

Phenomenology I - QED

It is useful to examine some common QED processes. There are many important processes, we will concentrate only on a few. Let us first consider high-energy μ^+ - μ^- pair production from e^-e^+ annihilation.

$$\underline{e^-e^+ \rightarrow \mu^-\mu^+}$$

We are interested in the unpolarized differential and total cross-sections. This is often the desired quantity, given that the spins of the incoming particles are arbitrary (beams are unpolarized), and one does not measure the spins of the final state.

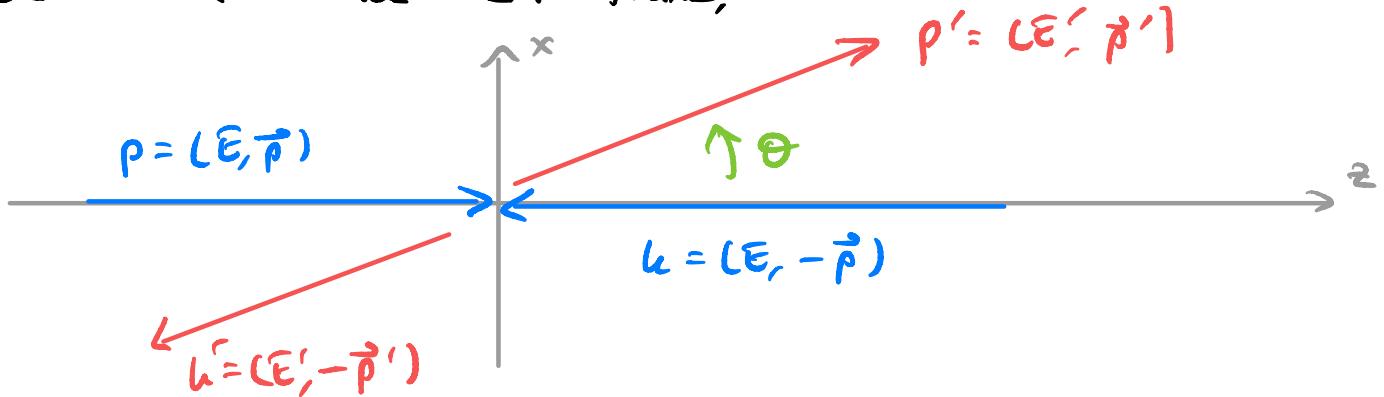
Let us define some kinematics

momentum, $p = (E, \vec{p})$

$$e_s^-(p) + e_r^+(k) \rightarrow \mu_s^-(p') + \mu_r^+(k')$$

↑ spin projection

Let's work in the CM frame,



To simplify the analysis, let's consider the ultrarelativistic limit, $\sqrt{s} \gg m_\mu \gg m_e$. So, we have (expose)

$$E = E' = \frac{\sqrt{s}}{2}, \quad |\vec{p}| = |\vec{p}'| \sim \frac{\sqrt{s}}{2}.$$

At leading order in α , there is a single Feynman diagram

$$\begin{aligned}
 iM &= \text{Feynman diagram} \\
 &= \overbrace{\bar{u}_{sr}(p')(-iq\gamma^\mu)v_{r'}(l')}^{\text{nucleon vertex}} \frac{-iq_{\mu\nu}}{(p+l)^2 + i\epsilon} \overbrace{\bar{v}_r(l)(-iq\gamma^\nu)u_s(p)}^{\text{electron vertex}} \\
 &\quad \text{photon propagator} \\
 &= -\frac{iq^2}{s} a^\mu b_\mu \\
 &= -i \frac{4\pi\alpha}{s} a^\mu b_\mu
 \end{aligned}$$

Now, the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \int d\Omega \langle |M|^2 \rangle$$

$$\text{where } \langle |M|^2 \rangle = \frac{1}{4} \sum_{r,s} \sum_{r',s'} |M|_r^2$$

$$\text{So, } |M|^2 = \mu^* \mu = \mu^+ \mu^-$$

$$= \left(\frac{4\pi\alpha}{s}\right)^2 [a^\mu a^\nu] [b_\mu b_\nu]$$

Now,

$$a^\mu = \bar{u} \gamma^\mu v \Rightarrow a^\mu = (\bar{u} \gamma^\mu v)^+$$

$$= v^\mu \gamma^\nu \bar{u}^\nu = v^\mu \gamma^\nu \gamma^\rho \gamma^\sigma (\bar{u}^\rho \gamma^\sigma)^+$$

$$\quad \quad \quad \downarrow \gamma^\mu = \gamma^\nu \gamma^\rho \gamma^\sigma$$

$$= \bar{v} \gamma^\mu \gamma^\nu \gamma^\rho u = \bar{v} \gamma^\mu u$$

$$\quad \quad \quad \gamma^\mu = \gamma^\nu \gamma^\rho \gamma^\sigma = 1$$

$$\& b^\mu = \bar{v} \gamma^\mu u \Rightarrow b^\mu = (\bar{v} \gamma^\mu u)^+ = \bar{u} \gamma^\mu v$$

So, we find

$$\sum_{s,r} b_\mu b_\nu^+ = \sum_{s,r} \bar{v}_r(u) \gamma_\mu u_s(p) \bar{u}_{s(p)} \gamma_\nu v_r(u)$$

$$= \sum_{s,r} \text{tr} [\bar{v}_r(u) \bar{u}_{s(p)} \gamma_\mu u_s(p) \bar{u}_{s(p)} \gamma_\nu]$$

$$= \text{tr} \left[\underbrace{\sum_r v_r(u) \bar{v}_r(u)}_{= k - n_c \rightarrow k} \gamma_\mu \underbrace{\sum_s u_s(p) \bar{u}_s(p)}_{= p + n_c \rightarrow p} \gamma_\nu \right]$$

$$= \text{tr} [k \gamma_\mu p \gamma_\nu] + O(n_c^2/s)$$

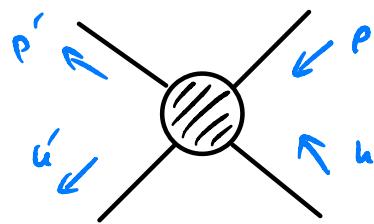
$$= 4(k_\mu p_\nu + k_\nu p_\mu - g_{\mu\nu} k \cdot p)$$

Recall the Mandelbar variables

$$S = (P + h)^2 = (P' + h')^2$$

$$t = (\rho - \rho')^2 = (h - h')^2$$

$$u = (p - u')^2 = (u - p')^2$$



$$f_s \quad s \gg h_r^2 \gg h_c^2 \Rightarrow \quad s \approx 2p \cdot h, \quad t \approx -2p \cdot p'$$

$$S_{\alpha\gamma} \sum_{\sigma,\tau} b_\mu b_\nu^+ = 4(k_m p_\nu + k_\nu p_m - g_{\mu\nu} k \cdot p) \\ = 4(k_\gamma p_\nu + k_\nu p_\gamma - g_{\mu\nu} S_{1/2})$$

$$\begin{aligned} \text{Similarly, } \sum_{\sigma, \tau} u^\sigma u^{\tau*} &= \text{tr}[p' \gamma^\mu p' \gamma^\nu] \\ &= 4(p'^\mu u^\nu + p'^\nu u^\mu - g^{\mu\nu} s_{12}) \end{aligned}$$

$$\begin{aligned}
 S_{\perp} & \quad \text{Sph aug.} \\
 \langle |m|^2 \rangle &= \left(\frac{4\pi\alpha}{s}\right)^2 \frac{1}{4} \sum_{s,r} b_r b_v + \sum_{s,r,r'} a^r a^{v^+} \\
 &= 4 \left(\frac{4\pi\alpha}{s}\right)^2 (h_r p_v + h_v p_r - g_{rv} s_{1/2}) (p'^r h'^v + p'^v h'^r - g'^{rv} s_{1/2}) \\
 &= 4 \left(\frac{4\pi\alpha}{s}\right)^2 \left[2 h \cdot p' p \cdot h' + 2 h \cdot h' p \cdot p' - \underbrace{s h \cdot p}_{-\overline{s} \left(\frac{s}{2}\right)} - \underbrace{s h' \cdot p'}_{-\overline{s} \left(\frac{s}{2}\right)} + \overline{s}^2 \right] \\
 &= 2 \left(\frac{4\pi\alpha}{s}\right)^2 (u^2 + t^2)
 \end{aligned}$$

$$b\bar{u}, \quad s+t+u=0 \quad \Rightarrow \quad u = -s-t$$

$$\Rightarrow \langle |M|^2 \rangle = 2 \left(\frac{4\pi\alpha}{s} \right)^2 \left(t^2 + (s+t)^2 \right)$$

$$\text{So, } \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}'|}{|\vec{p}|} \langle 1/m^2 \rangle$$

↑ since $s \rightarrow \infty$

$$\begin{aligned}
 &= \frac{1}{64\pi^2 s} \cdot 2 \left(\frac{4\pi\alpha}{s} \right)^2 \left(t^2 + (s+t)^2 \right) \\
 &= \frac{\alpha^2}{2s^3} \left(t^2 + (s+t)^2 \right) \\
 &= \frac{\alpha^2}{2s} \left[\left(\frac{t}{s} \right)^2 + \left(1 + \frac{t}{s} \right)^2 \right]
 \end{aligned}$$

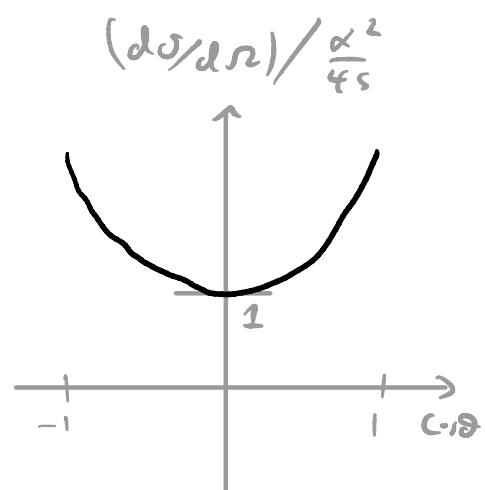
$$\begin{aligned}
 \text{Now, } t &= -2|\vec{p}'|/|\vec{p}| (1 - \cos\theta) \\
 &= -\frac{s}{2} (1 - \cos\theta) \Rightarrow \frac{t}{s} = -\frac{1}{2} + \frac{1}{2} \cos\theta
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } \frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{2s} \left[\frac{1}{4} (1 - \cos\theta)^2 + \frac{1}{4} (1 + \cos\theta)^2 \right] \\
 &= \frac{\alpha^2}{8s} \left[1 + \cos^2\theta - \cancel{\cos\theta} + 1 + \cos^2\theta + \cancel{\cos\theta} \right] \\
 &= \frac{\alpha^2}{4s} (1 + \cos^2\theta)
 \end{aligned}$$

