j$ are changing with time

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_{j=1}^{3} \frac{dQ_j}{dt}\,\hat{e}_j + \sum_{j=1}^{3} Q_j \left(\frac{d\hat{e}_j}{dt}\right)_{S_0}$$

Now, $\hat{e}_j$ is fixed in S, rotating w/ $\vec{\Omega}$ relative to S_0

$$\Rightarrow \boxed{\left(\frac{d\hat{e}_j}{dt}\right)_{S_0} = \vec{\Omega} \times \hat{e}_j}$$

So,
$$\sum_j Q_j \left(\frac{d\hat{e}_j}{dt}\right)_{S_0} = \sum_j Q_j \,(\vec{\Omega} \times \hat{e}_j)$$

$$= \vec{\Omega} \times \sum_j Q_j\,\hat{e}_j$$

$$= \vec{\Omega} \times \vec{Q}$$

Thus, we find

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}}$$

With this, we can relate NII in inertial frames to rotating frames.

Notice that if $\vec{Q} = \vec{\Omega}$

$$\left(\frac{d\vec{\Omega}}{dt}\right)_{S_0} = \left(\frac{d\vec{\Omega}}{dt}\right)_S \qquad \text{since } \vec{\Omega} \times \vec{\Omega} = \vec{0}$$

So, rate of change of $\vec{\Omega}$ is frame independent

Newton's Law in Rotating Frame

For simplicity, let's assume that $\vec{\Omega} = \text{const.}$, which from above we see $\vec{\Omega}$ is constant in all reference frames: $(\vec{\Omega})_{S_0} = (\vec{\Omega})_S$.

Consider a particle of mass m & position $\vec{r}$.

NII in S_0 is

$$\boxed{m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}}$$

Forces in IF S_0

To get NII in S, we use the relation

$$\left(\frac{d\vec{r}}{dt}\right)_{s_0} = \left(\frac{d\vec{r}}{dt}\right)_{s} + \vec{\Omega}\times\vec{r}$$

and

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{s_0} = \left(\frac{d}{dt}\right)_{s_0}\left(\frac{d\vec{r}}{dt}\right)_{s_0}$$

$$= \left(\frac{d}{dt}\right)_{s_0}\left[\left(\frac{d\vec{r}}{dt}\right)_{s} + \vec{\Omega}\times\vec{r}\right]$$

$$= \left(\frac{d}{dt}\right)_{s}\left[\left(\frac{d\vec{r}}{dt}\right)_{s} + \vec{\Omega}\times\vec{r}\right]$$

$$+ \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_{s} + \vec{\Omega}\times\vec{r}\right]$$

For notational simplicity, let $\dot{\vec{Q}} = \left(\frac{d\vec{Q}}{dt}\right)_s$

$$\Rightarrow \left(\frac{d^2\vec{r}}{dt^2}\right)_{s_0} = \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \quad , \quad \dot{\vec{\Omega}} = \vec{0}$$

So, NII in S is given by

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{s_0} = \vec{F}$$

$$\Rightarrow m\ddot{\vec{r}} + 2m\vec{\Omega}\times\dot{\vec{r}} + m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \vec{F}$$

Rearranging, & using $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

NII in S

$$\boxed{m\ddot{\vec{r}} = \vec{F} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}}$$

As before, $\vec{F}$ are the usual forces in an inertial frame, & the extra two terms are pseudo forces which are a consequence of the accelerating frame

Coriolis Force

$$\boxed{\vec{F}_{cor} = 2m\dot{\vec{r}} \times \vec{\Omega}}$$

Centrifugal Force

$$\boxed{\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}}$$

$$\boxed{m\ddot{\vec{r}} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf}}$$