Physics 303

Classical Mechanics II

Rigid Body Motion

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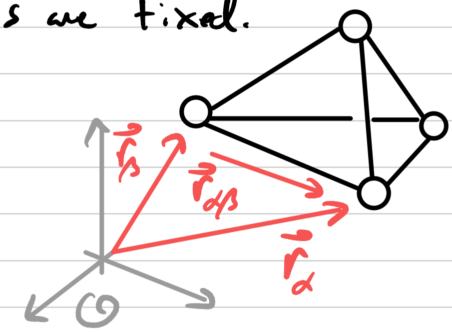
Rigid Bodies

A rigid body is an abstract notion of a collection of particles / objects that move together in such a way to maintain their shape, i.e., their relative positions are fixed.



$$\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$$

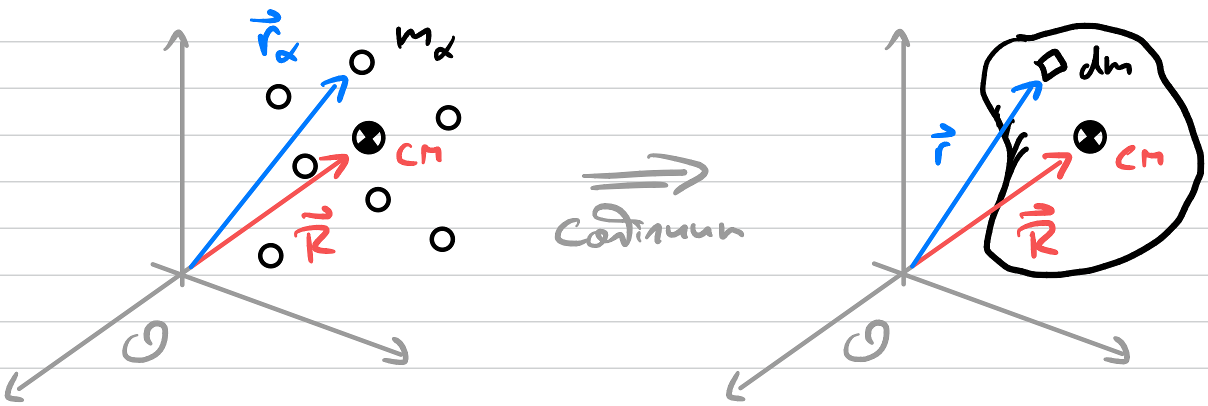
$$\Rightarrow \quad |\vec{r}_{\alpha\beta}| = \text{constant}$$

This is an idealization, as atoms and molecules vibrate meaning no object is completely rigid. However, this is a good starting point to build on.

Since the distances between particles are fixed, the system is highly constrained. For N particles, there are $3N$ coordinates needed. But, since the distances between particles is fixed, the rigid body only needs 6 degrees of freedom

- 3 to specify CM
- 3 to specify orientation

Consider system of N particles $\alpha = 1, \dots, N$ with masses m_α and positions $\vec{r}_\alpha$ measured w.r.t. O



The CM is $\boxed{\vec{R} = \dfrac{1}{M} \displaystyle\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha}$, $\boxed{M = \displaystyle\sum_\alpha m_\alpha}$

If the particles are small and numerous in a small volume, we can define a density $\rho(\vec{r})$ as

$$\Delta m = \rho(\vec{r})\, \Delta x \Delta y \Delta z$$

then we can consider the rigid body of a continuous distribution of mass $\boxed{M = \int dm = \int \rho(\vec{r})\, dV}$

and CM $\boxed{\vec{R} = \dfrac{1}{M} \displaystyle\int \vec{r}\, dm}$

We will switch between a discrete and continuous picture as needed.

Momentum & Angular Momentum

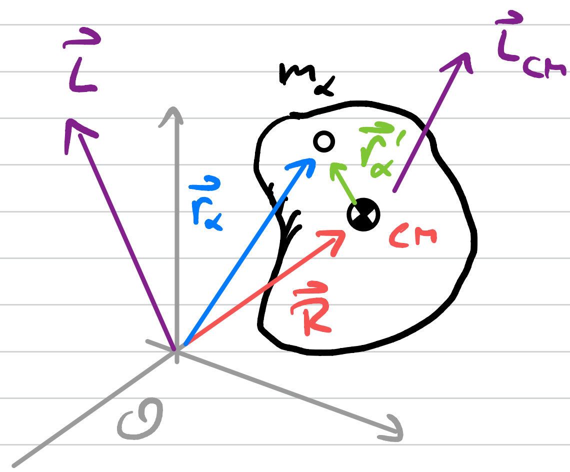
The total momentum of the system is

$$\vec{P} = \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$

If the system is exposed to an external force $\vec{F}^{ext}$, then NII for the CM is

$$\boxed{\dot{\vec{P}} = \vec{F}^{ext} = M\ddot{\vec{R}}}$$

Next we consider angular momentum.
Let $\vec{L}$ be the angular momentum of the system wrt O.
We want to split $\vec{L}$ into an $\vec{L}_{CM}$, the angular momentum of the body about the CM, and the $\vec{L}_{orb}$, the angular momentum of the CM.



The angular momentum of α about O is

$$\vec{l}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha = \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$$

So the total angular momentum is $\vec{L} = \sum_\alpha \vec{l}_\alpha$

So, $$\vec{L} = \sum_\alpha \vec{l}_\alpha = \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$$

Now, let $\vec{r}_\alpha'$ be location of α w.r.t. CM

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha'$$

So, find

$$\vec{L} = \sum_\alpha (\vec{R} + \vec{r}_\alpha') \times m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}_\alpha')$$
$$= \sum_\alpha \vec{R} \times m_\alpha \dot{\vec{R}} + \sum_\alpha \vec{R} \times m_\alpha \dot{\vec{r}}_\alpha'$$
$$+ \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{R}} + \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'$$

Recall $M = \sum_\alpha m_\alpha$

$$\Rightarrow \vec{L} = \vec{R} \times M\dot{\vec{R}} + \vec{R} \times \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'$$
$$+ \left(\sum_\alpha m_\alpha \vec{r}_\alpha'\right) \times \dot{\vec{R}} + \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'$$

Now, $\sum_\alpha m_\alpha \vec{r}_\alpha' = \vec{0}$ since this is location of CM relative to CM (of course)

likewise, $$\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha' = \vec{0}$$

So, $$\vec{L} = \vec{R} \times \vec{P} + \sum_\alpha \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'$$

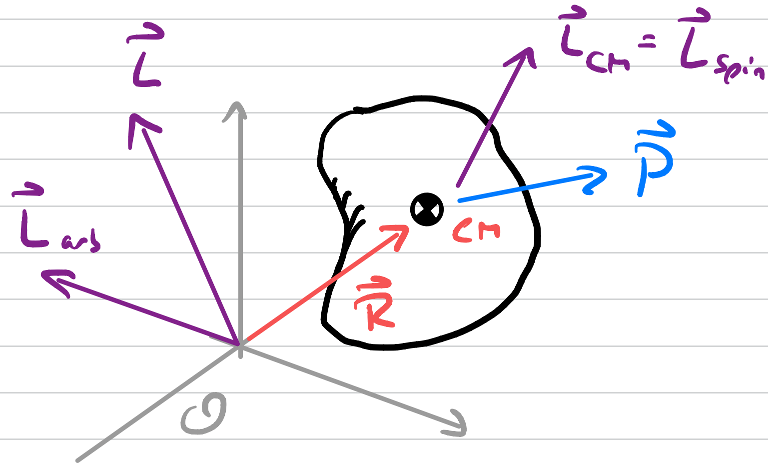
angular momentum of CM relative to O

angular momentum relative to the CM

Define $$\boxed{\vec{L}_{CM} = \vec{L}_{spin} = \sum_\alpha \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'}$$

$$\boxed{\vec{L}_{orb} = \vec{R} \times \vec{P}}$$

$$\Rightarrow \boxed{\vec{L} = \vec{L}_{orb} + \vec{L}_{CM}}$$



This separation is often useful as both are approximately conserved

$$\dot{\vec{L}}_{orb} = \cancelto{0}{\dot{\vec{R}} \times \vec{P}} + \vec{R} \times \dot{\vec{P}} = \vec{R} \times \vec{F}^{ext} \; ; \; \vec{F}^{ext} = \sum_\alpha \vec{F}_\alpha^{ext}$$

We know $\dot{\vec{L}} = \vec{\Gamma}^{ext}$, the external torque relative to O

So, $$\dot{\vec{L}}_{CM} = \dot{\vec{L}} - \dot{\vec{L}}_{orb} = \vec{\Gamma}^{ext} - \vec{R} \times \vec{F}^{ext}$$
$$= \sum_\alpha (\vec{r}_\alpha - \vec{R}) \times \vec{F}_\alpha^{ext} = \vec{\Gamma}_{CM}^{ext}$$

external torque relative to CM

Kinetic & Potential Energy

The total kinetic energy of N particles is

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$$

As before, write $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}'$, $\vec{r}_\alpha{}'$ position relative to CM

$$\Rightarrow \quad \dot{\vec{r}}_\alpha^{\,2} = \dot{\vec{R}}^2 + \dot{\vec{r}}_\alpha{}'^2 + 2\dot{\vec{R}} \cdot \dot{\vec{r}}_\alpha{}'$$

$$\Rightarrow \quad T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{R}}^2 + \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'^2 + \dot{\vec{R}} \cdot \underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'}_{\substack{=0 \\ \text{as before}}}$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'^2$$

Define KE relative to CM $\boxed{T_{CM} = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'^2}$

