Physics 303

Classical Mechanics II

Two-Body Systems

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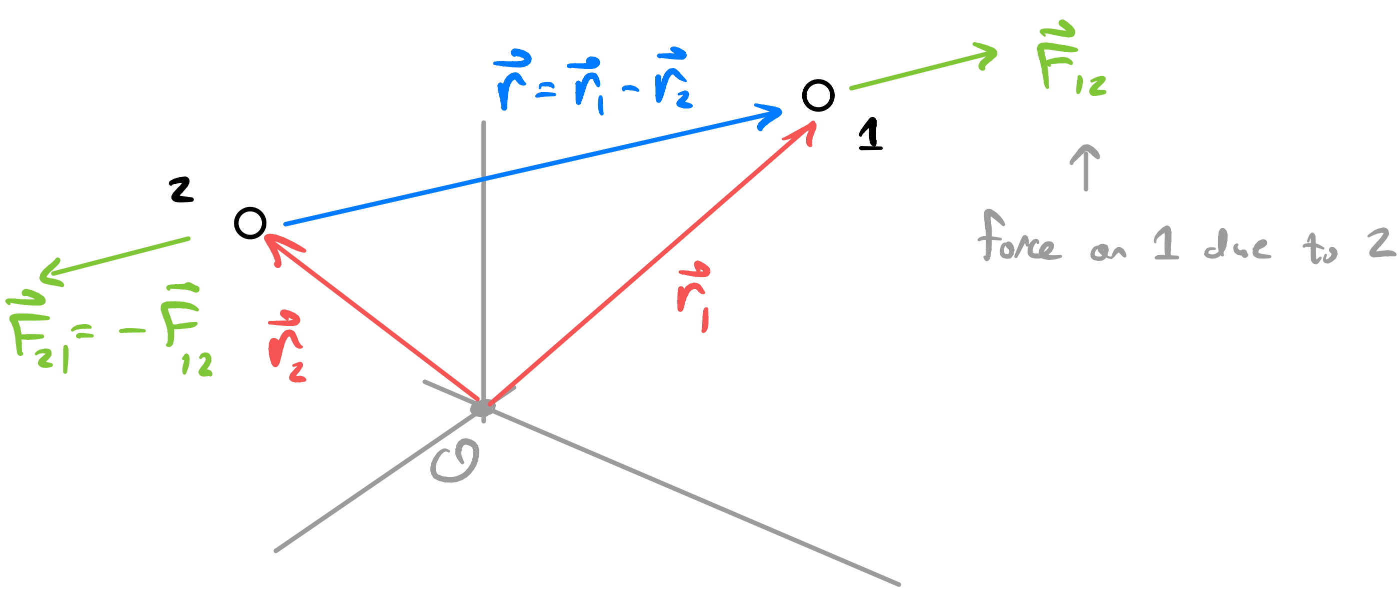
Two-Body Systems

Here we examine in detail the motion of two-body systems. Two-body systems are prevalent in the study of physics, such as the orbit of a planet about a star & the physics of interacting electron & proton in the hydrogen atom. Our focus will be on central force problems, that is each body exhibits a mutual force on each other without any external forces.

Central Forces

Consider two objects, considered as point-particles, with masses m_1 & m_2. The forces considered are $\vec{F}_{12} = -\vec{F}_{21}$, assumed conservative & central.



A central force has the functional form

$$\vec{F}_{12}(\vec{r}_1, \vec{r}_2) = \vec{F}_{12}(|\vec{r}_1 - \vec{r}_2|)$$
$$= -\vec{F}_{21}(|\vec{r}_1 - \vec{r}_2|)$$

Here, $\vec{r}_1$ & $\vec{r}_2$ are the positions of objects 1 & 2 in a coordinate system O.

An example of such a force is Newton's Law of Gravitation,

$$\boxed{\vec{F}_{12} = -G m_1 m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}}$$

Gravitational constant, $G = 6.67 \times 10^{-11} \, \frac{N \cdot m^2}{kg^2}$

Since the force is conservative ($\vec{\nabla} \times \vec{F} = \vec{0}$), we can describe it by a potential energy function,

$$\boxed{\vec{F}_{12}(\vec{r}_1, \vec{r}_2) = -\vec{\nabla}_1 \, U(\vec{r}_1, \vec{r}_2)}$$

w/ $\vec{\nabla}_1 = \frac{\partial}{\partial x_1}\hat{x}_1 + \frac{\partial}{\partial y_1}\hat{y}_1 + \frac{\partial}{\partial z_1}\hat{z}_1$

An isolated system is translationally invariant, & since the force is conservative, we have

$$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$$

Let us introduce the relative position $\vec{r}$,

$$\boxed{\vec{r} \equiv \vec{r}_1 - \vec{r}_2}$$

↳ position of body 1 relative to body 2

With this definition,

$$\vec{F}_{12} = -G m_1 m_2 \frac{\vec{r}}{r^3} = -\vec{\nabla}_r U(r)$$

with $r = |\vec{r}| = \sqrt{\vec{r}^{\,2}} = \sqrt{\vec{r}\cdot\vec{r}}$,

and the potential is $U = U(r)$

For gravitation, $U(r) = -G \dfrac{m_1 m_2}{r}$

The dynamical system of the two bodies is described by the Lagrangian

$$\boxed{\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)}$$

The Newtonian formulation is

$$\boxed{\ddot{\vec{r}}_1 = \frac{1}{m_1}\vec{F}_{12} \quad , \quad \ddot{\vec{r}}_2 = \frac{1}{m_2}\vec{F}_{21}}$$

We will use the Lagrangian approach to generate equations of motion in a more suitable coordinate system.

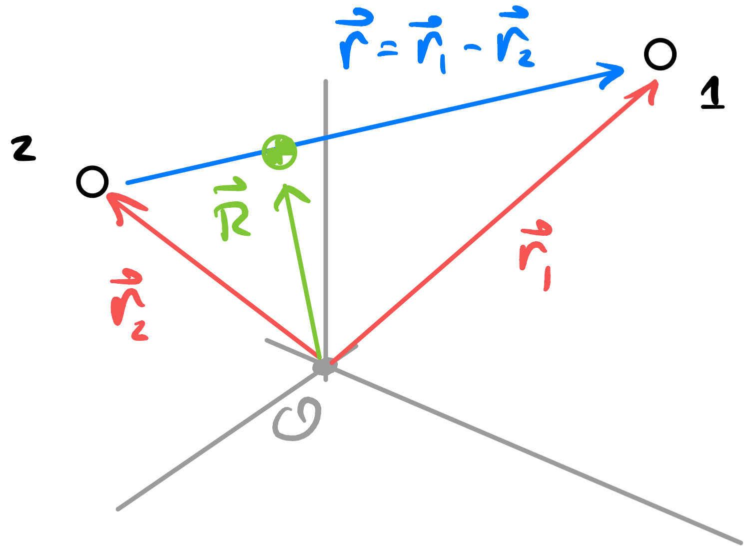
Center of Mass & Relative Coordinates

It is difficult to solve the system for $\vec{r}_1$ & $\vec{r}_2$ separately. However, since the potential is central, $U = U(r)$, this indicates that there is a better set of coordinates involving the relative position $\vec{r} = \vec{r}_1 - \vec{r}_2$. We have $3+3=6$ d.o.f. between $\vec{r}_1$ & $\vec{r}_2$, and $\vec{r}$ has 3 d.o.f., so we need 3 more.

