WILLIAM & MARY
CHARTERED 1693

PHYS 772 – The Standard Model of Particle Physics

Problem Set 6 – Solution

Due: Tuesday, March 25 at 4:00pm

Term: Spring 2025

Instructor: Andrew W. Jackura

1. Given the generators X^j for a Lie algebra $[X^j, X^k] = c_{jkl}X^l$, normalized such that $\mathrm{tr}(X^jX^k) = \mu_r\delta_{jk}$, show that the structure constants can be computed with

$$c_{jkl} = \frac{1}{\mu_r}\,\mathrm{tr}([X^j, X^k]X^l)\,.$$

Show that c_{jkl} are antisymmetric under interchange of any two indices.

Solution: Multiply the Lie bracket on the right by X^n, $[X^j, X^k]X^n = c_{jkl}X^lX^n$. Next, take the trace

$$\mathrm{tr}\left([X^j, X^k]X^n\right) = c_{jkl}\,\mathrm{tr}\left(X^lX^n\right) = c_{jkl}\,\mu_r\delta_{ln} = \mu_r c_{jkn}\,.$$

Isolating c_{jkn}, we find the desired relation,

$$c_{jkl} = \frac{1}{\mu_r}\,\mathrm{tr}([X^j, X^k]X^l)\,.$$

Note that from the cyclic properties of the trace, we find

$$\begin{aligned}
\mathrm{tr}([X^j, X^k]X^l) &= \mathrm{tr}(X^jX^kX^l - X^kX^jX^l)\,,\\
&= \mathrm{tr}(X^jX^kX^l - X^jX^lX^k)\,,\\
&= \mathrm{tr}(X^j[X^k, X^l]) = \mathrm{tr}([X^k, X^l]X^j)\,,
\end{aligned}$$

where the cyclic property was used on the second term of the second line, and

$$\begin{aligned}
\mathrm{tr}([X^j, X^k]X^l) &= \mathrm{tr}(X^jX^kX^l - X^kX^jX^l)\,,\\
&= \mathrm{tr}(X^kX^lX^j - X^kX^jX^l)\,,\\
&= \mathrm{tr}(X^k[X^l, X^j]) = \mathrm{tr}([X^l, X^j]X^k)\,,
\end{aligned}$$

where again the cyclic property was used on the first term of the second line. Thus, the structure constants are given by

$$c_{jkl} = \frac{1}{\mu_r}\,\mathrm{tr}([X^j, X^k]X^l) = \frac{1}{\mu_r}\,\mathrm{tr}([X^k, X^l]X^j) = \frac{1}{\mu_r}\,\mathrm{tr}([X^l, X^j]X^k)\,.$$

Since the Lie bracket is antisymmetric, $[X^j, X^k] = -[X^k, X^j]$, we see that c_{jkl} is antisymmetric under the interchange of any pair of indices (j, k), (k, l), and (l, j). Thus,

$$c_{jkl} = -c_{jlk}\,, \quad c_{jkl} = -c_{kjl}\,, \quad c_{jkl} = -c_{lkj}\,.$$

2. Compute the non-zero structure constants f_{abc} for the $\mathfrak{su}(3)$ algebra $[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c$, where λ_a are the Gell-Mann matrices. **Hint:** It is convenient to use a symbolic algebra software like `Mathematica`.

Solution: The normalization of the Gell-Mann matrices are $\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. So, from Problem 1, we have

$$f_{abc} = \frac{1}{4i}\,\mathrm{tr}\left([\lambda_a, \lambda_b]\lambda_c\right)\ .$$

Using `Mathematica`, we can write f_{abc} for each $a = 1, \ldots, 8$ as a matrix in bc,

$$f_{1bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{2bc} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_{3bc} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{4bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \end{pmatrix},$$

$$f_{5bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, f_{6bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix},$$

$$f_{7bc} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}, \quad f_{8bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Therefore, the following entries are non-zero:

$$
\begin{aligned}
&f_{123} = 1\,, \\
&f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = 1/2\,, \\
&f_{458} = f_{678} = \frac{\sqrt{3}}{2}\,,
\end{aligned}
$$

where other non-zero entries are given by the interchange of any pair of indices.