So,

$$\boxed{T = \frac{1}{2} M \dot{\vec{R}}^2 + T_{CM}}$$

KE of CM

For conservative forces, can write potential energy and decompose as

$$U = U_{ext} + U_{int}$$

$\uparrow$ external PE $\qquad$ $\uparrow$ internal PE

where $U_{int} = \sum_{\alpha < \beta} U_{\alpha\beta}\left(|\vec{r}_\alpha - \vec{r}_\beta|\right)$

assuming central forces

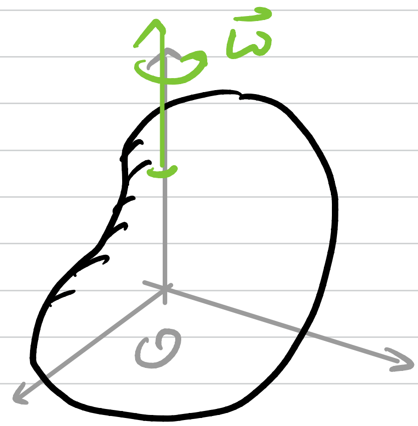
since $|\vec{r}_{\alpha\beta}| = \text{const.}$,

$$\Rightarrow \quad U_{int} = \text{const.}$$

$\therefore$ U_{int} is irrelevant for rigid body dynamics

Rotation about a Fixed Axis

Here we consider the rotation of a rigid body about some fixed axis. Since the axis is fixed, let us define it as the z-axis



$$\Rightarrow \vec{\omega} = (0, 0, \omega)$$

If the body consist of N particles, then

$$\vec{L} = \sum_\alpha \vec{l}_\alpha$$

$$= \sum_\alpha \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

Since the axis of rotation is fixed, $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha$,

So, with $\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$

$$\Rightarrow \vec{v}_\alpha = (-\omega y_\alpha, \omega x_\alpha, 0)$$

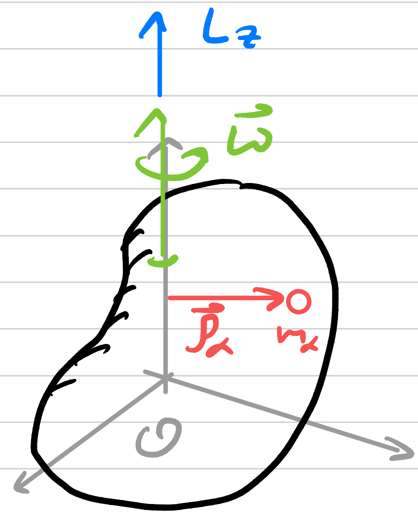
$$\therefore \vec{l}_\alpha = m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

$$= m_\alpha \omega (-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2)$$

Thus, $\vec{L} = \sum_\alpha \vec{l}_\alpha$ is total angular momentum.

Let's examine components of $\vec{L}$.

The z-component is



$$L_z = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega$$

Notice that

$$\rho_\alpha^2 = x_\alpha^2 + y_\alpha^2$$

with ρ_α being the distance to any point from the z-axis.

Thus, $$L_z = \sum_\alpha m_\alpha \rho_\alpha^2 \,\omega \equiv I_z\,\omega$$

where $$\boxed{I_z = \sum_\alpha m_\alpha \rho_\alpha^2}$$

is the moment of inertia about the z-axis.

The kinetic energy is then

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 .$$

Since $v_\alpha = \rho_\alpha\,\omega$ for a rotation about fixed z-axis,

$$\Rightarrow T = \frac{1}{2}\sum_\alpha m_\alpha \rho_\alpha^2\,\omega^2 = \frac{1}{2} I_z\,\omega^2 .$$

These should be familiar results from Phys 101.

Notice though that there is non-zero components for L_x & L_y,

$$L_x = -\sum_\alpha m_\alpha x_\alpha z_\alpha \omega$$

$$L_y = -\sum_\alpha m_\alpha y_\alpha z_\alpha \omega$$

we define the products of inertia about the z-axis as

$$L_x = I_{xz}\omega \quad , \quad L_y = I_{yz}\omega$$

with $$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$$

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

Obviously, $\vec{L}$ is not parallel to $\vec{\omega}$!

$$\vec{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$$

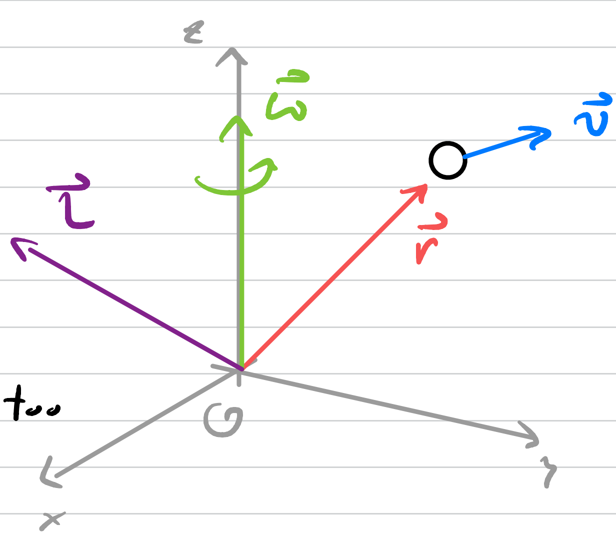
with $I_{zz} \equiv I_z$. Consider a single point particle,

$$\vec{L} = \vec{r} \times m\vec{v}$$

If $\vec{v} \parallel -\hat{x}$

$\vec{r}$ lies in yz plane,

find $\vec{L}$ lies in yz plane too

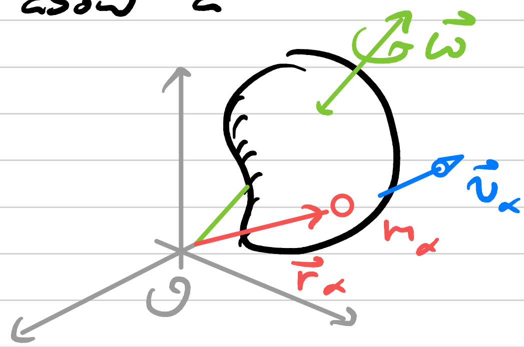


The Inertia Tensor

We saw that $\vec{L} \neq I\vec{\omega}$ with I being a number.

In general, I is a 3x3 symmetric tensor.

Let's see by consider a rotation about a general fixed axis $\vec{\omega}$.



$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

$$= \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

Recall identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$$\Rightarrow \vec{L} = \sum_\alpha m_\alpha \left[\vec{r}_\alpha^{\,2} \vec{\omega} - (\vec{r}_\alpha \cdot \vec{\omega}) \vec{r}_\alpha \right]$$

Let's look at i^{th}-component,

$$L_i = \sum_\alpha m_\alpha \left[\vec{r}_\alpha^{\,2} \omega_i - \left(\sum_j r_{\alpha,j} \omega_j \right) r_{\alpha,i} \right]$$

$$= \sum_j \left[\sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} \delta_{ij} - r_{\alpha,i} r_{\alpha,j} \right) \right] \omega_j$$

$$= \sum_j I_{ij} \omega_j$$

We define the inertia tensor as $\mathbb{I}$ with matrix elements

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} \delta_{ij} - r_{\alpha,i} r_{\alpha,j} \right)}$$

In terms of a continuous distribution,

$$\boxed{I_{ij} = \int dm \left(\vec{r}^{\,2} \delta_{ij} - r_i r_j \right)}$$

By inspection, $\mathbb{I}$ is symmetric, $\mathbb{I}^T = \mathbb{I}$ or $I_{ij} = I_{ji}$.

It characterizes an object's resistance to change in rotational motion

$$\boxed{\vec{L} = \mathbb{I} \cdot \vec{\omega} \quad \text{or} \quad L_i = \sum_j I_{ij} \omega_j}$$

The Cartesian components are

$$\mathbb{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

with $I_{xx} \equiv I_x = \sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} - x_\alpha^2 \right) = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2 \right)$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$$

etc...

Explicitly, $\vec{L} = \mathbb{I}\cdot\vec{\omega}$ is

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The inertia tensor is a 3×3 symmetric matrix which must transform as $\mathbb{I}' = R\,\mathbb{I}\,R^T$

rotation matrix

or $I'_{ij} = \sum_{m,n} R_{im} R_{jn} I_{mn}$

this is what makes $\mathbb{I}$ a tensor

To see this, note that $\vec{L}$ & $\vec{\omega}$ are physical vectors which must transform as $\vec{L}' = R\cdot\vec{L}$, $\vec{\omega}' = R\cdot\vec{\omega}$ under a rotation R.

$$\therefore \quad \vec{L}' = R\cdot\vec{L} = R\cdot\mathbb{I}\cdot\vec{\omega}$$

insert $\mathbb{1} = R^{-1}\cdot R$
$= R^T\cdot R$ since R is orthogonal

$$= R\cdot\mathbb{I}\cdot R^T R\,\vec{\omega}$$
$$= (R\cdot\mathbb{I}\cdot R^T)\cdot\vec{\omega}'$$
$$= \mathbb{I}'\cdot\vec{\omega}'$$

$\Rightarrow$ Require $\mathbb{I}' = R\cdot\mathbb{I}\cdot R^T$ if $\vec{L}, \vec{\omega}$ are physical vectors.

The kinetic energy is

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2}\sum_\alpha m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \sum_i (\vec{\omega}\times\vec{r}_\alpha)_i (\vec{\omega}\times\vec{r}_\alpha)_i$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \sum_i \sum_{j,k} \epsilon_{ijk}\,\omega_j\, r_{\alpha,k} \sum_{l,m} \epsilon_{ilm}\,\omega_l\, r_{\alpha,m}$$

Note the relation $\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{kl}\delta_{jm}$

$$\Rightarrow T = \frac{1}{2}\sum_\alpha m_\alpha \sum_{j,k}\sum_{l,m} (\delta_{jl}\delta_{km} - \delta_{kl}\delta_{jm})\,\omega_j\,\omega_l\, r_{\alpha,k}\, r_{\alpha,m}$$

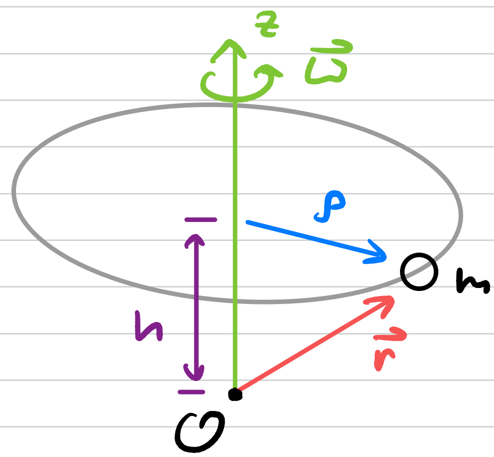
$$= \frac{1}{2}\sum_{i,j} \omega_i \left[\sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2}\delta_{ij} - r_{\alpha,i}\, r_{\alpha,j}\right)\right]\omega_j$$

$$= \frac{1}{2}\sum_{i,j}\omega_i\, I_{ij}\,\omega_j$$

$$\Rightarrow \boxed{T = \frac{1}{2}\vec{\omega}^T\cdot \mathbb{I}\cdot\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}}$$

Example – Consider a point-particle with mass m rotating around z-axis at a constant radius ρ, height h above origin, & angular velocity $\vec{\omega}$.