Consider the center-of-mass $\vec{R}$

$$\boxed{\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}}$$

Consider some limits



- $\frac{m_1}{m_2} \ll 1$

$$\Rightarrow \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{\frac{m_1}{m_2}\vec{r}_1 + \vec{r}_2}{1 + \frac{m_1}{m_2}} \simeq \vec{r}_2 + \left(\frac{m_1}{m_2}\right)\vec{r} + \mathcal{O}\left(\left(\frac{m_1}{m_2}\right)^2\right)$$

CM is close to $\vec{r}_2$

- $\frac{m_2}{m_1} \ll 1$

CM close to $\vec{r}_1$

$$\Rightarrow \vec{R} = \frac{\vec{r}_1 + \frac{m_2}{m_1}\vec{r}_2}{1 + \frac{m_2}{m_1}} \simeq \vec{r}_1 + \left(\frac{m_2}{m_1}\right)\vec{r} + \mathcal{O}\left(\left(\frac{m_2}{m_1}\right)^2\right)$$

- $m_1 = m_2 \equiv m$

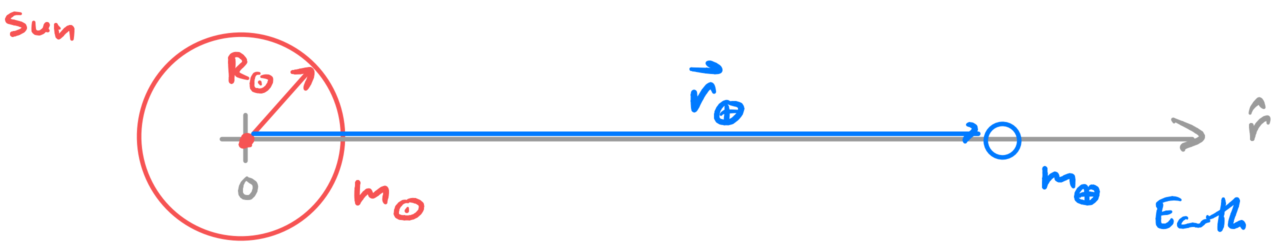
$$\Rightarrow \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$$ ← half-way between $\vec{r}_1$ & $\vec{r}_2$

$$= \vec{r}_1 - \frac{1}{2}\vec{r}$$

$$= \vec{r}_2 + \frac{1}{2}\vec{r}$$

Example

Consider the Earth-Sun system. Where is the CM using a coordinate system with the origin at the center of the sun.



$$\vec{R} = \frac{m_\odot \vec{r}_\odot + m_\oplus \vec{r}_\oplus}{m_\odot + m_\oplus} = \frac{m_\oplus}{m_\odot + m_\oplus} r_\oplus \hat{r}$$

$$\simeq \frac{m_\oplus}{m_\odot} \frac{1}{1 + \frac{m_\oplus}{m_\odot}} r_\oplus \hat{r}$$

$$\simeq \frac{m_\oplus}{m_\odot} r_\oplus \hat{r} + \mathcal{O}\left(\left(\frac{m_\oplus}{m_\odot}\right)^2\right)$$

Now, $m_\oplus \simeq 3\times10^{-6} m_\odot$

$\langle r_\oplus \rangle \simeq 200 R_\odot$

solar radius

$\Rightarrow \boxed{\langle R \rangle \simeq 6\times10^{-4} R_\odot}$ ■

The total momentum of the system $\vec{P}$ is given by

$$\vec{P} = (m_1 + m_2)\dot{\vec{R}} = M\dot{\vec{R}}$$

total mass of system

Recall that the total momentum of a closed system is constant. Therefore,

$$\vec{P} = \text{const} \Rightarrow \dot{\vec{R}} = \text{const}$$

$$\text{Let } \vec{V} \equiv \dot{\vec{R}} \quad \Rightarrow \quad \vec{R} = \vec{R}_0 + \vec{V}t$$

initial CM position wrt $\mathcal{O}$.

Given CM & relative coordinates $(\vec{R}, \vec{r})$, can derive relations for individual positions $(\vec{r}_1, \vec{r}_2)$,

$$\boxed{\begin{aligned} \vec{R} &= \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2) \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned}} \iff \boxed{\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{M}\vec{r} \end{aligned}}$$

Recall the Lagrangian

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - U(r)$$

Let us transform the kinetic energies to $(\vec{R}, \vec{r})$

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} \\ &= \frac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2 \\ &= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\,m_1\frac{m_2^2}{M^2}\dot{\vec{r}}^{\,2} + \frac{1}{2}\,m_2\frac{m_1^2}{M^2}\dot{\vec{r}}^{\,2} \\ &= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{M}\dot{\vec{r}}^{\,2} \end{aligned}$$

Let us define a parameter, the reduced mass μ

$$\boxed{\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}}$$

Consider limit

- $\frac{m_1}{m_2} \ll 1 \Rightarrow \mu = \frac{m_1}{1+\frac{m_1}{m_2}} \simeq m_1 - \left(\frac{m_1}{m_2}\right)m_1 + \mathcal{O}\left(\left(\frac{m_1}{m_2}\right)^2\right)$
- $m_1 = m_2 \equiv m \Rightarrow \mu = \frac{m^2}{2m} = \frac{m}{2}$

Thus, the kinetic energy is

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2$$

KE of CM   KE of relative motion

So, Lagrangian,

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)$$

$$\equiv \mathcal{L}_{cm} + \mathcal{L}_{rel}$$

depends only on $\vec{R}$ depends only on $\vec{r}$

Equations of Motion

We can generate the EOM for $\vec{R}$ & $\vec{r}$. Consider the Euler-Lagrange eqns. for $\vec{R}$,

