WILLIAM & MARY
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PHYS 303 – Classical Mechanics of Particles and Waves II

Problem Set 2

Due: Thursday, September 12 at 5:00pm

Term: Fall 2024

Instructor: Andrew W. Jackura

Readings

Read sections 8.6–8.8 and 14.1–14.4 of Taylor.

Problems

Problem 1. [15 pts.] – Relative Coordinates

Two particles with masses m_1 and m_2 are located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, with respect to some inertial frame $\mathcal{O}$. Define the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the center-of-mass (CM) position $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$, where and M is the total mass $M = m_1 + m_2$.

(a) **[5 pts.]** Verify that the positions of two particles can be written in terms of the CM and relative positions as $\mathbf{r}_1 = \mathbf{R} + m_2\mathbf{r}/M$ and $\mathbf{r}_2 = \mathbf{R} - m_1\mathbf{r}/M$. Hence, confirm that the total kinetic energy of the two particles can be expressed as $T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$, where μ denotes the reduced mass $\mu = m_1 m_2/M$.

(b) **[5 pts.]** The momentum $\mathbf{p}$ conjugate to the relative position $\mathbf{r}$ is defined with components $p_i = \partial\mathcal{L}/\partial\dot{r}_i$ for $i = 1, 2, 3$ or x, y, z. Prove that $\mathbf{p} = \mu\dot{\mathbf{r}}$. Prove also that in the CM frame, $\mathbf{p}$ is the same as $\mathbf{p}_1$ the momentum of particle 1 (and also $-\mathbf{p}_2$).

(c) **[5 pts.]** Show that in the CM frame, the angular momentum $\boldsymbol{\ell}_1$ of particle 1 is related to the total angular momentum $\mathbf{L}$ by $\boldsymbol{\ell}_1 = (m_2/M)\mathbf{L}$ and likewise $\boldsymbol{\ell}_2 = (m_1/M)\mathbf{L}$. Since $\mathbf{L}$ is conserved, this shows that the same is true of $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ separately in the CM frame.

Problem 2. [20 pts.] – Stability of Circular Orbits

Recall from lecture that the effective potential energy for a two-body system interacting via gravity is

$$U_{\text{eff}}(r) = -\frac{GM\mu}{r} + \frac{\ell^2}{2\mu r^2}\,,$$

where M is the total mass and μ the relative mass. Let's work with a sun-planet system, so that M is approximately the mass of the sun and μ is approximately the mass of the planet.

(a) **[10 pts.]** By examining the the effective potential energy, find the radius at which a planet with angular momentum ℓ can orbit the sun in a circular orbit with fixed radius. *Hint:* Look at $\mathrm{d}U_{\text{eff}}/\mathrm{d}r$.

(b) **[10 pts.]** Show that this circular orbit is stable, in the sense that a small radial nudge will cause only small radial oscillations. Show that the period of oscillations is equal to the planet's orbital period. *Hint:* Look at $\mathrm{d}^2U_{\text{eff}}/\mathrm{d}r^2$.

Problem 3. [15 pts.] – Effective Potential Energy of a Spring Force

Two particles whose reduced mass is μ interact via a potential energy $U = \frac{1}{2}kr^2$, where r is the distance between them and $k > 0$ is the force constant.

1. **[5 pts.]** Make a sketch showing $U(r)$, the centrifugal potential energy $U_{\text{cf}}(r)$, and the effective potential energy $U_{\text{eff}}(r)$. Treat the angular momentum ℓ as a known, non-zero constant.
2. **[5 pts.]** Find the equilibrium separation r_0, that is the distance at which the two particles can circle each other with constant separation.
3. **[5 pts.]** Make a Taylor expansion of $U_{\text{eff}}(r)$ about the equilibrium point r_0 and neglect all terms $\mathcal{O}(\epsilon^3)$ where $\epsilon = r - r_0$ is the deviation from equilibrium. Find the frequency for small oscillations about the circular orbit if the particles are disturbed a little from separation r_0.

Problem 4. [25 pts.] – Two-Body Systems in a Uniform Gravity Field

Consider two masses m_1 and m_2 moving in a uniform gravitational field $\mathbf{g}$ and interacting via a potential energy $U(r)$.

(**a**) **[5 pts.]** Show that the Lagrangian can be decomposed into a center-of-mass (CM) Lagrangian and a relative Lagrangian as $\mathcal{L} = \mathcal{L}_{\text{CM}} + \mathcal{L}_{\text{rel}}$.

(**b**) **[5 pts.]** Write down the Euler-Lagrange equations for the three CM coordinates $\mathbf{R} = (X, Y, Z)$ and describe its motion. Take Z to be the vertical component.

(**c**) **[5 pts.]** Write down the Euler-Lagrange equations for the relative coordinates $\mathbf{r}$ and show clearly that the motion is the same as that of a single particle of mass equal to the reduced mass μ with position $\mathbf{r}$ and potential energy $U(r)$.

(**d**) **[10 pts.]** Let the potential energy $U(r)$ describe the force of a massless spring of natural length L and force constant k. Initially, m_2 is resting on a table and I am holding m_1 vertically above m_2 at a height L. At time $t = t_0 = 0$, I project m_1 vertically upward with initial velocity v_0. Find the positions of the two masses at any subsequent time t (before either mass returns to the table) and describe the motion.

Problem 5. [25 pts.] – Logarithmic Spiral

The relative motion of a two-body system with reduced mass μ is that of a logarithmic spiral given by $r = ke^{\alpha\phi}$, where k and α are constants.

(**a**) **[5 pts.]** Find the central force law responsible for this relative motion in terms of k, α, μ, and the angular momentum ℓ. *Hint:* Recall that in cylindrical polar coordinates (r, φ) the equations of motion are

$$\mu r^2 \dot{\phi} = \ell\,, \qquad \mu\ddot{r} = \mu r\dot{\varphi}^2 - \frac{\partial U(r)}{\partial r}\,,$$

where $U(r)$ is the potential energy of the central force.

(**b**) **[10 pts.]** Unlike inverse-square force laws, we can solve for the time dependence of both r and ϕ. If given that $\phi = 0$ at $t = 0$, determine $\phi(t)$ and $r(t)$.

(**c**) **[10 pts.]** Determine the total energy E of the orbit, assuming we define $U(r \to \infty) = 0$.