Compute the elements of the inertia tensor.



$$\vec{\omega} = (0, 0, \omega)$$

& position

$$\vec{r}(t) = \rho \cos\omega t\, \hat{x} + \rho \sin\omega t\, \hat{y} + h\,\hat{z}$$

Since the rotation is about z, there are only 3 - non-zero components, I_z, $I_{xz} = I_{zx}$, $I_{yz} = I_{zy}$

$$I_z = m(x^2+y^2) = m\rho^2$$
$$I_{xz} = -mxz = -mh\rho\cos\omega t$$
$$I_{yz} = -myz = -mh\rho\sin\omega t$$

So, $\vec{L} = \mathbb{I}\cdot\vec{\omega} = I_{xz}\omega\,\hat{x} + I_{yz}\omega\,\hat{y} + I_z\omega\,\hat{z}$

$$= -mh\rho\omega(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}) + m\rho^2\omega\,\hat{z}$$

Exercise: Compare $\vec{L}$ to $\vec{L} = \vec{r}\times m\vec{v}$.

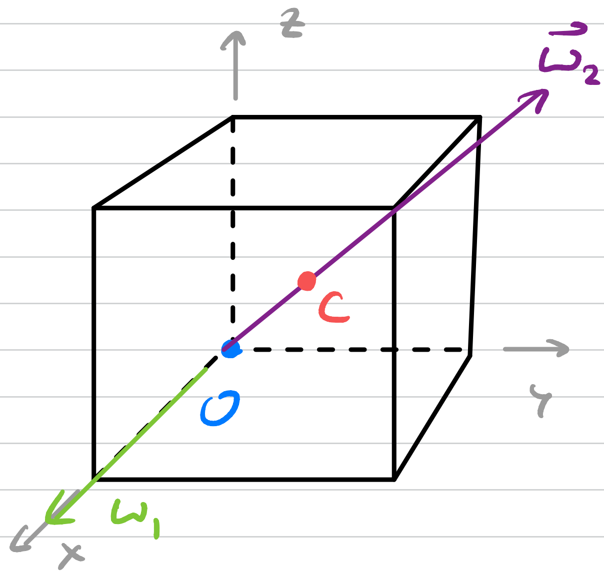
Notice that if $h=0$, i.e., the origin is in the plane of rotation

$$\Rightarrow \vec{L}_{h=0} = I_z \vec{\omega}$$

which is the result from Phys 101. $\blacksquare$

Example - Compute the inertia tensor of a solid cube of mass M and side length a about (a) the corner, (b) the center.
Compute $\vec{L}$ for both cases given $\vec{\omega}_1 = \omega(1,0,0)$ & $\vec{\omega}_2 = \frac{\omega}{\sqrt{3}}(1,1,1)$.

For a continuous distribution



$$I_{ij} = \int (\vec{r}^{\,2}\delta_{ij} - r_i r_j)\rho \, dV$$

where $\rho = \frac{M}{a^3}$

- Corner (point O) $\Rightarrow (0,0,0)$
- Center (point C) $\Rightarrow (\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$

(a) for point O,

$$I_x(O) = \int_0^a dx \int_0^a dy \int_0^a dz \; \underbrace{\rho}_{\text{constant}} \,(y^2 + z^2)$$

$$= \rho\left(\int_0^a dx\right)\left(\int_0^a dy\, y^2\right)\left(\int_0^a dz\right)$$

$$+ \rho\left(\int_0^a dx\right)\left(\int_0^a dy\right)\left(\int_0^a dz\, z^2\right)$$

$$= \rho \cdot a \cdot \frac{a^3}{3} \cdot a + \rho \cdot a \cdot a \cdot \frac{a^3}{3}$$

$$= \frac{2}{3}\rho a^5 = \frac{2}{3}\left(\frac{M}{a^3}\right)a^5$$

$$\Rightarrow \quad I_x(O) = \frac{2}{3} M a^2$$

By inspection, find $I_x(O) = I_y(O) = I_z(O) = \frac{2}{3} M a^2$

The product of inertia is

$$I_{xy}(O) = -\rho \int_0^a dx \int_0^a dy \int_0^a dz \cdot xy$$

$$= -\left(\frac{M}{a^3}\right)\frac{a^2}{2} \cdot \frac{a^2}{2} \cdot a = -\frac{1}{4} M a^2$$

By inspection, find all products of inertia are equal

So,

$$\mathbb{I}(0) = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

(b) for point C,

$$I_x(C) = \rho \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \, (y^2 + z^2)$$

$$= \frac{M}{a^3} \cdot a \cdot a \cdot \frac{1}{3} \cdot 2 \left[\left(\frac{a}{2}\right)^3 + \left(\frac{a}{2}\right)^3 \right]$$

$$= \frac{1}{6} M a^2$$

Likewise, $I_y(C) = I_z(C) = \frac{1}{6} M a^2$

$$I_{xy}(C) = \rho \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \; xy = 0$$

↳ odd integrand over even interval

$\Rightarrow$ All $I_{ij} = 0$ for $i \neq j$

$$\Rightarrow \quad \mathbb{I}(C) = \frac{1}{6} M a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{6} M a^2 \mathbb{1} .$$

The angular momenta are

$$\vec{L}_1(0) = \mathbb{I}(0) \cdot \vec{\omega}_1$$

$$= Ma^2\omega \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{12} Ma^2\omega\,(8, -3, -3)$$

$$\vec{L}_2(0) = \mathbb{I}(0) \cdot \vec{\omega}_2$$

$$= \frac{1}{\sqrt{3}} Ma^2\omega \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6\sqrt{3}} Ma^2\omega\,(1,1,1) = \frac{1}{6} Ma^2 \vec{\omega}_2$$

For point C,

$$\vec{L}_1(C) = \mathbb{I}(C) \cdot \vec{\omega}_1 = \frac{1}{6} Ma^2\, \mathbb{1} \cdot \vec{\omega}_1 = \frac{1}{6} Ma^2 \vec{\omega}_1$$

$$\vec{L}_2(C) = \mathbb{I}(C) \cdot \vec{\omega}_2 = \frac{1}{6} Ma^2\, \mathbb{1} \cdot \vec{\omega}_2 = \frac{1}{6} Ma^2 \vec{\omega}_2$$

$\blacksquare$

The previous example shows something interesting,
For a particular choice of origin and/or axis of rotation, the relation $\vec{L} = \mathbb{I} \cdot \vec{\omega}$ simplifies such that $\vec{L} \parallel \vec{\omega}$.

This particular set of axis are called principal axes. The moment of inertia about the principal axes is called the principal moments, and generally the moment of inertia is

$$\mathbb{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

so that $\vec{L} = \lambda \vec{\omega}$.

Principal axes are associated with some symmetry axis.

Theorem: Existence of Principal Axes

For any rigid body and point O, $\exists$ three perpendicular axes through O s.t. $\mathbb{I}$ is diagonal.

$\Rightarrow \mathbb{I} \vec{\omega} = \lambda \vec{\omega}$ and $\vec{L} \parallel \vec{\omega}$

To prove this, we need to recall some Linear Algebra...

Diagonalizing a Real-Symmetric Matrix

Let us remind ourself of some aspects of linear algebra, namely eigensystems & solutions.

Consider a real, symmetric $n \times n$ matrix $\mathbb{A}$.
We'd like to solve the eigenvalue equation

$$\mathbb{A}\vec{v} = \lambda \vec{v} \quad , \quad \lambda = \text{eigenvalue}$$

$$(\mathbb{I}\vec{\omega} = \lambda\vec{\omega}) \qquad \text{number} \qquad \vec{v} = \text{eigenvector}$$

This is equivalent to $(\mathbb{A} - \lambda \mathbb{1})\vec{v} = 0$

From linear algebra, we know that this has a nontrivial solution $(\vec{v} \neq \vec{0})$ iff $\det(\mathbb{A} - \lambda\mathbb{1}) = 0$.

characteristic eqn.

This is a polynomial of degree n. In general, it has n complex solutions.

For each solution λ_α, $\det(\mathbb{A} - \lambda_\alpha \mathbb{1}) = 0$, so $\exists$ a null vector $\vec{v}_\alpha$ of $\mathbb{A} - \lambda_\alpha \mathbb{1}$, i.e.,

$$\mathbb{A}\vec{v}_\alpha = \lambda_\alpha \vec{v}_\alpha \qquad (1)$$

and $\vec{v}_\alpha$ an eigenvector.