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{cm}}{\partial \dot{R}_j} - \frac{\partial \mathcal{L}_{cm}}{\partial R_j} = 0 \quad , \; j = 1,2,3$$

Since $\mathcal{L}_{cm} = \mathcal{L}_{cm}(\dot{R}_j) = \frac{1}{2} M \sum_j \dot{R}_j^2$,

the coordinate R_j is ignorable, $\Rightarrow \frac{\partial \mathcal{L}_{cm}}{\partial R_j} = 0$

Thus the EOM are

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{cm}}{\partial \dot{R}_j} = \frac{d}{dt}\frac{\partial}{\partial \dot{R}_j}\left(\frac{1}{2} M \sum_k \dot{R}_k^2\right)$$

$$= \frac{d}{dt}\left(M \sum_k \dot{R}_k \delta_{jk}\right)$$

$$= \frac{d}{dt}\left(M \dot{R}_j\right)$$

$$= M \ddot{R}_j$$

or, $\boxed{M\ddot{\vec{R}} = \vec{0}}$

The center of mass moves as a "free particle", as we expect for isolated-closed systems.

The solution is straightforward

$$\boxed{\vec{R}(t) = \vec{R}_0 + \vec{V}\,(t - t_0)}$$

with $\vec{R}_0 = \vec{R}(t_0)$, $\vec{V} = \dot{\vec{R}}(t_0)$

The relative motion is more complicated

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{rel}}{\partial \dot{r}_j} - \frac{\partial \mathcal{L}_{rel}}{\partial r_j} = 0 \qquad , \; j = 1, 2, 3$$

The relative Lagrangian is of a particle of mass μ interacting with a potential $U(r)$.

$$\frac{\partial}{\partial r_j} \mathcal{L}_{rel} = \frac{\partial}{\partial r_j}\left(\frac{1}{2}\mu \sum_k \dot{r}_k^2 - U(r) \right)$$
$$= -\frac{\partial}{\partial r_j} U(r)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{rel}}{\partial \dot{r}_j} = \frac{d}{dt}\frac{\partial}{\partial \dot{r}_j}\left(\frac{1}{2}\mu \sum_k \dot{r}_k^2 \right)$$
$$= \frac{d}{dt}\left(\mu \sum_k \dot{r}_k \delta_{jk} \right)$$
$$= \mu \ddot{r}_j$$

So, EOM $\Rightarrow$ $\mu \ddot{r}_j = -\frac{\partial}{\partial r_j} U(r)$

or,

$$\boxed{\mu \ddot{\vec{r}} = -\vec{\nabla}_r U(r)}$$

EOM of particle of mass μ in potential $U(r)$

The Center-of-Mass frame

We can simplify our problem further by choosing a special inertial reference frame. Since $\dot{\vec{R}} =$ const., we can choose a frame called the CM frame, where the CM is at rest, $\vec{R}(t) = \vec{0}$ $\forall$ t.

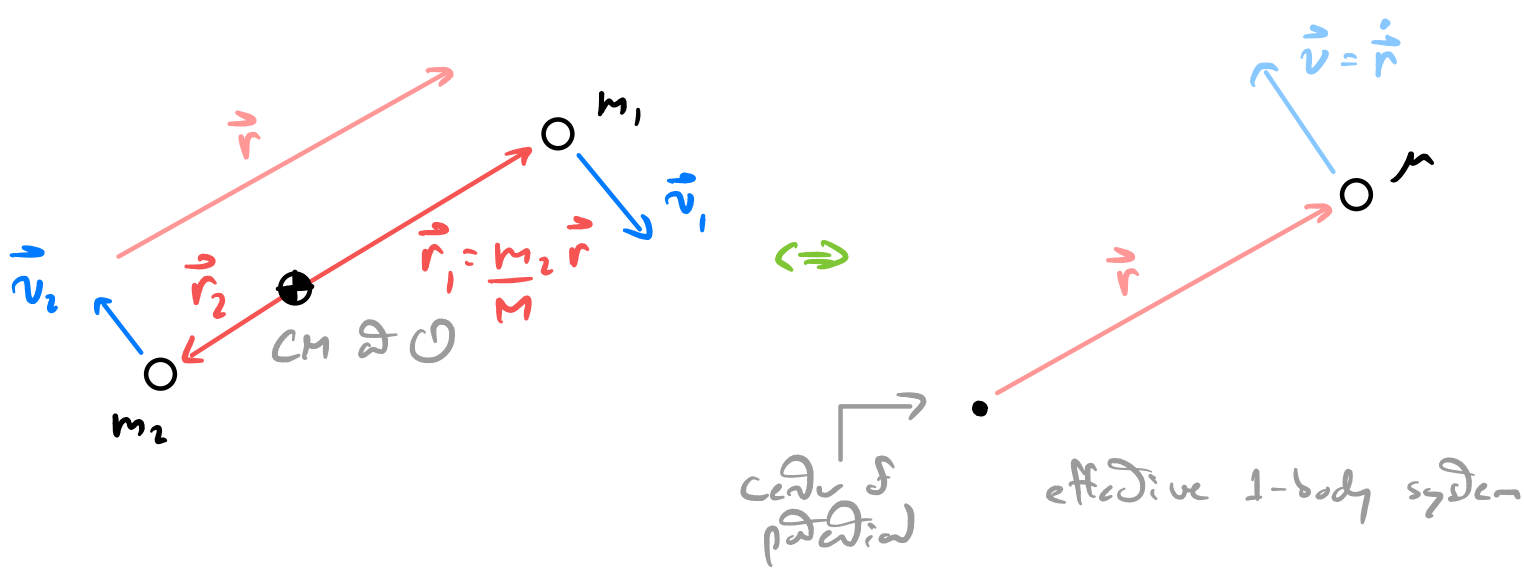
Thus, $\dot{\vec{R}} = \vec{0}$ $\Rightarrow$ $\mathcal{L}_{cm} = 0$

So, the Lagrangian is

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - U(r)$$

CM frame

This is an effective 1-body problem



We have reduced a problem in 6 variables to 3 variables in the CM frame. Using conservation of angular momentum, we can further simplify the problem. The total angular momentum $\vec{L}$ is

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$= m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

In the CM frame, $\vec{r}_1 = \frac{m_2}{M}\vec{r}$ & $\vec{r}_2 = -\frac{m_1}{M}\vec{r}$

So,
$$\vec{L} = \frac{m_1 m_2}{M^2}\left(m_2 \vec{r} \times \dot{\vec{r}} + m_1 \vec{r} \times \dot{\vec{r}} \right)$$
$$= \mu \, \vec{r} \times \dot{\vec{r}}$$

Since total angular momentum is conserved,

$$\dot{\vec{L}} = \vec{0}$$

$\Rightarrow$ $\vec{L}$ = const.

