1. Given the generators X^j for a Lie algebra $[X^j, X^k] = c_{jkl}X^l$, normalized such that $\mathrm{tr}(X^jX^k) = \mu_r\delta_{jk}$, show that the structure constants can be computed with

$$c_{jkl} = \frac{1}{\mu_r}\,\mathrm{tr}([X^j, X^k]X^l)\,.$$

Show that c_{jkl} are antisymmetric under interchange of any two indices.

Solution: Multiply the Lie bracket on the right by X^n, $[X^j, X^k]X^n = c_{jkl}X^lX^n$. Next, take the trace

$$\mathrm{tr}\left([X^j, X^k]X^n\right) = c_{jkl}\,\mathrm{tr}\left(X^lX^n\right) = c_{jkl}\,\mu_r\delta_{ln} = \mu_r c_{jkn}\,.$$

Isolating c_{jkn}, we find the desired relation,

$$c_{jkl} = \frac{1}{\mu_r}\,\mathrm{tr}([X^j, X^k]X^l)\,.$$

Note that from the cyclic properties of the trace, we find

$$\begin{aligned}\mathrm{tr}([X^j, X^k]X^l) &= \mathrm{tr}(X^jX^kX^l - X^kX^jX^l)\,,\\ &= \mathrm{tr}(X^jX^kX^l - X^jX^lX^k)\,,\\ &= \mathrm{tr}(X^j[X^k, X^l]) = \mathrm{tr}([X^k, X^l]X^j)\,,\end{aligned}$$

where the cyclic property was used on the second term of the second line, and

$$\begin{aligned}\mathrm{tr}([X^j, X^k]X^l) &= \mathrm{tr}(X^jX^kX^l - X^kX^jX^l)\,,\\ &= \mathrm{tr}(X^kX^lX^j - X^kX^jX^l)\,,\\ &= \mathrm{tr}(X^k[X^l, X^j]) = \mathrm{tr}([X^l, X^j]X^k)\,,\end{aligned}$$

where again the cyclic property was used on the first term of the second line. Thus, the structure constants are given by

$$c_{jkl} = \frac{1}{\mu_r}\,\mathrm{tr}([X^j, X^k]X^l) = \frac{1}{\mu_r}\,\mathrm{tr}([X^k, X^l]X^j) = \frac{1}{\mu_r}\,\mathrm{tr}([X^l, X^j]X^k)\,.$$

Since the Lie bracket is antisymmetric, $[X^j, X^k] = -[X^k, X^j]$, we see that c_{jkl} is antisymmetric under the interchange of any pair of indices (j, k), (k, l), and (l, j). Thus,

$$c_{jkl} = -c_{jlk}, \quad c_{jkl} = -c_{kjl}, \quad c_{jkl} = -c_{lkj}\,.$$

2. Compute the non-zero structure constants f_{abc} for the $\mathfrak{su}(3)$ algebra $[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c$, where λ_a are the Gell-Mann matrices. **Hint:** It is convenient to use a symbolic algebra software like `Mathematica`.

Solution: The normalization of the Gell-Mann matrices are $\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. So, from Problem 1, we have

$$f_{abc} = \frac{1}{4i}\,\mathrm{tr}\left([\lambda_a, \lambda_b]\lambda_c\right) .$$

Using `Mathematica`, we can write f_{abc} for each $a = 1, \ldots, 8$ as a matrix in bc,

$$f_{1bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{2bc} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_{3bc} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{4bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \end{pmatrix},$$

$$f_{5bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, f_{6bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix},$$

$$f_{7bc} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}, \quad f_{8bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Therefore, the following entries are non-zero:

$$f_{123} = 1\,,$$
$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = 1/2\,,$$
$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}\,,$$

where other non-zero entries are given by the interchange of any pair of indices.

3. The Gell-Mann matrices also satisfy the relation

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} I_3 + 2 d_{abc}\lambda_c\,,$$

where d_{abc} are symmetric under the interchange of any two indices. Compute the non-zero values of d_{abc}. **Hint:** It is convenient to use a symbolic algebra software like `Mathematica`.