3. The Gell-Mann matrices also satisfy the relation

$$
\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} I_3 + 2 d_{abc}\lambda_c\,,
$$

where d_{abc} are symmetric under the interchange of any two indices. Compute the non-zero values of d_{abc}. **Hint:** It is convenient to use a symbolic algebra software like `Mathematica`.

Solution: To isolate d_{abc}, multiply the anticommutator on the right by λ_e and take the trace,

$$
\begin{aligned}
\operatorname{tr}\left(\{\lambda_a, \lambda_b\}\lambda_e\right) &= \frac{4}{3}\delta_{ab}\operatorname{tr}(\lambda_e) + 2 d_{abc}\operatorname{tr}\left(\lambda_c\lambda_e\right)\,, \\
&= 4 d_{abe}\,,
\end{aligned}
$$

where we used that $\operatorname{tr}(\lambda_a) = 0$ and $\operatorname{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. Thus,

$$
d_{abc} = \frac{1}{4}\operatorname{tr}\left(\{\lambda_a, \lambda_b\}\lambda_c\right)\,.
$$

Using `Mathematica`, we can write d_{abc} for each $a = 1, \ldots, 8$ as a matrix in bc,

$$d_{1bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_{2bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d_{3bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_{4bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d_{5bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{pmatrix}, \quad d_{6bc} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 \end{pmatrix},$$

$$d_{7bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 \end{pmatrix},$$

$$d_{8bc} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

Therefore, the non-zero values of d_{abc} are

$$d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}\,,$$

$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}\,,$$

$$d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}\,,$$

where other non-zero entries are given by the interchange of any pair of indices.

4. Show that the $\mathbf{3}^*$ of $\mathfrak{su}(3)$ is inequivalent to the $\mathbf{3}$ of $\mathfrak{su}(3)$. **Hint:** Show that $(-\lambda_a^*)$ cannot be transformed to λ_a by a unitary transformation for every $a = 1, 2, \ldots, 8$.

Solution: Recall that a unitary transformation preserves the spectrum of a matrix. Suppose there exists a unitary matrix U such that $U^{-1}\lambda_a U = \Lambda_a$, where $\Lambda_a = \mathrm{diag}(\lambda_a^{(1)}, \lambda_a^{(2)}, \lambda_a^{(3)})$, where $\lambda_a^{(j)}$ with $j = 1, 2, 3$ are the eigenvalues of λ_a. If there exists another unitary matrix V such that $V^{-1}(-\lambda_a^*)V = \lambda_a$, then the spectrum of $(-\lambda_a^*)$ must be identical to λ_a for each $a = 1, \ldots, 8$. Consider λ_8,

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,,$$

which since it is diagonal the eigenvalues are $\{1/\sqrt{3}, 1/\sqrt{3}, -2/\sqrt{3}\}$. Now, consider $(-\lambda_8^*)$,

$$-\lambda_8^* = -\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,,$$

from which we see the eigenvalues are $\{-1/\sqrt{3}, -1/\sqrt{3}, 2/\sqrt{3}\}$. We see that the eigenvalues of $(-\lambda_8^*)$ are not the same as λ_8. Thus, there is no such unitary transformation V, and we conclude that the $\mathbf{3}^*$ is inequivalent to the $\mathbf{3}$ of $\mathfrak{su}(3)$.

5. Perform the Clebsch-Gordan decomposition for the following $\mathfrak{su}(3)$ products using Young Tableau, labeling the dimension of each representation: (a) $\mathbf{3} \times \mathbf{3} \times \mathbf{8}$, and (b) $\mathbf{3} \times \mathbf{3}^* \times \mathbf{8}$.

Solution: For (a), we first, let us look at $\mathbf{3} \times \mathbf{8}$,

$$
\begin{aligned}
\mathbf{3} \times \mathbf{8} &= \ydiagram{1} \times \ydiagram{2,1}\,, \\
&= \ydiagram{2,2} + \ydiagram{3,1} + \ydiagram{2,1,1}\,, \\
&= \ydiagram{2,2} + \ydiagram{3,1} + \ydiagram{1}\,, \\
&= \mathbf{6}^* + \mathbf{15} + \mathbf{3}\,,
\end{aligned}
$$

where for the last diagram we used that

$$
\ydiagram{1,1,1} = \bullet = \mathbf{1}\,,
$$

and found the dimension of the tableaux by using the dimension formula

$$
N(a_1, a_2) = \frac{1}{2}(a_1 + 1)(a_2 + 1)(a_1 + a_2 + 2)\,,
$$

where

$$
\begin{aligned}
a_1 &= \text{the number of boxes the first row exceeds the second row}\,, \\
a_2 &= \text{the number of boxes in the second row}\,.
\end{aligned}
$$