Therefore,
$$\vec{L} = \mu \vec{r} \times \dot{\vec{r}} = \text{const.}$$

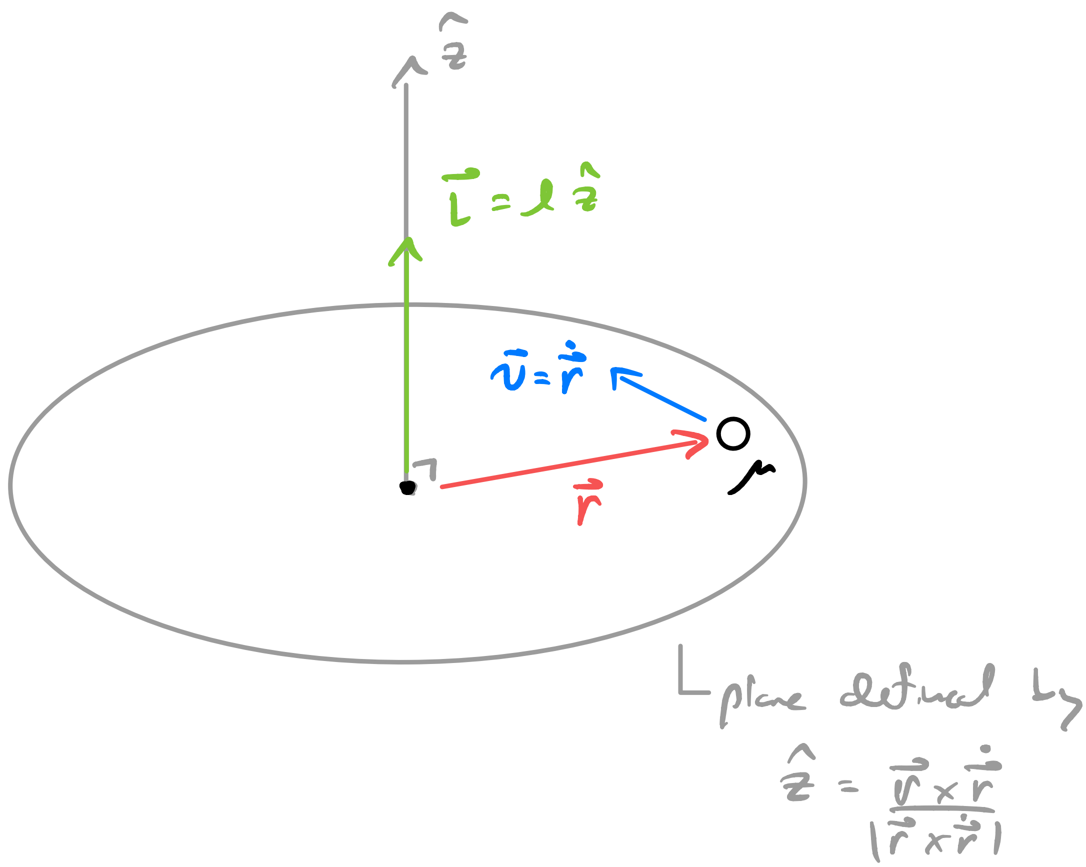
So, the direction $\vec{r} \times \dot{\vec{r}}$ = const.

Thus, we can write

$$\vec{L} \equiv \ell\,\hat{z} = \text{const.}$$

where $\hat{z} = \dfrac{\vec{r}\times\dot{\vec{r}}}{|\vec{r}\times\dot{\vec{r}}|}$ & $\ell \equiv \mu\,|\vec{r}\times\dot{\vec{r}}|$

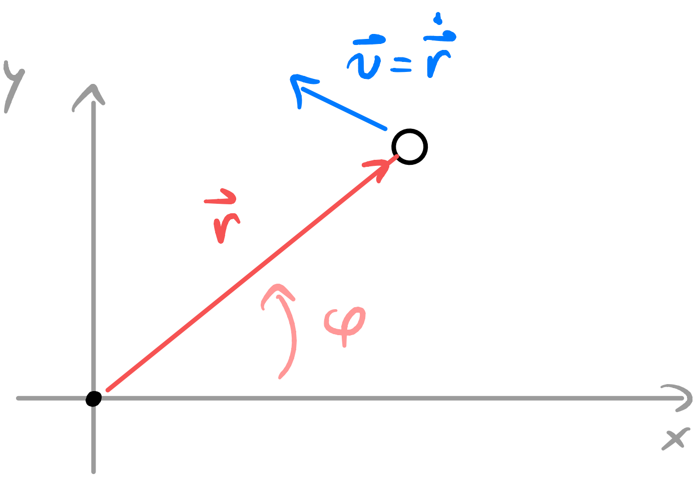
Thus, the motion of the system lies in a plane, effectively reducing 3 coordinates to 2.



Let us derive the equations of motion for the remaining 2 variables. Let us choose to work with cylindrical polar coordinates (r, φ)

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$$

$$\Rightarrow \dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2\dot{\varphi}^2$$



So,

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

Notice that $\mathcal{L}$ is independent of φ $\Rightarrow$ $\frac{\partial \mathcal{L}}{\partial \varphi} = 0$

So, $$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \boxed{\mu r^2 \dot{\varphi} = \ell = \text{const.}}$$ angular eqn.

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\ell} = 0$$

Recall: $\vec{L} = \mu\vec{r}\times\dot{\vec{r}} = \mu r^2\dot{\varphi}\,\hat{r}\times\hat{\varphi} = \mu r^2\dot{\varphi}\,\hat{z}$
$\equiv \ell\,\hat{z}$

So, the φ eqn. is simply a statement of conservation of angular momentum.

Now let's consider the radial eqn.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

So,

$$\frac{\partial L}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2}\mu r^2\dot{\varphi}^2 - U(r)\right)$$

$$= \mu r\dot{\varphi}^2 - \frac{\partial U}{\partial r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}\mu \dot{r}^2\right)$$

$$= \mu \ddot{r}$$

$$\Rightarrow \boxed{\mu\ddot{r} = \mu r\dot{\varphi}^2 - \frac{\partial U}{\partial r}}$$ radial eqn.

Given $U(r)$, we wish to solve for r.