Solution: To isolate d_{abc}, multiply the anticommutator on the right by λ_e and take the trace,

$$\begin{aligned}\mathrm{tr}\left(\{\lambda_a, \lambda_b\}\lambda_e\right) &= \frac{4}{3}\delta_{ab}\,\mathrm{tr}(\lambda_e) + 2d_{abc}\,\mathrm{tr}\left(\lambda_c\lambda_e\right)\,,\\ &= 4 d_{abe}\,,\end{aligned}$$

where we used that $\mathrm{tr}(\lambda_a) = 0$ and $\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. Thus,

$$d_{abc} = \frac{1}{4}\,\mathrm{tr}\left(\{\lambda_a, \lambda_b\}\lambda_c\right)\,.$$

Using `Mathematica`, we can write d_{abc} for each $a = 1, \ldots, 8$ as a matrix in bc,

$$d_{1bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_{2bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d_{3bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_{4bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d_{5bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{pmatrix}, \quad d_{6bc} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 \end{pmatrix},$$

$$d_{7bc} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 \end{pmatrix},$$

$$d_{8bc} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

Therefore, the non-zero values of d_{abc} are

$$d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}} \,,$$

$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2} \,,$$

$$d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \,,$$

where other non-zero entries are given by the interchange of any pair of indices.

4. Show that the $\mathbf{3}^*$ of $\mathfrak{su}(3)$ is inequivalent to the $\mathbf{3}$ of $\mathfrak{su}(3)$. **Hint:** Show that $(-\lambda_a^*)$ cannot be transformed to λ_a by a unitary transformation for every $a = 1, 2, \ldots, 8$.

Solution: Recall that a unitary transformation preserves the spectrum of a matrix. Suppose there exists a unitary matrix U such that $U^{-1}\lambda_a U = \Lambda_a$, where $\Lambda_a = \text{diag}(\lambda_a^{(1)}, \lambda_a^{(2)}, \lambda_a^{(3)})$, where $\lambda_a^{(j)}$ with $j = 1, 2, 3$ are the eigenvalues of λ_a. If there exists another unitary matrix V such that $V^{-1}(-\lambda_a^*)V = \lambda_a$, then the spectrum of $(-\lambda_a^*)$ must be identical to λ_a for each $a = 1, \ldots, 8$. Consider λ_8,

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \,,$$

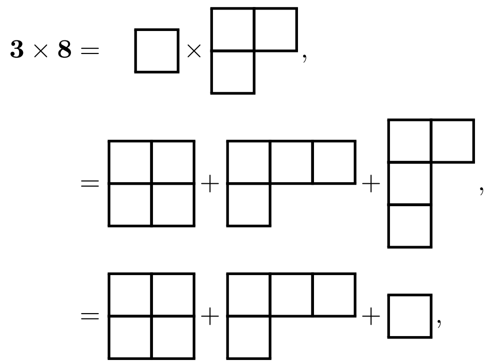
which since it is diagonal the eigenvalues are $\{1/\sqrt{3}, 1/\sqrt{3}, -2/\sqrt{3}\}$. Now, consider $(-\lambda_8^*)$,

$$-\lambda_8^* = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \,,$$

from which we see the eigenvalues are $\{-1/\sqrt{3}, -1/\sqrt{3}, 2/\sqrt{3}\}$. We see that the eigenvalues of $(-\lambda_8^*)$ are not the same as λ_8. Thus, there is no such unitary transformation V, and we conclude that the $\mathbf{3}^*$ is inequivalent to the $\mathbf{3}$ of $\mathfrak{su}(3)$.

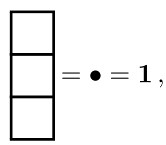
5. Perform the Clebsch-Gordan decomposition for the following $\mathfrak{su}(3)$ products using Young Tableau, labeling the dimension of each representation: (a) $\mathbf{3} \times \mathbf{3} \times \mathbf{8}$, and (b) $\mathbf{3} \times \mathbf{3}^* \times \mathbf{8}$.

Solution: For (a), we first, let us look at $\mathbf{3} \times \mathbf{8}$,



$$= \mathbf{6}^* + \mathbf{15} + \mathbf{3}\,,$$

where for the last diagram we used that



and found the dimension of the tableaux by using the dimension formula

$$N(a_1, a_2) = \frac{1}{2}(a_1 + 1)(a_2 + 1)(a_1 + a_2 + 2)\,,$$

where

$$a_1 = \text{the number of boxes the first row exceeds the second row}\,,$$

$$a_2 = \text{the number of boxes in the second row}\,.$$

Using this result, we can now take the product $\mathbf{3}\times\mathbf{3}\times\mathbf{8}$,

$$
\begin{aligned}
\mathbf{3}\times\mathbf{3}\times\mathbf{8} &= \ydiagram{1}\times\ydiagram{1}\times\ydiagram{2,1}\,, \\
&= \ydiagram{1}\times\left(\ydiagram{1}\times\ydiagram{2,1}\right), \\
&= \ydiagram{1}\times\left(\ydiagram{2,2}+\ydiagram{3,1}+\ydiagram{1}\right), \\
&= \left(\ydiagram{1}\times\ydiagram{2,2}\right)+\left(\ydiagram{1}\times\ydiagram{3,1}\right)+\left(\ydiagram{1}\times\ydiagram{1}\right), \\
&= \left(\ydiagram{2,2,1}+\ydiagram{3,2}\right)+\left(\ydiagram{3,1,1}+\ydiagram{3,2}+\ydiagram{4,1}\right) \\
&\qquad +\left(\ydiagram{1,1}+\ydiagram{2}\right).
\end{aligned}
$$

