# Physics 303

## Classical Mechanics II

# Two-Body Systems

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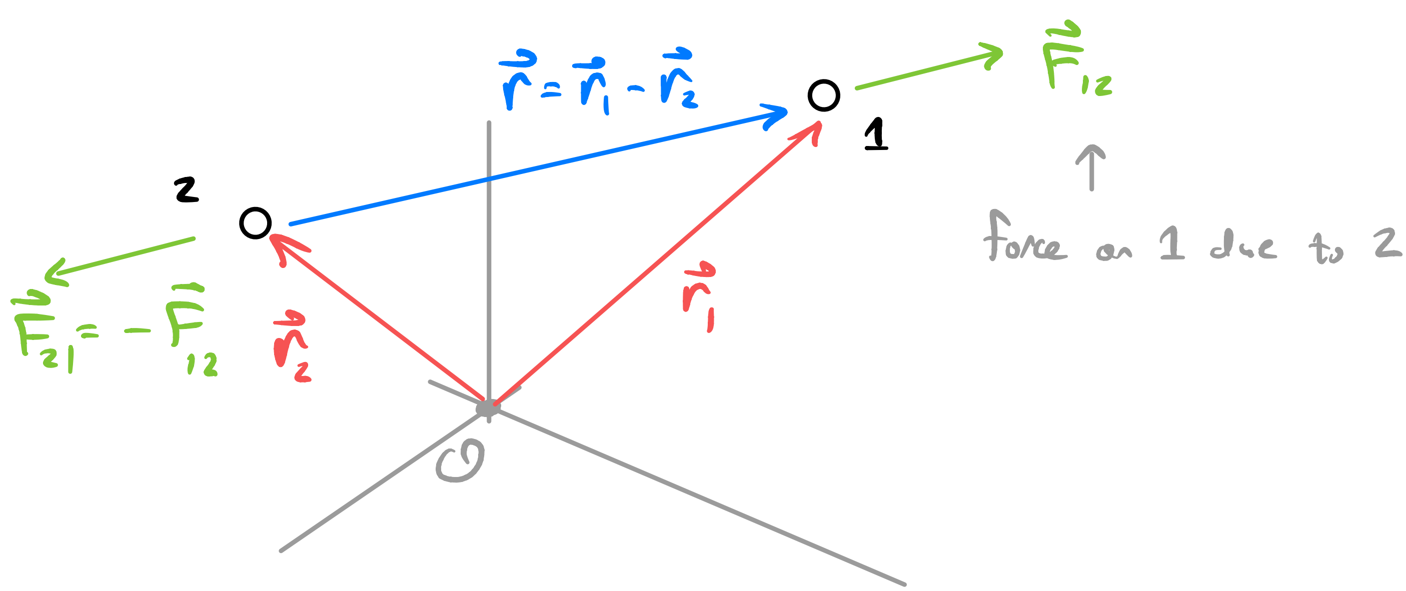
# Two-Body Systems

Here we examine in detail the motion of two-body systems. Two-body systems are prevalent in the study of physics, such as the orbit of a planet about a star & the physics of interacting electron & proton in the hydrogen atom. Our focus will be on central force problems, that is each body exhibits a mutual force on each other without any external forces.

## Central Forces

Consider two objects, considered as point-particles, with masses $m_1$ & $m_2$. The forces considered are $\vec{F}_{12} = -\vec{F}_{21}$, assumed conservative & central.



A central force has the functional form

$$\vec{F}_{12}(\vec{r}_1, \vec{r}_2) = \vec{F}_{12}(|\vec{r}_1 - \vec{r}_2|)$$

$$= -\vec{F}_{21}(|\vec{r}_1 - \vec{r}_2|)$$

Here, $\vec{r}_1$ & $\vec{r}_2$ are the positions of objects 1 & 2 in a coordinate system $\mathcal{O}$.

An example of such a force is Newton's Law of Gravitation,

$$\boxed{\vec{F}_{12} = -G m_1 m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}}$$

Gravitational constant, $G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$

Since the force is conservative ($\vec{\nabla} \times \vec{F} = \vec{0}$), we can describe it by a potential energy function,

$$\boxed{\vec{F}_{12}(\vec{r}_1, \vec{r}_2) = -\vec{\nabla}_1 \, U(\vec{r}_1, \vec{r}_2)}$$

w/ $\vec{\nabla}_1 = \frac{\partial}{\partial x_1}\hat{x}_1 + \frac{\partial}{\partial y_1}\hat{y}_1 + \frac{\partial}{\partial z_1}\hat{z}_1$

An isolated system is translationally invariant, & since the force is conservative, we have

$$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$$

Let us introduce the relative position $\vec{r}$,

$$\boxed{\vec{r} \equiv \vec{r}_1 - \vec{r}_2}$$

↳ position of body 1 relative to body 2

With this definition,

$$\vec{F}_{12} = -G m_1 m_2 \frac{\vec{r}}{r^3} = -\vec{\nabla}_r U(r)$$

with $r = |\vec{r}| = \sqrt{\vec{r}^{\,2}} = \sqrt{\vec{r}\cdot\vec{r}}$,

and the potential is $U = U(r)$

For gravitation, $U(r) = -G \frac{m_1 m_2}{r}$

The dynamical system of the two bodies is described by the Lagrangian

$$\boxed{\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)}$$

The Newtonian formulation is

$$\boxed{\ddot{\vec{r}}_1 = \frac{1}{m_1}\vec{F}_{12} \quad , \quad \ddot{\vec{r}}_2 = \frac{1}{m_2}\vec{F}_{21}}$$

We will use the Lagrangian approach to generate equations of motion in a more suitable coordinate system.

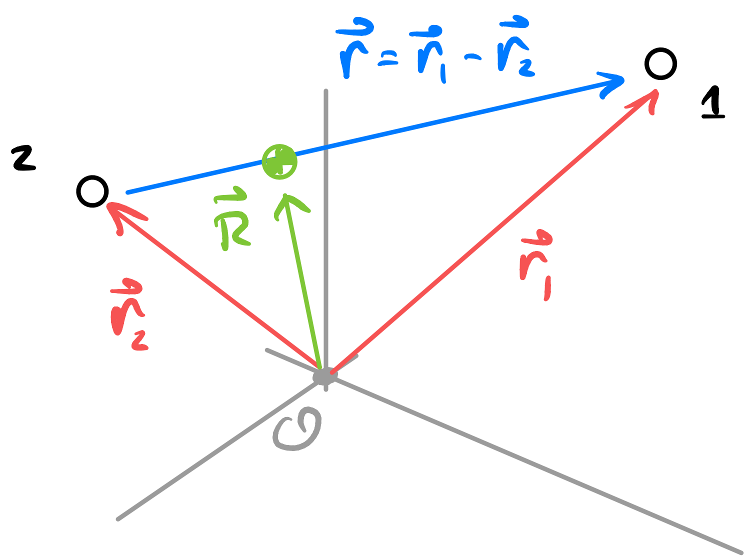
## Center of Mass & Relative Coordinates

It is difficult to solve the system for $\vec{r}_1$ & $\vec{r}_2$ separately. However, since the potential is central, $U = U(r)$, this indicates that there is a better set of coordinates involving the relative position $\vec{r} = \vec{r}_1 - \vec{r}_2$. We have $3+3=6$ d.o.f. between $\vec{r}_1$ & $\vec{r}_2$, and $\vec{r}$ has 3 d.o.f., so we need 3 more.

Consider the center-of-mass $\vec{R}$

$$\boxed{\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}}$$

Consider some limits




- $\frac{m_1}{m_2} \ll 1$

$$\Rightarrow \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$= \frac{\frac{m_1}{m_2} \vec{r}_1 + \vec{r}_2}{1 + \frac{m_1}{m_2}}$$

$$\simeq \vec{r}_2 + \left(\frac{m_1}{m_2}\right) \vec{r} + O\left(\left(\frac{m_1}{m_2}\right)^2\right)$$

CM is close to $\vec{r}_2$

- $\frac{m_2}{m_1} \ll 1$

CM close to $\vec{r}_1$

$$\Rightarrow \vec{R} = \frac{\vec{r}_1 + \frac{m_2}{m_1} \vec{r}_2}{1 + \frac{m_2}{m_1}} \simeq \vec{r}_1 + \left(\frac{m_2}{m_1}\right) \vec{r} + O\left(\left(\frac{m_2}{m_1}\right)^2\right)$$

- $m_1 = m_2 \equiv m$

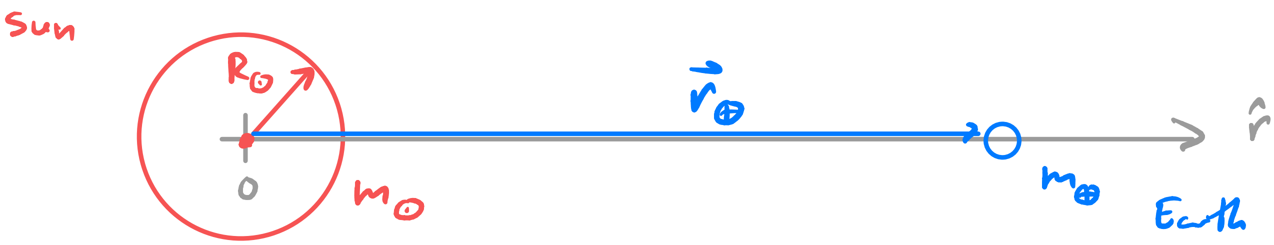
$$\Rightarrow \vec{R} = \frac{1}{2} (\vec{r}_1 + \vec{r}_2) \leftarrow \text{half-way between } \vec{r}_1 \,\&\, \vec{r}_2$$

$$= \vec{r}_1 - \frac{1}{2} \vec{r}$$

$$= \vec{r}_2 + \frac{1}{2} \vec{r}$$

## Example

Consider the Earth-Sun system. Where is the CM using a coordinate system with the origin at the center of the sun.




$$\vec{R} = \frac{m_\odot \vec{r}_\odot + m_\oplus \vec{r}_\oplus}{m_\odot + m_\oplus} = \frac{m_\oplus}{m_\odot + m_\oplus} r_\oplus \hat{r}$$

$$\simeq \frac{m_\oplus}{m_\odot} \frac{1}{1 + \frac{m_\oplus}{m_\odot}} r_\oplus \hat{r}$$

$$\simeq \frac{m_\oplus}{m_\odot} r_\oplus \hat{r} + \mathcal{O}\left(\left(\frac{m_\oplus}{m_\odot}\right)^2\right)$$

Now, $m_\oplus \simeq 3 \times 10^{-6} m_\odot$

$\langle r_\oplus \rangle \simeq 200 R_\odot$

solar radius

$$\Rightarrow \boxed{\langle R \rangle \simeq 6 \times 10^{-4} R_\odot}$$

■

The total momentum of the system $\vec{P}$ is given by

$$\vec{P} = (m_1 + m_2)\dot{\vec{R}} = M\dot{\vec{R}}$$

total mass of system

Recall that the total momentum of a closed system is constant. Therefore,

$$\vec{P} = \text{const} \Rightarrow \dot{\vec{R}} = \text{const}$$

$$\text{Let } \vec{V} \equiv \dot{\vec{R}} \Rightarrow \vec{R} = \vec{R}_0 + \vec{V}t$$

initial CM position wrt $O$.