Effective Potentials

Before specifying a potential $U(r)$, let us examine the effective one-dimensional problem. The equations of motion are

$$\boxed{\begin{aligned} \mu r^2 \dot{\varphi} &= \ell \qquad (1) \\ \& \quad \mu \ddot{r} &= \mu r \dot{\varphi}^2 - \frac{\partial U}{\partial r} \qquad (2) \end{aligned}}$$

Since $\ell = \text{const.}$, the φ equation is thus fixed from initial conditions since given

$$r_0 = r(t_0) \quad , \quad \varphi_0 = \varphi(t_0)$$
$$\dot{r}_0 = \dot{r}(t_0) \quad , \quad \dot{\varphi}_0 = \dot{\varphi}(t_0)$$

$$\Rightarrow \boxed{\ell = \mu r_0^2 \dot{\varphi}_0}$$

So, let us write (1) as

$$\dot{\varphi} = \frac{\ell}{\mu r^2} \quad \left(= \left(\frac{r_0}{r}\right)^2 \dot{\varphi}_0 \right)$$

and eliminate $\dot{\varphi}$ from (2)

$$\boxed{\mu \ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}} \qquad (3)$$

Eqn. 3 is an equivalent 1-dimensional problem, only involving the unknown r.

$$\mu \ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

central force

"fictitious" centrifugal force

Let $F_{cf} = \frac{\ell^2}{\mu r^3}$ be the centrifugal force.

We can define a centrifugal potential energy

$$F_{cf} = -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2}\right) \equiv -\frac{\partial}{\partial r} U_{cf}$$

where $\boxed{U_{cf}(r) = \frac{\ell^2}{2\mu r^2}}$

So, the radial eqn can be written as

$$\mu \ddot{r} = -\frac{\partial}{\partial r}\left(U(r) + U_{cf}(r)\right)$$

$$\equiv -\frac{\partial}{\partial r} U_{eff}$$

We have defined the effective potential

$$U_{eff}(r) = U(r) + U_{cf}(r)$$
$$= U(r) + \frac{\ell^2}{2\mu r^2}$$

It's effectively as if a single particle is moving in 1-dimension in a potential $U_{eff}(r)$.

Let's look at gravitational interactions as an example,

$$U(r) = -G\frac{m_1 m_2}{r}$$

Recall $\mu = \frac{m_1 m_2}{M}$ $\Rightarrow$ $U(r) = -G\frac{\mu M}{r}$

So,

$$U_{eff}(r) = -G\frac{\mu M}{r} + \frac{\ell^2}{2\mu r^2}$$

For $\ell \neq 0$, $U_{eff} \sim -G\frac{\mu M}{r}$ as $r \to \infty$

$$U_{eff} \sim \frac{\ell^2}{2\mu r^2} \quad \text{as } r \to 0$$

Let r_0 be the location of the minimum value of U_{eff} for $\ell \neq 0$.

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = 0$$

So,

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = +\frac{G\mu M}{r_0^2} - \frac{\ell^2}{\mu r_0^3} = 0$$

$$\Rightarrow \quad r_0 = \frac{\ell^2}{GM\mu^2}$$

$\ell^2 = GM\mu^2 r_0$

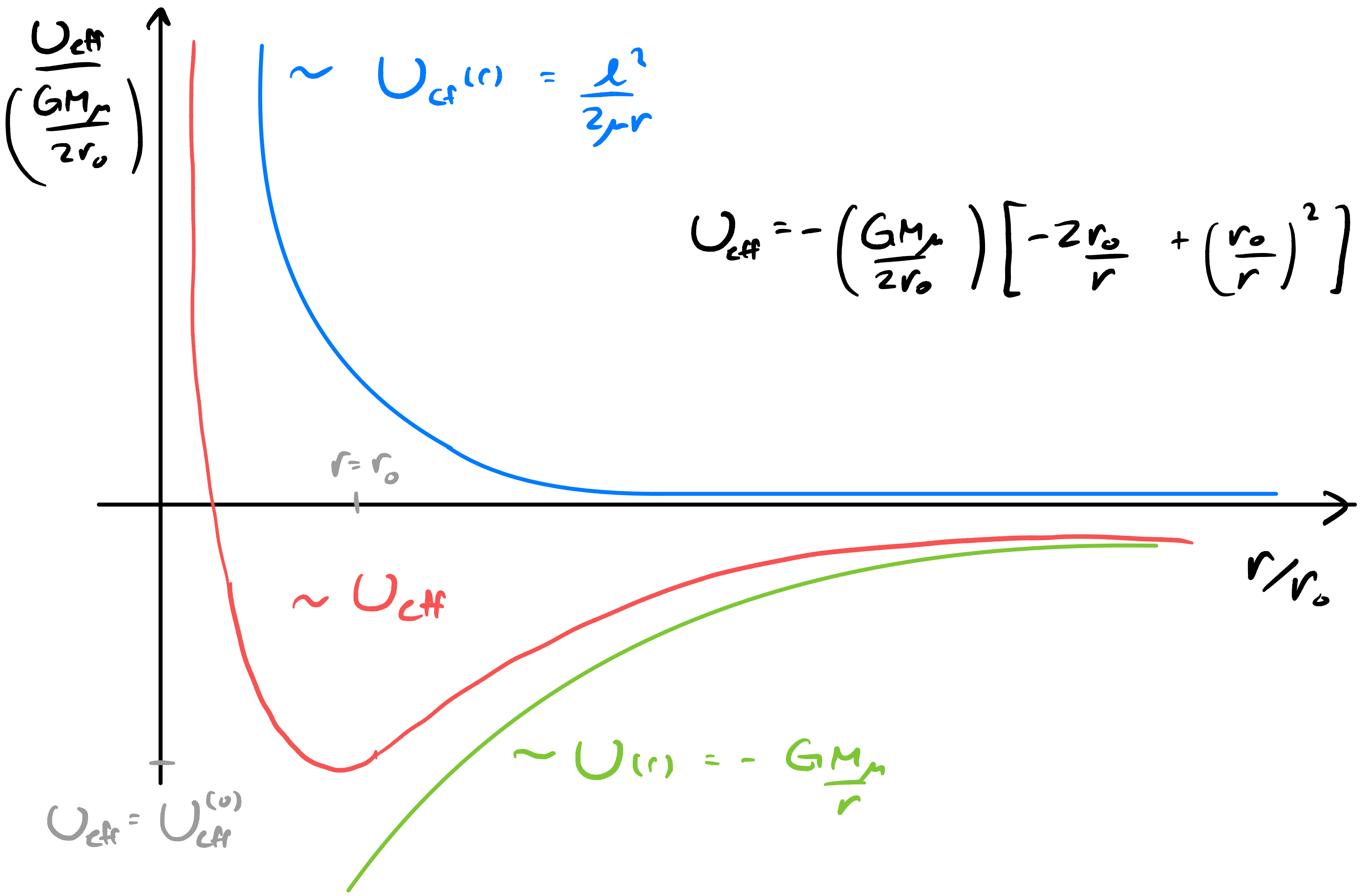
At the minimum, $U_{eff}^{(0)} = -\frac{GM\mu}{r_0} + \frac{\ell^2}{2\mu r_0^2}$