We now use the fact that

$$\ydiagram{1,1,1} = \bullet = \mathbf{1}\,,$$

so that

$$
\begin{aligned}
\mathbf{3}\times\mathbf{3}\times\mathbf{8} &= \left(\ydiagram{1,1}+\ydiagram{3,2}\right)+\left(\ydiagram{2}+\ydiagram{3,2}+\ydiagram{4,1}\right) \\
&\qquad +\left(\ydiagram{1,1}+\ydiagram{2}\right), \\
&= (\mathbf{3}^*+\mathbf{15}^*)+(\mathbf{6}+\mathbf{15}^*+\mathbf{24})+(\mathbf{3}^*+\mathbf{6})\,.
\end{aligned}
$$

So, the Clebsch-Gordan decomposition is $\mathbf{3}\times\mathbf{3}\times\mathbf{8}=\mathbf{3}^*+\mathbf{3}^*+\mathbf{6}+\mathbf{6}+\mathbf{15}^*+\mathbf{15}^*+\mathbf{24}$.

For (b), we first, let us look at $\mathbf{3}^* \times \mathbf{8}$,

$$
\begin{aligned}
\mathbf{3}^* \times \mathbf{8} &= \ydiagram{1,1} \times \ydiagram{2,1}\,, \\
&= \ydiagram{1,1} + \ydiagram{2} + \ydiagram{3,2}\,, \\
&= \mathbf{3}^* + \mathbf{6} + \mathbf{15}^*\,,
\end{aligned}
$$

where we used the Language and shape rules to eliminate invalid diagrams. Taking the product $\mathbf{3} \times \mathbf{3}^* \times \mathbf{8}$, we have

$$
\begin{aligned}
\mathbf{3} \times \mathbf{3}^* \times \mathbf{8} = \quad & \ydiagram{1} \times \ydiagram{1,1} \times \ydiagram{2,1}\,, \\
= \quad & \ydiagram{1} \times \left(\ydiagram{1,1} \times \ydiagram{2,1} \right)\,, \\
= \quad & \ydiagram{1} \times \left(\ydiagram{1,1} + \ydiagram{2} + \ydiagram{3,2} \right)\,, \\
= \quad & \left(\ydiagram{1} \times \ydiagram{1,1} \right) + \left(\ydiagram{1} \times \ydiagram{2} \right) + \left(\ydiagram{1} \times \ydiagram{3,2} \right)\,, \\
= \quad & \left(\ydiagram{1,1,1} + \ydiagram{2,1} \right) + \left(\ydiagram{2,1} + \ydiagram{3} \right) \\
& + \left(\ydiagram{3,2,1} + \ydiagram{3,3} + \ydiagram{4,2} \right)\,, \\
= \quad & \left(\bullet + \ydiagram{2,1} \right) + \left(\ydiagram{2,1} + \ydiagram{3} \right) \\
& + \left(\ydiagram{2,1} + \ydiagram{3,3} + \ydiagram{4,2} \right)\,, \\
= \quad & (\mathbf{1} + \mathbf{8}) + (\mathbf{8} + \mathbf{10}) + (\mathbf{8} + \mathbf{10}^* + \mathbf{27})\,.
\end{aligned}
$$

So, the Clebsch-Gordan decomposition is $\mathbf{3} \times \mathbf{3}^* \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{8} + \mathbf{10} + \mathbf{10}^* + \mathbf{27}$.

6. Using the current *Review of Particle Physics* particle listings or the summary tables (Particle Data Group, https://pdg.lbl.gov), complete Table 1 for some typical light and strange *mesons*. For hadrons without an explicit charge index, label all possible charges in the multiplet. For hadrons with multiple decay modes, we list principle ones as those with branching ratios greater than 1%.

> **Solution:** See table below. For hadrons with widths reported in the Review of Particle Physics, we use $\tau = \hbar/\Gamma$ to estimate lifetimes, with $\hbar \approx 5.58 \times 10^{-22}\,\mathrm{MeV}\cdot s$. For multiplet states, we average the widths between all charge states. For hadrons with multiple decay modes, we list principle ones as those with branching ratios greater than 1%. For the neutral kaons $K^0, \bar{K}^0$ are mass eigenstates, and flavor oscillations mean that these hadrons decay via K_S, K_L, which are not eigenstates.