Given CM & relative coordinates $(\vec{R}, \vec{r})$, can derive relations for individual positions $(\vec{r}_1, \vec{r}_2)$,

$$\boxed{\begin{aligned} \vec{R} &= \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2) \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned}} \Longleftrightarrow \boxed{\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{M}\vec{r} \end{aligned}}$$

Recall the Lagrangian

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)$$

Let us transform the kinetic energies to $(\vec{R}, \vec{r})$

$$\begin{aligned}
T &= T_1 + T_2 \\
&= \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} \\
&= \frac{1}{2} m_1 \left( \dot{\vec{R}} + \frac{m_2}{M} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left( \dot{\vec{R}} - \frac{m_1}{M} \dot{\vec{r}} \right)^2 \\
&= \frac{1}{2} (m_1 + m_2) \dot{\vec{R}}^2 + \frac{1}{2} m_1 \frac{m_2^2}{M^2} \dot{\vec{r}}^2 + \frac{1}{2} m_2 \frac{m_1^2}{M^2} \dot{\vec{r}}^2 \\
&= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{M} \dot{\vec{r}}^2
\end{aligned}$$

Let us define a parameter, the reduced mass $\mu$

$$\boxed{\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}}$$

Consider limit

- $\frac{m_1}{m_2} \ll 1 \Rightarrow \mu = \frac{m_1}{1 + \frac{m_1}{m_2}} \simeq m_1 - \left(\frac{m_1}{m_2}\right) m_1 + \mathcal{O}\left(\left(\frac{m_1}{m_2}\right)^2\right)$
- $m_1 = m_2 \equiv m \Rightarrow \mu = \frac{m^2}{2m} = \frac{m}{2}$

Thus, the kinetic energy is

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2$$

KE of CM

KE of relative motion

So, Lagrangian,

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)$$

$$= \mathcal{L}_{cm} + \mathcal{L}_{rel}$$

depends only on $\vec{R}$

depends only on $\vec{r}$

## Equations of Motion

We can generate the EOM for $\vec{R}$ & $\vec{r}$. Consider the Euler-Lagrange eqns. for $\vec{R}$,

$$\frac{d}{dt} \frac{\partial \mathcal{L}_{cm}}{\partial \dot{R}_j} - \frac{\partial \mathcal{L}_{cm}}{\partial R_j} = 0 \qquad , j = 1, 2, 3$$

Since $\mathcal{L}_{cm} = \mathcal{L}_{cm}(\dot{R}_j) = \frac{1}{2} M \sum_j \dot{R}_j^2$,

the coordinate $R_j$ is ignorable, $\Rightarrow \frac{\partial \mathcal{L}_{cm}}{\partial R_j} = 0$

Thus the EOM are

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{cm}}{\partial \dot{R}_j} = \frac{d}{dt}\frac{\partial}{\partial \dot{R}_j}\left(\frac{1}{2} M \sum_k \dot{R}_k^2\right)$$

$$= \frac{d}{dt}\left(M \sum_k \dot{R}_k \delta_{jk}\right)$$

$$= \frac{d}{dt}\left(M \dot{R}_j\right)$$

$$= M \ddot{R}_j$$

or, $\boxed{M\ddot{\vec{R}} = \vec{0}}$

The center of mass moves as a "free particle", as we expect for isolated - closed systems.

The solution is straightforward

$$\boxed{\vec{R}(t) = \vec{R}_0 + \vec{V}(t - t_0)}$$

with $\vec{R}_0 = \vec{R}(t_0)$ , $\vec{V} = \dot{\vec{R}}(t_0)$

The relative motion is more complicated

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{rel}}{\partial \dot{r}_j} - \frac{\partial \mathcal{L}_{rel}}{\partial r_j} = 0 \qquad , j = 1, 2, 3$$

The relative Lagrangian is of a particle of mass $\mu$ interacting with a potential $U(r)$.

$$\frac{\partial}{\partial r_j} \mathcal{L}_{rel} = \frac{\partial}{\partial r_j}\left( \frac{1}{2}\mu \sum_k \dot{r}_k^2 - U(r) \right)$$
$$= -\frac{\partial}{\partial r_j} U(r)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{rel}}{\partial \dot{r}_j} = \frac{d}{dt}\frac{\partial}{\partial \dot{r}_j}\left( \frac{1}{2}\mu \sum_k \dot{r}_k^2 \right)$$
$$= \frac{d}{dt}\left( \mu \sum_k \dot{r}_k \delta_{jk} \right)$$
$$= \mu \ddot{r}_j$$

so, EOM $\Rightarrow$ $\mu \ddot{r}_j = -\frac{\partial}{\partial r_j} U(r)$

or,

$$\boxed{\mu \ddot{\vec{r}} = -\vec{\nabla}_r U(r)}$$

EOM of particle of mass $\mu$ in potential $U(r)$

# The Center-of-Mass frame

We can simplify our problem further by choosing a special Inertial reference frame. Since $\dot{\vec{R}}$ = const., we can choose a frame called the CM frame, where the CM is at rest, $\vec{R}(t) = \vec{0} \quad \forall \ t$.

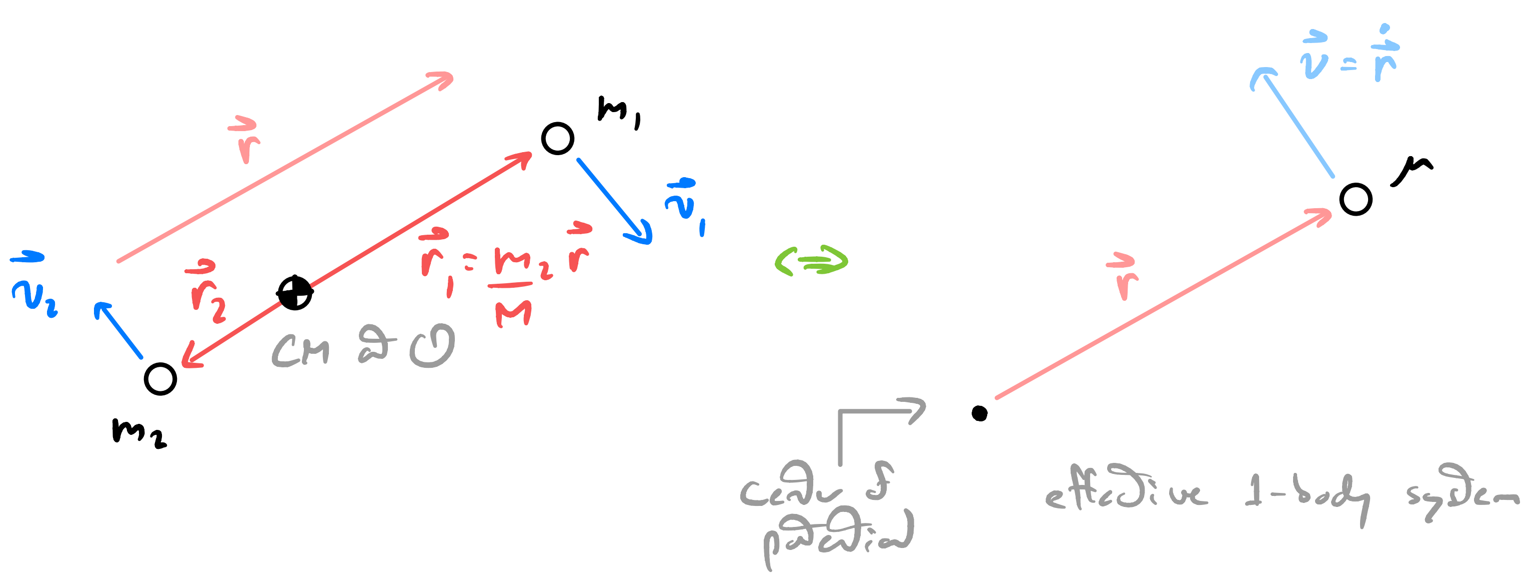
Thus, $\dot{\vec{R}} = \vec{0} \quad \Rightarrow \quad \mathcal{L}_{cm} = 0$

So, the Lagrangian is

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

CM frame

This is an effective 1-body problem



We have reduced a problem in 6 variables to 3 variables in the CM frame. Using conservation of angular momentum, we can further simplify the problem. The total angular momentum $\vec{L}$ is

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$= m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

In the CM frame, $\vec{r}_1 = \frac{m_2}{M}\vec{r}$ & $\vec{r}_2 = -\frac{m_1}{M}\vec{r}$

So,
$$\vec{L} = \frac{m_1 m_2}{M^2}\left( m_2 \vec{r} \times \dot{\vec{r}} + m_1 \vec{r} \times \dot{\vec{r}} \right)$$
$$= \mu \vec{r} \times \dot{\vec{r}}$$

Since total angular momentum is conserved,

$$\dot{\vec{L}} = \vec{0}$$

$$\Rightarrow \vec{L} = \text{const.}$$

Therefore,
$$\vec{L} = \mu \vec{r} \times \dot{\vec{r}} = \text{const.}$$

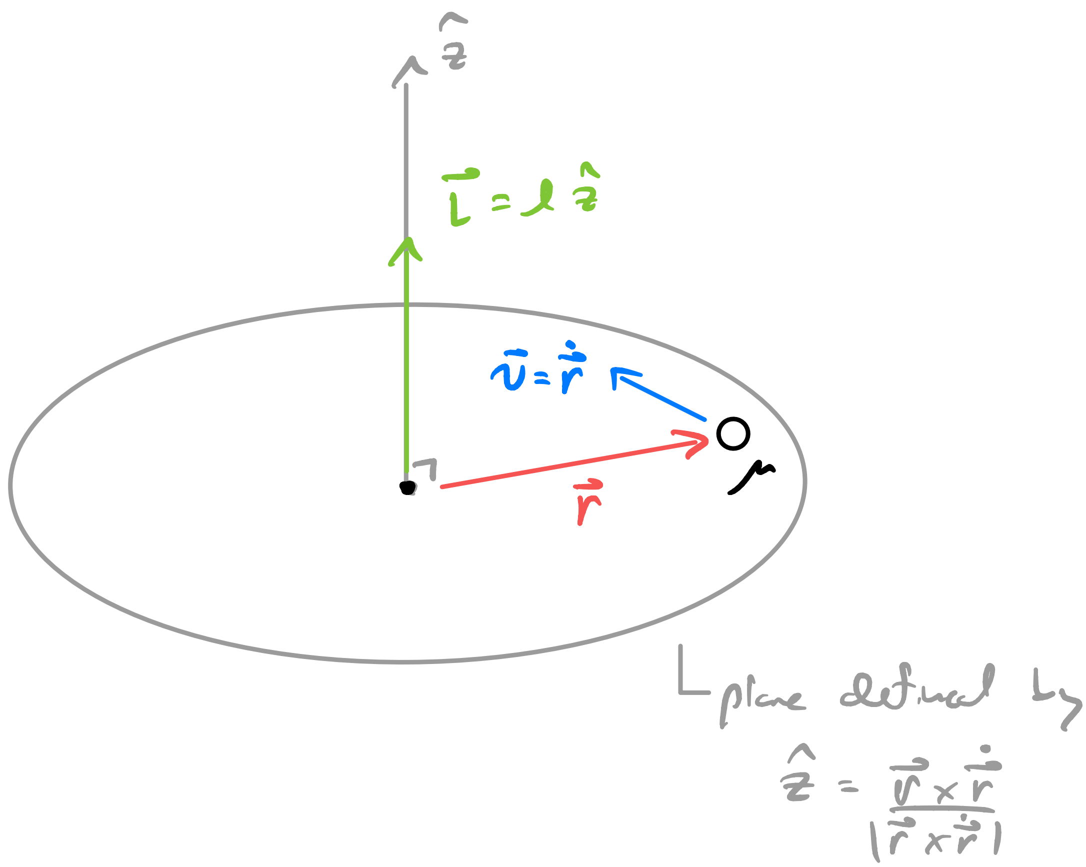
So, the direction $\vec{r} \times \dot{\vec{r}} = \text{const.}$