$$= -\frac{GM\mu}{r_0} + \frac{GM\mu}{2r_0} = -\frac{1}{2}\frac{GM\mu}{r_0}$$

We can then write U_{eff} as

$$U_{eff} = -\frac{GM\mu}{r} + \frac{1}{2} GM\mu \frac{r_0}{r^2}$$

$$= U_{eff}^{(0)} \left[2\frac{r_0}{r} - \frac{r_0^2}{r^2} \right]$$



Let us consider the consequences of conservation of energy. Take the EOM & multiply by $\dot r$,

$$\dot r \mu \ddot r = -\dot r \frac{\partial}{\partial r} U_{eff} \qquad \left(\dot r = \frac{dr}{dt}\right)$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}\mu \dot r^2\right) = -\frac{d}{dt} U_{eff}$$

This means that

$$\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = \text{const.}$$

$$\Rightarrow \quad \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = \text{const.}$$

But, recall $T_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2$

& $\dot{\varphi}^2 = \frac{\ell^2}{\mu^2 r^4} \quad \Rightarrow \quad T_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2}$

Therefore, $T_{rel} + U(r) = \text{const}$

This is just a statement of total energy,

$$\boxed{E = T_{rel} + U(r)} = \text{const.}$$

which is conserved, $\frac{dE}{dt} = 0$.

Let's again look at the motion of a particle of mass μ in a 1-dim effective system,

$$\boxed{\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = E}$$

Notice that $\frac{1}{2}\mu\dot{r}^2 \geq 0$ always,
thus

$$E \geq U_{eff}$$

The points such that $\dot{r} = 0$ are turning points in the reduced particles trajectory.

Now, U_{eff} can in general be positive or negative, thus we have two cases to consider: $E \geq 0$ & $E < 0$.

Lets look at $E \geq 0$ case, for an object, such as a comet, in a gravitational well, $U(r) = -G\frac{M\mu}{r}$, where $l \neq 0$.

If $E = 0$ & $E \geq U_{eff}$, we have $U_{eff} \leq 0$ or,

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r) \leq 0$$

For gravity, $U(r) = -\frac{GM\mu}{r}$, and $l \neq 0$, this gives

$$\left(\frac{l^2}{2\mu r^2} - \frac{GM\mu}{r}\right) = 0$$

$$\Rightarrow r_{max} \to \infty \quad \text{or} \quad r_{min} = \frac{l^2}{2GM\mu^2}$$

So, there is only 1 turning point, $\dot{r}=0$, at

$$r_{min} = \frac{\ell^2}{2GM\mu^2}$$

Thus, if a comet comes in from $r \to \infty$, it turns around at r_{min}, and moves back toward $r \to \infty$.

As a function of $E \geq 0$,

we can determine turning points, $\dot{r}=0$.

$$E \geq U_{eff}(r_\pm)$$

this gives $$E = \frac{\ell^2}{2\mu r^2} - \frac{GM\mu}{r}$$ (take $E = U_{eff}$ case)

$$\Rightarrow \quad r^2 + \frac{GM\mu}{E} r - \frac{\ell^2}{2\mu E} = 0$$

$$\Rightarrow \quad r_\pm = -\frac{GM\mu}{2E} \pm \frac{1}{2}\sqrt{\left(\frac{GM\mu}{E}\right)^2 + \frac{2\ell^2}{\mu E}}$$

$$= -\frac{GM\mu}{2E} \pm \frac{GM\mu}{2E}\sqrt{1 + \frac{2\ell^2 E}{G^2M^2\mu^3}}$$

Now, since $r \geq 0$, r_- is an unphysical solution for $E \geq 0$. Therefore, $E \geq U_{eff}(r_{min})$ with

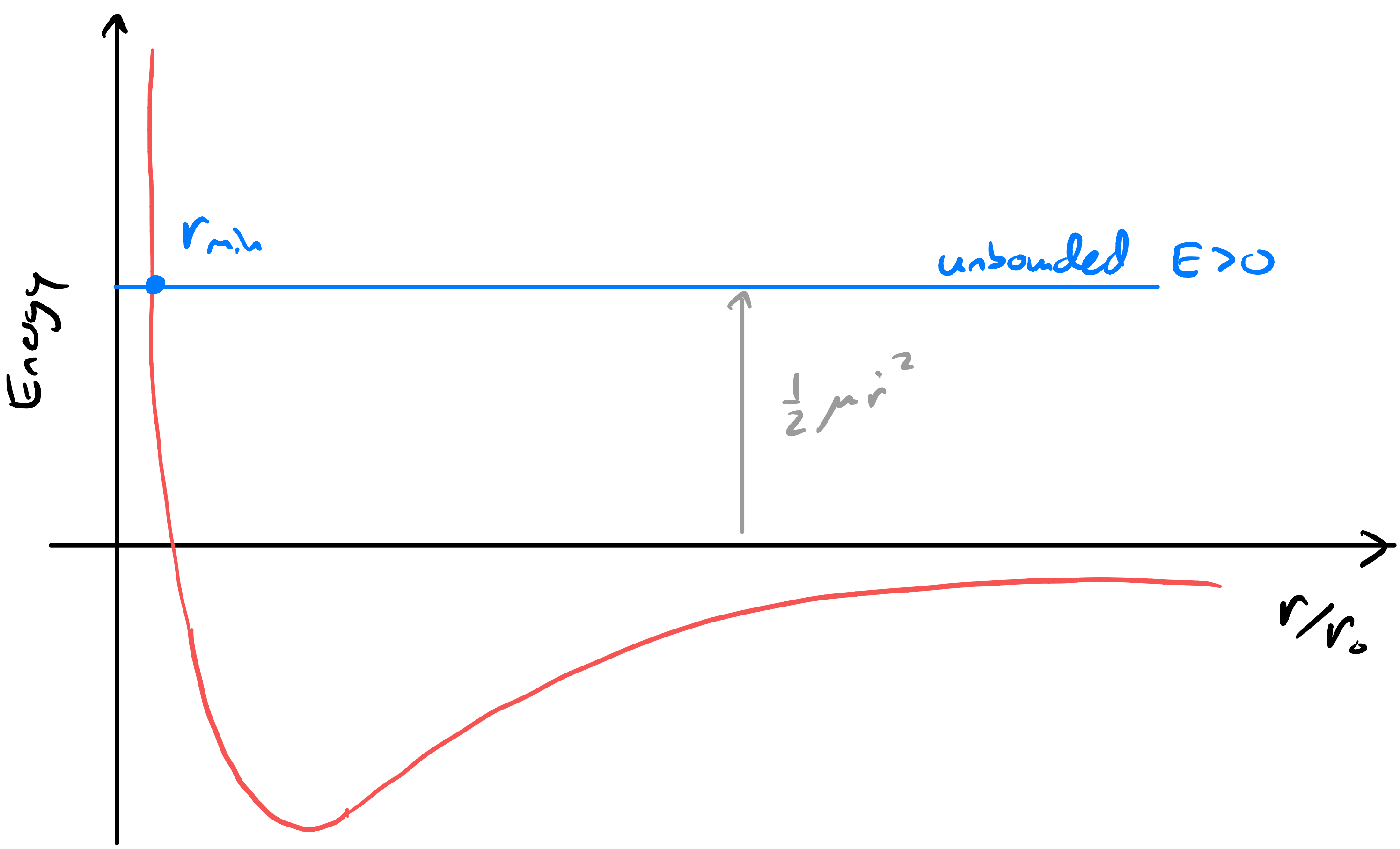
$$\boxed{r_{min} \equiv r_+ = -\frac{GM\mu}{2E} + \frac{GM\mu}{2E}\sqrt{1 + \frac{2\ell^2 E}{G^2M^2\mu^3}}}$$