7. Using the current *Review of Particle Physics* particle listings or the summary tables (Particle Data Group, https://pdg.lbl.gov), complete Table 2 for some typical light and strange *baryons*. Note that for some listings, the decay width is reported as $\Gamma = -2\,\mathrm{Im}\,(\text{pole position})$. For hadrons without an explicit charge index, label all possible charges in the multiplet.

> **Solution:** See table below. For hadrons with widths reported in the Review of Particle Physics, we use $\tau = \hbar/\Gamma$ to estimate lifetimes. For multiplet states, we average the widths between all charge states.

8. Classify the following observed reactions into strong, electromagnetic, and weak processes:

(a) $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$,

> **Solution:** The presence of the leptons in the final state indicates a non-strong process. Looking at isospin I_3 conservation,
>
> $$\begin{aligned} &\pi^- \to \pi^0 + e^- + \bar{\nu}_e\,, \\ I_3: \quad &-1 \to 0 + 0 + 0\,, \end{aligned}$$
>
> we see that since $\Delta I_3 = 0 - (-1) \neq 0$, the interaction must be *weak* process.

(b) $\gamma + p \to \pi^+ + n$,

> **Solution:** The photo-production of a a pion on a nuclear target is not a pure strong process. Again, isospin I_3 conservation again gives
>
> $$\begin{aligned} &\gamma + p \to \pi^+ + n\,, \\ I_3: \quad &0 + \frac{1}{2} \to 1 + \left(-\frac{1}{2}\right)\,. \end{aligned}$$
>
> We see that $\Delta I_3 = 0$, and we conclude that this is an *electromagnetic* process.

(c) $p + \bar{p} \to \pi^+ + \pi^- + \pi^0$,

Solution: Without the presence of leptons, we suspect that this is a strong processes. As before, isospin I_3 conservation. yields

$$p + \bar{p} \to \pi^+ + \pi^- + \pi^0 \,,$$

$$I_3 : \quad \frac{1}{2} + \left(-\frac{1}{2}\right) \to 1 + (-1) + 0 \,,$$

and we see that $\Delta I_3 = 0$. Moreover, the total I is conserved as $p + \bar{p}$ gives either $I = 0$ or 1, and three pions can have $I = 0, 1, 2, 3$. Therefore, this reaction can occur through either $I = 0$ or 1 modes. We conclude that this reaction is a *strong* process.

(d) $D^- \to K^+ + 2\pi^-$,

Solution: Checking if isospin I_3 is conserved,

$$D^{-1} \to K^+ + \pi^- + \pi^- \,,$$

$$I_3 : \quad \left(-\frac{1}{2}\right) \to \frac{1}{2} + (-1) + (-1) \,,$$

we see that $\Delta I_3 = -3/2 - 1/2 \neq 0$, therefore this decay must be a *weak* process. Another indication that this is a weak process is to look at ΔC and ΔS, which are both non-zero in this reaction, indicating flavor changing which is mediated by the weak interaction.

(e) $\Lambda^0 + p \to K^- + 2p$,

Solution: With no leptons in the reactants or products, and isospin I_3 being conserved $\Delta I_3 = 0$,

$$\Lambda^0 + p \to K^- + p + p \,,$$

$$I_3 : \quad 0 + \frac{1}{2} \to \left(-\frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2} \,,$$

as well as strangeness being conserved $\Delta S = 0$, we find that this process is flavor preserving, and conclude that this reaction is a *strong* process.

(f) $\pi^- + p \to n + e^+ + e^-$.

Solution: Total isospin must be violated due to the production of leptons. Checking if isospin I_3 is conserved,

$$\pi^- + p \to n + e^+ + e^- \,,$$

$$I_3 : \quad (-1) + 1/2 \to (-1/2) + 0 + 0 \,,$$

we see that indeed $\Delta I_3 = 0$, so this reaction is not mediated by the weak interaction. Therefore this reaction is an *electromagnetic* process.

9. Both the ρ^0 meson and the ω meson are vector mesons, $J^{PC} = 1^{--}$. However, the ρ^0 is observed to strongly decay predominately into 2π, while the ω is observed to decay into 3π. Why this is so?

Solution: While both hadrons are vector mesons with $J^{PC} = 1^{--}$, note that the ρ^0 is an isovector $I^G = 1^+$ while the ω^0 is an isoscalar $I^G = 0^-$, which can be seen from the Review of Particle Physics. The G-parity of an n-pion state is $G_{n\pi} = (-1)^n$ since $G_\pi = -$. So, if isospin is exact, $\omega^0 \to 3\pi$ is only allowed since $G_\omega = -$, while $\rho^0 \to 2\pi$ is allowed since $G_\rho = +$. Since isospin is broken, but mildly, this means that these decay modes are dominant.