Therefore,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2\theta)$$

Notice: Cross-section is symmetric about $\cos\theta$



The cross-section is then

$$\begin{aligned}
 \sigma &= \int d\Omega \frac{d\sigma}{d\Omega} \\
 &= 2\pi \int_{-1}^1 d\cos\theta \left[\frac{\alpha^2}{4s} (1 + \cos^2\theta) \right] \\
 &= \frac{\pi\alpha^2}{2s} \left(\cos\theta \Big|_{-1}^1 + \frac{\cos^3\theta}{3} \Big|_{-1}^1 \right) \\
 &= \frac{\pi\alpha^2}{2s} \left(2 + \frac{2}{3} \right) \\
 &= \frac{4}{3} \frac{\pi\alpha^2}{s} \quad \Rightarrow \quad \boxed{\sigma = \frac{4}{3} \frac{\pi\alpha^2}{s}}
 \end{aligned}$$

This reaction ($e^-e^+ \rightarrow \mu^-\mu^+$) sets the scale for many high-energy e^-e^+ -reactions. Let us define R as

$$R = \frac{\sigma(e^-e^+ \rightarrow \text{hadrons})}{\sigma(e^-e^+ \rightarrow \mu^-\mu^+)}$$

So, $\sigma(e^-e^+ \rightarrow \mu^-\mu^+)$ sets the scale.

$$\begin{aligned}
 \sigma(e^-e^+ \rightarrow \mu^-\mu^+) &= \frac{4\pi\alpha^2}{3s} (\text{tc})^2 \\
 &= \frac{4\pi}{3s} \left(\frac{1}{137}\right)^2 (0.197 \text{ GeV fm})^2 \left(\frac{1 \text{ b}}{100 \text{ fm}^2}\right)^{\cancel{=1}} \\
 &\approx 86.8 \text{ nb} \left(\frac{\text{GeV}^2}{s}\right)
 \end{aligned}$$

Quark - Antiquark production

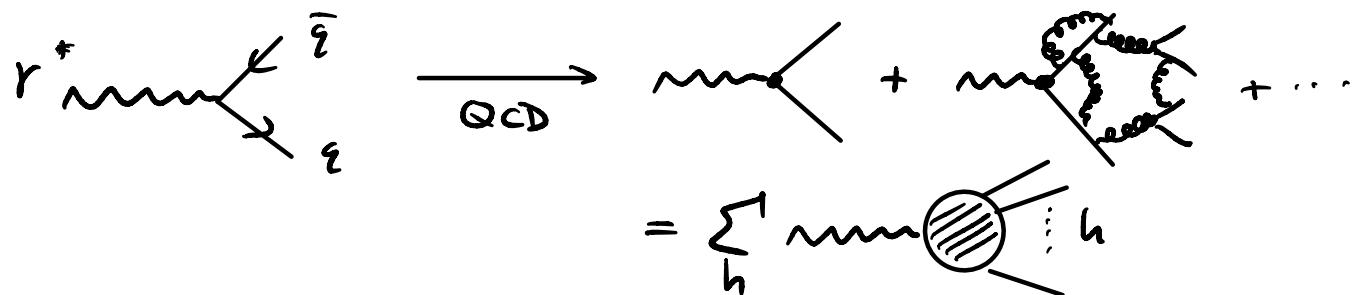
From the previous analysis, we get directly the cross-section for $q\bar{q}$ -final states. We only need to replace

$$e \rightarrow Q_i e$$

along with multiplying by N_c to account for $N_c = 3$ colors.

$$\Rightarrow \sigma_{e^-e^+ \rightarrow q\bar{q}} = 3 Q_i^2 \sigma_{e^-e^+ \rightarrow \mu^-\mu^+}$$

Of course, free quarks are not observed, but hadrons

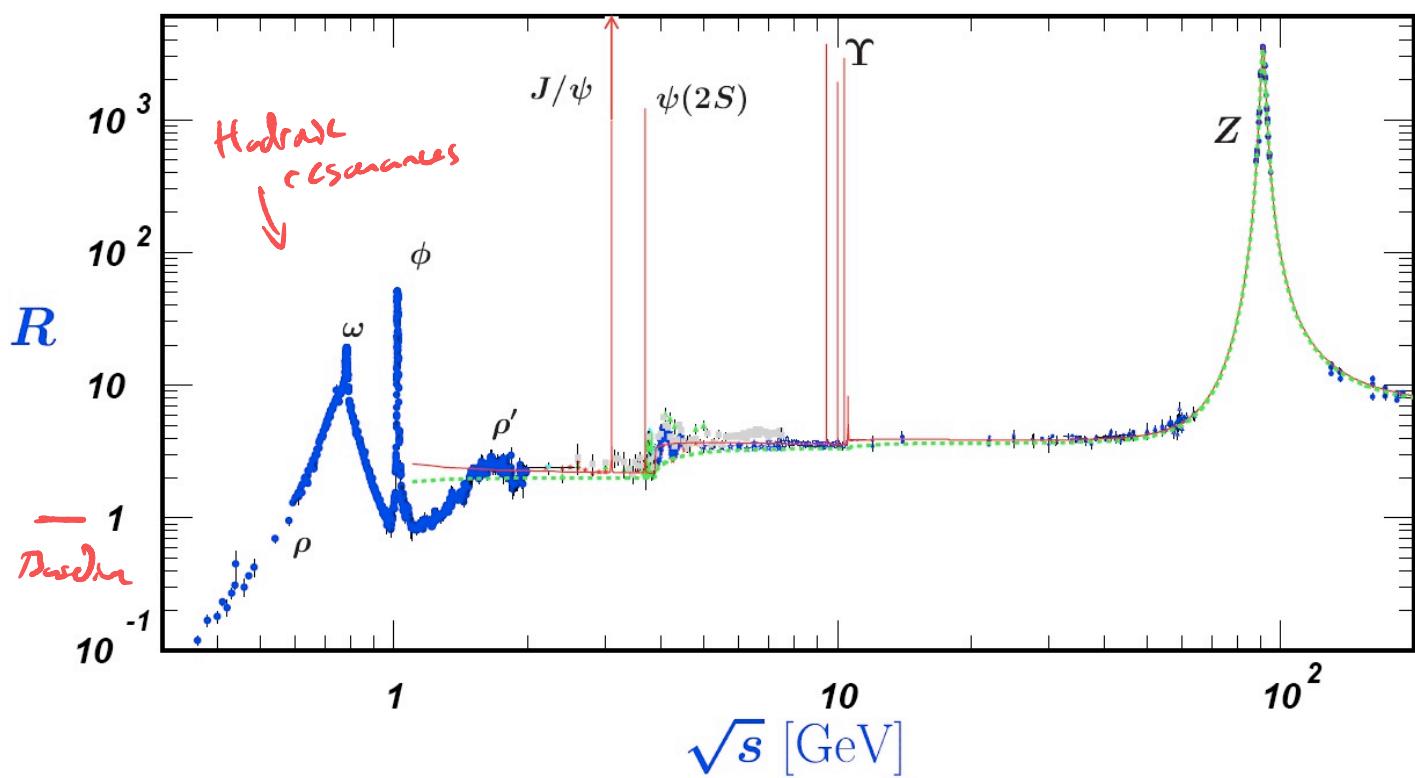


In experiment, the $q\bar{q}$ final state is observed as two back-to-back jets. The fact that the cross-section for high-energies can be computed from

$$\sigma_{e^-e^+ \rightarrow \text{hadrons}} \xrightarrow{\sqrt{s} \rightarrow \infty} \left(3 \sum_{f=1}^{N_f} Q_f^2 \right) \sigma_{e^-e^+ \rightarrow \mu^-\mu^+}$$

We will see that this is a consequence of an important property of QCD - Asymptotic freedom.

At high energies, the photons "see" through the cloud of QCD virtual states surrounding each quark.



Lepton Anomalous Magnetic Moment

The anomalous magnetic moment, $g-2$, of the electron is one of the crowning achievements of QED.

Experimental measurements and theoretical calculations agree to one part in a trillion! The $g-2$ of the muon is a strong test of the SM as it is sensitive to states beyond the SM. Measuring $g-2$ of the muon provides a probe into new BSIM physics.

Here, I'll focus on the first radiative correction of the lepton $g-2$.

Recall that g is a measure of a leptons susceptibility to magnetic fields,

$$\bar{\mu} = g \frac{e}{2m} \vec{s}$$

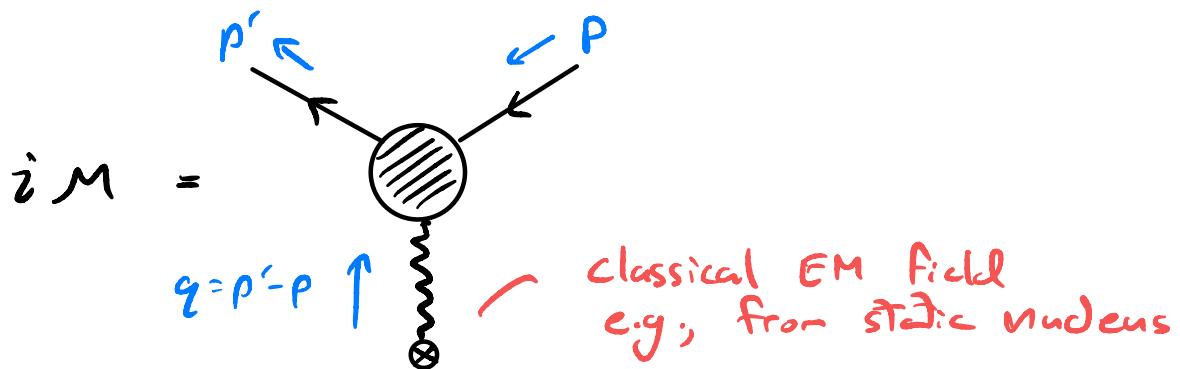
gyromagnetic ratio spin operator

To calculate g in QED, consider a lepton (here first focus on electron) in a Classical (or background) EM field

$$Z_{\text{int}} = -e \bar{\gamma} \gamma^{\mu} (A_{\mu} + A_{\mu}^{\text{cl.}})$$

quantum field classical field

We want to compute the one-body scattering amplitude and connect it to the non-relativistic potential $V = -\vec{\mu} \cdot \vec{B}$.



One-body amplitude has generic structure

$$\begin{aligned} \langle p', s' | iT | p, s \rangle &\equiv 2\pi \delta(E' - E) iM \\ &= \bar{u}(p', s') [-ie\Gamma^\mu(p', p)] u(p, s) \tilde{A}_\mu^{\text{cl.}}(q) \end{aligned}$$

↗ Momentum not conserved in background field.