In general, $\vec{v}_\alpha \in \mathbb{C}^n$. Let's act (1) on the left by $\vec{v}_\alpha^{\,+} \equiv (v_\alpha^T)^*$ ($+ \equiv$ conjugate transpose)

$$\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha = \lambda_\alpha \vec{v}_\alpha^{\,+} \vec{v}_\alpha$$

Note that $\vec{v}_\alpha^{\,+} \vec{v}_\alpha \equiv |\vec{v}_\alpha|^2 \in \mathbb{R}$

$$\Rightarrow \quad \lambda_\alpha = \frac{\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha}{|\vec{v}_\alpha|^2}$$

Recall that, in general, $\lambda_\alpha \in \mathbb{C}$, so it is a 1×1 matrix and is symmetric, $\lambda_\alpha^T = \lambda_\alpha$

Similarly, $|\vec{v}_\alpha|^2 \in \mathbb{R} \Rightarrow (|\vec{v}_\alpha|^2)^T = |\vec{v}_\alpha|^2$

So, take transpose,

$$\lambda_\alpha = \lambda_\alpha^T = \frac{(\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha)^T}{|\vec{v}_\alpha|^2}$$

Recall $(ABC)^T = C^T B^T A^T$

$$\Rightarrow (\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha)^T = \vec{v}_\alpha^{\,T} \mathbb{A}^T \vec{v}_\alpha^{\,*}$$
$$= v_\alpha^{+*} \mathbb{A}^T \vec{v}_\alpha^{\,*}$$

Now, $\mathbb{A}$ is real and symmetric $\Rightarrow \mathbb{A}^T = \mathbb{A} = \mathbb{A}^*$

So,

$$\lambda_\alpha = \frac{\vec{V}_\alpha^{+*} \mathbb{A}^* \vec{V}_\alpha^*}{|\vec{V}_\alpha|^2}$$

$$= \left(\frac{\vec{V}_\alpha^{+} \mathbb{A} \vec{V}_\alpha}{|\vec{V}_\alpha|^2} \right)^* = \lambda_\alpha^*$$

$$\therefore \quad \lambda_\alpha = \lambda_\alpha^* \Rightarrow \boxed{\lambda_\alpha \in \mathbb{R}}$$

for real, symmetric matrix $\mathbb{A}$

Notice also

$$\mathbb{A}^* \vec{V}_\alpha^* = \lambda_\alpha^* \vec{V}_\alpha^* \Rightarrow \mathbb{A} \vec{V}_\alpha^* = \lambda_\alpha \vec{V}_\alpha^*$$

So, $\vec{V}_\alpha^*$ is also an eigenvector w/ same eigenvalue as $\vec{V}_\alpha$. $\Rightarrow$ Can take $\vec{V}_\alpha + \vec{V}_\alpha^*$, this must also be an eigenvector with eigenvalue λ_α.

But, $\vec{V}_\alpha + \vec{V}_\alpha^* = 2\mathcal{Re}(\vec{V}_\alpha) \in \mathbb{R}^n$.

$\Rightarrow$ Through suitable manipulations, all eigenvectors can be chosen to be real.

We may also normalize the eigenvectors

$$\vec{V}_\alpha \rightarrow \frac{\vec{V}_\alpha}{\sqrt{\vec{V}_\alpha^T \vec{V}_\alpha}}, \quad \text{so that } \vec{V}_\alpha^T V_\alpha = 1.$$

From now on, assume $\vec{v}_\alpha$ is normalized.

Finally, consider two eigenvalues $\lambda_\alpha \neq \lambda_\beta$.

Then,

$$\mathbb{A}\,\vec{v}_\alpha = \lambda_\alpha \vec{v}_\alpha$$

$$\& \quad \mathbb{A}\,\vec{v}_\beta = \lambda_\beta \vec{v}_\beta \quad \Rightarrow \quad \vec{v}_\beta^{\,T} \mathbb{A} = \lambda_\beta \vec{v}_\beta^{\,T}$$

take transpose

Act second eqn. on $\vec{v}_\alpha$

$$\Rightarrow \quad \vec{v}_\beta^{\,T} \underbrace{\mathbb{A}\,\vec{v}_\alpha} = \lambda_\beta \vec{v}_\beta \cdot \vec{v}_\alpha$$

$$\vec{v}_\beta^{\,T} \cdot (\lambda_\alpha \vec{v}_\alpha) = \lambda_\alpha \vec{v}_\beta^{\,T} \cdot \vec{v}_\alpha$$

$$\therefore \quad (\lambda_\alpha - \lambda_\beta)\, \vec{v}_\beta \cdot \vec{v}_\alpha = 0 \quad \Rightarrow \quad \vec{v}_\beta^{\,T} \cdot \vec{v}_\alpha = 0$$

We conclude

$$\boxed{\vec{v}_\alpha^{\,T} \cdot \vec{v}_\beta = \delta_{\alpha\beta}} \qquad \text{(orthonormality)}$$

With all this, we can now show that $\mathbb{A}$ can be diagonalized as $\mathbb{A} = \mathbb{V}\,\mathbb{D}\,\mathbb{V}^T$ where $\mathbb{D}$ is diagonal matrix with the eigenvalues on the diagonal and $\mathbb{V}$ is an orthogonal matrix formed by placing $\vec{v}_\alpha$ at column α in the same order as λ_α in $\mathbb{D}$.

Proof. Since $\vec{v}_\alpha$ are orthonormal, they form a complete basis & we just need to show

$$\mathbb{A}\,\vec{v}_\alpha = (\mathbb{V}\,\mathbb{D}\,\mathbb{V}^T)\,\vec{v}_\alpha$$

In the usual basis of $\mathbb{A}$, $\vec{e}_1 = (1,0,0,\dots,0)$

$$\vec{e}_2 = (0,1,0,\dots,0)$$

$$\vdots$$

$$\vec{e}_\alpha = (0,0,\dots,\underset{\substack{\uparrow \\ \alpha^{\text{th}}\text{ element}}}{1},\dots,0)$$

We can write the β-element of the α-basis vector $(\vec{e}_\alpha)_\beta = \delta_{\alpha\beta}$.

With this basis, we can expand the eigenvector as

$$\vec{v}_\alpha = \sum_\beta v_{\alpha,\beta}\,\vec{e}_\beta$$

Then, $\mathbb{V}_{\alpha\beta} = v_{\alpha,\beta}$ is an orthogonal matrix $\mathbb{V}^{-1} = \mathbb{V}^T$.
To see this, $\mathbb{V}\,\vec{e}_\alpha = \vec{v}_\alpha$, also

$$(\mathbb{V}^T\vec{v}_\alpha)_\beta = \sum_\gamma (\mathbb{V}^T)_{\beta\gamma}(\vec{v}_\alpha)_\gamma$$

$$= \sum_\gamma (\mathbb{V})_{\gamma\beta}(\vec{v}_\alpha)_\gamma = \sum_\gamma v_{\beta,\gamma}\,v_{\alpha,\gamma}$$

$$= \vec{v}_\beta^{\,T}\cdot\vec{v}_\alpha = \delta_{\alpha\beta}$$

$$\Rightarrow \mathbb{V}^T \vec{v}_\alpha = \vec{e}_\alpha \Rightarrow \mathbb{V}^T \mathbb{V} \vec{e}_\alpha = \vec{e}_\alpha$$

$$\Rightarrow \mathbb{V}^T \mathbb{V} = \mathbb{1} \quad \forall \alpha$$

Finally,

$$(\mathbb{V} \mathbb{D} \mathbb{V}^T) \vec{v}_\alpha = \mathbb{V} \mathbb{D} \vec{e}_\alpha = \lambda_\alpha \mathbb{V} \vec{e}_\alpha = \lambda_\alpha \vec{v}_\alpha \equiv \mathbb{A} \vec{v}_\alpha$$

$$\therefore \quad \boxed{\mathbb{A} = \mathbb{V} \mathbb{D} \mathbb{V}^T}$$

Principle Axes & Principal Moments

For the inertia tensor, a real symmetric matrix, we can diagonalize it, i.e., choose a set of axes, such that $\vec{L} \parallel \vec{\omega}$.

The diagonal elements are given by the eigenvalues of $\mathbb{I}$, i.e., the solutions to

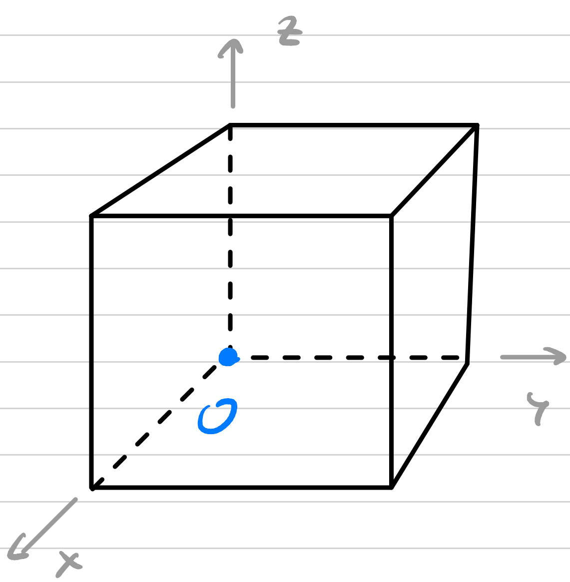
$$\boxed{\det(\mathbb{I} - \lambda \mathbb{1}) = 0}$$

Example - Principal axes of cube about corner?

Recall from previous example

$$\mathbb{I}_0 = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

$$\equiv \mu \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$



with $\mu = \frac{1}{12} Ma^2$

So, we want to solve $\det(\mathbb{I}_0 - \lambda \mathbb{1}) = 0$

$$\Rightarrow \quad \det \begin{bmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{bmatrix} = 0$$

To solve, note the following:

- Replacing a column (row) of a matrix with the sum of that column (row) & a multiple of another column (row) does NOT change det.
- Multiplying a column (row) by number, multiplies det by same number
- $$\det \begin{bmatrix} a_1 & \cdot & \cdot \\ 0 & a_2 & \cdot \\ 0 & 0 & a_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & 0 & 0 \\ \cdot & a_2 & 0 \\ \cdot & \cdot & a_3 \end{bmatrix} = a_1 a_2 a_3$$

So, take

$$0 = \det \begin{bmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{bmatrix}$$

take col 1 → col 1 − col 2

$$\Rightarrow \quad 0 = \det \begin{bmatrix} 11\mu - \lambda & -3\mu & -3\mu \\ -11\mu + \lambda & 8\mu - \lambda & -3\mu \\ 0 & -3\mu & 8\mu - \lambda \end{bmatrix}$$

row2 → row 2 + row 1

$$= \det \begin{bmatrix} 11\mu - \lambda & -3\mu & -3\mu \\ 0 & 5\mu - \lambda & -6\mu \\ 0 & -3\mu & 8\mu - \lambda \end{bmatrix}$$

col 2 → col 2 - col 3

$$= \det \begin{bmatrix} 11\mu - \lambda & 0 & -3\mu \\ 0 & 11\mu - \lambda & -6\mu \\ 0 & -11\mu + \lambda & 8\mu - \lambda \end{bmatrix}$$

row 3 → row3 + row 2

$$= \det \begin{bmatrix} 11\mu - \lambda & 0 & -3\mu \\ 0 & 11\mu - \lambda & -6\mu \\ 0 & 0 & 2\mu - \lambda \end{bmatrix}$$

So, find

$$\det[\dots] = (11\mu - \lambda)^2 (2\mu - \lambda) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 2\mu \\ \lambda_2 = \lambda_3 = 11\mu \end{cases}$$

So, the principal moments are

$$\mathbb{I}_0' = \mathbb{D} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

What about the axes?