10. Consider πN scattering at the $\Delta(1232)$ resonance, i.e., at center-of-momentum energies $\sqrt{s} \sim 1232\,\text{MeV}$. For this reaction, $\pi N \to \Delta(1232) \to \pi N$, focus on the following three processes:

(**a**) $\pi^+ p \to \pi^+ p$ *elastic scattering* via the Δ^{++} resonance,
(**b**) $\pi^- p \to \pi^- p$ *elastic scattering* via the Δ^0 resonance,
(**c**) $\pi^- p \to \pi^0 n$ *charge exchange* via the Δ^0 resonance.

Estimate the relative cross sections $\sigma_a : \sigma_b : \sigma_c$.

Solution: Since $m_{\pi^\pm} \approx m_{\pi^0}$ $m_p \approx m_n$, and $m_{\Delta^{++}} \approx m_{\Delta^0}$, the approximate isospin symmetry can be considered a good symmetry for this reaction. Let us therefore assume m_π as the mass of the $I = 1$ pion multiplet, m_N as the mass of the $I = 1/2$ nucleon doublet, and m_Δ as the mass of the $I = 3/2$ delta multiplet.

The cross-section in the Δ-region has the structure

$$\sigma_\Delta \propto |\langle f|\, T_\Delta \,|i\rangle|^2 \times (\text{kinematic factors})\,,$$

where T_Δ is the T matrix with Δ quantum numbers, and $|i\rangle$ and $|f\rangle$ are the initial and final states of the processes of interest. Since a particular state can be expressed in terms of the isospin states $|II_3\rangle$, where $I = I_3 = 3/2$ for the Δ^{++} channel and $I = 3/2$, $I_3 = -1/2$ for the Δ^0 channel, then the ratio of the cross sections will involve only ratios of the isospin Clebsch-Gordan coefficients associated with overlaps of either $|\Delta(3/2, 3/2)\rangle$ or $|\Delta(3/2, -1/2)\rangle$. In terms of πN isospin states, the processes are given by

$$|\pi^+ p\rangle = |\pi(1, +1)\rangle \otimes |N(1/2, +1/2)\rangle = |\pi N(3/2, +3/2)\rangle\ ,$$

$$|\pi^- p\rangle = |\pi(1, -1)\rangle \otimes |N(1/2, +1/2)\rangle = \sqrt{\frac{1}{3}}\,|\pi N(3/2, -1/2)\rangle - \sqrt{\frac{2}{3}}\,|\pi N(1/2, -1/2)\rangle\ ,$$

$$|\pi^0 n\rangle = |\pi(1, 0)\rangle \otimes |N(1/2, -1/2)\rangle = \sqrt{\frac{2}{3}}\,|\pi N(3/2, -1/2)\rangle + \sqrt{\frac{1}{3}}\,|\pi N(1/2, -1/2)\rangle\ .$$

Therefore, the ratios of the cross-sections are

$$\begin{aligned}\sigma_a : \sigma_b : \sigma_c &= \Big|1 \cdot 1\Big|^2 : \left|\sqrt{\frac{1}{3}} \cdot \sqrt{\frac{1}{3}}\right|^2 : \left|\sqrt{\frac{1}{3}} \cdot \sqrt{\frac{2}{3}}\right|^2 ,\\ &= 1 : \frac{1}{9} : \frac{2}{9}\,,\\ &= 9 : 1 : 2\,,\end{aligned}$$

in qualitative agreement with experiment.

11. Given the plot of the πN total cross-sections shown in Fig. 1, identify potential resonances and estimate their mass and decay widths, as well as their charge, strange, and baryon quantum numbers. Further, identity their potential spin and isospin quantum numbers. Referring to the *Review of Particle Physics*, can you identify candidates for these unstable states?

Solution:
From the total cross-section alone, it is difficult to rigorously identify resonances, one needs to do a *partial wave analysis* on differential cross sections and other angular observables to get a more complete spectroscopic picture. Indeed, in this energy region, for this system, there are about 14 observed resonances in the πN spectrum between threshold and $\sqrt{s} \sim 1.8\,\text{MeV}$, about 5 excited Δ states and 9 N states.