Thus, we can write

$$\vec{L} \equiv \ell \, \hat{z} = \text{const.}$$

where $\quad \hat{z} = \dfrac{\vec{r} \times \dot{\vec{r}}}{|\vec{r} \times \dot{\vec{r}}|} \quad \& \quad \ell \equiv \mu \, |\vec{r} \times \dot{\vec{r}}|$

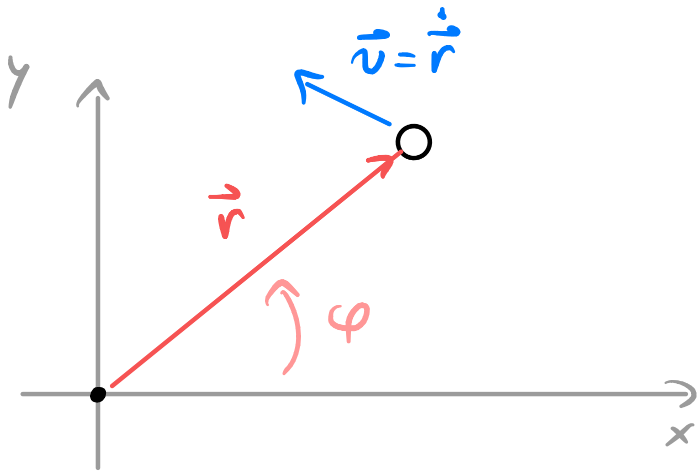
Thus, the motion of the system lies in a plane, effectively reducing 3 coordinates to 2.



Let us derive the equations of motion for the remaining 2 variables. Let us choose to work with cylindrical polar coordinates $(r, \varphi)$

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\varphi}\,\hat{\varphi}$$

$$\Rightarrow \quad \dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2\dot{\varphi}^2$$




So,

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

Notice that $\mathcal{L}$ is independent of $\varphi$ $\Rightarrow$ $\dfrac{\partial \mathcal{L}}{\partial \varphi} = 0$

So,

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \boxed{\mu r^2 \dot{\varphi} = \ell = \text{const.}}$$ angular eqn.

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\ell} = 0$$

Recall: $\vec{L} = \mu\,\vec{r} \times \dot{\vec{r}} = \mu r^2\dot{\varphi}\;\hat{r} \times \hat{\varphi} = \mu r^2 \dot{\varphi}\;\hat{z}$
$\equiv \ell\,\hat{z}$

So, the $\varphi$ eqn. is simply a statement of conservation of angular momentum.

Now let's consider the radial eqn.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0$$

So,
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2}\mu r^2 \dot{\varphi}^2 - U(r)\right)$$
$$= \mu r \dot{\varphi}^2 - \frac{\partial U}{\partial r}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}\mu \dot{r}^2\right)$$
$$= \mu \ddot{r}$$

$$\Rightarrow \boxed{\mu \ddot{r} = \mu r \dot{\varphi}^2 - \frac{\partial U}{\partial r}}$$ radial eqn.

Given $U(r)$, we wish to solve for $r$.

# Effective Potentials

Before specifying a potential $U(r)$, let us examine the effective one-dimensional problem. The equations of motion are

$$\boxed{\begin{aligned} \mu r^2 \dot{\varphi} &= \ell & (1) \\ \& \quad \mu \ddot{r} &= \mu r \dot{\varphi}^2 - \frac{\partial U}{\partial r} & (2) \end{aligned}}$$

Since $\ell = \text{const.}$, the $\varphi$ equation is thus fixed from initial conditions since given

$$r_0 = r(t_0) \quad , \quad \varphi_0 = \varphi(t_0)$$
$$\dot{r}_0 = \dot{r}(t_0) \quad , \quad \dot{\varphi}_0 = \dot{\varphi}(t_0)$$

$$\Rightarrow \boxed{\ell = \mu r_0^2 \dot{\varphi}_0}$$

So, let us write (1) as

$$\dot{\varphi} = \frac{\ell}{\mu r^2} \quad \left( = \left(\frac{r_0}{r}\right)^2 \dot{\varphi}_0 \right)$$

and eliminate $\dot{\varphi}$ from (2)

$$\boxed{\mu \ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}} \qquad (3)$$

Eqn. 3 is an equivalent 1-dimensional problem, only involving the unknown $r$.

$$\mu \ddot{r} = \underbrace{\frac{\ell^2}{\mu r^3}}_{\text{"fictitious" centrifugal force}} - \underbrace{\frac{\partial U}{\partial r}}_{\text{central force}}$$

Let $F_{cf} = \frac{\ell^2}{\mu r^3}$ be the centrifugal force.

We can define a centrifugal potential energy

$$F_{cf} = -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2}\right) \equiv -\frac{\partial}{\partial r} U_{cf}$$

where $\boxed{U_{cf}(r) = \frac{\ell^2}{2\mu r^2}}$

So, the radial eqn can be written as

$$\mu \ddot{r} = -\frac{\partial}{\partial r}\left(U(r) + U_{cf}(r)\right)$$

$$\equiv -\frac{\partial}{\partial r} U_{eff}$$

We have defined the effective potential

$$U_{eff}(r) = U(r) + U_{cf}(r)$$
$$= U(r) + \frac{\ell^2}{2\mu r^2}$$

It's effectively as if a single particle is moving in 1-dimension in a potential $U_{eff}(r)$.

Let's look at gravitational interactions as an example,

$$U(r) = -G\frac{m_1 m_2}{r}$$

Recall $\mu = \frac{m_1 m_2}{M} \Rightarrow U(r) = -G\frac{\mu M}{r}$

So,

$$U_{eff}(r) = -G\frac{\mu M}{r} + \frac{\ell^2}{2\mu r^2}$$

For $\ell \neq 0$, $U_{eff} \sim -G\frac{\mu M}{r}$ as $r \to \infty$

$$U_{eff} \sim \frac{\ell^2}{2\mu r^2} \quad \text{as } r \to 0$$

Let $r_0$ be the location of the minimum value of $U_{eff}$ for $\ell \neq 0$.

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = 0$$

So,

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = +\frac{G\mu M}{r_0^2} - \frac{\ell^2}{\mu r_0^3} = 0$$

$$\Rightarrow \quad r_0 = \frac{\ell^2}{GM\mu^2}$$

$\ell^2 = GM\mu^2 r_0$

At the minimum,

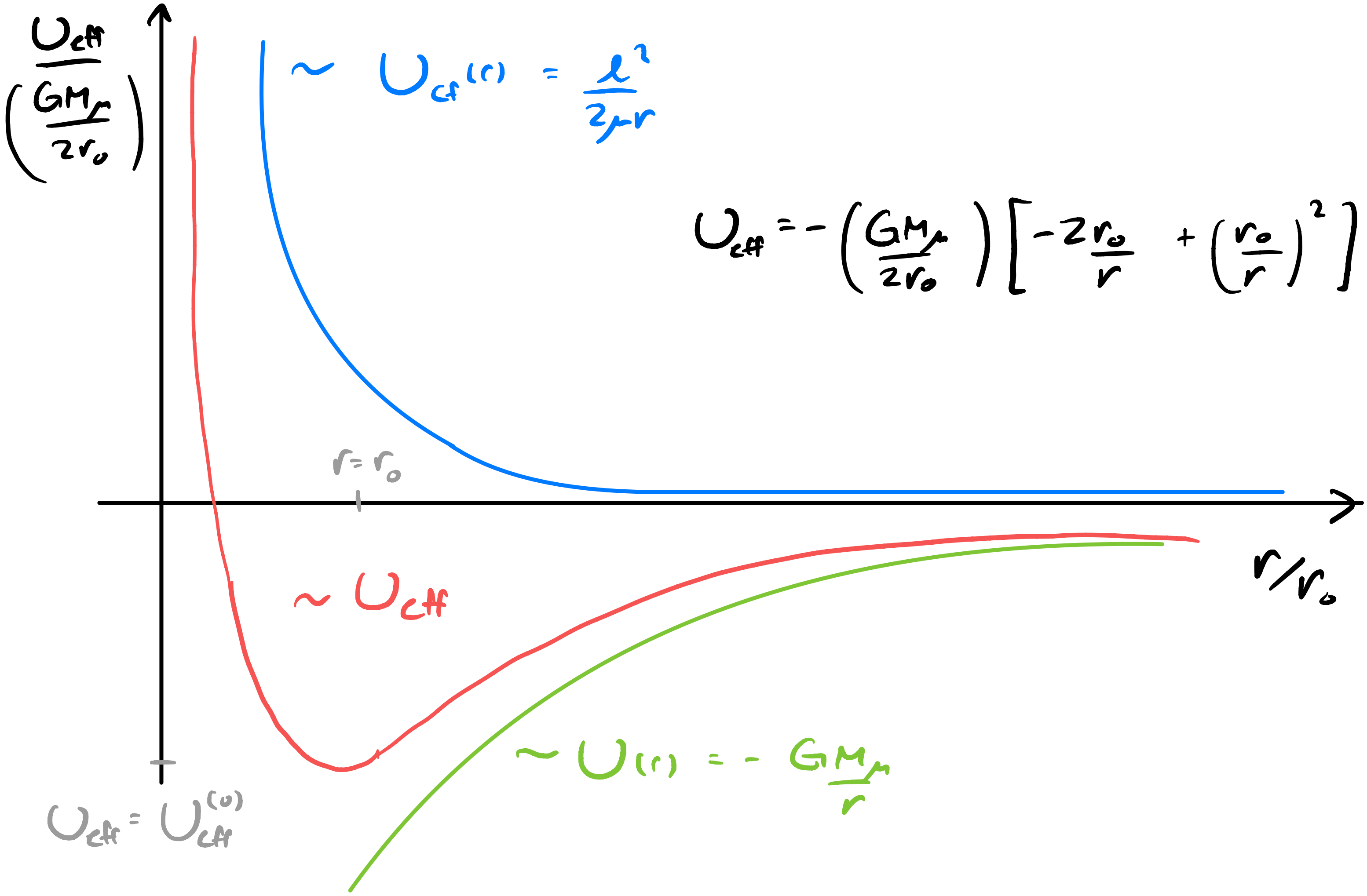
$$U_{eff}^{(0)} = -\frac{GM\mu}{r_0} + \frac{\ell^2}{2\mu r_0^2}$$

$$= -\frac{GM\mu}{r_0} + \frac{GM\mu}{2r_0} = -\frac{1}{2}\frac{GM\mu}{r_0}$$

We can then write $U_{eff}$ as

$$U_{eff} = -\frac{GM\mu}{r} + \frac{1}{2}GM\mu\frac{r_0}{r^2}$$

$$= U_{eff}^{(0)}\left[2\frac{r_0}{r} - \frac{r_0^2}{r^2}\right]$$




Let us consider the consequences of conservation of energy. Take the EOM & multiply by $\dot{r}$,

$$\dot{r}\mu\ddot{r} = -\dot{r}\frac{\partial}{\partial r}U_{eff} \qquad \left(\dot{r} = \frac{dr}{dt}\right)$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}U_{eff}$$

This means that

$$\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = \text{const.}$$

$$\Rightarrow \quad \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = \text{const.}$$

But, recall $T_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2$

& $\dot{\varphi}^2 = \frac{\ell^2}{\mu^2 r^4} \Rightarrow T_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2}$

Therefore, $T_{rel} + U(r) = \text{const}$

This is just a statement of total energy,

$$\boxed{E = T_{rel} + U(r)} = \text{const.}$$

which is conserved, $\frac{dE}{dt} = 0$.