Want $\hat{e}_1, \hat{e}_2, \hat{e}_3$

First, find $\vec{\omega}_1 = \omega \hat{e}_1$ associated w/ λ_1

Solve $(\mathbb{I}_0 - \lambda_1 \mathbb{1})\vec{\omega}_1 = 0$

$$\Rightarrow \quad \mu \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix} \begin{pmatrix} \omega_{1,x} \\ \omega_{1,y} \\ \omega_{1,z} \end{pmatrix} = 0$$

$$\Rightarrow \quad 2\omega_{1,x} - \omega_{1,y} - \omega_{1,z} = 0 \qquad (a)$$

$$-\omega_{1,x} + 2\omega_{1,y} - \omega_{1,z} = 0 \qquad (b)$$

$$-\omega_{1,x} - \omega_{1,y} + 2\omega_{1,z} = 0 \qquad (c)$$

Take $(a)-(b) \Rightarrow \omega_{1,x} = \omega_{1,y}$

From $(a) \Rightarrow \omega_{1,x} = \omega_{1,y} = \omega_{1,z}$

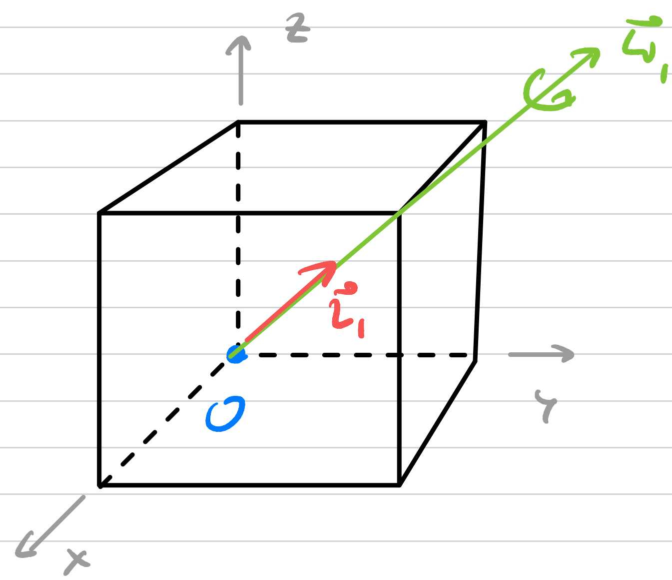
So, $\vec{\omega} \parallel (1,1,1) \Rightarrow$ Normalize to find

$$\hat{e}_1 = \frac{1}{\sqrt{3}}(1,1,1)$$

This means that if $\vec{\omega}_1 = \omega \hat{e}_1$

that $\quad \vec{L}_1 = \mathbb{I}\,\vec{\omega}_1 = \omega\,\mathbb{I}\,\hat{e}_1 = \omega \lambda_1 \hat{e}_1 = \lambda_1 \vec{\omega}_1$

$$\Rightarrow \quad \vec{L}_1 = \lambda_1 \vec{\omega}_1$$



For $\vec{\omega}_2$ & $\vec{\omega}_3$, $\lambda_2 = \lambda_3$

solve

$$(\mathbb{I}_0 - \lambda_2 \mathbb{1})\,\vec{\omega} = 0$$

$$\Rightarrow \mu \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\Rightarrow \quad \omega_x + \omega_y + \omega_z = 0$$

Notice that this is equal to $\vec{\omega} \cdot \hat{e}_1 = 0$

$\Rightarrow$ $\vec{\omega}$ needs to be orthogonal to $\hat{e}_1$

$\Rightarrow$ $\hat{e}_2$ & $\hat{e}_3$ need to be orthogonal to $\hat{e}_1$

Two such solutions are

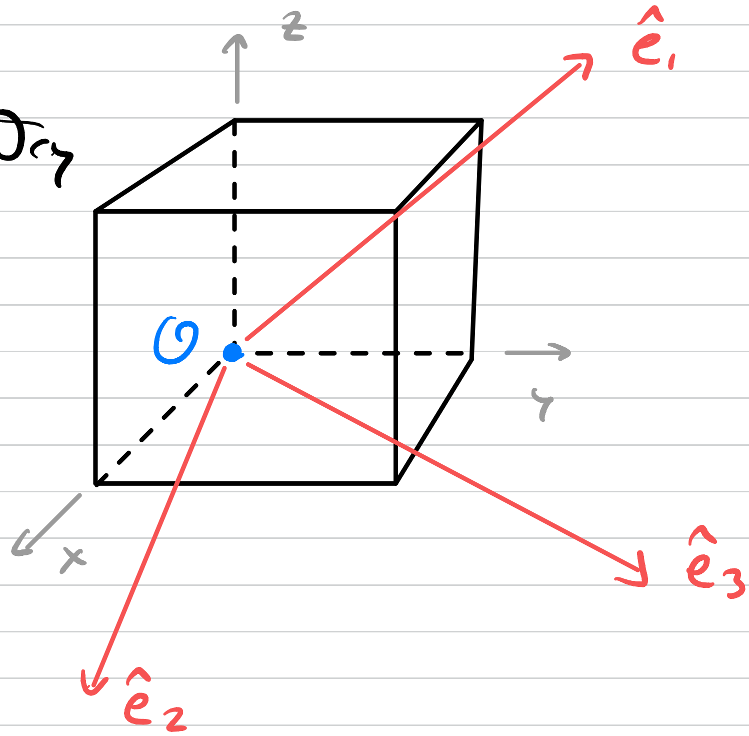
$$\hat{e}_2 = \frac{1}{\sqrt{2}}(1, 0, -1), \quad \hat{e}_3 = \frac{1}{\sqrt{6}}(-1, 2, -1)$$

(verify)

So, principal axes (eigenvectors) are

$$\hat{e}_1 = \frac{1}{\sqrt{3}}(1,1,1)\,,\quad \hat{e}_2 = \frac{1}{\sqrt{2}}(1,0,-1)\,,\quad \hat{e}_3 = \frac{1}{\sqrt{6}}(-1,2,-1)$$

The principal axes correspond to symmetry of the cube about the body diagonal with center at O.



Can decompose $\mathbb{I}_0 = \mathbb{V}\,\mathbb{D}_0\,\mathbb{V}^T$

with
$$\mathbb{D}_0 = \frac{1}{12} M a^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$$\mathbb{V} = \frac{1}{\sqrt{6}} \begin{array}{c} \begin{pmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & -1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{array} \end{array}$$

$\blacksquare$

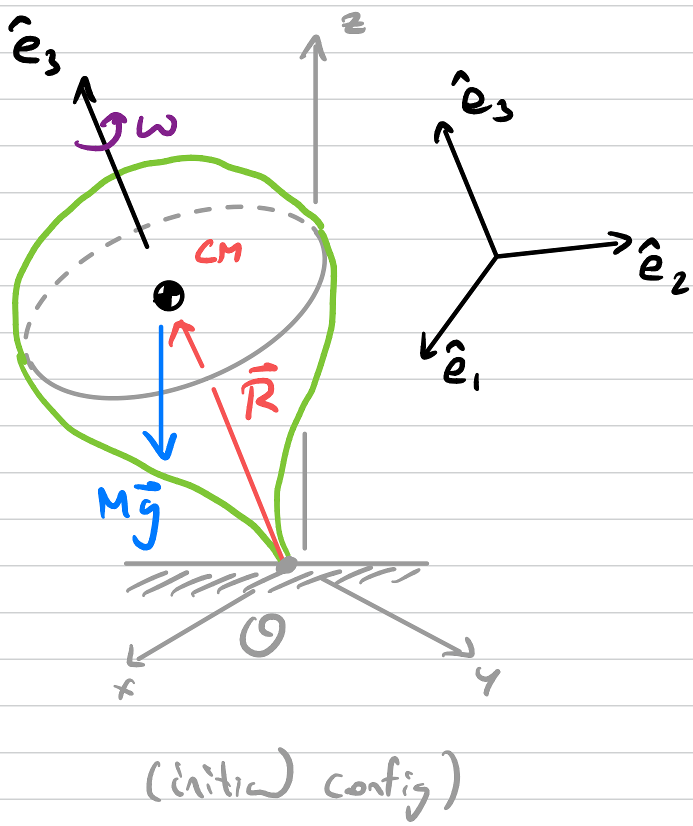
Precession of Symmetric Top due to Weak Torque

Having all the tools in our arsenal, let's apply them for a "simple" physics problem.

Consider a symmetric top with mass M and inertia tensor $\mathbb{I} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ in the basis of its principal axes, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$.

It rotates freely with its tip pivoted at a fixed point O in a lab frame (inertial)

Its CM is at $\vec{R}$.



We assume its angular velocity is initially along its symmetry axis,

$$\vec{\omega} = \omega \hat{e}_3$$

The angular momentum is initially

$$\vec{L} = \mathbb{I}\vec{\omega} = \lambda_3 \omega \hat{e}_3$$

If we release the top, gravity will exert a torque, and cause a change of angular momentum.

The torque due to gravity is $\vec{\Gamma} = \vec{R} \times M\vec{g}$, at the CM wrt fixed point O.

- If $\vec{\Gamma} = \vec{0}$ (e.g., $\vec{R} \times \vec{g} = \vec{0}$)

  $\Rightarrow \vec{L} = \text{const.}$

To see, take

$$\left(\frac{d\vec{L}}{dt}\right)_{lab} = \lambda_3 \frac{d}{dt}\left(\omega \hat{e}_3\right)_{lab}$$

$$= \lambda_3\left(\dot{\omega}\hat{e}_3 + \omega\left(\frac{d\hat{e}_3}{dt}\right)_{lab}\right) = 0$$

Since $\vec{\omega} \parallel \hat{e}_3 \Rightarrow \left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \vec{\omega} \times \hat{e}_3 = \vec{0}$

That is, $\hat{e}_3$ is fixed in the lab frame too!