However, a rough estimate may get us some idea of the what the spectral content is of some reaction. Here, we identify four 'strong' bumps, one in $\pi^+ p$, and three in $\pi^- p$. The bump in $\pi^+ p$ peaks around $\sqrt{s} \sim 1.23\,\text{GeV}$, and the bumps in $\pi^- p$ peak at $\sqrt{s} \sim 1.23\,\text{GeV}$, $1.52\,\text{GeV}$, and $1.68\,\text{GeV}$. It is reasonable to assume that the two bumps at $\sqrt{s} \sim 1.23\,\text{MeV}$ are different isospin states of the same resonance. Therefore, we can "easily" identify three resonances, which we call R_1 ($m_1 \sim 1.23\,\text{GeV}$), R_2 ($m_2 \sim 1.52\,\text{GeV}$), and R_3 ($m_3 \sim 1.68\,\text{GeV}$). Assuming a Breit-Wigner form for each resonance,

$$\sigma_R \propto \frac{1}{(s - m_R^2)^2 + m_R^2 \Gamma_R^2} \,,$$

where m_R is the mass of the resonance and Γ_R is the width of the resonance, we that the full-width at half-maximum for the first peak is $\Gamma_{R_1} \sim (1.28-1.18)\,\text{GeV} = 0.10\,\text{GeV}$, the second peak is $\Gamma_{R_2} \sim (1.55-1.47)\,\text{GeV} = 0.08\,\text{GeV}$, and the third peak is $\Gamma_{R_3} \sim (1.72-1.64)\,\text{GeV} = 0.08\,\text{GeV}$. For the higher resonance, we measure with respect to the background cross-section.

Since these states are resonances in $N\pi$, the strangeness for every resonance is $S = 0$ and the baryon number is $B_n = 1$. Since the N is an isospinor, and π is an isovector, the $N\pi$ state can be either $I = 1/2$ or $3/2$. Moreover, since the N is a spin-1/2 object, and the π is spinless, the total spin of the $N\pi$ system is $s = 1/2$. So, the total angular moment J must be $|\ell - 1/2| \le J \le \ell + 1/2$, where $\ell = 0, 1, 2, \ldots$ is the orbital angular momentum of the $N\pi$ system.

The isospin quantum numbers for $\pi^+ p$ must be $I = I_3 = 3/2$. Since the first resonance in $\pi^+ p$ must have charge $Q = +2$, we identify this as the Δ^{++}, $R_1 \to \Delta^{++}$, which has a mass $m_{\Delta^{++}} \approx 1.21\,\text{GeV}$ and width $\Gamma_{\Delta^{++}} \approx 0.1\,\text{GeV}$, which agrees with are rough estimate. Since $J^P = 3/2^+$, and the total spin of $N\pi$ is $s = 1/2$, we conclude that the orbital angular momentum of the state is $\ell = 1$, or a P-wave resonance, since the parity of the $N\pi$ state is always $P = (-1)^{\ell+1}$.

For the $\pi^- p$ cross-section, we can have either $I = 1/2$ or $I = 3/2$, which means R_2 and R_3 are either an excited N or Δ state. The charge of R_2 and R_3 is $Q = 0$. Looking at the RPP, we find the following candidates for R_2: $\Delta(1600)$ (with $J^P = 3/2^+$, $m = 1.52\,\text{GeV}$, and $\Gamma = 0.28\,\text{GeV}$), $N(1520)$ (with $J^P = 3/2^-$, $m = 1.51\,\text{GeV}$, and $\Gamma = 0.11\,\text{GeV}$), and $N(1535)$ (with $J^P = 1/2^-$,

$m = 1.51\,\text{GeV}$, and $\Gamma = 0.11\,\text{GeV}$). Given the estimated width is $\Gamma_{R_2} \approx 0.08\,\text{GeV}$, we postulate that R_2 is either $N(1520)$ with $J^P = 3/2^-$ or the $N(1535)$ with $J^P = 1/2^-$. This means that the isospin of R_2 is $I = 1/2$. Since $\ell = J \pm 1/2$, and $P = -1 = (-1)^{\ell+1}$, then for the $J = 1/2$ case we have an S wave state ($\ell = 0$), while for the $J = 3/2$ state it is a D wave reaction ($\ell = 2$).

For the R_3 resonance, the possible states are $N(1650)$ (with $J^P = 1/2^-$, $m = 1.67\,\text{GeV}$, and $\Gamma = 0.14\,\text{GeV}$), $N(1675)$ (with $J^P = 5/2^-$, $m = 1.66\,\text{GeV}$, and $\Gamma = 0.14\,\text{GeV}$), $N(1680)$ (with $J^P = 5/2^+$, $m = 1.67\,\text{GeV}$, and $\Gamma = 0.12\,\text{GeV}$), and $\Delta(1700)$ (with $J^P = 3/2^-$, $m = 1.66\,\text{GeV}$, and $\Gamma = 0.25\,\text{GeV}$). Since the estimated width is $\Gamma_{R_3} \sim 0.08\,\text{GeV}$, we postulate that R_3 is either $N(1650)$ with $J^P = 1/2^-$, $N(1675)$ with $J^P = 5/2^-$, or $N(1680)$ with $J^P = 5/2^+$. Again, the isospin of R_3 is $I = 1/2$. For the $J^P = 1/2^-$ state, the partial wave is $\ell = 0$, for the $J^P = 5/2^-$ state it is $\ell = 2$, and for $J^P = 5/2^+$ it is $\ell = 3$.