Using this result, we can now take the product $\mathbf{3}\times\mathbf{3}\times\mathbf{8}$,

$$
\begin{aligned}
\mathbf{3}\times\mathbf{3}\times\mathbf{8} &= \ydiagram{1}\times\ydiagram{1}\times\ydiagram{2,1}\,,\\
&= \ydiagram{1}\times\left(\ydiagram{1}\times\ydiagram{2,1}\right),\\
&= \ydiagram{1}\times\left(\ydiagram{2,2}+\ydiagram{3,1}+\ydiagram{1}\right),\\
&= \left(\ydiagram{1}\times\ydiagram{2,2}\right)+\left(\ydiagram{1}\times\ydiagram{3,1}\right)+\left(\ydiagram{1}\times\ydiagram{1}\right),\\
&= \left(\ydiagram{2,2,1}+\ydiagram{3,2}\right)+\left(\ydiagram{3,1,1}+\ydiagram{3,2}+\ydiagram{4,1}\right)\\
&\qquad+\left(\ydiagram{1,1}+\ydiagram{2}\right).
\end{aligned}
$$

We now use the fact that

$$\ydiagram{1,1,1} = \bullet = \mathbf{1}\,,$$

so that

$$
\begin{aligned}
\mathbf{3}\times\mathbf{3}\times\mathbf{8} &= \left(\ydiagram{1,1}+\ydiagram{3,2}\right)+\left(\ydiagram{2}+\ydiagram{3,2}+\ydiagram{4,1}\right)\\
&\qquad+\left(\ydiagram{1,1}+\ydiagram{2}\right),\\
&= (\mathbf{3}^*+\mathbf{15}^*)+(\mathbf{6}+\mathbf{15}^*+\mathbf{24})+(\mathbf{3}^*+\mathbf{6})\,.
\end{aligned}
$$

So, the Clebsch-Gordan decomposition is $\mathbf{3}\times\mathbf{3}\times\mathbf{8}=\mathbf{3}^*+\mathbf{3}^*+\mathbf{6}+\mathbf{6}+\mathbf{15}^*+\mathbf{15}^*+\mathbf{24}$.

For (b), we first, let us look at $\mathbf{3}^* \times \mathbf{8}$,

$$
\begin{aligned}
\mathbf{3}^* \times \mathbf{8} &= \ydiagram{1,1} \times \ydiagram{2,1}\,, \\
&= \ydiagram{1,1} + \ydiagram{2} + \ydiagram{3,2}\,, \\
&= \mathbf{3}^* + \mathbf{6} + \mathbf{15}^*\,,
\end{aligned}
$$

where we used the Language and shape rules to eliminate invalid diagrams. Taking the product $\mathbf{3} \times \mathbf{3}^* \times \mathbf{8}$, we have

$$
\begin{aligned}
\mathbf{3} \times \mathbf{3}^* \times \mathbf{8} &= \ydiagram{1} \times \ydiagram{1,1} \times \ydiagram{2,1}\,, \\
&= \ydiagram{1} \times \left(\ydiagram{1,1} \times \ydiagram{2,1} \right)\,, \\
&= \ydiagram{1} \times \left(\ydiagram{1,1} + \ydiagram{2} + \ydiagram{3,2} \right)\,, \\
&= \left(\ydiagram{1} \times \ydiagram{1,1} \right) + \left(\ydiagram{1} \times \ydiagram{2} \right) + \left(\ydiagram{1} \times \ydiagram{3,2} \right)\,, \\
&= \left(\ydiagram{1,1,1} + \ydiagram{2,1} \right) + \left(\ydiagram{2,1} + \ydiagram{3} \right) \\
&\quad + \left(\ydiagram{3,2,1} + \ydiagram{3,3} + \ydiagram{4,2} \right)\,, \\
&= \left(\bullet + \ydiagram{2,1} \right) + \left(\ydiagram{2,1} + \ydiagram{3} \right) \\
&\quad + \left(\ydiagram{2,1} + \ydiagram{3,3} + \ydiagram{4,2} \right)\,, \\
&= (\mathbf{1} + \mathbf{8}) + (\mathbf{8} + \mathbf{10}) + (\mathbf{8} + \mathbf{10}^* + \mathbf{27})\,.
\end{aligned}
$$