$\Rightarrow \dot{\omega} = 0$

- If $\vec{\Gamma} \neq \vec{0}$, but ω_1, ω_2 are small, so $\vec{\omega} \simeq \omega_3 \hat{e}_3$

$\Rightarrow \vec{\Gamma} \perp \vec{\omega}$ and $\dot{\omega}$ remains small

So, EOM gives $\left(\frac{d\vec{L}}{dt}\right)_{lab} = \vec{\Gamma}$

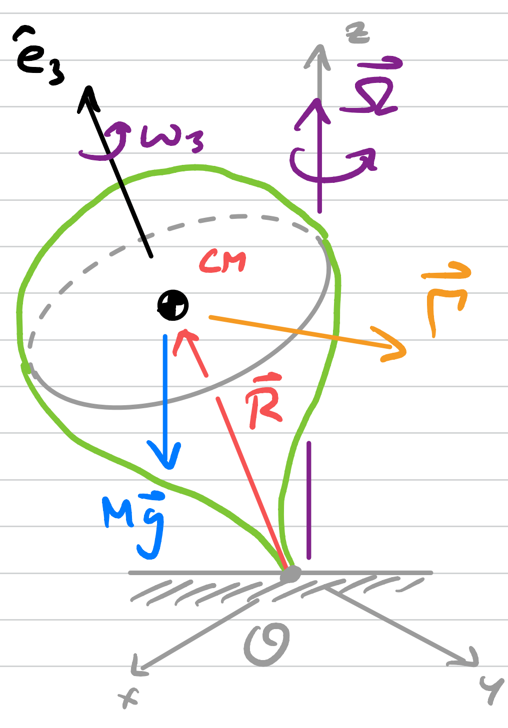
$$\Rightarrow \lambda\left(\cancelto{0}{\dot{\omega}}\hat{e}_3 + \omega\left(\frac{d\hat{e}_3}{dt}\right)_{lab}\right) = \vec{R} \times M\vec{g}$$

Since $\dot{\omega} \simeq 0$, $\left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \frac{1}{\lambda\omega}\vec{R} \times M\vec{g}$

Now, $\vec{R} = R\hat{e}_3$, $\vec{g} = -g\hat{z}$ $\Rightarrow \vec{R} \times \vec{g} = Rg\,\hat{z} \times \hat{e}_3$

$$\Rightarrow \boxed{\left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \left(\frac{MRg}{\lambda\omega}\right)\hat{z} \times \hat{e}_3}$$

Define $\vec{\Omega} = \frac{MRg}{\lambda\omega}\hat{z}$



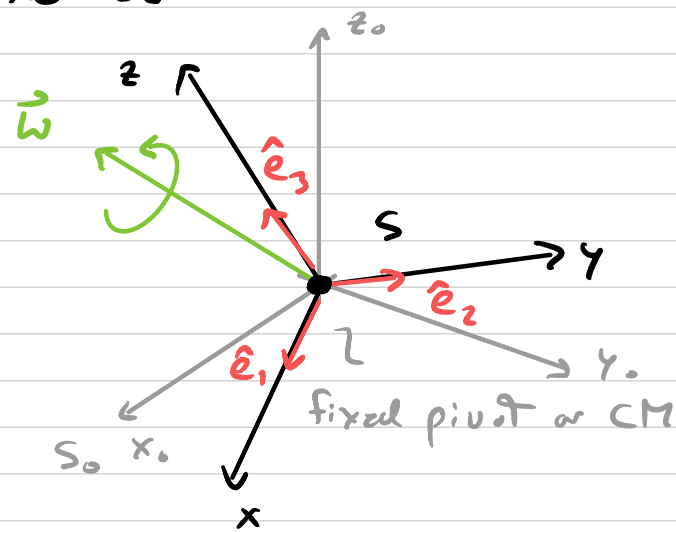
$\Rightarrow$ the symmetry axis of the top rotates w/ $\Omega = \frac{MRg}{\lambda\omega}$ about $\hat{z}$ axis

$\Rightarrow$ precession

Eulers Equations

Newton's Law's of motion for a body rotating in some inertial (space-fixed) frame S_0

$$\left(\frac{d\vec{L}}{dt}\right)_{S_0} = \vec{\Gamma}$$



Recall that

$$\left(\frac{d\vec{L}}{dt}\right)_{S_0} = \left(\frac{d\vec{L}}{dt}\right)_{S} + \vec{\omega} \times \vec{L}$$

Where S is a frame fixed to the body.

Therefore, we find

$$\boxed{\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}}$$ in body-fixed frame S

Notice: If $\vec{\Gamma} = \vec{0} \Rightarrow \vec{L}$ is conserved in S_0

But NOT in S!

The use of the body-fixed frame makes it possible to choose the principle axes of the moment of inertia, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, as the coordinate axes.

Therefore,

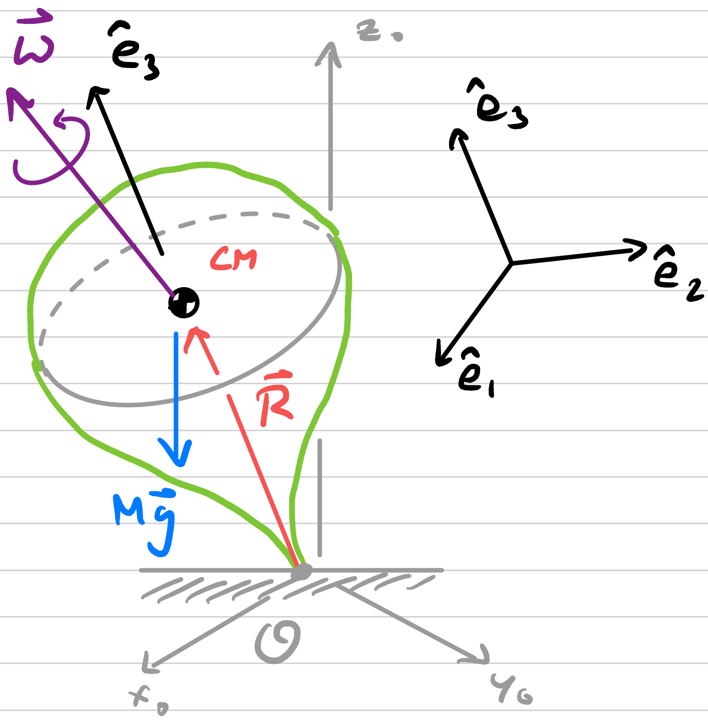
$$\vec{L} = \mathbb{I}\,\vec{\omega} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

So, $\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}$

$$\boxed{\begin{aligned} \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2 \omega_3 &= \Gamma_1 \\ \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3 \omega_1 &= \Gamma_2 \\ \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1 \omega_2 &= \Gamma_3 \end{aligned}}$$

These are called the Euler Equations of rotational motion

An example is the symmetric spinning top we considered before



$\lambda_1 = \lambda_2$

& $\Gamma_3 = 0 \Rightarrow \lambda_3 \dot{\omega}_3 = 0$

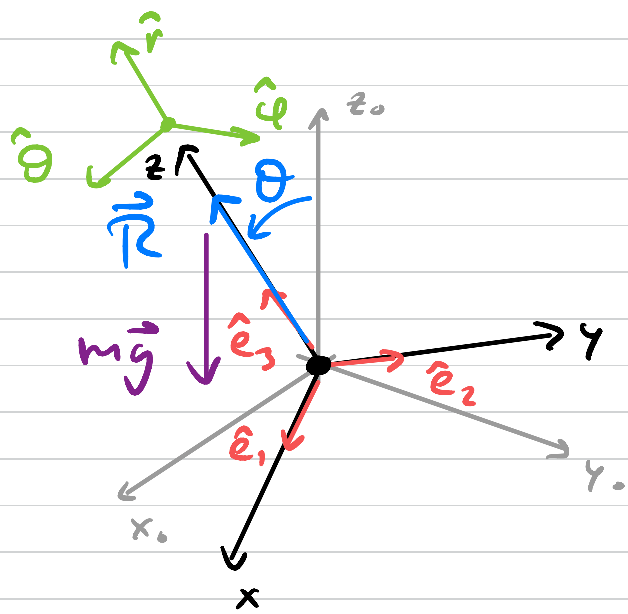
The other two Euler eqns. are

$$\begin{cases} \dot{\omega}_1 = \dfrac{1}{\lambda_1}\Gamma_1 - \left(\dfrac{\lambda_3}{\lambda_1} - 1\right)\omega_2 \omega_3 \\ \dot{\omega}_2 = \dfrac{1}{\lambda_2}\Gamma_2 + \left(\dfrac{\lambda_3}{\lambda_1} - 1\right)\omega_1 \omega_3 \end{cases}$$

Take the axis ($\hat{e}_3$) to be at spherical angles (θ, φ) in the space frame, we find $\vec{\Gamma} \parallel \hat{\varphi}$.

Let's choose the axes at time $t=0$ such that

$\hat{e}_2$? $\hat{e}_1 \parallel \vec{\Gamma} \parallel \hat{y}_0$ and $\hat{e}_3$ is in the xz plane.