To distinguish these states further, one needs to do an angular analysis to determine the spin-parity quantum numbers.

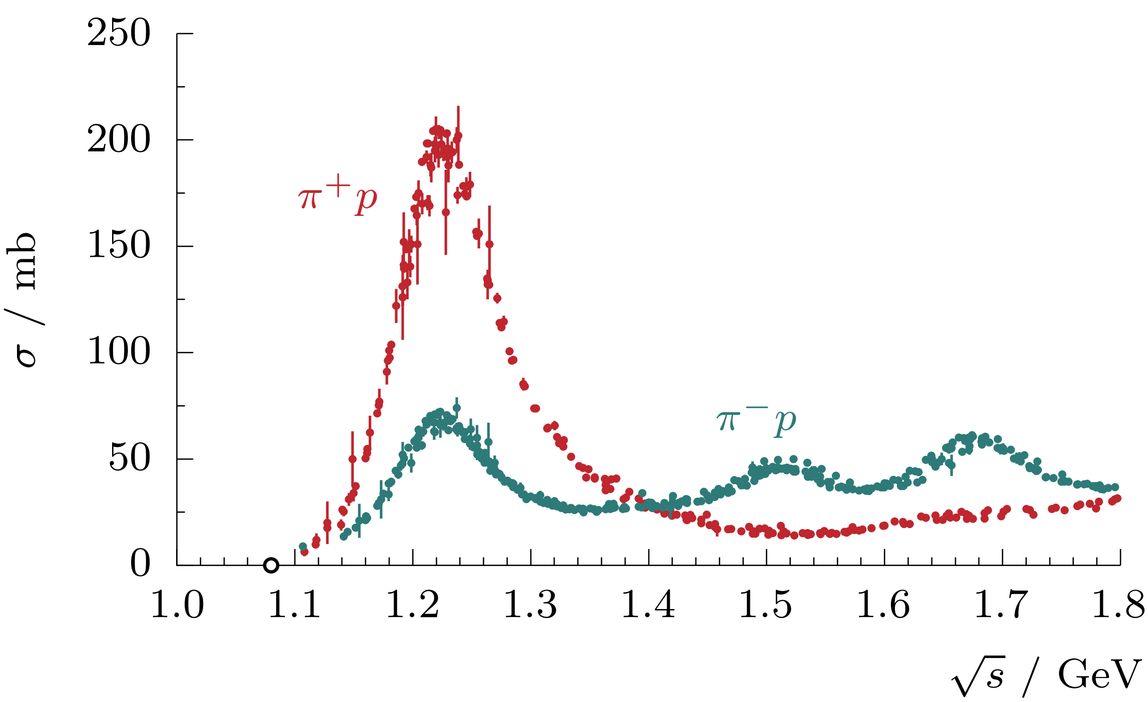


Figure 1: Total πN cross-sections as a function of center-of-momentum frame energy $\sqrt{s}$. Data taken from the *Review of Particle Physics* by the Particle Data Group.

Table 1: Light and Strange Mesons.