Let's again look at the motion of a particle of mass $\mu$ in a 1-dim effective system,

$$\boxed{\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = E}$$

Notice that $\frac{1}{2}\mu\dot{r}^2 \geq 0$ always, thus

$$E \geq U_{eff}$$

The points such that $\dot{r} = 0$ are turning points in the reduced particles trajectory.

Now, $U_{eff}$ can in general be positive or negative, thus we have two cases to consider: $E \geq 0$ & $E < 0$.

Lets look at $E \geq 0$ case, for an object, such as a comet, in a gravitational well, $U(r) = -G\frac{M\mu}{r}$, where $\ell \neq 0$.

If $E = 0$ & $E \geq U_{eff}$, we have $U_{eff} \leq 0$ or,

$$U_{eff} = \frac{\ell^2}{2\mu r^2} + U(r) \leq 0$$

For gravity, $U(r) = -\frac{GM\mu}{r}$, and $\ell \neq 0$, this gives

$$\left(\frac{\ell^2}{2\mu r^2} - \frac{GM\mu}{r}\right) = 0$$

$$\Rightarrow r_{max} \to \infty \quad \text{or} \quad r_{min} = \frac{\ell^2}{2GM\mu^2}$$

So, there is only 1 turning point, $\dot{r}=0$, at

$$r_{min} = \frac{\ell^2}{2GM\mu^2}$$

Thus, if a comet comes in from $r \to \infty$, it turns around at $r_{min}$, and moves back toward $r \to \infty$.

As a function of $E \geq 0$,

we can determine turning points, $\dot{r}=0$,

$$E \geq U_{eff}(r_\pm)$$

this gives $\quad E = \frac{\ell^2}{2\mu r^2} - \frac{GM\mu}{r} \quad$ (take $E = U_{eff}$ case)

$$\Rightarrow \quad r^2 + \frac{GM\mu}{E} r - \frac{\ell^2}{2\mu E} = 0$$

$$\Rightarrow \quad r_\pm = -\frac{GM\mu}{2E} \pm \frac{1}{2}\sqrt{\left(\frac{GM\mu}{E}\right)^2 + \frac{2\ell^2}{\mu E}}$$

$$= -\frac{GM\mu}{2E} \pm \frac{GM\mu}{2E}\sqrt{1 + \frac{2\ell^2 E}{G^2M^2\mu^3}}$$

Now, since $r \geq 0$, $r_-$ is an unphysical solution for $E \geq 0$. Therefore, $E \geq U_{eff}(r_{min})$ with

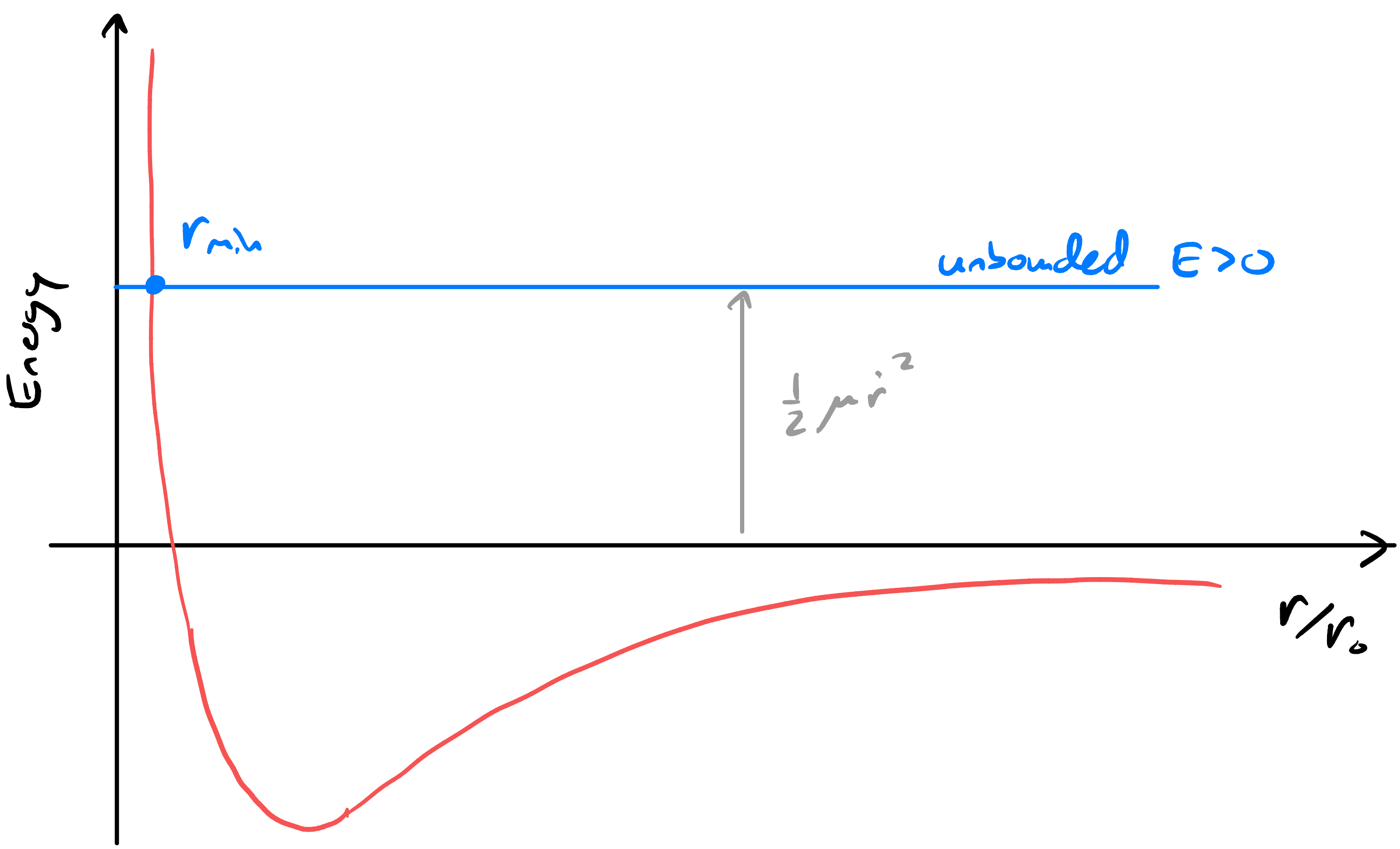
$$\boxed{r_{min} \equiv r_+ = -\frac{GM\mu}{2E} + \frac{GM\mu}{2E}\sqrt{1 + \frac{2\ell^2 E}{G^2M^2\mu^3}}}$$

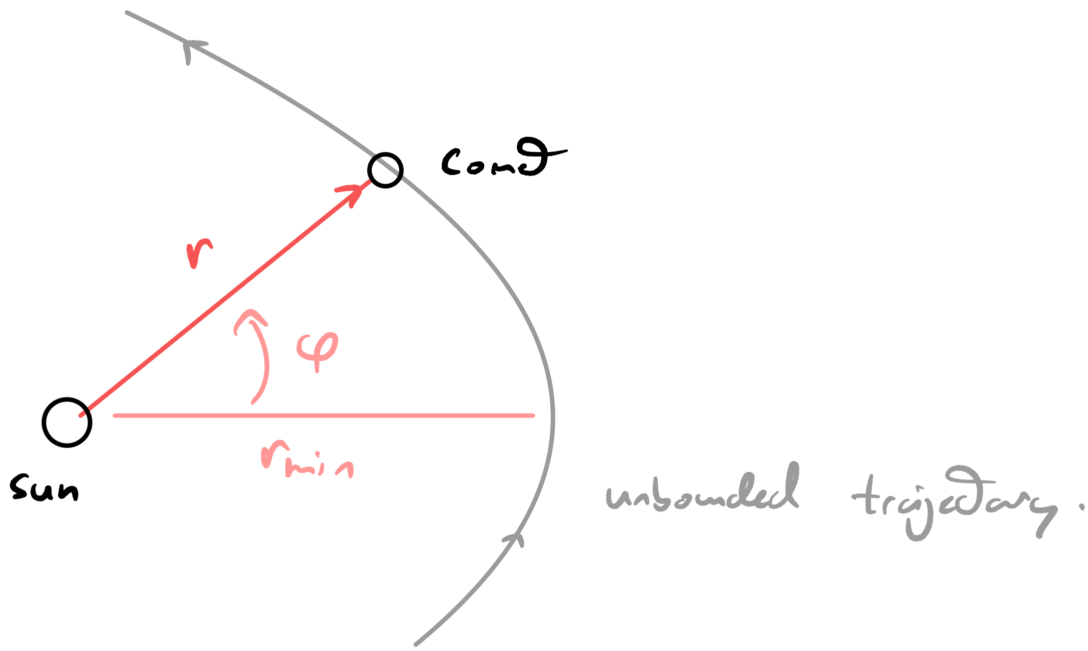
Let us expand the solution for small $E$, $E/\mu \ll 1$,

$$\Rightarrow r_{min} = -\frac{GM\mu}{2E} + \frac{GM\mu}{2E}\left(1 + \frac{l^2 E}{G^2M^2\mu^3} + \mathcal{O}\left(\frac{E}{\mu}\right)\right)$$

$$= \frac{l^2}{2GM\mu^2} + \mathcal{O}\left(\frac{E}{\mu}\right)$$

Graphically, this is shown in blue on the effective potential plot. This $E \geq 0$ scenario is an unbounded orbit.






Now, consider $E < 0$. Let $E \equiv -\mathcal{E}$, $\mathcal{E} > 0$.

The turning points are now,

$$-\mathcal{E} \geq U_{eff}(r_{\substack{min \\ max}})$$

or, $\quad \mathcal{E} \leq -U_{eff}(r_{\substack{min \\ max}})$.