The CM is at $\vec{R} = R\,\hat{r}$

where $m\vec{g}$ is given by

$$m\vec{g} = mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

At time t, $\hat{e}_1$ will have rotated by an angle $\omega_3 t$ around $\hat{e}_3$ (since $\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{const.}$)

Therefore,
$$\vec{\Gamma}_1 = \Gamma \sin\omega_3 t \,, \quad \vec{\Gamma}_2 = \Gamma \cos\omega_3 t$$

where
$$\Gamma = |\vec{\Gamma}| = MgR\sin\theta$$

To solve the remaining Euler eqns., we introduce a complex variable

$$\eta = \omega_1 + i\omega_2$$

Therefore,

$$\dot{\eta} = \dot{\omega}_1 + i\,\dot{\omega}_2$$

$$= \frac{1}{\lambda_1}\left(\Gamma_1 + i\,\Gamma_2\right) + \left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3\left(-\omega_2 + i\,\omega_1\right)$$

$$\Rightarrow \quad \dot{\eta} = \frac{i\,\Gamma}{\lambda_1}\, e^{-i\omega_3 t} + i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3\,\eta\,.$$

We can write the solution as $\eta = \eta_h + \eta_p$

particular solution

The homogeneous eqn is

homogeneous general solution

$$\dot{\eta} = i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3\,\eta\,,$$

which by direct integration gives homogeneous solution

$$\eta_h = A\exp\left[\,i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3 t\right]$$

TBD from ICs

The particular solution is $\eta_p = \eta_0\, e^{-i\omega_3 t}$

Fix η_0 by substituting into EOM,

$$-i\omega_3\,\eta_0 = \frac{i\,\Gamma}{\lambda_1} + i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3\,\eta_0$$

$$\Rightarrow \quad \eta_0 = -\frac{\Gamma}{\lambda_3\,\omega_3}$$

To fix A, the IC's are $\omega_1 = \omega_2 = 0$ @ $t = 0$

$$\Rightarrow \quad \eta(0) = 0 = A + \eta_0 \quad \Rightarrow \quad A = -\eta_0$$

So, altogether

$$\eta(t) = \frac{\Gamma}{\lambda_3 \omega_3} \left[e^{i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3 t} - e^{-i\omega_3 t} \right]$$

Writing $\vec{\omega} = \vec{\omega}_\perp + \omega_3 \hat{e}_3$, the small torque condition is

$$|\eta| = |\vec{\omega}_\perp| \ll \omega_3 \qquad \& \quad \theta \ll 1$$

$$\Rightarrow \quad \frac{\Gamma}{\lambda_3 \omega_3} \ll \omega_3 \quad \Rightarrow \quad \underbrace{\frac{MgR}{\lambda_3 \omega_3}}_{\equiv \Omega = \text{the precession frequency}} \ll \omega_3$$

Zero Torque

The zero-torque Euler eqns. are

$$\boxed{\begin{aligned} \lambda_1 \dot{\omega}_1 &= (\lambda_2 - \lambda_3)\omega_2 \omega_3 \\ \lambda_2 \dot{\omega}_2 &= (\lambda_3 - \lambda_1)\omega_3 \omega_1 \\ \lambda_3 \dot{\omega}_3 &= (\lambda_1 - \lambda_2)\omega_1 \omega_2 \end{aligned}}$$

Let's consider again the "free" motion of a symmetric top, $\lambda_1 = \lambda_2$ w/ $\hat{e}_3$ being the symmetry axis.

Then $\quad \lambda_3 \dot{\omega}_3 = 0$

& $\quad \dot{\omega}_1 = -\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_2 \omega_3 \,, \quad \dot{\omega}_2 = +\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_1 \omega_3$

As before, introduce $\eta = \omega_1 + i\omega_2$

$$\Rightarrow \quad \dot{\eta} = i\Omega_b \eta \quad \text{where} \quad \Omega_b = \left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3$$

The general solution is $\eta = A e^{i\Omega_b t}$

So,

free precession

$$\boxed{\vec{\omega} = A\cos\Omega_b t \, \hat{e}_1 + A\sin\Omega_b t \, \hat{e}_2 + \omega_3 \hat{e}_3}$$

Notice that since $\vec{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$

$$= \lambda_1 A \cos\Omega_b t \, \hat{e}_1 + \lambda_2 A \sin\Omega_b t \, \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

But, $\lambda_1 = \lambda_2 \quad \Rightarrow \quad \vec{L} = \lambda_1 A \left(\cos\Omega_b t \, \hat{e}_1 + \sin\Omega_b t \, \hat{e}_2\right) + \lambda_3 \omega_3 \hat{e}_3$

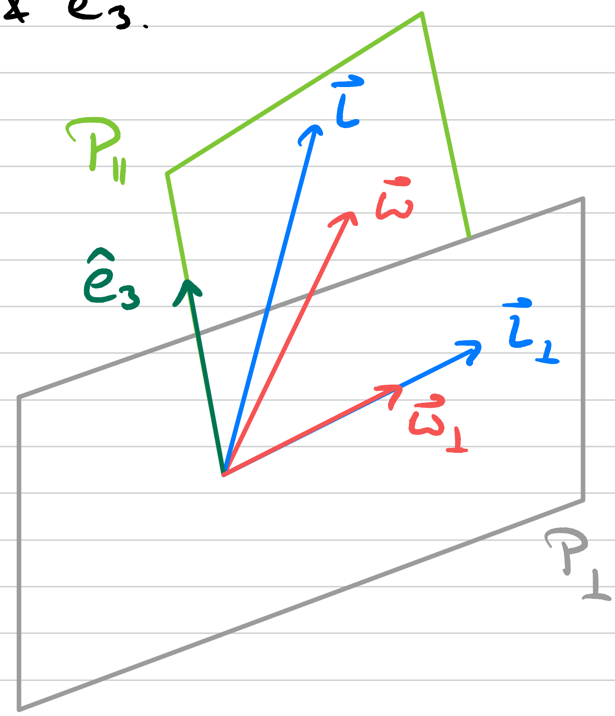
Define $\vec{\omega}_\perp$ via $\vec{\omega} = \vec{\omega}_\perp + \omega_3 \hat{e}_3$

where $\vec{\omega}_\perp \cdot \hat{e}_3 = 0$

Then,

$$\vec{L} = \mathbb{I} \cdot \vec{\omega} = \lambda_1 \vec{\omega}_\perp + \lambda_3 \omega_3 \hat{e}_3$$

Geometrically, since the perpendicular component of $\vec{L}$, $\vec{L}_\perp = \lambda_1 \vec{\omega}_\perp$, is parallel to $\vec{\omega}_\perp$, $\vec{L}$ should be in the same plane as $\vec{\omega}$ & $\hat{e}_3$.



Analytically, we can show three vectors, $\vec{A}, \vec{B}, \vec{C}$ are in the same plane iff $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

$$\vec{L}\cdot(\hat{e}_3\times\vec{\omega})$$
$$= \lambda_1 \vec{\omega}_\perp \cdot (\hat{e}_3 \times \vec{\omega}_\perp) + \lambda_3 \omega_3 \hat{e}_3 \cdot (\hat{e}_3 \times \vec{\omega}_\perp)$$
$$= 0$$

Thus, we see that $\vec{\omega}$, $\vec{L}$, & $\hat{e}_3$ all lie in the same plane.

What if $\lambda_1 \neq \lambda_2$?

For a body whose principal moments of inertia are all different, we may assume $\lambda_1 < \lambda_3 < \lambda_2$ to find

$$\begin{cases} \dot{\omega}_1 = \left(\dfrac{\lambda_2 - \lambda_3}{\lambda_1}\omega_3\right)\omega_2 \\ \dot{\omega}_2 = \left(\dfrac{\lambda_3 - \lambda_1}{\lambda_2}\omega_3\right)\omega_1 \end{cases}$$

As long as ω_1 & ω_2 are small, $\dot{\omega}_3$ remains small & we can take ω_3 to be constant. Then, the two coupled equations for ω_1 & ω_2 can be solved easily

$$\ddot{\omega}_1 = \left(\frac{\lambda_2-\lambda_3}{\lambda_1}\,\omega_3\right)\dot{\omega}_2 = \left[\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\,\omega_3^2\right]\omega_1$$

Since $(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)>0$, this solution will have a real exponential solution — which is unstable & grows rapidly.

In fact, a very strong statement about this instability can be shown to be true:

There is no purely decaying solution $\sim e^{-\alpha t}$, $\alpha>0$ for any initial conditions.

To see this, note that $\mathrm{sgn}(\dot{\omega}_1) = \mathrm{sgn}(\omega_3)\cdot\mathrm{sgn}(\omega_2)$

For an initial value $\omega_1(t=0) = \omega_{1,0}$, we have

$\mathrm{sgn}(\dot{\omega}_2(t=0)) = \mathrm{sgn}(\omega_3)\cdot\mathrm{sgn}(\omega_{1,0})$.

Shortly after $t=0$, $\mathrm{sgn}(\omega_2) = \mathrm{sgn}(\dot{\omega}_2(t=0)) = \mathrm{sgn}(\omega_3)\cdot\mathrm{sgn}(\omega_{1,0})$

So,

$$\operatorname{sgn}(\dot{\omega}_1(t=0)) = \operatorname{sgn}(\omega_3)\cdot\operatorname{sgn}(\omega_2) = \operatorname{sgn}(\omega_{1,0}).$$

This is a constraint that must be satisfied by all initial conditions & solutions.