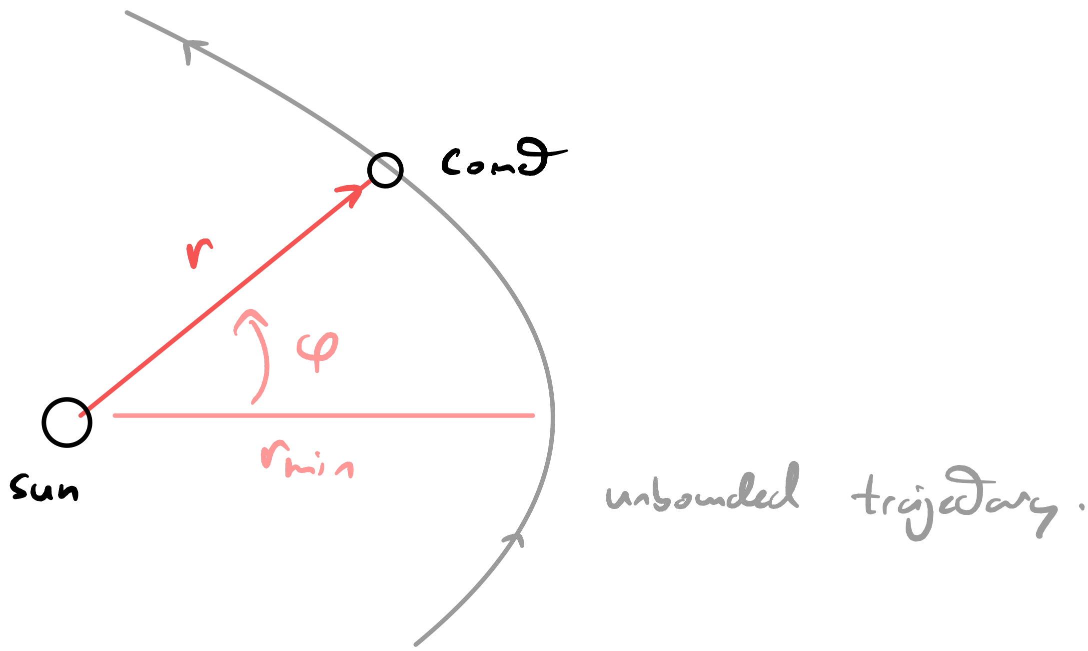
Let us expand the solution for small E, $E/\mu \ll 1$,

$$\Rightarrow r_{min} = -\frac{GM\mu}{2E} + \frac{GM\mu}{2E}\left(1 + \frac{l^2 E}{G^2 M^2 \mu^3} + \mathcal{O}\left(\frac{E}{\mu}\right)\right)$$

$$= \frac{l^2}{2GM\mu^2} + \mathcal{O}\left(\frac{E}{\mu}\right)$$

Graphically, this is shown in blue on the effective potential plot. This $E \geq 0$ scenario is an unbounded orbit.





Now, consider $E < 0$. Let $E \equiv -\mathcal{E}$, $\mathcal{E} > 0$.

The turning points are now,

$$-\mathcal{E} \geq U_{eff}(r_{\substack{min\\max}})$$

or, $\mathcal{E} \leq -U_{eff}(r_{\substack{min\\max}})$.

Solving for the turning points, for $\ell \neq 0$ & gravity

$$-\frac{\ell^2}{2\mu r^2} + \frac{GM\mu}{r} \geq \mathcal{E}$$

or, $$r^2 - \frac{GM\mu}{\mathcal{E}} r + \frac{\ell^2}{2\mu \mathcal{E}} = 0$$

which has solutions

$$\boxed{r_\pm = \frac{GM\mu}{2\mathcal{E}} \pm \frac{GM\mu}{2\mathcal{E}}\sqrt{1 - \frac{2\ell^2\mathcal{E}}{G^2M^2\mu^3}}}$$