Solving for the turning points, for $\ell \neq 0$ & gravity

$$-\frac{\ell^2}{2\mu r^2} + \frac{GM\mu}{r} \geq \mathcal{E}$$

or, $\quad r^2 - \frac{GM\mu}{\mathcal{E}} r + \frac{\ell^2}{2\mu\mathcal{E}} = 0$

which has solutions

$$\boxed{r_\pm = \frac{GM\mu}{2\mathcal{E}} \pm \frac{GM\mu}{2\mathcal{E}}\sqrt{1 - \frac{2\ell^2\mathcal{E}}{G^2M^2\mu^3}}}$$

To get a sense of the solution, let $\mathcal{E}/\mu \ll 1$,

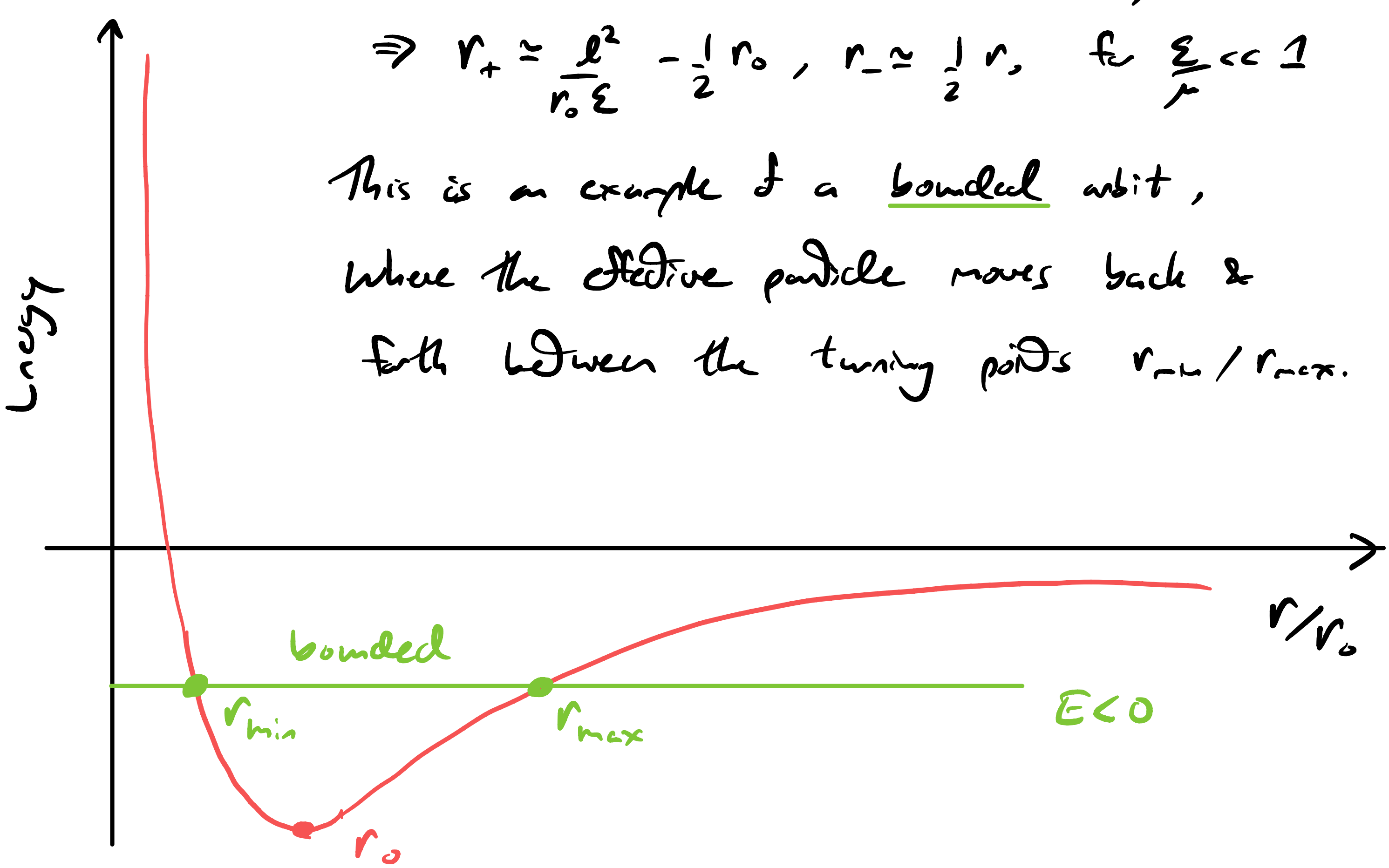
$$\Rightarrow \quad r_{\pm} = \frac{GM\mu}{2\mathcal{E}} \pm \frac{GM\mu}{2\mathcal{E}}\left(1 - \frac{\ell^2 \mathcal{E}}{G^2M^2\mu^3} + \mathcal{O}\left(\frac{\mathcal{E}}{\mu}\right)\right)$$

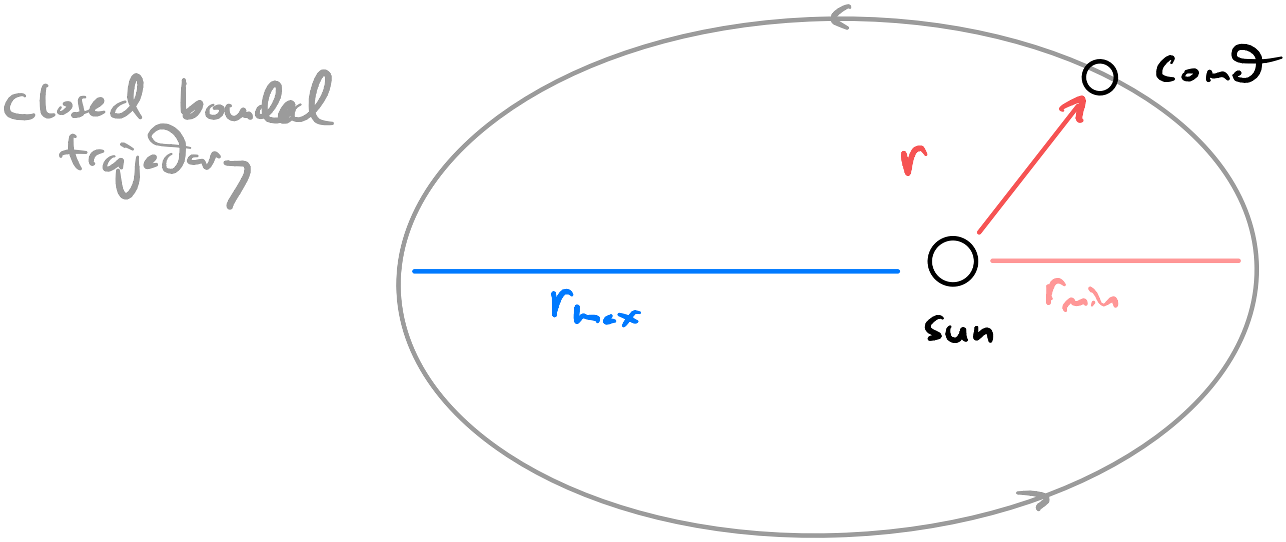
$$= \begin{cases} \dfrac{GM\mu}{\mathcal{E}} - \dfrac{\ell^2}{2GM\mu^2} + \mathcal{O}\left(\dfrac{\mathcal{E}}{\mu}\right) \\[2ex] +\dfrac{\ell^2}{2GM\mu^2} + \mathcal{O}\left(\dfrac{\mathcal{E}}{\mu}\right) \end{cases}$$

So, $r_{min} \equiv r_-$ & $r_{max} \equiv r_+$. Recall that the minimum of the effective potential is at $r_0 = \dfrac{\ell^2}{GM\mu^2}$,

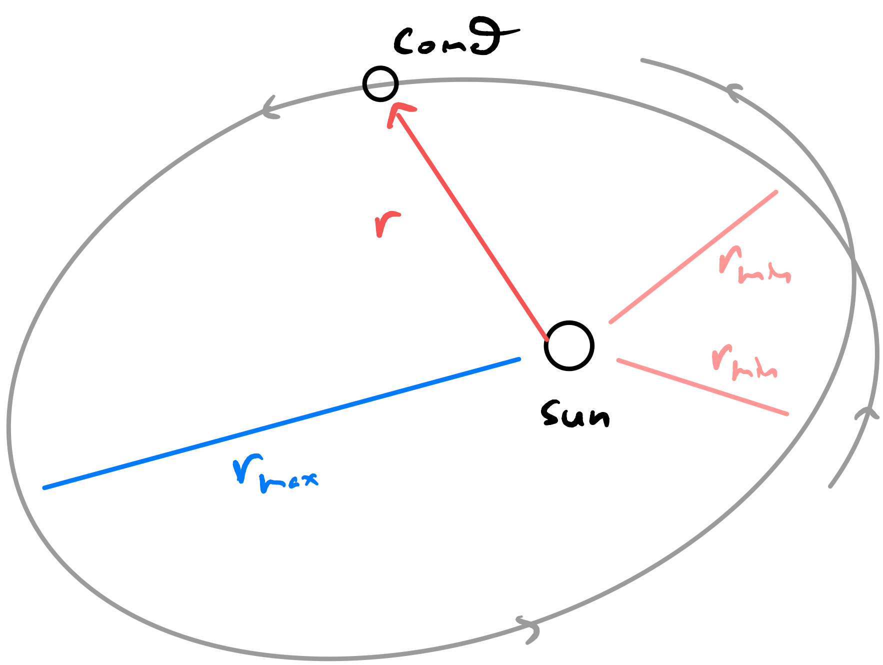
$$\Rightarrow r_+ \simeq \frac{\ell^2}{r_0 \mathcal{E}} - \frac{1}{2} r_0 \, , \quad r_- \simeq \frac{1}{2} r_0 \quad \text{for } \frac{\mathcal{E}}{\mu} \ll 1$$

This is an example of a bounded orbit, where the effective particle moves back & forth between the turning points $r_{min} / r_{max}$.






The arguments we made work for general central potentials, but inverse-square-laws like gravitation result in closed bounded orbits. One can show that most other force laws have open bounded orbits, that is they precess.



## Equation of Orbit

Let us now move toward understanding the details of the geometry of the trajectory. Recall the equations of motion,

$$\mu r^2 \dot{\varphi} = \ell \quad (= \text{const.})$$

$$\mu \ddot{r} = \frac{\ell^2}{\mu r^3} + F(r)$$

where $F(r) = -\frac{\partial U}{\partial r}$ is the central force.

In general, solving for $r = r(t)$ & $\varphi = \varphi(t)$ is very complicated, and in general requires numerical solutions. But, we can learn something about the geometry of the orbit in a relatively simple way.