So, the Clebsch-Gordan decomposition is $\mathbf{3} \times \mathbf{3}^* \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{8} + \mathbf{10} + \mathbf{10}^* + \mathbf{27}$.

6. Using the current *Review of Particle Physics* particle listings or the summary tables (Particle Data Group, https://pdg.lbl.gov), complete Table 1 for some typical light and strange *mesons*. For hadrons without an explicit charge index, label all possible charges in the multiplet. For hadrons with multiple decay modes, we list principle ones as those with branching ratios greater than 1%.

> **Solution:** See table below. For hadrons with widths reported in the Review of Particle Physics, we use $\tau = \hbar/\Gamma$ to estimate lifetimes, with $\hbar \approx 5.58 \times 10^{-22}\,\mathrm{MeV} \cdot s$. For multiplet states, we average the widths between all charge states. For hadrons with multiple decay modes, we list principle ones as those with branching ratios greater than 1%. For the neutral kaons $K^0, \bar{K}^0$ are mass eigenstates, and flavor oscillations mean that these hadrons decay via K_S, K_L, which are not eigenstates.

7. Using the current *Review of Particle Physics* particle listings or the summary tables (Particle Data Group, https://pdg.lbl.gov), complete Table 2 for some typical light and strange *baryons*. Note that for some listings, the decay width is reported as $\Gamma = -2\,\mathrm{Im}\,(\text{pole position})$. For hadrons without an explicit charge index, label all possible charges in the multiplet.

> **Solution:** See table below. For hadrons with widths reported in the Review of Particle Physics, we use $\tau = \hbar/\Gamma$ to estimate lifetimes. For multiplet states, we average the widths between all charge states.

Table 1: Light and Strange Mesons.

Meson	**Quark Content**	$J^{P(C)}$	$I^{(G)}$	**Charge**	**Mass / MeV**	**Lifetime / s**	**Principle Decay Modes**
$\pi^\pm$	$u\bar{d}, d\bar{u}$	0^-	1^-	±1	139.57	2.60×10^{-8}	$\mu^+\bar{\nu}$
π^0	$u\bar{u} - d\bar{d}$	0^{-+}	1^-	0	134.98	8.42×10^{-17}	$\gamma\gamma$
$K^\pm$	$u\bar{s},\ \bar{u}s$	0^-	$1/2$	±1	493.68	1.24×10^{-8}	$\mu^+\nu_\mu,\ \pi^+\pi^0$
$K^0, \bar{K}^0$	$d\bar{s},\ \bar{d}s$	0^-	$1/2$	0	497.61	—	—
K_S	$d\bar{s},\ \bar{d}s$	0^-	$1/2$	0	—	8.95×10^{-11}	$\pi^0\pi^0,\ \pi^+\pi^-$
K_L	$d\bar{s},\ \bar{d}s$	0^-	$1/2$	0	—	5.11×10^{-8}	$3\pi^0,\ \pi^+\pi^-\pi^0,$ $\pi^\pm e^\mp\nu_e,\ \pi^\pm\mu^\mp\nu_\mu$
η	$u\bar{u},\ d\bar{d},\ s\bar{s}$	0^{-+}	0^+	0	547.86	4.26×10^{-19}	$\pi^+\pi^-\pi^0,\ 3\pi^0,\ 2\gamma$
η'	$u\bar{u},\ d\bar{d},\ s\bar{s}$	0^{-+}	0^+	0	957.78	2.97×10^{-21}	$\pi^+\pi^-\eta,\ \rho^0\gamma,\ 2\pi^0\eta$
$\rho(770)$	$u\bar{d},\ u\bar{u} - d\bar{d},$ $d\bar{u}$	1^{--}	1^+	$\pm1, 0$	763	3.72×10^{-24}	$\pi\pi$
$\omega(782)$	$u\bar{u},\ d\bar{d},\ s\bar{s}$	1^{--}	0^-	0	782.66	6.43×10^{-23}	$\pi^+\pi^-\pi^0$
$K^*(892)$	$u\bar{s},\ d\bar{s},\ \bar{d}s,\ \bar{u}s$	1^-	$1/2$	$\pm1, 0$	890	2.1×10^{-23}	$K\pi$
$f_0(500)$	$u\bar{u},\ d\bar{d}$	0^{++}	0^+	0	500	2.03×10^{-24}	$\pi\pi$
$f_0(1370)$	$u\bar{u},\ d\bar{d},\ s\bar{s}$	0^{++}	0^+	0	1345	3.1×10^{-24}	$\pi\pi,\ 4\pi$
$a_0(980)$	$u\bar{d},\ u\bar{u} - d\bar{d},$ $d\bar{u}$	0^{++}	1^-	$\pm1, 0$	980	1.1×10^{-23}	$\eta\pi$
$a_1(1260)$	$u\bar{d},\ u\bar{u} - d\bar{d},$ $d\bar{u}$	1^{++}	1^-	$\pm1, 0$	1230	1.3×10^{-24}	3π
$a_2(1320)$	$u\bar{d},\ u\bar{u} - d\bar{d},$ $d\bar{u}$	2^{++}	1^-	$\pm1, 0$	1318.2	5.3×10^{-24}	$3\pi,\ \eta\pi,\ \omega\pi\pi$
$\pi_1(1600)$	$u\bar{d},\ u\bar{u} - d\bar{d},$ $d\bar{u}$	1^{-+}	1^-	$\pm1, 0$	1580	3.7×10^{-24}	$3\pi,\ b_1\pi,\ \eta\pi,\ \eta'\pi$

