1. Given the generators X^j for a Lie algebra $[X^j, X^k] = c_{jkl}X^l$, normalized such that $\mathrm{tr}(X^j X^k) = \mu_r \delta_{jk}$, show that the structure constants can be computed with

$$c_{jkl} = \frac{1}{\mu_r}\,\mathrm{tr}([X^j, X^k]X^l)\,.$$

Show that c_{jkl} are antisymmetric under interchange of any two indices.

2. Compute the non-zero structure constants f_{abc} for the $\mathfrak{su}(3)$ algebra $[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c$, where λ_a are the Gell-Mann matrices. **Hint:** It is convenient to use a symbolic algebra software like `Mathematica`.

3. The Gell-Mann matrices also satisfy the relation

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab}I_3 + 2d_{abc}\lambda_c\,,$$

where d_{abc} are symmetric under the interchange of any two indices. Compute the non-zero values of d_{abc}. **Hint:** It is convenient to use a symbolic algebra software like `Mathematica`.

4. Show that the $\mathbf{3}^*$ of $\mathfrak{su}(3)$ is inequivalent to the $\mathbf{3}$ of $\mathfrak{su}(3)$. **Hint:** Show that $(-\lambda_a^*)$ cannot be transformed to λ_a by a unitary transformation for every $a = 1, 2, \ldots, 8$.

5. Perform the Clebsch-Gordan decomposition for the following $\mathfrak{su}(3)$ products using Young Tableau, labeling the dimension of each representation: (a) $\mathbf{6} \times \mathbf{6}$, (b) $\mathbf{3} \times \mathbf{3} \times \mathbf{8}$, and (c) $\mathbf{3} \times \mathbf{3}^* \times \mathbf{8}$.

6. Using the current *Review of Particle Physics* particle listings or the summary tables (Particle Data Group, https://pdg.lbl.gov), complete Table 1 for some typical light and strange *mesons*. For hadrons without an explicit charge index, label all possible charges in the multiplet.

7. Using the current *Review of Particle Physics* particle listings or the summary tables (Particle Data Group, https://pdg.lbl.gov), complete Table 2 for some typical light and strange *baryons*. Note that for some listings, the decay width is reported as $\Gamma = -2\,\mathrm{Im}\,(\text{pole position})$. For hadrons without an explicit charge index, label all possible charges in the multiplet.

8. Given the plot of the πN total cross-sections shown in Fig. 1, identify potential resonances and estimate their mass and decay widths, as well as their charge, strange, and baryon quantum numbers. Further, identity their potential spin and isospin quantum numbers. Referring to the *Review of Particle Physics*, can you identify candidates for these unstable states?

9. Classify the following observed reactions into strong, electromagnetic, and weak processes:
   - (a) $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$,
   - (b) $\gamma + p \to \pi^+ + n$,
   - (c) $p + \bar{p} \to \pi^+ + \pi^- + \pi^0$,
   - (d) $D^- \to K^+ + 2\pi^-$,
   - (e) $\Lambda^0 + p \to K^- + 2p$,
   - (f) $\pi^- + p \to n + e^+ + e^-$.

10. Both the ρ^0 meson and the ω meson are vector mesons, $J^{PC} = 1^{--}$. However, the ρ^0 is observed to strongly decay predominately into 2π, while the ω is observed to decay into 3π. Why this is so?

11. Consider πN scattering at the $\Delta(1232)$ resonance, i.e., at center-of-momentum energies $\sqrt{s} \sim 1232\,\mathrm{MeV}$. For this reaction, $\pi N \to \Delta(1232) \to \pi N$, focus on the following three processes:

(**a**) $\pi^+ p \to \pi^+ p$ *elastic scattering* via the Δ^{++} resonance,

(**b**) $\pi^- p \to \pi^- p$ *elastic scattering* via the Δ^0 resonance,

(**c**) $\pi^- p \to \pi^0 n$ *charge exchange* via the Δ^0 resonance.