First, let us perform a variable change,

$$r = \frac{1}{u} \quad \sim \quad u = \frac{1}{r}$$

so that the EOM for the radial component is

$$\frac{d^2}{dt^2}\left(\frac{1}{u}\right) = \frac{\ell^2}{\mu^2} u^3 + \frac{1}{\mu} F\left(\frac{1}{u}\right)$$

Now, we trade $t \to \varphi$ as

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi}$$

but, $\dot{\varphi} = \frac{\ell}{\mu r^2} = \frac{\ell}{\mu} u^2$ from the angular EOM.

$$\Rightarrow \quad \frac{d}{dt} = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}$$

Note that $\frac{d}{dt}\left(\frac{1}{u}\right) = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}\left(\frac{1}{u}\right)$

$$= -\frac{\ell}{\mu}\frac{du}{d\varphi}$$

and

$$\frac{d^2}{dt^2}\left(\frac{1}{u}\right) = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}\left(-\frac{\ell}{\mu}\frac{du}{d\varphi}\right)$$

$$= -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\varphi^2}$$

So, the radial eqn. is

$$-\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\varphi^2} = \frac{\ell^2 u^3}{\mu^2} + \frac{1}{\mu}F\left(\frac{1}{u}\right)$$

or,

$$\boxed{\frac{d^2u}{d\varphi^2} = -u - \frac{\mu}{\ell^2 u^2}F\left(\frac{1}{u}\right)}$$

So, we have an ODE for $u(\varphi)$, from which we can find $r(\varphi) = 1/u(\varphi)$, given a force $F$.

**Example**

Consider a free particle, $F=0$. Find its orbit, $r(\varphi)$.

For $F=0$, the EOM is
$$\frac{d^2u}{d\varphi^2} = -u$$

This is the eqn of a SHO

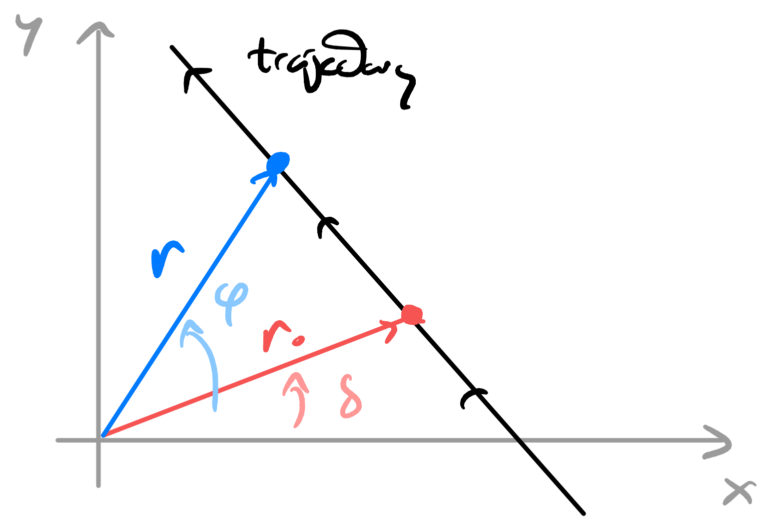
$$\Rightarrow \quad u(\varphi) = A\cos(\varphi - \delta)$$

where $A$ & $\delta$ are constants to be fixed by initial conditions.

So, $r(\varphi)$ is simply

$$\boxed{r(\varphi) = \frac{r_0}{\cos(\varphi - \delta)}}$$

where $r_0 \equiv 1/A$. ∎




Exercise: Show that is is an eqn. of a straight line!

# Kepler Orbits

Let us now specify the force as an inverse square law,

$$F(r) = -\frac{\gamma}{r^2}$$

with $\gamma > 0$ by assumption. For gravity, $\gamma = GM\mu$, while for Coulombic forces $\gamma = kq_1q_2$ (which can be + or −).

As a function of $u$, $F(u) = -\gamma u^2$.

So, the radial equation takes the form

$$\frac{d^2u}{d\varphi^2} = -u + \frac{\gamma\mu}{\ell^2}$$

constant

To solve, let $w = u - \frac{\gamma\mu}{\ell^2} \Rightarrow \frac{d^2w}{d\varphi^2} = \frac{d^2u}{d\varphi^2}$

$$\Rightarrow \frac{d^2w}{d\varphi^2} = -w$$

whose solution is $w(\varphi) = A\cos(\varphi - \delta)$

where $A$ & $\delta$ are to be fixed from initial conditions.

We can choose ICs such that $\delta = 0$, effectively choosing the axis where $\varphi = 0$.

$$\Rightarrow \quad u(\varphi) = \frac{\gamma\mu}{\ell^2} + A\cos\varphi$$
$$\equiv \frac{\gamma\mu}{\ell^2}\left(1 + \epsilon\cos\varphi\right)$$

where $\epsilon = \frac{A\ell^2}{\gamma\mu} > 0$ is a dimensionless constant.

Notice that $\left[\frac{\ell^2}{\gamma\mu}\right] = L$, let $c \equiv \frac{\ell^2}{\gamma\mu}$

So, the orbit is

$$\boxed{r(\varphi) = \frac{c}{1 + \epsilon\cos\varphi}}$$

## Bounded orbits

Let's explore the features of bounded orbits.

There two sectors of $\epsilon$, $\epsilon < 1$ & $\epsilon \geq 1$

if $\epsilon < 1$, then $1 + \epsilon\cos\varphi$ never vanishes

as $\cos 0 = +1$, $\cos\pi = -1$, so, $1+\epsilon$, $1-\epsilon > 0$.

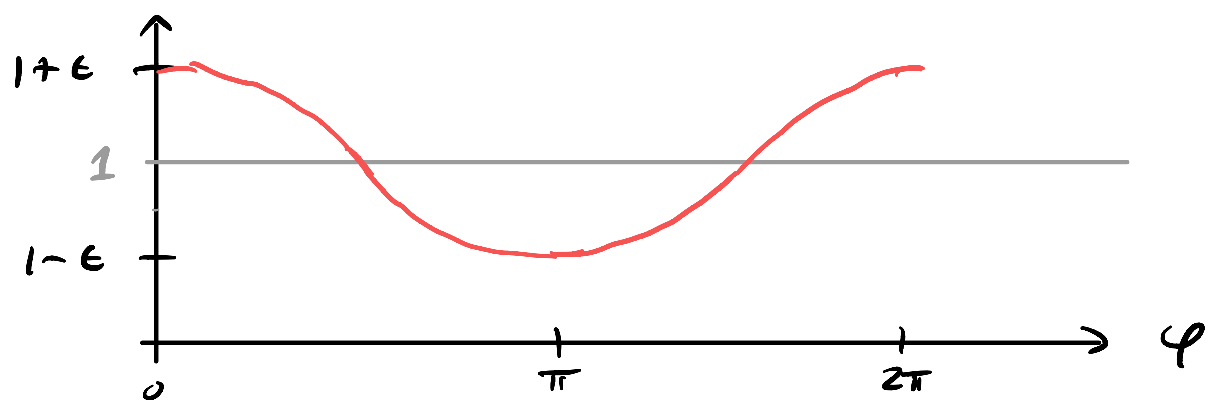
$\Rightarrow$ orbit remains bounded $\forall\, \varphi$.

so, $r(\varphi)$ oscillates between

$$r_{min} = \frac{c}{1+\epsilon} \quad \text{and} \quad r_{max} = \frac{c}{1-\epsilon}$$

Here, $r = r_{min}$ is called the periapsis when $\varphi = 0$ (or perihelion if object orbiting the sun), and $r = r_{max}$ is called apoapsis when $\varphi = \pi$ (or aphelion if orbiting the sun).

Notice that $r(\varphi)$ is periodic, $r(0) = r(2\pi)$, thus the orbit is closed.




The geometry of the orbit is an ellipse.
Recall the standard* form for the ellipse,

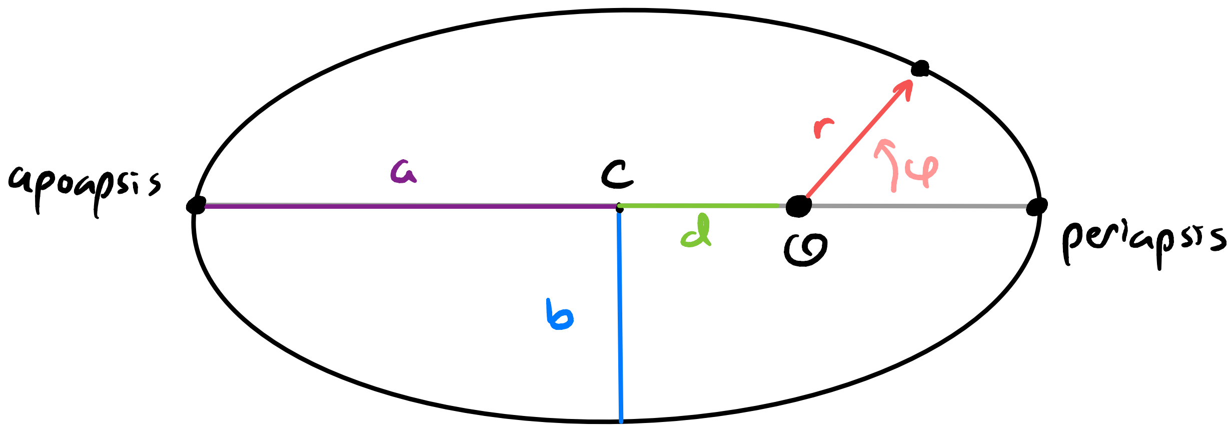
$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

semi-major axis

semi-minor axis

one can show that

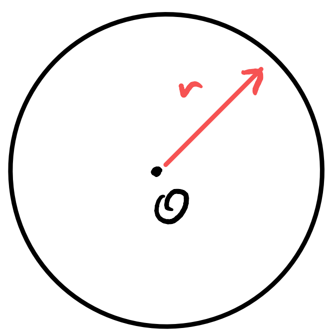
$$a = \frac{c}{1-\epsilon^2}, \quad b = \frac{c}{\sqrt{1-\epsilon^2}}, \quad \text{and} \quad d = a\epsilon$$




Notice that the ratio $\frac{b}{a} = \sqrt{1-\epsilon^2}$

eccentricity of ellipse

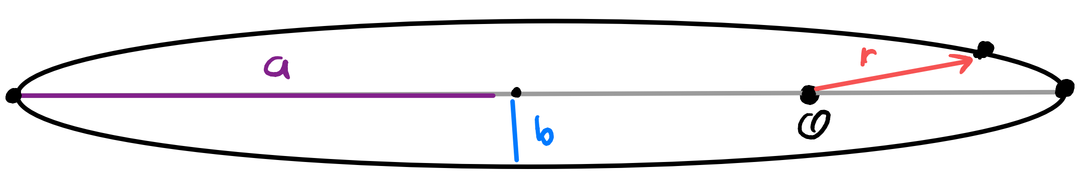
If $a = b \Rightarrow \epsilon = 0$ (a circle)




$a = b = c, \quad d = 0$

$\Rightarrow \boxed{r(\varphi) = c}$

If $\frac{b}{a} \to 0 \Rightarrow \epsilon \to 1$ (elongated ellipse)




Singularity at $\epsilon = 1 \Rightarrow$ transition from bounded to unbounded

The CM is located at $d = a\epsilon$ from the center.

This is the focus of an ellipse.