$\Rightarrow$ It is NOT satisfied by a purely decaying solution:

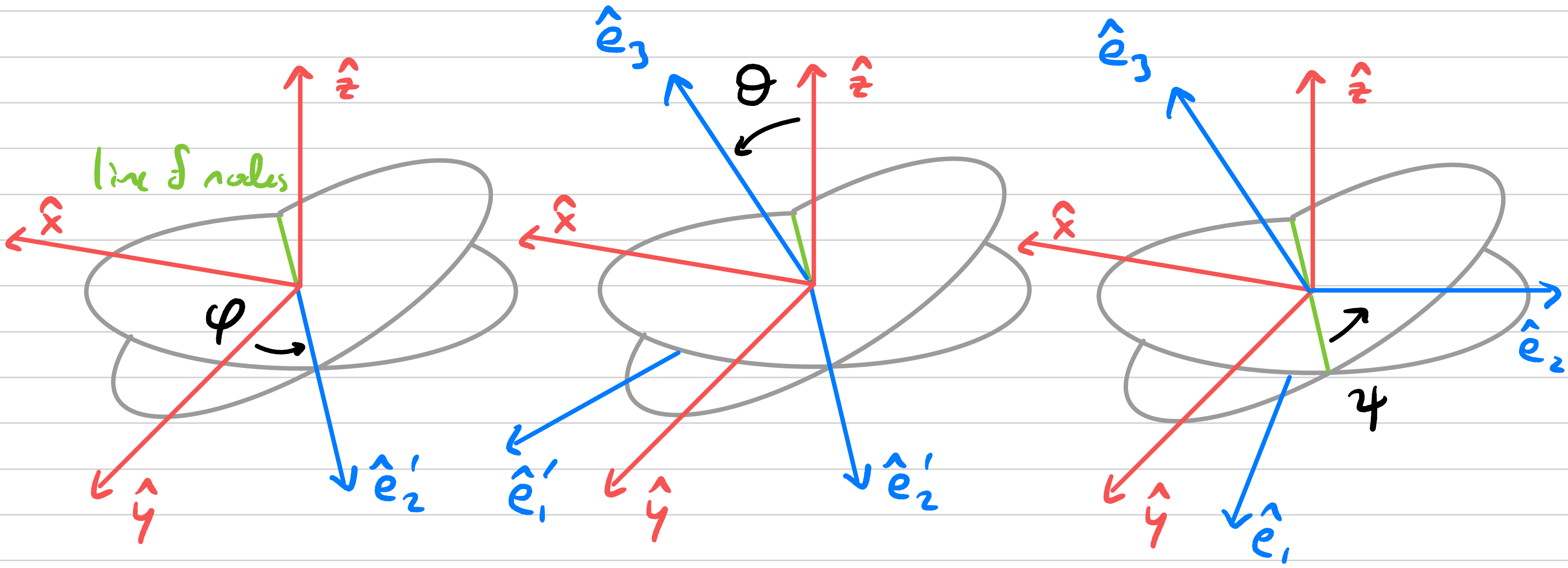
$$\omega_1 \sim e^{-\alpha t} \Rightarrow \dot{\omega}_{1,0} = -\alpha\,\omega_{1,0}$$

$$\Rightarrow \operatorname{sgn}(\dot{\omega}_{1,0}) = -\operatorname{sgn}(\omega_{1,0})$$

Which is a contradiction.

Euler Angles

The Euler angles are a special set of angles which are frequently used to describe rotational motion



From some fixed coordinate system (x, y, z) [zyz convention]

1. Rotate about $\hat{z}$ by φ
2. Rotate about $\hat{e}_2'$ by θ
3. Rotate about $\hat{e}_3$ by ψ

By definition, $\vec{\omega} = \dot{\varphi}\,\hat{z} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$

Can decompose $\hat{z}$, $\hat{e}_1'$, & $\hat{e}_2'$ into $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$$\begin{cases} \hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1' \\ \hat{e}_1' = \cos\psi\,\hat{e}_1 - \sin\psi\,\hat{e}_2 \\ \hat{e}_2' = \sin\psi\,\hat{e}_1 + \cos\psi\,\hat{e}_2 \end{cases}$$

Therefore

$$\boxed{\begin{aligned}\vec{\omega} = &(-\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\hat{e}_1 \\ &+ (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{e}_2 \\ &+ (\dot{\varphi}\cos\theta + \dot{\psi})\hat{e}_3\end{aligned}}$$

We can also express $\vec{\omega}$ in terms of $\hat{x}, \hat{y}, \hat{z}$

$$\begin{cases} \hat{e}_2' = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} \\ \hat{e}_3 = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} \end{cases}$$

Therefore,

$$\boxed{\begin{aligned}\vec{\omega} = &(-\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi)\hat{x} \\ &+ (\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi)\hat{y} \\ &+ (\dot{\varphi} + \dot{\psi}\cos\theta)\hat{z}\end{aligned}}$$

Note that for a body w/ & axis of symmetry parallel to $\hat{e}_3$, then $\lambda_1 = \lambda_2$ for any chosen axes $\perp$ to $\hat{e}_3$.

For example, $\hat{e}_1'$ & $\hat{e}_2'$ work just as well as $\hat{e}_1$ & $\hat{e}_2$

$$\Rightarrow \quad \vec{\omega} = -\dot{\varphi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\varphi}\cos\theta + \dot{\psi})\hat{e}_3$$

We find then ($\lambda_1 = \lambda_2$)

$$\vec{L} = \mathbb{I}\,\vec{\omega}$$
$$= (-\lambda_1 \dot{\varphi} \sin\theta)\,\hat{e}_1' + \lambda_1 \dot{\theta}\,\hat{e}_2' + \lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi})\,\hat{e}_3$$

Note that

$$\hat{e}_1' \cdot \hat{z} = -\sin\theta \,, \quad \hat{e}_2' \cdot \hat{z} = 0 \,, \quad \hat{e}_3 \cdot \hat{z} = \cos\theta$$

$$\Rightarrow \begin{cases} L_z = \vec{L} \cdot \hat{z} = \lambda_1 \dot{\varphi} \sin^2\theta + L_3 \cos\theta \\ L_3 = \lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi}) \end{cases}$$

Finally, the kinetic energy can be written

$$\boxed{T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}\lambda_1 (\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi})^2}$$

Spinning Top

We are now in position to the solve for the full motion of a symmetric spinning top.

The Lagrangian is

$$\mathcal{L} = T - U$$
$$= \frac{1}{2}\lambda_1 (\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

Notice that φ is cyclic

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot\varphi} = \lambda_1 \dot\varphi \sin^2\theta + \lambda_3 \cos\theta (\dot\varphi \cos\theta + \dot\psi)$$

$$= \text{const.} \equiv L_z$$

Notice that also ψ is cyclic

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = \lambda_3 (\dot\varphi \cos\theta + \dot\psi) = \text{const.} \equiv L_3$$

finally, for θ

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} \Rightarrow \lambda_1 \ddot\theta = \lambda_1 \dot\varphi^2 \sin\theta\cos\theta - \lambda_3 \dot\varphi \sin\theta (\dot\varphi\cos\theta + \dot\psi) + MgR\sin\theta$$

The total energy is conserved

$$E = T + U$$
$$= \frac{1}{2}\lambda_1 (\dot\varphi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3 (\dot\varphi\cos\theta + \dot\psi)^2 + MgR\cos\theta$$

Writing in terms of L_z & L_3

$$E = \frac{1}{2}\lambda_1 \dot\theta^2 + \frac{1}{2}\lambda_1 \left(\frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}\right)^2 \sin^2\theta + \frac{1}{2}\lambda_3 \left(\frac{L_3}{\lambda_3}\right)^2 + MgR\cos\theta$$

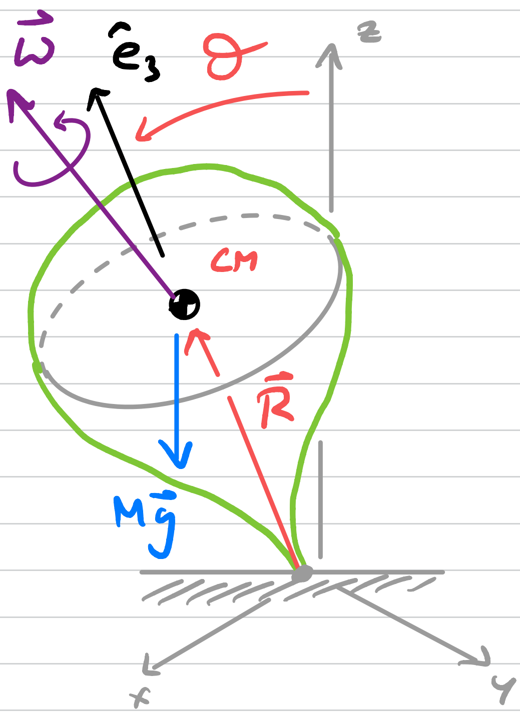
We can write this as

$$E = \frac{1}{2}\lambda_1 \dot{\theta}^2 + U_{eff}(\theta)$$

where the effective potential is

$$\boxed{U_{eff}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}$$

Depending on the energy of the top, the angle θ = angle between the symmetry axis & the z-direction will vary between two limiting values



If the energy is exactly matched to be at the minimum of U_{eff}, θ will be fixed. An especially simple example arises, since

$$\dot{\varphi} = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta} = \text{const.}$$

Since $\dot{\varphi} \equiv \Omega$ is constant, the top will precess at a fixed angle θ & constant angular velocity.

From the expression for energy,

$$\frac{dU_{eff}}{d\theta} = 0 \Rightarrow \frac{dU_{eff}}{d\cos\theta} = 0$$

$$\Rightarrow \quad \frac{(L_z - L_3\cos\theta)^2}{\lambda_1 \sin^4\theta}\cos\theta - L_3 \cdot \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta} + MgR = 0$$

Since $\Omega = \dfrac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}$, we find

$$\underbrace{\lambda_1 \cos\theta}_{a}\, \Omega^2 \underbrace{- L_3}_{b}\, \Omega + \underbrace{MgR}_{c} = 0$$

$$\Rightarrow \quad a\Omega^2 + b\Omega + c = 0$$

There are two solutions: $\Omega_\pm = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

When $b^2 \gg ac$, we expand $\sqrt{b^2 - 4ac} \simeq b\left(1 - \dfrac{2ac}{b^2}\right)$

So,

$$\begin{cases} \Omega_+ \simeq -\dfrac{c}{b} = \dfrac{MgR}{L_3} = \dfrac{MgR}{\lambda_3 \omega_3} & \leftarrow \text{precession} \\[2ex] \Omega_- \simeq -\dfrac{b}{a} = \dfrac{L_3}{\lambda_1 \cos\theta} & \leftarrow \text{free precession} \end{cases}$$

For the angular momentum, we have

$$\vec{L} = -\lambda_1 \dot{\varphi} \sin\theta \, \hat{e}_1' + L_3 \hat{e}_3$$

Since $\hat{z}$, $\hat{e}_1'$, & $\hat{e}_3$ are all on the same plane, the horizontal component of $\vec{L}$ is

$$L_h = -\lambda_1 \dot{\varphi} \sin\theta \cos\theta + L_3 \sin\theta$$

For the large solution $\Omega_- \simeq -\frac{b}{a} = + \frac{L_3}{\lambda_1 \cos\theta}$, we find

$$L_h \simeq -\lambda_1 \cdot \frac{L_3}{\lambda_1 \cos\theta} \sin\theta \cos\theta + L_3 \sin\theta = 0$$

That is, $\vec{L}$ is nearly vertical.