$\tilde{A}_\mu^{\text{cl.}}(q) = \int d^4x e^{iq \cdot x} \tilde{A}_\mu^{\text{cl.}}(x)$

The vertex function, Γ^μ in general contains 12 tensors formed from momenta and gamma matrices. We can simplify things by considering on-shell leptons only, that is we use the Dirac eqn.

$$(p - m) u(p, s) = 0 \quad \text{with} \quad p^2 = m^2$$

$$\bar{u}(p', s')(p' - m) = 0 \quad \text{with} \quad p'^2 = m^2$$

This reduces the number of terms. From Lorentz invariance and C,P,T symmetry (recall GED is invariant under C,P,T), we can write generally

$$\Gamma^\mu = A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu$$

where A, B, C are scalar functions of

$$Q^2 = -q^2 = -(p'-p)^2 = 2p \cdot p - 2m^2$$

The EM current is conserved, so from Ward identity

$$\begin{aligned} 0 &= q_\mu \Gamma^\mu \\ &= q_\mu (A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu) \\ &= Aq + Bq \cdot p + Cq^2 \\ &\xrightarrow{\text{q} \cdot p = p'^2 - p^2 = 0} Cq^2 \end{aligned}$$

$$\Rightarrow C = 0$$

Also, by convention, we use the Gordan Identity,

$$\bar{u}\gamma^\mu u = \bar{u} \left[\frac{(p'+p)^\mu}{2m} + i \frac{\sigma^{\mu\nu}}{2m} q_\nu \right] u$$

$$\text{where } \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

to write the vertex function Γ in the form

$$\Gamma^{\mu}(\rho, p) = \gamma^{\mu} F_1(Q^2) + i \frac{\sigma^{\mu\nu}}{2m} \epsilon_{\nu} F_2(Q^2)$$

F_1 is the Dirac Form-factor

F_2 is the Pauli Form-factor

The Form-factors contain complete information about the EM fields influence on the lepton.

To gain an understanding of their physical meaning,

let us consider time-independent field configurations

$$A_{\mu}^{cl.}(x) = A_{\mu}^{cl.}(\vec{x})$$

or,

$$\tilde{A}_{\mu}^{cl.}(q) = 2\pi \delta(q^0) \tilde{A}_{\mu}^{cl.}(\vec{q})$$

Electric Coupling

Consider a static electric source $A_{\mu}^{cl.}(x) = (\varphi(x), \vec{0})$

$$\Rightarrow \tilde{A}_{\mu}^{cl.}(\vec{q}) = (\tilde{\varphi}(\vec{q}), \vec{0})$$

Must recover in non-relativistic limit

$$V(\vec{x}) = e\varphi(\vec{x})$$

so,

$$\begin{aligned} iM &= -ie\bar{u}(\rho, s') \Gamma^{\mu}(\rho, p) u(\rho, s) \tilde{\varphi}(\vec{q}) \\ &\stackrel{\text{Suppressing}}{=} -ie\bar{u}(\rho, s') \left\{ \gamma^{\mu} F_1 + i \frac{\sigma^{\mu\nu}}{2m} \epsilon_{\nu} F_2 \right\} u(\rho, s) \tilde{\varphi}(\vec{q}) \end{aligned}$$

We want to examine the non-relativistic limit,
 $\vec{q} = (\vec{p}' - \vec{p}) \rightarrow 0$ and $\vec{p} \rightarrow \vec{0}$.

So, $\bar{u}(\rho', s') \Gamma^0 (\rho', \rho) u(\rho, s) = \bar{u}(\rho', s') \gamma^0 u(\rho, s) F_i$
 $= u^+(\rho', s') u(\rho, s) F_i$

Recall the Dirac spinors in Chiral basis

$$u(\rho, s) = \begin{pmatrix} \sqrt{\rho \cdot \sigma'} & \xi_s \\ \sqrt{\rho \cdot \bar{\sigma}} & \xi_s \end{pmatrix} \quad \text{with } \begin{array}{l} \sigma = (1, \vec{\sigma}) \\ \bar{\sigma} = (1, -\vec{\sigma}) \end{array}$$

and $\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In the non-relativistic limit

$$\begin{aligned} \sqrt{\rho \cdot \sigma'} &= \sqrt{m - \vec{p} \cdot \vec{\sigma}'} \\ &\approx \sqrt{m} \left(1 - \frac{\vec{p} \cdot \bar{\sigma}}{2m} \right) \end{aligned}$$

$$\begin{aligned} \sqrt{\rho \cdot \bar{\sigma}} &= \sqrt{m + \vec{p} \cdot \bar{\sigma}} \\ &\approx \sqrt{m} \left(1 + \frac{\vec{p} \cdot \bar{\sigma}}{2m} \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow u^+(\rho', s') u(\rho, s) &= m (\xi_{s'}^+, \xi_{s'}^-) \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} + O(\vec{p}, \vec{p}') \\ &= 2m \xi_{s'}^+ \xi_s + O(\vec{p}, \vec{p}') \\ &= 2m \delta_{s's} + O(\vec{p}, \vec{p}') \end{aligned}$$

↳ spin preserving

Therefore, the T-matrix element is

$$iM \approx -ie F_1(0) \tilde{\varphi}(\vec{q}) \cdot 2m \delta_{ss'}$$

Let us compare to the Born amplitude in non-relativistic quantum mechanics

$$\langle \vec{p}' | iT | \vec{p} \rangle = -i \tilde{V}(\vec{q}) \cdot 2\pi \delta(E' - E)$$

↳ states & NRQM

normalized as $\langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{p})$

So, conclude

$$\begin{aligned} \tilde{V}(\vec{q}) &= -\frac{1}{2m} M \\ &= e F_1(0) \tilde{\varphi}(\vec{q}) \cdot \delta_{ss'} \quad \text{keep implicit in } V(\vec{x}) \end{aligned}$$

Fourier transform

$$V(\vec{x}) = e F_1(0) \varphi(\vec{x})$$

$$\Rightarrow F_1(0) = 1 \quad \text{to all orders in perturbation theory!}$$

At zero momentum transfer, the Dirac form-factor is fixed to 1. This is known as the charge renormalization condition. In other words, "e" is a free parameter in QED, and we fix it by requiring $F_1(0) = 1$.

Magnetic Coupling

Let us repeat the previous analysis for a static magnetic field. The vector potential of a static magnetic field is $A_\mu = (0, \vec{A})$ with $\vec{B} = \vec{\nabla} \times \vec{A}$

Consider the k^{th} -component

$$B^k = (\vec{\nabla} \times \vec{A})^k = \epsilon^{ijk} \partial_i A_j$$

Now, $B^k(\vec{x}) = \int \frac{d^3 \vec{q}}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}} \tilde{B}^k(\vec{q})$

$$= \epsilon^{ijk} \frac{\partial}{\partial x^i} \int \frac{d^3 \vec{q}}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}} \tilde{A}_j(\vec{q})$$

$$= \int \frac{d^3 \vec{q}}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}} \epsilon^{ijk} \frac{\partial}{\partial x^i} (iq^l x^l) \tilde{A}_j(\vec{q})$$

so, $\tilde{B}^k(\vec{q}) = \epsilon^{ijk} \cdot iq^l \delta_{il} \tilde{A}_j(\vec{q})$
 $= i \epsilon^{kij} q_i \tilde{A}_j(\vec{q}) = i \epsilon^{kij} q_i \tilde{A}_j(\vec{q})$
 $= + i \epsilon^{ijk} q_i \tilde{A}_j(\vec{q}) \quad \checkmark \quad \epsilon^{kij} = - \epsilon^{ikj} = + \epsilon^{ijk}$

$$\Rightarrow \tilde{B}^k(\vec{q}) = i \epsilon^{ijk} q_i \tilde{A}_j^{\text{cl.}}(\vec{q})$$

we want to keep terms proportional to \vec{q}

Note: Peskin & Schroeder report

$$\tilde{B}_k(\vec{q}) = -i \epsilon^{ijk} q_i \tilde{A}_j^{\text{cl.}}(\vec{q})$$

The scattering amplitude is then

$$\begin{aligned} iM &= -ie\bar{u}(\rho', s') \Gamma^{\mu}(\rho', \rho) u(\rho, s) \tilde{A}_{\mu}^{cl.}(\vec{q}) \\ &= +ie\bar{u}(\rho', s') \Gamma^{\mu}(\rho', \rho) u(\rho, s) \tilde{A}_{\mu}^{cl.}(\vec{q}) \\ &= +ie\bar{u}(\rho', s') \left[\gamma^{\mu} F_1 + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m} F_2 \right] u(\rho, s) \tilde{A}_{\mu}^{cl.}(\vec{q}) \end{aligned}$$

It is useful to use the Gordon Identity

$$\bar{u} \gamma^{\mu} u = \bar{u} \left[\frac{(\rho' + \rho)^{\mu}}{2m} + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m} \right] u$$

such that

$$iM = ie\bar{u}(\rho', s') \left[\frac{(\rho' + \rho)^{\mu}}{2m} F_1 + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m} (F_1 + F_2) \right] u(\rho, s) \tilde{A}_{\mu}^{cl.}(\vec{q})$$

\uparrow spin-independent
 \Rightarrow contributes only to kinetic energy $\sim \vec{p} \cdot \vec{A}$

Keeping only the spin-dependent piece

$$iM = ie [F_1 + F_2] \bar{u}(\rho', s') \frac{i\sigma^{\mu\nu} q_{\nu}}{2m} u(\rho, s) \tilde{A}_{\mu}^{cl.}(\vec{q})$$

We now take the non-relativistic limit.