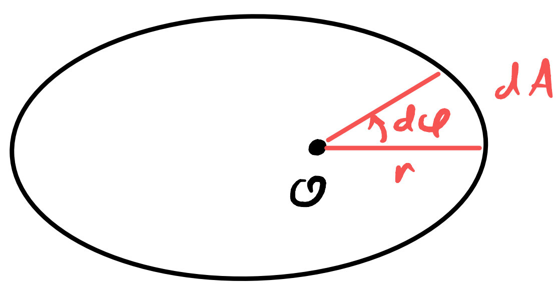
$\Rightarrow$ We have proven Kepler's 1st law

## Orbital Period

Obtaining the time-dependence of the orbit is in general very difficult. However, we can get info on the orbital period, $\tau$.

Kepler's 2nd law states $\dot{A} = \frac{\ell}{2\mu}$

To see this, look at $dA$




$$dA = \frac{1}{2} r^2 d\varphi$$

$$\Rightarrow \dot{A} = \frac{1}{2} r^2 \dot{\varphi}$$

But, recall $\varphi$ EOM : $\mu r^2 \dot{\varphi} = \ell$

$$\Rightarrow \dot{A} = \frac{\ell}{2\mu} = \text{constant!}$$

Area of ellipse $= \pi ab$

The period is then $\tau = \int_0^\tau dt = \int_0^{A_{ell}} dA \, \frac{1}{\dot{A}} = \frac{\pi a b}{\ell / 2\mu}$

constant

So,
$$\tau = \frac{2\pi ab}{\ell}\mu$$

Square both sides, & recall $b = \sqrt{1-\epsilon^2}\,a$

$$\Rightarrow \tau^2 = \frac{4\pi^2 a^4 (1-\epsilon^2)\mu^2}{\ell^2} \leftarrow a = \frac{c}{1-\epsilon^2}$$
$$= 4\pi^2 a^3 \frac{c\mu^2}{\ell^2}$$

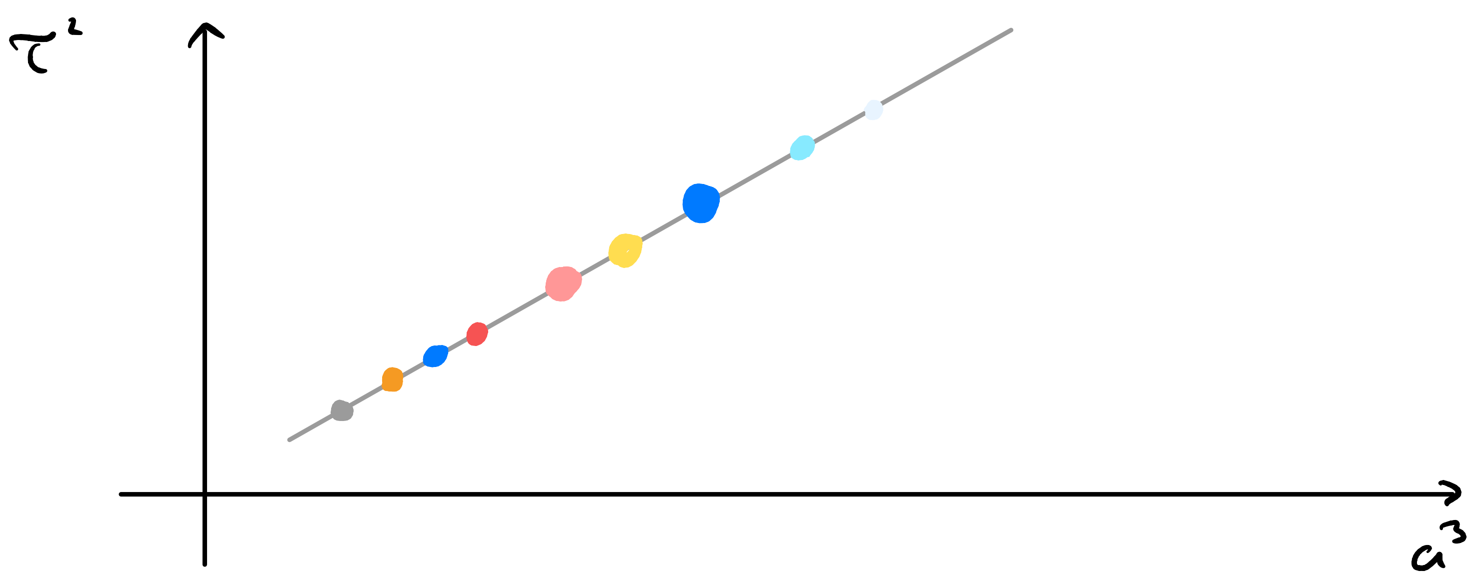
we defined $c = \frac{\ell^2}{\gamma\mu} \Rightarrow \tau^2 = \frac{4\pi^2 a^3 \mu}{\gamma}$

For gravity, $\gamma = G m_1 m_2 = G\mu M$

$$\Rightarrow \boxed{\tau^2 = \frac{4\pi^2}{GM}a^3}$$ Kepler's third law

For the solar system, $m_1 = m_{planet}$, $m_2 = M_{sun} = M_\odot$

$\Rightarrow \mu \simeq m_p, \; M \simeq M_\odot$



# Eccentricity & Energy

The measurement of eccentricity $\epsilon$ gives information on the energy of the orbiting objects.

Recall that at closest approach, $r_{min} = \dfrac{c}{1+\epsilon}$.

At this point, $\dot{\vec{r}} = 0$, and

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$
$$= \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right)$$

Recall $c = \ell^2/\gamma\mu \Rightarrow r_{min} = \dfrac{\ell^2}{\gamma\mu(1+\epsilon)}$

$$\Rightarrow \quad E = \frac{\gamma\mu(1+\epsilon)}{2\ell^2}\left(\gamma(1+\epsilon) - 2\gamma\right)$$
$$= \frac{\gamma^2\mu}{2\ell^2}\left(\epsilon^2 - 1\right)$$

Notice that since $0 \leq \epsilon < 1$ for bounded orbits, then $E < 0$, as expected!

# Time Dependence of Orbits

We have found the geometrical orbit $r = r(\varphi)$

$$r(\varphi) = \frac{c}{1+\epsilon\cos\varphi}$$

with $\delta = 0$ by choice. For astronomical research, would also like to have $\varphi = \varphi(t)$.

↑ called the true anomaly

Recall the EOM $\mu r^2 \dot{\varphi} = l$

$$\Rightarrow \quad t = \int_0^t dt' = \int_0^{\varphi} d\varphi' \, \frac{\mu r^2}{l}$$

But, recall $\tau = \frac{\pi a b}{l/2\mu}$ $\Rightarrow$ $t = \frac{1}{2}\frac{\tau}{\pi a b}\int_0^{\varphi} d\varphi' \, r(\varphi')^2$

or,

$$\frac{\pi a b}{\tau} t = \frac{c^2}{2}\int_0^{\varphi} d\varphi' \frac{1}{(1+\epsilon\cos\varphi)^2}$$

Can show (challenge) the result is Kepler's equation

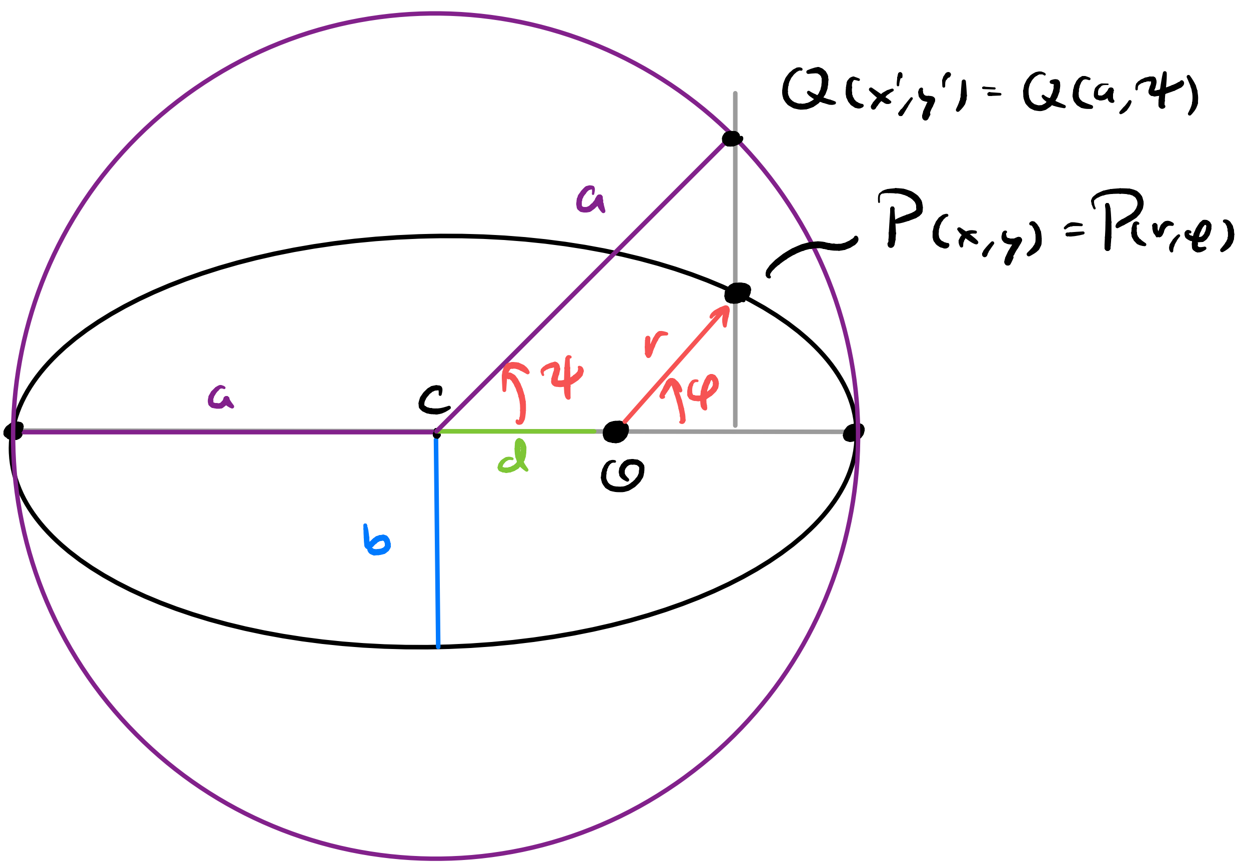
$$\frac{2\pi}{\tau} t = 2\tan^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\,\tan\frac{\varphi}{2}\right) - \frac{\epsilon\sqrt{1-\epsilon^2}\,\sin\varphi}{1+\epsilon\cos\varphi}$$

This is ... complicated

We would like to invert $t = t(\varphi) \Rightarrow \varphi = \varphi(t)$, but impossible analytically. Can construct series expansion, or numerically solve.

However, can do some semi-analytic approximations by introducing some new geometric quantities.

Let us circumscribe a circle of radius $a$ on the ellipse




$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$ Eq. of ellipse (Measured from O)

$$\frac{x'}{a^2} + \frac{y'}{a^2} = 1$$ Eq. of reference circle (Measured from C)

We introduce $\psi$ as the eccentric anomaly, which is the angle of point Q, which is the projection of point P on the reference circle.