To get a sense of the solution, let $\mathcal{E}/\mu \ll 1$,

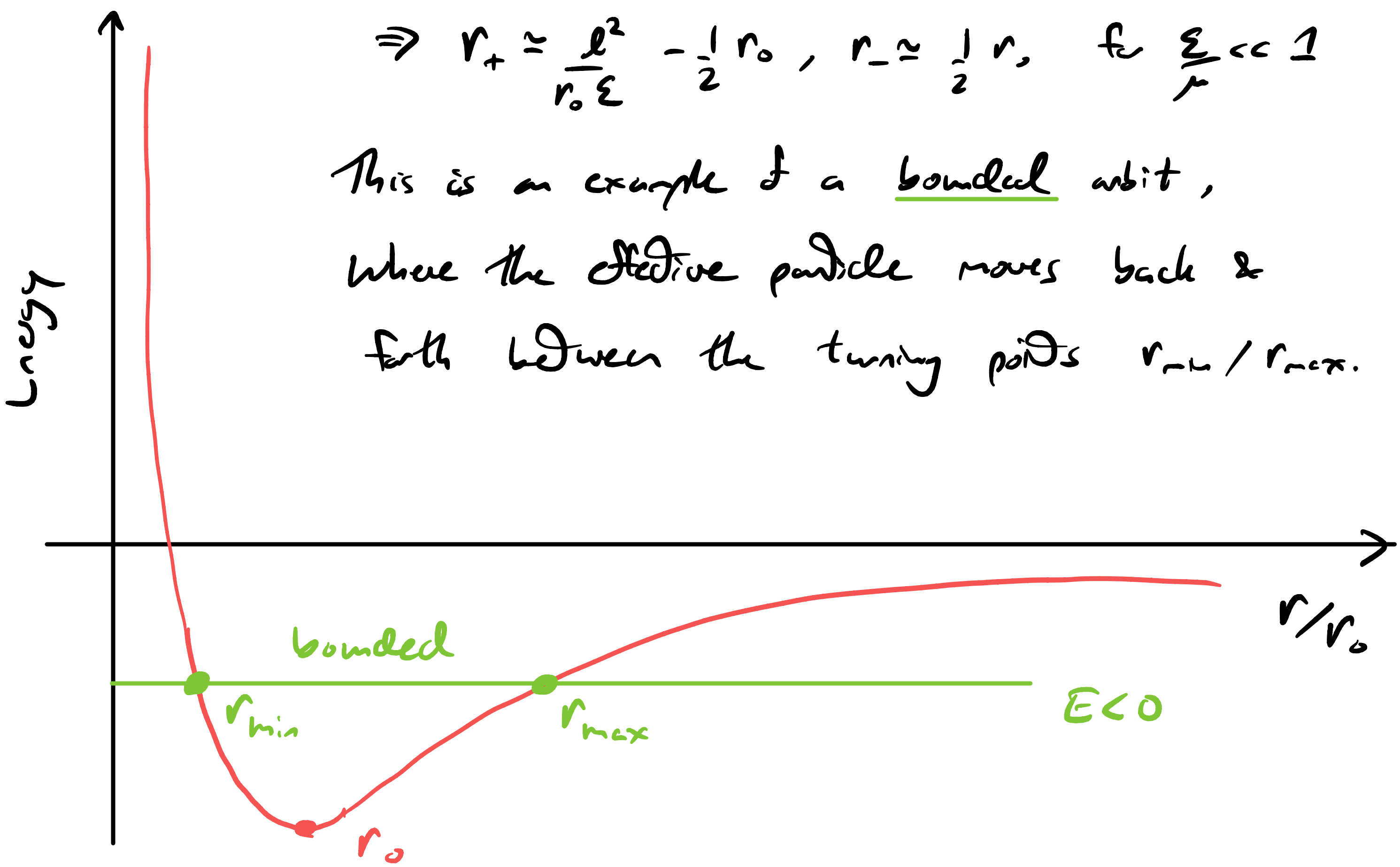
$$\Rightarrow \quad r_\pm = \frac{GM\mu}{2\mathcal{E}} \pm \frac{GM\mu}{2\mathcal{E}}\left(1 - \frac{\ell^2\mathcal{E}}{G^2M^2\mu^3} + \mathcal{O}\left(\frac{\mathcal{E}}{\mu}\right)\right)$$

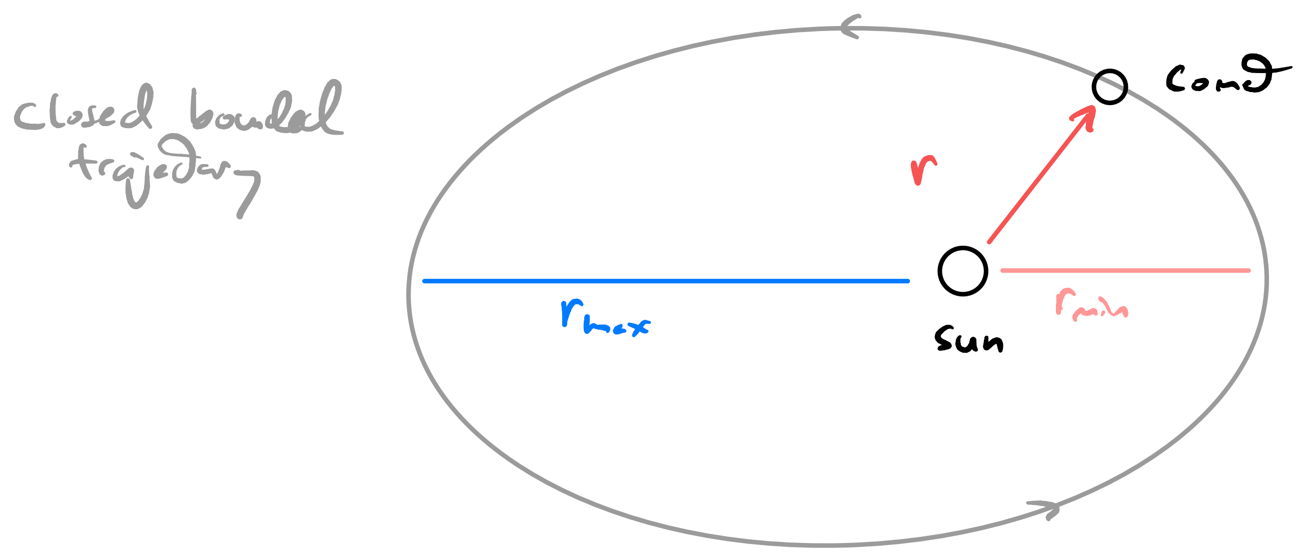
$$= \begin{cases} \dfrac{GM\mu}{\mathcal{E}} - \dfrac{\ell^2}{2GM\mu^2} + \mathcal{O}\left(\dfrac{\mathcal{E}}{\mu}\right) \\[2ex] +\dfrac{\ell^2}{2GM\mu^2} + \mathcal{O}\left(\dfrac{\mathcal{E}}{\mu}\right) \end{cases}$$

So, $r_{min} \equiv r_-$ & $r_{max} \equiv r_+$. Recall that the minimum of the effective potential is at $r_0 = \dfrac{\ell^2}{GM\mu^2}$,

$$\Rightarrow r_+ \simeq \frac{\ell^2}{r_0\mathcal{E}} - \frac{1}{2}r_0, \quad r_- \simeq \frac{1}{2}r_0 \quad \text{for } \frac{\mathcal{E}}{\mu} \ll 1$$

This is an example of a bounded orbit, where the effective particle moves back & forth between the turning points r_{min}/r_{max}.





The arguments we made work for general central potentials, but inverse-square-laws like gravitation result in closed bounded orbits. One can show that most other force laws have open bounded orbits, that is they precess.