Recall

$$u(\rho, s) \approx \sqrt{m} \begin{pmatrix} \left(1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right) \xi_s \\ \left(1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right) \xi_s \end{pmatrix} = \sqrt{m} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$$

Since we are already working linear in q in the vertex function.

$$\begin{aligned}
 \text{So, } iM &= ie[F_1(\sigma) + F_2(\sigma)] u^+(\varphi, s) i \frac{\gamma^\mu \sigma^{\mu\nu}}{2m} q_\nu u(\varphi, s) \tilde{A}_\mu^{cl.}(\vec{q}) \\
 &\simeq ie[F_1(\sigma) + F_2(\sigma)] m(\xi_s^+, \xi_s^+) i \frac{\gamma^\mu \sigma^{\mu\nu}}{2m} q_\nu \left(\frac{\xi_s}{\xi_s} \right) \tilde{A}_\mu^{cl.}(\vec{q})
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \sigma^{\mu\nu} &= \frac{i}{2} [\gamma^\mu, \gamma^\nu] \\
 &= \frac{i}{2} (\gamma^\mu \gamma^\nu - \underbrace{\gamma^\nu \gamma^\mu}_{\gamma^\mu \gamma^\nu \cancel{3} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}}) \\
 &= i\gamma^\mu \gamma^\nu - ig^{\mu\nu}
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } \gamma^\mu \sigma^{\mu\nu} q_\nu &= i\gamma^\mu (\gamma^\nu - g^{\mu\nu}) q_\nu \\
 &= i\gamma^\mu \gamma^k \gamma^v q_v - i\gamma^\mu \gamma^k \gamma^j q_j + i\gamma^\mu q^k \\
 &\stackrel{\cancel{-\gamma^0 \gamma^k}}{=} -i\gamma^\mu \gamma^k q_v + i\gamma^\mu q^k - i\gamma^\mu \gamma^k \gamma^j q_j \\
 &= \cancel{i\gamma^\mu \gamma^k} q_v + i\gamma^\mu q^k - i\gamma^\mu \gamma^k \gamma^j q_j \\
 &\quad \text{Drop as } \vec{q} \rightarrow 0 \\
 &\quad \text{In non-relativistic limit, } q_v \rightarrow 0
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \gamma^\mu \sigma^{\mu\nu} q_\nu &= -i\gamma^\mu \gamma^k \gamma^j q_j \\
 &= -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} q_j \\
 &= -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sigma^k \sigma^j & 0 \\ 0 & -\sigma^k \sigma^j \end{pmatrix} q_j \\
 &= i \begin{pmatrix} 0 & \sigma^k \sigma^j \\ \sigma^k \sigma^j & 0 \end{pmatrix} q_j
 \end{aligned}$$

So, the T-matrix element is

$$\begin{aligned}
 iM &\simeq ie [F_1(\omega) + F_2(\omega)] m(\vec{\xi}_s^+, \vec{\xi}_s^+) i \frac{e}{2m} \sigma^{uv} q_v \left(\vec{\xi}_s \right) \tilde{A}_u^{cl.}(\vec{q}) \\
 &= ie [F_1(\omega) + F_2(\omega)] m\left(\frac{i}{2m}\right) \\
 &\quad \times (\vec{\xi}_s^+, \vec{\xi}_s^+) i \begin{pmatrix} 0 & \sigma^u \sigma^j \\ \sigma^u \sigma^j & 0 \end{pmatrix} \left(\vec{\xi}_s \right) q_j \tilde{A}_u^{cl.}(\vec{q}) \\
 &= -i \left(\frac{e}{2m} \right) m [F_1(\omega) + F_2(\omega)] \vec{\xi}_s^+ (\sigma^u \sigma^j + \sigma^u \sigma^j) \vec{\xi}_s q_j \tilde{A}_u^{cl.}(\vec{q}) \\
 &= -i \left(\frac{e}{m} \right) 2m [F_1(\omega) + F_2(\omega)] \vec{\xi}_s^+ \sigma^u \sigma^j \vec{\xi}_s q_j \tilde{A}_u^{cl.}(\vec{q})
 \end{aligned}$$

Recall

$$\sigma^u \sigma^j = \delta^{ju} \mathbb{1} + i \epsilon^{ujl} \sigma_l$$

therefore

$$\begin{aligned}
 \sigma^u \sigma^j q_j &= \underbrace{q^u \mathbb{1}}_{\text{Drop as } \vec{q} \rightarrow 0} + i \epsilon^{ujl} \sigma_l q_j
 \end{aligned}$$

$$\Rightarrow iM = +i \left(\frac{e}{2m} \right) 2m [F_1(\omega) + F_2(\omega)] \underbrace{i \epsilon^{jkl} q_j}_{\tilde{B}^k(\vec{q})} \tilde{A}_u^{cl.}(\vec{q}) \vec{\xi}_s^+ \sigma_j \vec{\xi}_s$$

Recall

$$\langle \vec{s} \rangle = \vec{\xi}_s^+ \frac{\vec{\sigma}}{2} \vec{\xi}_s$$

Therefore, we find

$$iM = i(2m) \cdot 2 [F_1(\omega) + F_2(\omega)] \left(\frac{e}{2m} \right) \langle \vec{s} \rangle \cdot \tilde{\vec{B}}(\vec{q})$$

Compare with Born approximation

$$\begin{aligned}\tilde{V}(\vec{q}) &= -\frac{1}{2m} M \\ &= -\frac{e}{2m} \cdot 2[F_1(\omega) + F_2(\omega)] \langle \vec{s} \rangle \cdot \tilde{\vec{B}}(\vec{\xi})\end{aligned}$$

Fourier transform

$$\begin{aligned}V(x) &= -\underbrace{\frac{e}{2m} \cdot 2[F_1(\omega) + F_2(\omega)] \langle \vec{s} \rangle}_{\langle \vec{\mu} \rangle} \cdot \tilde{\vec{B}}(\vec{x}) \\ &= -\langle \vec{\mu} \rangle \cdot \tilde{\vec{B}}(\vec{x})\end{aligned}$$

Lepton magnetic moment

$$\boxed{\langle \vec{\mu} \rangle = 2[F_1(\omega) + F_2(\omega)] \frac{e}{2m} \langle \vec{s} \rangle}$$

Generally, $\vec{\mu} = g \frac{e}{2m} \vec{s}$

$\underbrace{\text{Lande' } g\text{-factor}}$

$$\text{so, } g = 2[F_1(\omega) + F_2(\omega)]$$

$$= 2[1 + F_2(\omega)]$$

\hookrightarrow charge renormalization

The Dirac eqn. predicts $g=2$, i.e., to leading order in QED

$$g = 2 + O(\alpha)$$

$$\Rightarrow F_2(\omega) = O(\alpha)$$

Unlike the electric charge, g is not a parameter of QED $\Rightarrow g$ is a pure prediction!

Therefore, we can compute the form-factor order-by-order in $\alpha = e^2/4\pi$.

$$F_2(\omega) = F_2^{(0)}(\omega) + F_2^{(1)}(\omega) + \dots$$

$O(\alpha)$ $O(\alpha^2)$

It is convention to define $\alpha_\ell = \alpha_g$, the anomalous magnetic moment of the lepton ℓ

$$\begin{aligned} \alpha_\ell &= F_2(\omega) \\ &= g - 2 \end{aligned}$$

A comment on the UV behavior of α_ℓ . Since g is not a parameter of the theory, it cannot be used to absorb UV divergences of radiative corrections. We would require an operator of the form

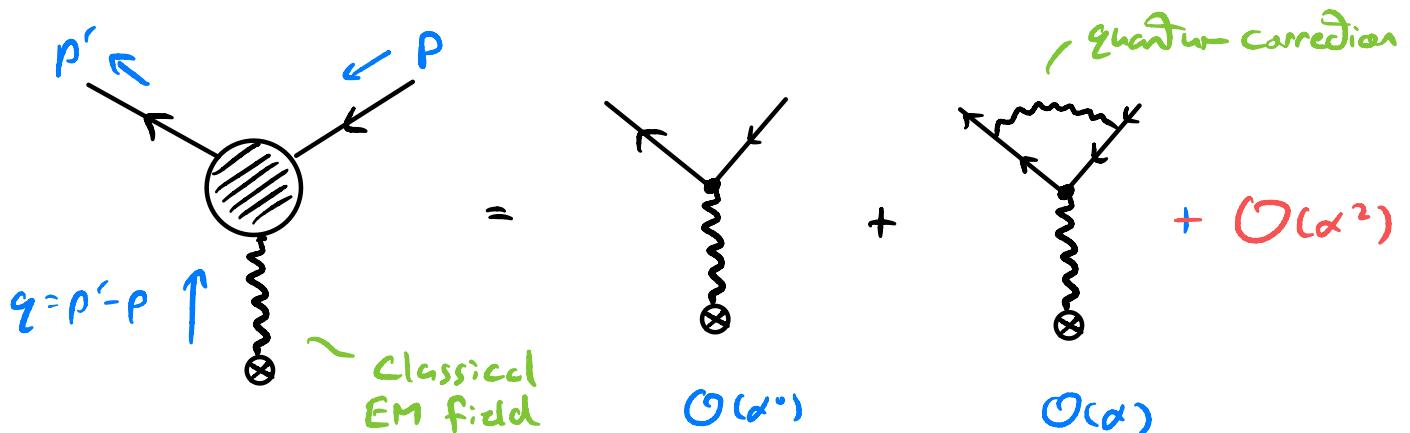
$$\mathcal{L} = g \cdot e \bar{\psi} i \frac{\sigma^\mu}{2m} F_{\mu\nu} \psi$$

\hookrightarrow this can cancel a divergence.

However, if QED is to be a renormalizable GFT, this operator is **NOT** allowed as it is a dimension-5 operator. Without such an operator, there can be no UV divergence, or the theory is not correct or not "renormalizable".

Perturbative Corrections to "g-2"

Let us begin the computation of the anomalous magnetic moment of the electron in QED. We will concentrate on the leading $\mathcal{O}(\alpha^0)$ and next-to-leading $\mathcal{O}(\alpha)$ perturbative corrections in $\alpha = e^2/2m$. From the Feynman rules for an electron in a classical EM field



$$-ie\Gamma^{(1)}(p', p) = -ie \sum_{n=0}^{\infty} \Gamma_n^{(1)}(p', p)$$

Coupling to classical field

The quantum corrections have the form

$$\Gamma_n^{(n)} = \left(\frac{\alpha}{\pi}\right)^n \tilde{\Gamma}_n^{(n)} \quad \begin{matrix} \text{no } \alpha\text{-dependence} \\ \text{conventional} \end{matrix}$$

The corresponding Form-factors have the expansion

$$F_j = \sum_{n=0}^{\infty} F_j^{(n)} \quad \text{for } j=1, 2.$$

Leading order

At leading order,

$$\begin{aligned} iM &= q = p' - p \quad \text{---} \quad -ie\gamma^\mu \quad + \mathcal{O}(\alpha) \\ &\quad \otimes \tilde{A}_\mu^{cl.}(\vec{q}) \\ &= -ie \bar{u}(p', s') [\gamma^\mu + \mathcal{O}(\alpha)] u(p, s) \tilde{A}_\mu^{cl.}(\vec{q}) \end{aligned}$$

so, we find $\Gamma_0^{(0)} = \gamma^\mu$. Comparing to the generic Lorentz decomposition, we conclude

$$F_1^{(0)} = 1, \quad F_2^{(0)} = 0.$$

So,

$$g = 2 + \mathcal{O}(\alpha)$$

Dirac's triumph!