For this projection to be true,

$$\frac{(x+a\epsilon)^2}{a^2} + \frac{y^2}{b^2} = 1 = \frac{x'^2}{a^2} + \frac{y'^2}{a^2} \qquad (*)$$

From geometry, we must have

$$\cos\psi = \frac{x'}{a} \quad \& \quad \sin\psi = \frac{y'}{a}$$

Now, $x' = d + r\cos\varphi = a\epsilon + x$

$$\Rightarrow \quad \cos\psi = \frac{x + a\epsilon}{a}$$

From $(*)$, we conclude $y' = \frac{a}{b}y \Rightarrow \sin\psi = \frac{y}{b}$

Inverting, $x = a(\cos\psi - \epsilon)$

$$y = b\sin\psi = a\sqrt{1-\epsilon^2}\,\sin\psi$$

In terms of $\varphi$, $x = r\cos\varphi$, $y = r\sin\varphi$

Can show explicit relation as (challenge)

$$\boxed{\tan\frac{\varphi}{2} = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{\psi}{2}} \qquad (1)$$

Inserting this into Kepler's equation (challenge) one obtains

$$\boxed{\frac{2\pi}{\tau}t = \psi - \epsilon\sin\psi} \qquad (2)$$

This is still a transcendental equation, but we can first approximate (2), & solve (1) for $\varphi$.

Let $M = \frac{2\pi t}{\tau}$ Mean anomaly

We want $\psi = \psi(M)$ as a solution.

Since $\psi - M = -\epsilon\sin\psi$ is an odd function on $[0,\pi]$, can expand in Fourier Sine series

$$\Rightarrow \psi(M) - M = \sum_{n=1}^{\infty} A_n(M)\sin(nM)$$

Invert,

$$A_n(M) = \frac{2}{\pi}\int_0^{\pi} dM\,(\psi(M)-M)\sin(nM)\,dM$$

$$= -\frac{2}{n\pi}\int_{-1}^{1} d(\cos(nM))\,[\psi(M)-M]$$

$$\Rightarrow \quad A_n = \cancelto{0}{-\frac{2}{n\pi}\left[\psi(M) - M\right]\cos nM \Big|_{M=0}^{\pi}} + \frac{2}{n\pi}\int_0^{\pi} dM \left[\psi'(M) - 1\right]\cos nM$$

$$= \frac{2}{n\pi}\int_0^{\pi} dM \; \psi'(M) \cos nM$$

$$= \frac{2}{n\pi}\int_0^{\pi} d(\psi(M)) \cos nM$$

Recall that $M = \psi - \epsilon \sin\psi$

Let $E = \psi(M)$

$$\Rightarrow \quad A_n = \frac{2}{n\pi}\int_0^{\pi} d(\psi(M)) \cos\left[n\psi(M) - n\epsilon \sin\psi(M)\right]$$

$$= \frac{2}{n}\left\{ \frac{1}{\pi}\int_0^{\pi} dE \cos\left[nE - n\epsilon \sin E\right] \right\}$$

$$\equiv \frac{2}{n} J_n(n\epsilon)$$ Bessel functions of 1st kind!

For small $\epsilon$, $\quad J_n(x) = x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+n}\, k!\,(k+n)!} \quad ; \quad x = n\epsilon$

So,

$$\boxed{\psi(M) = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\epsilon) \sin(nM) \qquad M = \frac{2\pi}{\tau} t}$$

For small $\epsilon$, few terms could be adequate to yield good approximation. For high eccentricity orbits, e.g., comets, often need very many terms, & thus numerical methods are preferred.

Once $\psi = \psi(t)$ is determined, either semi-analytically or numerically, then we can get $\varphi = \varphi(t)$ by

$$\varphi(t) = 2 \tan^{-1}\left[ \sqrt{\frac{1+\epsilon}{1-\epsilon}} \tan \frac{\psi(t)}{2} \right] \quad \text{mod } \pi$$

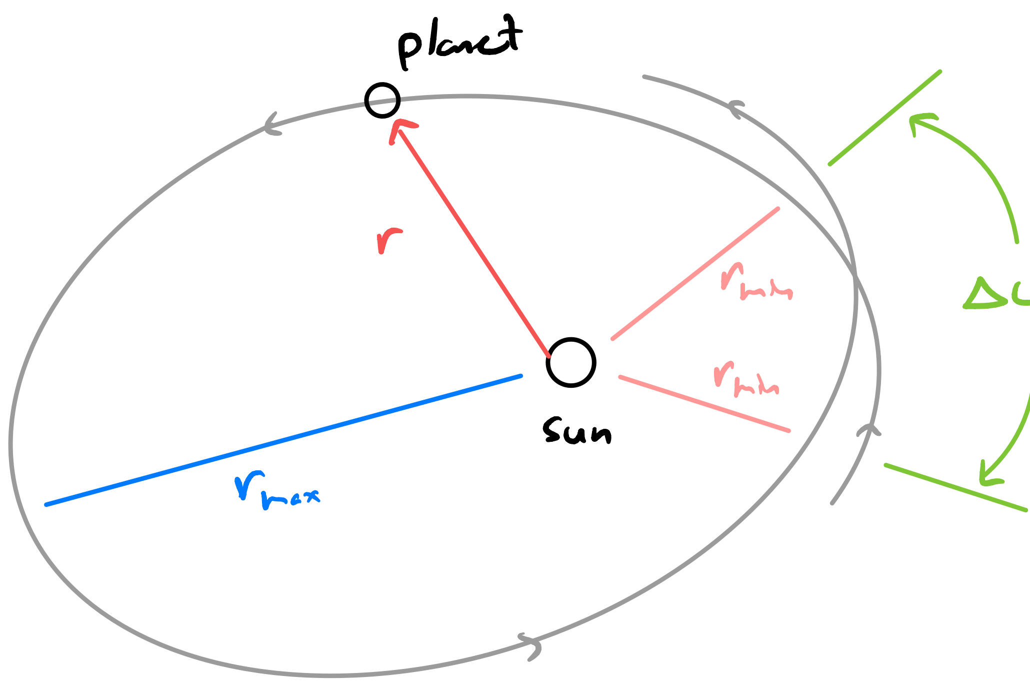
Finally, $r(t)$ is then

$$r(t) = \frac{c}{1 + \epsilon \cos \varphi(t)}$$

## Precession

Physical orbiting bodies in solar systems are rarely (never) two-body problems. Our solar system has 8 planets & many dwarf planets & other objects which all interact gravitationally.

These perturbations impact the orbit, & instead of being closed orbits, $\Delta\varphi \neq 0$, there is some deviation & the planet precesses.




These deviations can be computed for a given planet using Newtonian gravity. However, Mercury was historically seen to have issues, as its orbit precesses at an additional 43" of arc per century to the Newtonian theory.

The resolution came from Einstein's General Relativity.
General Relativity supercedes Newtonian gravity

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}$$

Spacetime Curvature Energy-Matter density

It can be shown that the relativistic corrections on an orbit appear as additional terms to the central force. Recall the equation for the orbit,

$$\frac{d^2u}{d\varphi^2} = -u + \frac{GM\mu^2}{\ell^2}$$

with $u = 1/r$. GR corrections are of the form

$$\frac{d^2u}{d\varphi^2} = -u + \frac{GM\mu^2}{\ell^2} + \frac{3GM}{c^2} u^2$$

speed of light

Since $3GM/c^2 \ll 1$, this is a small correction.
Let $\frac{1}{\alpha} \equiv \frac{GM\mu^2}{\ell^2}$ & $\delta = \frac{3GM}{c^2}$

$$\Rightarrow \quad \frac{d^2u}{d\varphi^2} = -u + \frac{1}{\alpha} + \delta u^2$$

To solve this, we construct a perturbation series in $\delta$,

$$u = u_0 + \delta u_1 + \mathcal{O}(\delta^2)$$

ignore

Expanding,

$$\frac{d^2 u_0}{d\varphi^2} + \delta \frac{d^2 u_1}{d\varphi^2} = -u_0 - \delta u_1 + \frac{1}{\alpha} + \delta (u_0 + \delta u_1)^2$$

$$= -u_0 - \delta u_1 + \frac{1}{\alpha} + \delta u_0^2 + \mathcal{O}(\delta^2)$$

Collecting powers of $\delta$,

$$\delta^0 : \quad \frac{d^2 u_0}{d\varphi^2} = -u_0 + \frac{1}{\alpha} \quad \Rightarrow \quad u_0 = \frac{1}{\alpha}(1 + \epsilon \cos\varphi)$$

As before

$$\delta^1 : \quad \frac{d^2 u_1}{d\varphi^2} = -u_1 + u_0^2$$

$$= -u_1 + \frac{1}{\alpha^2}(1 + 2\epsilon\cos\varphi + \epsilon^2 \cos^2\varphi)$$

To solve, $u_1 = u_1^{(h)} + u_1^{(p)}$

homogeneous particular

$$u_1^{(h)} = A\cos(\varphi - \psi)$$

But, ICs fix $u = u[\delta = 0] \Rightarrow A = \psi = 0$.

Only need to enforce particular solution.

Can show that $u_1^{(\varphi)}$ is

$$u_1 = u_1^{(\varphi)} = \frac{1}{\alpha^2}\left[\left(1+\frac{\epsilon^2}{2}\right) + \epsilon\varphi\sin\varphi - \frac{\epsilon^2}{6}\cos 2\varphi\right]$$

So, to $\mathcal{O}(\delta)$,

$$u(\varphi) = \frac{1}{\alpha}(1+\epsilon\cos\varphi) + \frac{\delta\epsilon}{\alpha^2}\varphi\sin\varphi$$
$$+ \underbrace{\frac{\delta}{\alpha^2}\left(1+\frac{\epsilon^2}{2}\right)}_{\text{small constant}} - \underbrace{\frac{\delta\epsilon^2}{6\alpha^2}\cos 2\varphi}_{\text{small periodic disturbance}}$$

For large timescales, the last two terms will average out. So, lets ignore these

$$\Rightarrow u \simeq \frac{1}{\alpha}\left[1+\epsilon\cos\varphi + \frac{\delta\epsilon}{\alpha}\varphi\sin\varphi\right]$$

For small $\delta$, $\cos\frac{\delta}{\alpha}\varphi \simeq 1$, $\sin\frac{\delta}{\alpha}\varphi \simeq \frac{\delta}{\alpha}\varphi$

$$\Rightarrow u \simeq \frac{1}{\alpha}\left[1 + \epsilon\cos\left(\varphi - \frac{\delta}{\alpha}\varphi\right)\right]$$

At $t=0$, $\varphi=0$ as chosen. At successive periods

$$\varphi - \frac{\delta}{\alpha}\varphi = 2\pi$$

Solving for $\varphi$,

$$\varphi = \frac{2\pi}{1-\delta/\alpha} \simeq 2\pi\left(1 + \frac{\delta}{\alpha}\right)$$

So, $\Delta\varphi \simeq \frac{2\pi\delta}{\alpha} = 6\pi\left(\frac{GM\mu}{c\ell}\right)^2$

$$\Rightarrow \boxed{\Delta\varphi \simeq \frac{6\pi GM}{ac^2(1-\epsilon^2)}}$$

For Mercury, $\Delta\varphi_{calc} = 43.03 \pm 0.03$

$\Delta\varphi_{obs} = 43.11 \pm 0.45$

Excellent agreement!