Meson	**Quark Content**	$J^{P(C)}$	$I^{(G)}$	**Charge**	**Mass** / MeV	**Lifetime** / s	**Principle Decay Modes**
$\pi^\pm$	$u\bar{d}, d\bar{u}$	0^-	1^-	±1	139.57	2.60×10^{-8}	$\mu^+\bar{\nu}$
π^0	$u\bar{u}-d\bar{d}$	0^{-+}	1^-	0	134.98	8.42×10^{-17}	$\gamma\gamma$
$K^\pm$	$u\bar{s}$, $\bar{u}s$	0^-	$1/2$	±1	493.68	1.24×10^{-8}	$\mu^+\nu_\mu$, $\pi^+\pi^0$
$K^0, \bar{K}^0$	$d\bar{s}$, $\bar{d}s$	0^-	$1/2$	0	497.61	—	—
K_S	$d\bar{s}$, $\bar{d}s$	0^-	$1/2$	0	—	8.95×10^{-11}	$\pi^0\pi^0$, $\pi^+\pi^-$
K_L	$d\bar{s}$, $\bar{d}s$	0^-	$1/2$	0	—	5.11×10^{-8}	$3\pi^0$, $\pi^+\pi^-\pi^0$, $\pi^\pm e^\mp\nu_e$, $\pi^\pm\mu^\mp\nu_\mu$
η	$u\bar{u}$, $d\bar{d}$, $s\bar{s}$	0^{-+}	0^+	0	547.86	4.26×10^{-19}	$\pi^+\pi^-\pi^0$, $3\pi^0$, 2γ
η'	$u\bar{u}$, $d\bar{d}$, $s\bar{s}$	0^{-+}	0^+	0	957.78	2.97×10^{-21}	$\pi^+\pi^-\eta$, $\rho^0\gamma$, $2\pi^0\eta$
$\rho(770)$	$u\bar{d}$, $u\bar{u}-d\bar{d}$, $d\bar{u}$	1^{--}	1^+	$\pm1, 0$	763	3.72×10^{-24}	$\pi\pi$
$\omega(782)$	$u\bar{u}$, $d\bar{d}$, $s\bar{s}$	1^{--}	0^-	0	782.66	6.43×10^{-23}	$\pi^+\pi^-\pi^0$
$K^*(892)$	$u\bar{s}$, $d\bar{s}$, $\bar{d}s$, $\bar{u}s$	1^-	$1/2$	$\pm1, 0$	890	2.1×10^{-23}	$K\pi$
$f_0(500)$	$u\bar{u}$, $d\bar{d}$	0^{++}	0^+	0	500	2.03×10^{-24}	$\pi\pi$
$f_0(1370)$	$u\bar{u}$, $d\bar{d}$, $s\bar{s}$	0^{++}	0^+	0	1345	3.1×10^{-24}	$\pi\pi$, 4π
$a_0(980)$	$u\bar{d}$, $u\bar{u}-d\bar{d}$, $d\bar{u}$	0^{++}	1^-	$\pm1, 0$	980	1.1×10^{-23}	$\eta\pi$
$a_1(1260)$	$u\bar{d}$, $u\bar{u}-d\bar{d}$, $d\bar{u}$	1^{++}	1^-	$\pm1, 0$	1230	1.3×10^{-24}	3π
$a_2(1320)$	$u\bar{d}$, $u\bar{u}-d\bar{d}$, $d\bar{u}$	2^{++}	1^-	$\pm1, 0$	1318.2	5.3×10^{-24}	3π, $\eta\pi$, $\omega\pi\pi$
$\pi_1(1600)$	$u\bar{d}$, $u\bar{u}-d\bar{d}$, $d\bar{u}$	1^{-+}	1^-	$\pm1, 0$	1580	3.7×10^{-24}	3π, $b_1\pi$, $\eta\pi$, $\eta'\pi$

Table 2: Light and Strange Baryons.

Baryon	**Quark Content**	J^P	I	**Charge**	**Mass / MeV**	**Lifetime / s**	**Principle Decay Modes**
p	uud	$1/2^+$	$1/2$	$+1$	938.27	stable	—
n	udd	$1/2^+$	$1/2$	0	939.57	878.4	$pe^-\bar{\nu}_e$
Λ^0	uds	$1/2^+$	0	0	1115.68	2.63×10^{-10}	$p\pi^-$, $n\pi^0$
$\Sigma^\pm$	uus, dds	$1/2^+$	1	± 1	1189.37	8.02×10^{-11}	$p\pi^0$, $n\pi^+$
Σ^0	uds	$1/2^+$	1	0	1192.64	74×10^{-21}	$\Lambda\gamma$
Ξ^-	dss	$1/2^+$	$1/2$	-1	1321.71	1.64×10^{-10}	$\Lambda\pi^-$
Ξ^0	uss	$1/2^+$	$1/2$	0	1314.86	2.90×10^{-10}	$\Lambda\pi^0$
$\Delta^{++}(1231)$	uuu	$3/2^+$	$3/2$	$+2$	1210	5.58×10^{-24}	$N\pi$
$\Delta^{\pm}(1231)$	uud, udd	$3/2^+$	$3/2$	± 1	1210	5.58×10^{-24}	$N\pi$
$\Delta^0(1231)$	udd	$3/2^+$	$3/2$	0	1210	5.58×10^{-24}	$N\pi$
$\Sigma(1385)$	uus, uds, dds	$3/2^+$	1	± 1, 0	1385	1.5×10^{-23}	$\Lambda\pi$, $\Sigma\pi$, $\Lambda\gamma$
$\Xi(1530)$	uss, dss	$3/2^+$	$1/2$	0, -1	1532.8	5.88×10^{-23}	$\Xi\pi$
Ω^-	sss	$1/2^+$	0	-1	1672.45	8.21×10^{-11}	ΛK^-, $\Xi^0\pi^-$, $\Xi^-\pi^0$
$N(1440)$	uud, udd	$1/2^+$	$1/2$	$+1$, 0	1370	2.94×10^{-24}	$N\pi$, $N\pi\pi$
$\Lambda(1405)$	uds	$1/2^-$	0	0	1405.1	1.12×10^{-23}	$\Sigma\pi$