Next-to-Leading Order

At next-to-leading order (NLO), we find four diagrams contributing to the amplitude at $\mathcal{O}(\alpha)$

$$iM = \begin{array}{c} \text{Diagram 1: A fermion loop with a vertical wavy line (mass insertion) and a crossed fermion line.} \\ + \end{array}$$
$$+ \mathcal{O}(\alpha^2)$$

The first two terms at $\mathcal{O}(\alpha)$ contribute to the mass and wavefunction renormalization of the electron. The third term contributes to the vacuum polarization of the EM field. The last diagram is the only correction to the vertex function.

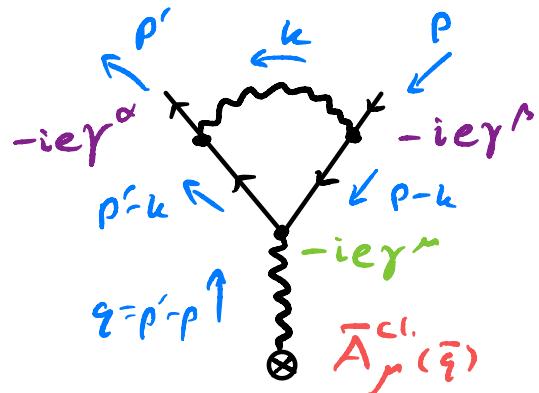
It is convenient to define $\Gamma^{\hat{m}} = \gamma^{\hat{m}} + \Lambda^{\hat{m}}$,

with the expansion

$$\Lambda^{\hat{m}} = \sum_{n=0}^{\infty} \Lambda_n^{\hat{m}}$$

Since $\Gamma_0^{\hat{m}} = \gamma^{\hat{m}}$, $\Rightarrow \Lambda_0^{\hat{m}} = 0$.

From the QED Feynman rules, the vertex correction is



$$-ie\bar{u}(p', s') \Lambda_i^{\hat{m}}(p', p) u(p, s) \bar{A}_\mu^{cl.}(\vec{q}) =$$

where $\Lambda_i^{\hat{m}}$ is

$$\begin{aligned} \Lambda_i^{\hat{m}}(p', p) &= (-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-ig\alpha s}{k^2} \gamma^\alpha \frac{i}{p'-k-m} \gamma^{\hat{m}} \frac{i}{p-k-m} \gamma^{\hat{m}} \\ &= -e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i^3 N^{\hat{m}}(p', p, k)}{k^2 [(p'-k)^2 - m^2] [(p-k)^2 - m^2]} \end{aligned}$$

$$\text{with } N^{\hat{m}} = \gamma_\nu [(\not{p}-\not{k})+m] \gamma^{\hat{m}} [(\not{p}-\not{k})+m] \gamma^\nu$$

Note that we will always be interested in the on-shell case, $p'^2 = p^2 = m^2$, and the Dirac spinors acting on $\Lambda_i^{\hat{m}}$.

Form-Factor Extraction

Our task is now to compute $\Gamma^{\mu\nu}$. In general, loop integrals are divergent and require regularization. We know that g is a finite prediction, which means that the contribution to F_2 must be finite and does not need regularization. It is convenient to isolate the contribution to F_2 , and that we can avoid the complications of regularization. Let us then construct projectors for the form-factors.

Recall

$$\Gamma^{\mu\nu} = g^{\mu\nu} F_1 + i \frac{\sigma^{\mu\nu}}{2m} q_\nu F_2$$

The vertex is evaluated on-mass-shell, $\bar{u}(p', s') \Gamma^{\mu\nu} u(p, s)$. The Dirac spinors themselves satisfy Dirac's eqn.

$$(p - m) u = 0 \quad \text{and} \quad \bar{u}(p - m) = 0$$

Since completeness relation is $\sum_s u(p, s) \bar{u}(p, s) = p + m$ we expect that $(p' + m) \Gamma^{\mu\nu} (p + m)$ has the same decomposition

$$(p' + m) \Gamma^{\mu\nu} (p + m) = (p' + m) \left[g^{\mu\nu} F_1 + i \frac{\sigma^{\mu\nu}}{2m} q_\nu F_2 \right] (p + m)$$

provided $p'^2 = p'^2 = m^2$.

We now multiply on the left and contract with both
 $P_\mu = (\rho' + p)_\mu^*$ and γ_μ , and then take the trace

- $\text{tr}[(\rho' + p) P_\mu \Gamma^\mu (\rho + p)] = \text{tr}[(\rho' + p) P_\mu (\rho + p)] F_1$
 $+ \text{tr}[(\rho' + p) P_\mu \sigma^{\mu\nu} g_\nu (\rho + p)] \frac{iF_2}{2m}$
- $\text{tr}[\gamma_\mu (\rho' + p) \Gamma^\mu (\rho + p)] = \text{tr}[\gamma_\mu (\rho' + p) \gamma^\mu (\rho + p)] F_1$
 $+ \text{tr}[\gamma_\mu (\rho' + p) \sigma^{\mu\nu} g_\nu (\rho + p)] \frac{iF_2}{2m}$

Recall the trace identities

$$\text{tr}[1] = 4$$

$$\text{tr}[\gamma_{\alpha_1} \cdots \gamma_{\alpha_{2n+1}}] = 0$$

$$\text{tr}[\gamma_\mu \gamma_\nu] = 4g_{\mu\nu}$$

$$\text{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 4(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})$$

$$\gamma_\mu \gamma^\mu = 41$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$$

$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4g^{\nu\rho}$$

$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\tau = -2\gamma^\tau \gamma^\rho \gamma^\sigma$$

We note some useful kinematic relations, with $p'^2 = p^2 = m^2$

$$\left. \begin{aligned} P^2 &= (p'+p)^2 = 2m^2 + 2p' \cdot p \\ q^2 &= (p'-p)^2 = 2m^2 - 2p' \cdot p \end{aligned} \right\} \quad P^2 + q^2 = 4m^2,$$

$$P \cdot q = (p'+p) \cdot (p'-p) = p'^2 - p^2 = 0$$

$$p' \cdot P = p \cdot P = 2m^2 - \frac{q^2}{2}$$

$$p' \cdot q = -p \cdot q = \frac{q^2}{2}$$

Evaluating the traces

$$\text{tr}[(p'+m)\bar{P}(p+m)] = m \text{tr}[p'\bar{P}] + m \text{tr}[\bar{P}p]$$

$$= 4m(p' \cdot \bar{P} + p \cdot \bar{P})$$

$$= 4m(4m^2 - q^2)$$

want to express
scalars in terms
of momentum transfer q^2

$$\text{tr}[(p'+m)\bar{P}_\mu \sigma^{\mu\nu} q_\nu (p+m)] = i \text{tr}[(p'+m)\bar{P} q (p+m)]$$

$$\text{recall } \sigma^{\mu\nu} = i\gamma^\mu \gamma^\nu - i\gamma^\nu \gamma^\mu$$

with,

$$-i \text{tr}[(p'+m)\bar{P} q (p+m)]$$

$$P \cdot q = 0$$

$$i \text{tr}[(p'+m)\bar{P} q (p+m)] = i \text{tr}[p' \bar{P} q p] + i m^2 \text{tr}[\bar{P} q]$$

$$= 4i(p' \cdot \bar{P} q \cdot p - p' \cdot q \bar{P} \cdot p + p' \cdot p \bar{P} q) + 4im^2 \bar{P} q$$

$$= -4i \frac{q^2}{2} [P \cdot p' + P \cdot p]$$

$$= -2iq^2(4m^2 - q^2)$$

$$\Rightarrow \text{tr}[(p'+m)\bar{P}_\mu \sigma^{\mu\nu} q_\nu (p+m)] = -2iq^2(4m^2 - q^2)$$

$$\begin{aligned}
 \text{tr}[\gamma_{\mu}(\rho'+m)\gamma^{\nu}(\rho+m)] &= \text{tr}[\gamma_{\mu}\rho'\gamma^{\nu}\rho] + m^2 \text{tr}[\gamma_{\mu}\gamma^{\nu}] \\
 &= -2\text{tr}[\rho'\rho] + 4m^2 \text{tr}[1] \\
 &= -8\rho' \cdot \rho + 16m^2 \\
 &= 4(q^2 + 2m^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{tr}[\gamma_{\mu}(\rho'+m)\overset{\sigma^{\mu\nu}}{\gamma}_{\nu}(\rho+m)] &= i\text{tr}[\gamma_{\mu}(\rho'+m)\gamma^{\nu}\gamma^{\rho}(\rho+m)] \\
 \sigma^{\mu\nu} = i\gamma^{\mu}\gamma^{\nu} - i\gamma^{\nu}\gamma^{\mu} &\quad -i\text{tr}[\gamma^{\rho}(\rho'+m)(\rho+m)] \\
 &= i\text{int}_r[\gamma_{\mu}\rho'\gamma^{\nu}\gamma^{\rho}] + i\text{int}_r[\gamma_{\mu}\gamma^{\nu}\gamma^{\rho}\rho] \\
 &\quad - i\text{int}_r[\gamma^{\rho}\rho] - i\text{int}_r[\gamma^{\rho}\rho'] \\
 &= -2i\text{int}_r[\rho'\gamma^{\rho}] + 4i\text{int}_r[\gamma^{\rho}\rho] \\
 &\quad - i\text{int}_r[\gamma^{\rho}\rho] - i\text{int}_r[\gamma^{\rho}\rho'] \\
 &= 3i\text{int}_r[\rho\gamma^{\rho}] - 3i\text{int}_r[\rho'\gamma^{\rho}] \\
 &= 12i\text{int}_r[(\rho-\rho') \cdot \gamma^{\rho}] \\
 &= -12imq^2
 \end{aligned}$$

Therefore, the relations

- $\text{tr}[(\rho' + m) \tilde{P}_\mu \Gamma^\mu (\rho + m)] = \text{tr}[(\rho' + m) \tilde{P}(\rho + m)] F_1 + \text{tr}[(\rho' + m) \tilde{P}_\mu \sigma^\mu q_\nu (\rho + m)] \frac{iF_2}{2m}$
- $\text{tr}[\gamma_\mu (\rho' + m) \Gamma^\mu (\rho + m)] = \text{tr}[\gamma_\mu (\rho' + m) \gamma^\mu (\rho + m)] F_1 + \text{tr}[\gamma_\mu (\rho' + m) \sigma^\mu q_\nu (\rho + m)] \frac{iF_2}{2m}$

Simplify to

- $\text{tr}[(\rho' + m) \tilde{P}_\mu \Gamma^\mu (\rho + m)] = 4m(4m^2 - q^2) F_1 - 2iq^2(4m^2 - q^2) \left(\frac{iF_2}{2m} \right)$
- $\text{tr}[\gamma_\mu (\rho' + m) \Gamma^\mu (\rho + m)] = 4(q^2 + 2m^2) F_1 - 12imq^2 \left(\frac{iF_2}{2m} \right)$
- or,
- $\text{tr}[(\rho' + m) \tilde{P}_\mu \Gamma^\mu (\rho + m)] = 4m(4m^2 - q^2) \left[F_1 + \frac{q^2}{4m^2} F_2 \right]$
- $\text{tr}[\gamma_\mu (\rho' + m) \Gamma^\mu (\rho + m)] = 4(q^2 + 2m^2) F_1 + 6q^2 F_2$