Estimate the relative cross sections $\sigma_a : \sigma_b : \sigma_c$.

12. Consider a $q\bar{q}$ meson within an exact flavor SU(3) quark model, i.e., $q = u, d, s$. Assume the meson is flavor neutral. A generic wave function for this meson is given by

$$|n\,{}^{2S+1}L_J, m_J\rangle_{q\bar{q}} = \sum_{m_L, m_S} \langle L m_L; S m_S | J m_J \rangle \sum_{s,\bar{s}} \langle \frac{1}{2} s; \frac{1}{2} \bar{s} | S m_S \rangle$$
$$\times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \varphi_{n,L}(p)\, Y_{L m_L}(\hat{\mathbf{p}})\, |q_s(\mathbf{p}) \bar{q}_{\bar{s}}(-\mathbf{p})\rangle \ ,$$

where n is the radial quantum number, S is the total intrinsic spin, L is the orbital angular momentum, J is the total angular momentum, m_J is the total angular momentum projection on some fixed z-axis, m_L is the orbital angular momentum projection, m_S is the total intrinsic spin projection, $\varphi_{n,L}$ is the momentum-space radial wave function, and $Y_{L m_L}$ are spherical harmonics. The quarks are spin-1/2 fermions with spin s and $\bar{s}$ for the q and $\bar{q}$, respectively. The two-quark state is defined in the center-of-momentum frame as the usual direct product $|q_s(\mathbf{p}) \bar{q}_{\bar{s}}(-\mathbf{p})\rangle \equiv |q_s(\mathbf{p})\rangle \otimes |\bar{q}_{\bar{s}}(-\mathbf{p})\rangle$.

(**a**) Determine the allowed values of S.

(**b**) Show that under parity $\mathcal{P}$, the $q\bar{q}$ meson has an eigenvalue

$$\mathcal{P}\,|n\,{}^{2S+1}L_J, m_J\rangle_{q\bar{q}} = (-1)^{L+1}\,|n\,{}^{2S+1}L_J, m_J\rangle_{q\bar{q}} \ .$$

Hint: Recall that $\mathcal{P}\,|q_s(\mathbf{p})\rangle = \eta_q\,|q_s(-\mathbf{p})\rangle$ and $\eta_{\bar{q}} \equiv -\eta_q$.

(**c**) Show that under charge conjugation $\mathcal{C}$, the $q\bar{q}$ meson has an eigenvalue

$$\mathcal{C}\,|n\,{}^{2S+1}L_J, m_J\rangle_{q\bar{q}} = (-1)^{L+S}\,|n\,{}^{2S+1}L_J, m_J\rangle_{q\bar{q}} \ .$$

Hint: Recall that $\mathcal{C}\,|q_s(\mathbf{p})\rangle = |\bar{q}_s(\mathbf{p})\rangle$, and under interchange $\mathrm{P}_{12}\,|q_1 q_2\rangle = -\,|q_2 q_1\rangle$.

(**d**) Determine *all* allowed J^{PC} quantum numbers for of the meson for $L \leq 3$. List all J^{PC} that are forbidden for $J \leq 3$ (observed mesons with these quantum numbers are called *exotic*, as they are not allowed in the quark model).

(**e**) List *one* example of an observed unflavored meson for each J^{PC} supermultiplet by searching the Particle Data Group database (https://pdglive.lbl.gov) for *light unflavored mesons*. Are there any examples of observed mesons with exotic quantum numbers?