Table 2: Light and Strange Baryons.

Baryon	**Quark Content**	$\boldsymbol{J^P}$	$\boldsymbol{I}$	**Charge**	**Mass / MeV**	**Lifetime / s**	**Principle Decay Modes**
p	uud	$1/2^+$	$1/2$	$+1$	938.27	stable	—
n	udd	$1/2^+$	$1/2$	0	939.57	878.4	$pe^-\bar{\nu}_e$
Λ^0	uds	$1/2^+$	0	0	1115.68	2.63×10^{-10}	$p\pi^-$, $n\pi^0$
$\Sigma^\pm$	uus, dds	$1/2^+$	1	±1	1189.37	8.02×10^{-11}	$p\pi^0$, $n\pi^+$
Σ^0	uds	$1/2^+$	1	0	1192.64	74×10^{-21}	$\Lambda\gamma$
Ξ^-	dss	$1/2^+$	$1/2$	-1	1321.71	1.64×10^{-10}	$\Lambda\pi^-$
Ξ^0	uss	$1/2^+$	$1/2$	0	1314.86	2.90×10^{-10}	$\Lambda\pi^0$
$\Delta^{++}(1231)$	uuu	$3/2^+$	$3/2$	$+2$	1210	5.58×10^{-24}	$N\pi$
$\Delta^\pm(1231)$	uud, udd	$3/2^+$	$3/2$	±1	1210	5.58×10^{-24}	$N\pi$
$\Delta^0(1231)$	udd	$3/2^+$	$3/2$	0	1210	5.58×10^{-24}	$N\pi$
$\Sigma(1385)$	uus, uds, dds	$3/2^+$	1	±1, 0	1385	1.5×10^{-23}	$\Lambda\pi$, $\Sigma\pi$, $\Lambda\gamma$
$\Xi(1530)$	uss, dss	$3/2^+$	$1/2$	0, -1	1532.8	5.88×10^{-23}	$\Xi\pi$
Ω^-	sss	$1/2^+$	0	-1	1672.45	8.21×10^{-11}	ΛK^-, $\Xi^0\pi^-$, $\Xi^-\pi^0$
$N(1440)$	uud, udd	$1/2^+$	$1/2$	$+1$, 0	1370	2.94×10^{-24}	$N\pi$, $N\pi\pi$
$\Lambda(1405)$	uds	$1/2^-$	0	0	1405.1	1.12×10^{-23}	$\Sigma\pi$