We then solve the resulting 2×2 system for F_1 and F_2

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{AD - BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$= \frac{1}{AD - BC} \begin{pmatrix} DT_1 - BT_2 \\ -CT_1 + AT_2 \end{pmatrix}$$

The determinant,

$$\begin{aligned}
 AD - BC &= 4mq^2(4m^2 - q^2) \left[6 - \frac{1}{4m^2} \cdot 4(q^2 + 2m^2) \right] \\
 &= 4mq^2(4m^2 - q^2)(4 - q^2/m^2) \\
 &= 4\frac{q^2}{m}(4m^2 - q^2)^2
 \end{aligned}$$

F_1 form-factor

$$\begin{aligned}
 F_1 &= \frac{1}{AD - BC} (DT_1 - BT_2) \\
 &= \frac{m}{4q^2(4m^2 - q^2)^2} \\
 &\quad \times \text{tr} \left[\left(6q^2 P_r - 4m(4m^2 - q^2) \frac{q^2}{4m^2} \gamma_r \right) (\rho' + m) \Gamma^\mu (\rho + m) \right] \\
 &= \frac{1}{4(q^2 - 4m^2)} \text{tr} \left[\left(\gamma_r - \frac{6m(\rho' + \rho)}{4m^2 - q^2} \right) (\rho' + m) \Gamma^\mu (\rho + m) \right]
 \end{aligned}$$

F_2 form-factor

$$\begin{aligned}
 F_2 &= \frac{1}{AD - BC} (-CT_1 + AT_2) \\
 &= \frac{m}{4q^2(4m^2 - q^2)^2} \\
 &\quad \times \text{tr} \left[\left(-4(q^2 + 2m^2) P_r + 4m(4m^2 - q^2) \gamma_r \right) (\rho' + m) \Gamma^\mu (\rho + m) \right] \\
 &= -\frac{m^2}{q^2(q^2 - 4m^2)} \text{tr} \left[\left(\gamma_r + \left(\frac{q^2 + 2m^2}{q^2 - 4m^2} \right) \left(\frac{\rho' + \rho}{m} \right)_r \right) (\rho' + m) \Gamma^\mu (\rho + m) \right]
 \end{aligned}$$

Therefore, we can project the form-factors F_1 and F_2 from Γ^m using

$$F_1 = \frac{1}{4(q^2 - 4m^2)} \text{tr} \left[\left(\gamma_\mu - \frac{6m(p' + p)_\mu}{q^2 - 4m^2} \right) (p' + m) \Gamma^m (p + m) \right]$$

$$F_2 = \frac{-m^2}{q^2(q^2 - 4m^2)} \text{tr} \left[\left(\gamma_\mu + \left(\frac{q^2 + 2m^2}{q^2 - 4m^2} \right) (p' + p)_\mu \right) (p' + m) \Gamma^m (p + m) \right]$$

Notice that these projectors are a non-perturbative result, and prove useful when evaluating higher-order diagrams. If we were to consider a regularization procedure, e.g., dimensional regularization, then we would need to derive projectors within that regularization, e.g., defining projectors in d -spacetime dimensions in dimensional regularization. This is especially important for finding F_1 .

Here, we are only interested in the magnetic moment anomaly, $a_2 = F_2(0)$. Therefore, it is useful to expand the projector about $q^2 = 0$, or more specifically $q^m = 0$.

To do the expansion, it is useful to take P, q as the independent kinematic variables instead of p, p' ,

$$\left. \begin{array}{l} P_\mu = p'_\mu + p_\mu \\ q_\mu = p'_\mu - p_\mu \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} p'_\mu = \frac{1}{2}(P+q)_\mu \\ p_\mu = \frac{1}{2}(P-q)_\mu \end{array} \right.$$

such that $\Gamma^{\hat{\mu}}(p', p) = \Gamma^{\hat{\mu}}(P, q)$. The form-factor is then

$$\begin{aligned} F_2(Q^2) &= \frac{-m^2}{q^2(q^2-4m^2)} \text{tr} \left[\left(Y_\mu + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) \tilde{P}_\mu \right) \right. \\ &\quad \times \left. \left(\frac{1}{2}(P+q) + m \right) \Gamma^{\hat{\mu}} \left(\frac{1}{2}(P-q) + m \right) \right] \end{aligned}$$

Now, as $q^{\hat{\mu}} \rightarrow 0$

$$\begin{aligned} \Gamma^{\hat{\mu}}(P, q) &= \Gamma^{\hat{\mu}}(P, 0) + q^\nu \frac{\partial}{\partial q^\nu} \Gamma^{\hat{\mu}}(P, q) \Big|_{q^\nu=0} + O(q^\nu q^\rho) \\ &= V^{\hat{\mu}}(P) + q^\nu \delta V_{\nu}{}^{\hat{\mu}}(P) \end{aligned}$$

Note that at $q^2=0$, $\tilde{P}^2 = 4m^2 - q^2 = 4m^2$. However, we will consider $\tilde{P}^2 \neq 0$ throughout the derivation, and take $\tilde{P}^2 \rightarrow 4m^2$ at the end.

Using the cyclic property of the trace, we write the form-factor as

$$F_2(Q^2) = -\frac{m^2}{q^2(q^2-4m^2)} \text{tr} \left\{ \left[\left(\frac{1}{2}(P-q) + m \right) \gamma_\mu \left(\frac{1}{2}(P+q) + m \right) \right. \right. \\ \left. \left. + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) \overline{P}_\mu \left(\frac{1}{2}(P-q) + m \right) \left(\frac{1}{2}(P+q) + m \right) \right] \Gamma^\mu \right\}$$

Now,

$$\left(\frac{1}{2}(P-q) + m \right) \left(\frac{1}{2}(P+q) + m \right) = \frac{1}{4} (P^2 - q^2 + [P, q]) \\ + \frac{1}{2} m (P-q) + \frac{1}{2} m (P+q) + m^2 \\ = \frac{1}{4} (P^2 - q^2) + \frac{1}{4} [P, q] + mP + m^2$$

Recall,

$$P^2 + q^2 = 4m^2 \Rightarrow P^2 - q^2 = 4m^2 - 2q^2$$

$$\text{and } [P, q] = Pq - qP = Pq + Pq - \cancel{P \cdot q}^0 = 2Pq$$

$$\text{so, } \left(\frac{1}{2}(P-q) + m \right) \left(\frac{1}{2}(P+q) + m \right) = m^2 - \frac{q^2}{2} + \frac{1}{2} Pq + mP + m^2 \\ = 2m^2 - \frac{q^2}{2} + P \left(m + \frac{q}{2} \right)$$

also,

$$\begin{aligned}
 & \left(\frac{1}{2}(P-Q) + m \right) r_m \left(\frac{1}{2}(P+Q) + m \right) \\
 &= \frac{1}{4} (P-Q) r_m (P+Q) + \frac{m}{2} (P-Q) r_m + \frac{m}{2} r_m (P+Q) + m^2 r_m \\
 &= \frac{1}{4} (P-Q) \left[2(P+Q) r_m - (P+Q) r_m \right] + m^2 r_m \\
 &\quad + \frac{m}{2} \left[(P-Q) r_m + 2(P+Q) r_m - (P+Q) r_m \right] \\
 &= \frac{1}{2} (P+Q) r_m (P-Q) - \frac{1}{4} (P-Q) (P+Q) r_m \\
 &\quad + m (P+Q) r_m - m Q r_m + m^2 r_m
 \end{aligned}$$

Now,

$$\begin{aligned}
 (P-Q)(P+Q) &= P^2 - Q^2 + [P, Q] \\
 &= P^2 - Q^2 + 2PQ = 4m^2 - 2Q^2 + 2PQ
 \end{aligned}$$

So,

$$\begin{aligned}
 & \left(\frac{1}{2}(P-Q) + m \right) r_m \left(\frac{1}{2}(P+Q) + m \right) \\
 &= \frac{1}{2} (P+Q) r_m (P-Q) - \cancel{m^2 r_m} + \frac{1}{2} Q^2 r_m - \frac{1}{2} PQ r_m \\
 &\quad + m (P+Q) r_m - m Q r_m + \cancel{m^2 r_m} \\
 &= (P+Q) r_m \left[\frac{1}{2}(P-Q) + m \right] + \left[\frac{1}{2} Q^2 - \frac{1}{2} PQ - m Q \right] r_m
 \end{aligned}$$

So, we find the form-factor takes the form

$$F_2(Q^2) = \frac{-m^2}{q^2(q^2 - 4m^2)} \times \text{tr} \left\{ \left[\left[\frac{1}{2}q^2 - \frac{1}{2}\not{P}_q - m\not{q} \right] \not{v}_r + (\not{P} + \not{q})_r \left[\frac{1}{2}(\not{P} - \not{q}) + m \right] + \left(\frac{q^2 + 2m^2}{q^2 - 4m^2} \right) \not{P}_r \left[2m^2 - \frac{\not{q}^2}{2} + \not{P} \left(m + \frac{\not{q}}{2} \right) \right] \right] (V^r + q^v \delta V_v^r) \right\}$$

Next, we can average over the spatial direction of \not{q}^r , since $\int d\Omega_q F_2(Q^2) = 4\pi F_2(Q^2)$.