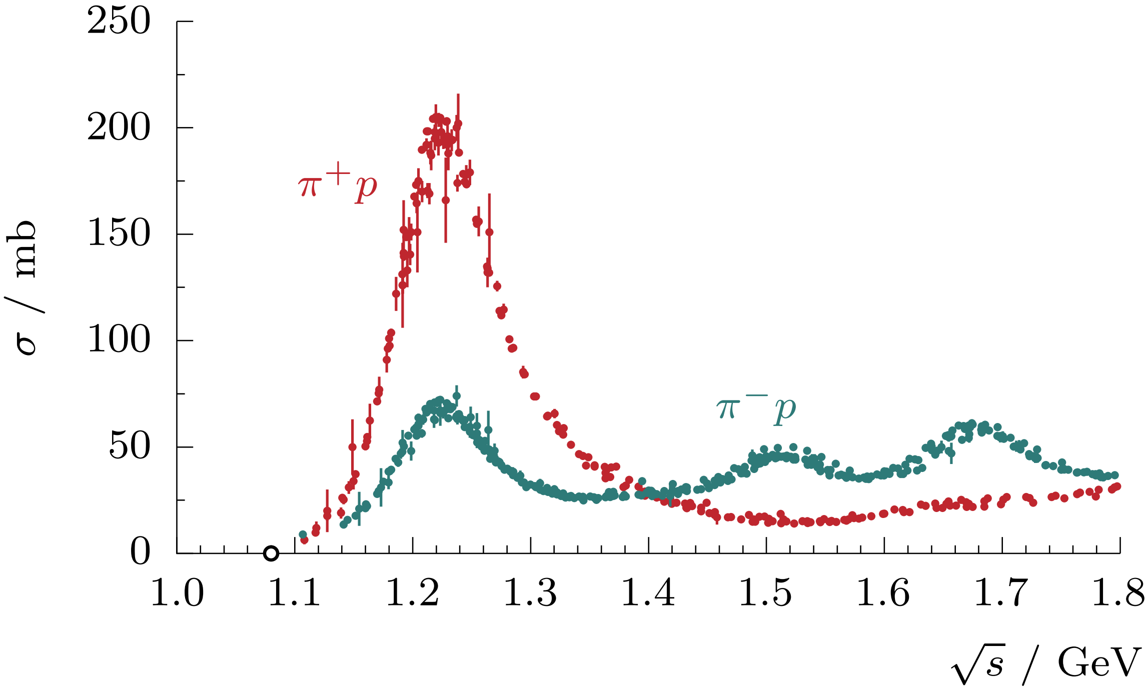

Figure 1: Total πN cross-sections as a function of center-of-momentum frame energy $\sqrt{s}$. Data taken from the *Review of Particle Physics* by the Particle Data Group.

Table 1: Light and Strange Mesons.

Meson	**Quark Content**	$J^{P(C)}$	$I^{(G)}$	**Charge**	**Mass** / MeV	**Lifetime** / s	**Principle Decay Modes**
$\pi^{\pm}$	$u\bar{d}, d\bar{u}$	0^-	1^-	± 1	139.57	2.60×10^{-8}	$\mu^+\bar{\nu}$
π^0	$u\bar{u} - d\bar{d}$	0^{-+}	1^-	0	134.98		
$K^{\pm}$							
$K^0, \bar{K}^0$							
K_S							
K_L							
η							
η'							
$\rho(770)$							
$\omega(782)$							
$K^*(892)$							
$f_0(500)$							
$f_0(1370)$							
$a_0(980)$							
$a_1(1260)$							
$a_2(1320)$							
$\pi_1(1600)$							

Table 2: Light and Strange Baryons.

Baryon	**Quark Content**	J^P	I	**Charge**	**Mass** / MeV	**Lifetime** / s	**Principle Decay Modes**
p	uud	$1/2^+$	$1/2$	$+1$	938.27	stable	—
n							
Λ^0							
$\Sigma^\pm$							
Σ^0							
Ξ^-							
Ξ^0							
$\Delta^{++}(1231)$							
$\Delta^{\pm}(1231)$							
$\Delta^0(1231)$							
$\Sigma(1385)$							
$\Xi(1530)$							
Ω^-							
$N(1440)$							
$\Lambda(1405)$							