For terms linear in \not{q}^r ,

$$\int \frac{d\Omega_q}{4\pi} \not{q}_r = A \not{P}_r$$

where A is undetermined, and \not{P}_r is the only four-vector remaining. But, $\not{P} \cdot \not{q} = 0$

$$\Rightarrow 0 = \int \frac{d\Omega_q}{4\pi} \not{P} \cdot \not{q} = A \not{P}^2$$

the only way this works is if $A = 0$

so,

$$\boxed{\int \frac{d\Omega_q}{4\pi} \not{q}_r = 0}$$

For term proportional to $q_\mu q_\nu$, we have

$$\int \frac{d\Omega_4}{4\pi} q_\mu q_\nu = A g_{\mu\nu} + B P_\mu P_\nu$$

where A and B are undetermined. Now, contract with $g^{\mu\nu}$,

$$g^{\mu\nu} \int \frac{d\Omega_4}{4\pi} q_\mu q_\nu = \int \frac{d\Omega_4}{4\pi} q^2 = q^2$$

$$= 4A + B P^2$$

$$\text{so, } q^2 = 4A + B P^2$$

Also, contract with $P^\mu P^\nu$, recalling $P \cdot q = 0$

$$\Rightarrow 0 = \int \frac{d\Omega_4}{4\pi} P \cdot q P \cdot q = A P^2 + B P^2 P^2$$

$$\text{so, } (A + B P^2) P^2 = 0$$

$$\Rightarrow B = -\frac{A}{P^2}$$

$$\text{which gives } q^2 = 4A + B P^2 = 4A - A = 3A$$

$$\Rightarrow A = q^2/3$$

therefore,

$$\boxed{\int \frac{d\Omega_4}{4\pi} q_\mu q_\nu = \frac{1}{3} q^2 \left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \right)}$$

Finally, note that terms with $q_r q_v q_p$ will also give a zero angular average, $\int d\Omega_q q_r q_v q_p = 0$
 So, averaging over the form factor

$$F_2(Q^2) = \int \frac{d\Omega_q}{4\pi} F_2(Q^2)$$

$$= \frac{-m^2}{q^2(q^2-4m^2)} \int \frac{d\Omega_q}{4\pi} \text{tr} \left[\left\{ \left(\frac{1}{2}q^2 - \left(\frac{P}{2} + m \right) q \right) r_r \right. \right.$$

$$+ P_r \left(\frac{1}{2} \cancel{(P-q)} + m \right) + Q_r \left(\frac{1}{2} \cancel{(P-q)} + m \right)$$

$$\left. \left. + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) \overline{P_m} \left[2m^2 - \frac{q^2}{2} + P \left(m + \frac{q}{2} \right) \right] \right\} V^m \right]$$

$$+ \frac{-m^2}{q^2(q^2-4m^2)} \int \frac{d\Omega_q}{4\pi} \text{tr} \left[\left\{ \left(\frac{1}{2}q^2 - \left(\frac{P}{2} + m \right) q \right) r_r \right. \right.$$

$$+ P_r \left(\frac{1}{2} \cancel{(P-q)} + m \right) + Q_r \left(\frac{1}{2} \cancel{(P-q)} + m \right)$$

$$\left. \left. + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) \overline{P_m} \left[2m^2 - \frac{q^2}{2} + P \left(m + \frac{q}{2} \right) \right] \right\} q^v \delta V_v \right]$$

$$\int \frac{d\Omega_q}{4\pi} = 1, \quad \int \frac{d\Omega_q}{4\pi} Q_r = 0, \quad \int \frac{d\Omega_q}{4\pi} Q_r Q_v = \frac{1}{3} q^2 \left(S_{rv} - \frac{P_r P_v}{P^2} \right)$$

$$\int \frac{d\Omega_q}{4\pi} Q_r Q_v Q_p = 0$$

$$F_2(Q^2) = \int d\Omega_q \frac{1}{4\pi} F_2(Q^2)$$

$$\begin{aligned}
 &= \frac{-m^2}{q^2(q^2-4m^2)} \int d\Omega_q \frac{1}{4\pi} \text{tr} \left[\left\{ \frac{1}{2} q^2 r_s + P_s \left(\frac{1}{2} P + m \right) - \frac{1}{2} q_s q_s r^s \right. \right. \\
 &\quad \left. \left. + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) P_s \left[2m^2 - \frac{q^2}{2} + P_m \right] \right\} V^s \right] \\
 &+ \frac{-m^2}{q^2(q^2-4m^2)} \int d\Omega_q \frac{1}{4\pi} \text{tr} \left[\left\{ - \left(\frac{P}{2} + m \right) q_s r^s r_s - \frac{1}{2} q_s P_s r^s \right. \right. \\
 &\quad \left. \left. + q_s \left(\frac{1}{2} P + m \right) + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) P_s \left[\frac{P}{2} r^s q_s \right] \right\} q^s \delta V_s \right]
 \end{aligned}$$

$$\begin{aligned}
 F_2(Q^2) &= \frac{-m^2}{q^2(q^2-4m^2)} \\
 &\times \text{tr} \left[\left\{ \frac{1}{2} q^2 r_s + P_s \left(\frac{P}{2} + m \right) - \frac{1}{2} \cdot \frac{1}{3} q^2 \left(r_s - \frac{P_s P}{P^2} \right) \right. \right. \\
 &\quad \left. \left. + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) P_s \left(2m^2 - \frac{q^2}{2} + P_m \right) \right\} V^s \right] \\
 &+ \frac{-m^2}{q^2(q^2-4m^2)} \text{tr} \left[\left\{ - \frac{1}{3} q^2 \left(\frac{P}{2} + m \right) \left(r^s - \frac{P^s P}{P^2} \right) r_s \right. \right. \\
 &\quad - \frac{1}{2} P_s \frac{1}{3} q^2 \left(r^s - \frac{P^s P}{P^2} \right) + \left(\frac{1}{2} P + m \right) \frac{1}{3} q^2 \left(r^s - \frac{P_s P^s}{P^2} \right) \\
 &\quad \left. \left. + \left(\frac{q^2+2m^2}{q^2-4m^2} \right) P_s \left(\frac{P}{2} \frac{1}{3} q^2 \left(r^s - \frac{P^s P}{P^2} \right) \right) \right\} \delta V_s \right]
 \end{aligned}$$

Notice that the δV_s term is proportional to q^2 . Let us manipulate two expressions* in the first term

Manipulating the two expressions

$$\frac{P}{m} \left(\frac{P}{2} + m \right) + \left(\frac{q^2 + 2m^2}{q^2 - 4m^2} \right) \frac{P}{m} \left(2m^2 + Pm \right)$$

$$= \frac{1}{q^2 - 4m^2} \frac{P}{m} \left[q^2 - 4m^2 + 2(q^2 + 2m^2) \right] \left(\frac{P}{2} + m \right)$$

$$= \frac{1}{q^2 - 4m^2} \frac{P}{m} \cdot 3 \left(\frac{P}{2} + m \right)$$

Thus, every term in the trace is proportional to q^2 , which cancels the $\frac{1}{q^2}$ pre-factor. Canceling this factor allows us to take the limit $q \rightarrow 0$.

$$\begin{aligned} F_2(Q^2) &= \frac{-m^2}{(q^2 - 4m^2)} \\ &\times \text{tr} \left[\left\{ \frac{1}{2} \gamma_\mu - \frac{1}{2} \cdot \frac{1}{3} \left(\gamma_\mu - \frac{P^\nu P_\nu}{P^2} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{q^2 - 4m^2} \frac{P}{m} \cdot 3 \left(\frac{P}{2} + m \right) - \frac{1}{2} \left(\frac{q^2 + 2m^2}{q^2 - 4m^2} \right) \frac{P}{m} \right\} V^\mu \right] \\ &+ \frac{-m^2}{(q^2 - 4m^2)} \text{tr} \left[\left\{ -\frac{1}{3} \left(\frac{P}{2} + m \right) \left(\gamma^\nu - \frac{P^\nu P_\nu}{P^2} \right) \gamma_\mu \right. \right. \\ &\quad \left. \left. - \frac{1}{2} P_\mu \frac{1}{3} \left(\gamma^\nu - \frac{P^\nu P_\nu}{P^2} \right) + \left(\frac{1}{2} P + m \right) \frac{1}{3} \left(g_\mu^\nu - \frac{P_\mu P_\nu}{P^2} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{q^2 + 2m^2}{q^2 - 4m^2} \right) \frac{P}{m} \left(\frac{P}{2} \frac{1}{3} \left(\gamma^\nu - \frac{P^\nu P_\nu}{P^2} \right) \right) \right\} \delta V_\nu \right] \end{aligned}$$

So, as $\eta \rightarrow 0$

recall $P^2 = 4n^2$

$$F_2(0) = +\frac{1}{4} \text{tr} \left[\left\{ \frac{1}{2} r_r - \frac{1}{6} \left(r_r - \frac{P_r P}{P^2} \right) \right. \right.$$

$$\left. \left. - \frac{3}{4n^2} P_r \left(\frac{P}{2} + n \right) + \frac{1}{4n} P_r \right\} V^r \right.$$

$$+ \left\{ -\frac{1}{3} \left(\frac{P}{2} + n \right) \left(r^v - \frac{P^v P}{P^2} \right) r_r + n \frac{1}{3} \left(g_r^v - \frac{P_r P^v}{P^2} \right) \right.$$

$$\left. - \frac{1}{2} P_r \frac{1}{3} \left(r^v - \frac{P^v P}{P^2} \right) + \frac{1}{2} P \frac{1}{3} \left(g_r^v - \frac{P_r P^v}{P^2} \right) \right]$$

$$\left. - \frac{1}{2} \frac{P}{n} \left(\frac{1}{3} \frac{P}{2} \left(r^v - \frac{P^v P}{P^2} \right) \right) \right\} \delta V_r]$$

$$= +\frac{1}{4} \text{tr} \left[\left\{ \frac{1}{3} r_r + \frac{1}{6} P_r \frac{P}{4n^2} - \frac{3}{8n^2} P_r P - \frac{3P_r}{4n} + \frac{1}{4n} P_r \right\} V^r \right.$$

$$+ \left\{ -\frac{1}{3} \left(\frac{P}{2} + n \right) r^v r_r + \frac{1}{3} \left(\frac{P}{2} + n \right) P^v \frac{P}{P^2} r_r + \frac{n}{3} \left(g_r^v - \frac{P_r P^v}{P^2} \right) \right.$$

$$\left. - \frac{1}{6} P_r r^v + \frac{1}{6} P g_r^v - \frac{1}{12n} P_r P r^v + \frac{1}{12n} P_r P^v \frac{P}{P^2} \right\} \delta V_r]$$

$$= +\frac{1}{12n^2} \text{tr} \left\{ \left[n^2 r_r - P_r P - \frac{3}{2} n P_r \right] V^r \right.$$

$$+ \left[-n^2 \left(\frac{P}{2} + n \right) r_r r_r + n^2 \left(\frac{P}{2} + n \right) P^v \frac{P}{4n^2} r_r \right.$$

$$+ n^3 \left(g_{rv} - \frac{P_r P^v}{4n^2} \right) - \frac{n^2}{2} P_r r_v + \frac{n^2}{2} P g_{rv} \left. \right]$$

$$\left. - \frac{n}{4} P_r P r_v + \frac{n}{4} P_r P_v \right] \delta V_{rv,r} \}$$

$$F_2^{(0)} = +\frac{1}{12m^2} \operatorname{tr} \left\{ \left[m^2 r_r - P_r P_r - \frac{3}{2} m P_r \right] V^r \right. \\ \left. + \left[-m^2 \left(\frac{P}{2} + m \right) r_v r_r + m^2 \left(\frac{P}{2} + m \right) \frac{PP_v}{4m^2} r_r \right. \right. \\ \left. \left. + m^3 g_{rv} - \frac{m^2}{2} P_r r_v + \frac{m^2}{2} P g_{rv} - \frac{m}{4} P_r P r_v \right] \delta V^{v,r} \right\}$$

For the second term, note

$$\{\gamma_r, \gamma_v\} = 2g_{\mu\nu}, \quad \text{and} \quad P^2 = p^2 = 4m^2$$

So,

One can show, e.g. Mather-Dice, that the second term
can be written as

$$Z^{\text{ad}} \text{ term} = \frac{m}{4} \left(\frac{P}{2} + m \right) [\gamma_m, \gamma_v] \left(\frac{P}{2} + m \right)$$

N.B. It's still @ analytical; proving this, 55

MatherJice can do the algebra faster than me...

So, we find that the anomalous magnetic moment can be found directly by

$$\begin{aligned}
 a &= F_2(0) \\
 &= \frac{1}{12m^2} \text{tr} \left[\left(m^2 g_r - P_r P_r - \frac{3}{2} m P_r \right) V^r \right. \\
 &\quad \left. + \frac{m}{4} \left(\frac{P}{2} + m \right) [g_r, g_r] \left(\frac{P}{2} + m \right) \delta V^{rr} \right]
 \end{aligned}$$

where we take $P^2 \rightarrow 4m^2$ after taking the trace, and where

$$V^r(P) = \Gamma^r(P, 0)$$

$$\delta V^{rr}(P) = \left. \frac{\partial \Gamma^r(P, \varepsilon)}{\partial g_r} \right|_{\varepsilon=0}$$

Thus, we only need to find the vertex function linear in g^r to compute a .

Radiative Corrections

We are now (finally) ready to compute the first-order radiative correction to the magnetic moment of the electron

$$a_e^{(1)} = F_2^{(0)}$$

$$= \frac{1}{12m^2} t_c \left[\left(m^2 \gamma_r - \frac{P_r P}{2} - \frac{3}{2} m P_r \right) V_1^r \right. \\ \left. + \frac{m}{4} \left(\frac{P}{2} + m \right) [\gamma_r, \gamma_v] \left(\frac{P}{2} + m \right) \delta V_1^v \right]$$

where $V_1^r(P) = \Lambda_1^r(P, 0)$

$$\delta V_1^v(P) = \left. \frac{\partial \Lambda_1^r(P, q)}{\partial \Sigma_v} \right|_{q=0}$$

where $\Lambda_1^r(P, q) \equiv \Lambda_1^r(p', p)$ is the correction to the vertex function

$$\Lambda_1^r(p', p) = -e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i^3 N^r(p', p, k)}{k^2 [(p' - k)^2 - m^2][(p - k)^2 - m^2]}$$

$$\text{with } N^r = \gamma_v [(\not{p} - \not{k}) + m] \gamma^r [(\not{p} - \not{k}) + m] \gamma^v$$

Let us now replace the kinematics

$$p' = \frac{1}{2}(P+q), \quad p = \frac{1}{2}(P-q)$$

S_1

$$\Lambda_1^{\mu}(P, q) = -ie^2 \int \frac{d^4 k}{(2\pi)^4} \frac{N^{\mu}(P, q, k)}{k^2 [((P+q)/2 - k)^2 - m^2] [((P-q)/2 - k)^2 - m^2]}$$

with

$$N^{\mu}(P, q, k) = \gamma_{\alpha} [(P+q)/2 - k + n] \gamma^{\mu} [(P-q)/2 - k + n] \gamma^{\alpha}$$

At $q=0$,

$$\Lambda_1^{\mu}(P, 0) = -ie^2 \int \frac{d^4 k}{(2\pi)^4} \frac{N^{\mu}(P, 0, k)}{k^2 [(P/2 - k)^2 - m^2]^2}$$

$$\text{with } N^{\mu}(P, 0, k) = \gamma_{\alpha} [P/2 - k + n] \gamma^{\mu} [P/2 - k + n] \gamma^{\alpha}$$

The derivative is given by

$$\begin{aligned} \left. \frac{\partial \Lambda_1^{\mu}}{\partial q_{\nu}} \right|_{q=0} &= -ie^2 \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{1}{k^2 [((P/2 - k)^2 - m^2)]^2} \frac{\partial N^{\mu}}{\partial q_{\nu}} \Big|_{q=0} \right. \\ &\quad + \left. \frac{N^{\mu}(P, 0, k)}{k^2} \times \frac{\partial}{\partial q_{\nu}} \left(\frac{1}{[((P+q)/2 - k)^2 - m^2] [((P-q)/2 - k)^2 - m^2]} \right) \right\} \end{aligned}$$

$$\begin{aligned} \text{Now, } \left. \frac{\partial N^{\mu}}{\partial q_{\nu}} \right|_{q=0} &= \gamma_{\alpha} \frac{\partial}{\partial q_{\nu}} [\gamma^{\mu} [(P+q)/2 - k + n]] \gamma^{\nu} [\gamma^{\mu} [(P-q)/2 - k + n]] \gamma^{\alpha} \\ &\quad + \gamma_{\alpha} [\gamma^{\mu} [(P/2 - k + n)] \gamma^{\nu} \frac{\partial}{\partial q_{\nu}} [\gamma^{\mu} [(P-q)/2 - k + n]]] \gamma^{\alpha} \end{aligned}$$

$$\text{with } \frac{\partial}{\partial q_{\nu}} [\gamma^{\mu} [(P+q)/2 - k + n]] = \frac{1}{2} \frac{\partial}{\partial q_{\nu}} q_{\mu} = \frac{1}{2}$$

$$\Rightarrow \left. \frac{\partial N^{\mu}}{\partial q_{\nu}} \right|_{q=0} = \gamma_{\alpha} \frac{1}{2} \gamma^{\nu} \gamma^{\mu} [\gamma^{\mu} [(P/2 - k + n)] \gamma^{\alpha} - \gamma_{\alpha} [\gamma^{\mu} [(P-q)/2 - k + n]] \gamma^{\nu}]$$

The derivative of the propagators is

$$\frac{\partial}{\partial \xi_\nu} \left(\frac{1}{[(P+q)_{1/2}-\mu]^2 - r^2} \frac{1}{[(P-q)_{1/2}-\mu]^2 - r^2} \right)_{\xi=0}$$

$$= \left[\frac{1}{[(P_{1/2}-\mu)^2 - r^2]} \frac{\partial}{\partial \xi_\nu} \left(\frac{1}{[(C(P+q))_{1/2}-\mu]^2 - r^2} \right)_{\xi=0} \right. \\ \left. + \frac{1}{[(C(P_{1/2}-\mu))^2 - r^2]} \frac{\partial}{\partial \xi_\nu} \left(\frac{1}{[(C(P-q))_{1/2}-\mu]^2 - r^2} \right)_{\xi=0} \right]$$

Now,

$$\frac{\partial}{\partial \xi_\nu} \left(\frac{1}{[(C(P\pm q))_{1/2}-\mu]^2 - r^2} \right)_{\xi=0} = \frac{1}{[(C(P_{1/2}-\mu))^2 - r^2]^2} \cdot \frac{\partial}{\partial \xi_\nu} ((P\pm q)_{1/2}-\mu)^2 \\ = \frac{1}{[(C(P_{1/2}-\mu))^2 - r^2]^2} \cdot 2(C(P_{1/2}-\mu))^{\alpha} \cdot \frac{1}{2} (\pm \delta_{\alpha}^{\nu}) \\ = \frac{\pm (C(P_{1/2}-\mu))^{\nu}}{[(C(P_{1/2}-\mu))^2 - r^2]^2}$$

↑ sign means that these terms
will sum to zero

$$\Rightarrow \frac{\partial}{\partial \xi_\nu} \left(\frac{1}{[(P+q)_{1/2}-\mu]^2 - r^2} \frac{1}{[(P-q)_{1/2}-\mu]^2 - r^2} \right)_{\xi=0} = 0 .$$

So, the derivative of the vertex function is

$$\frac{\partial \Lambda_i}{\partial q_\nu} \Big|_{q=0} = -ie^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 [(P_{12}-k)^2 - m^2]} \gamma^\mu \frac{\partial N^\mu}{\partial q_\nu} \Big|_{q=0}$$

with,

$$\frac{\partial N^\mu}{\partial q_\nu} \Big|_{q=0} = \gamma_\alpha \frac{\gamma^\nu}{2} \gamma^\mu [P_{12}-k+m] \gamma^\alpha - \gamma_\alpha [P_{12}-k+m] \gamma^\nu \frac{\gamma^\mu}{2} \gamma^\alpha$$

In the projection formula, we need to take traces.

For the first term, we need

$$T_1 = \text{tr} \left[\left(\underline{m^2 \gamma_\nu} - \underline{P_\nu P^\nu} - \frac{3}{2} \underline{m P_\nu} \right) N^\nu(P, 0, m) \right]$$

with

$$N^\nu(P, 0, m) = \gamma_\alpha [P_{12}-k+m] \gamma^\nu [P_{12}-k+m] \gamma^\alpha$$

so, the following components are

$$\gamma_\nu \gamma_\alpha \gamma^\alpha = 2 \gamma_\nu$$

$$\begin{aligned} & \cdot \text{tr} [\gamma_\nu \gamma_\alpha [P_{12}-k+m] \gamma^\nu [P_{12}-k+m] \gamma^\alpha] \\ &= -2 \text{tr} [\gamma_\nu [P_{12}-k+m] \gamma^\nu [P_{12}-k+m]] \\ &= -2 \text{tr} [\gamma_\nu (P_{12}-k) \gamma^\nu (P_{12}-k)] - 2m^2 \text{tr} [\gamma_\nu \gamma^\nu] \\ &= +4 \text{tr} [(P_{12}-k)(P_{12}-k)] - 8m^2 \text{tr} [\mathbb{1}] \\ &= 16(P_{12}-k)^2 - 32m^2 \\ &= 16 \left(\frac{P^2}{4} + k^2 - P \cdot k \right) - 32m^2 \\ &= 16(k^2 - P \cdot k - m^2) \end{aligned}$$

- $$\begin{aligned} & \text{tr}[\mathcal{P} \gamma_\alpha [\mathcal{P}_{12} - k + n] \mathcal{P} [\mathcal{P}_{12} - k + n] \gamma^\alpha] \\ &= -2 \text{tr}[\mathcal{P} [\mathcal{P}_{12} - k + n] \mathcal{P} [\mathcal{P}_{12} - k + n]] \\ &= -2 \text{tr}[\mathcal{P} (\mathcal{P}_{12} - k) \mathcal{P} (\mathcal{P}_{12} - k)] - 2n^2 \text{tr}[\mathcal{P} \mathcal{P}] \\ &= -8 \left[2 \mathcal{P} \cdot \left(\frac{\mathcal{P}}{2} - k \right) \mathcal{P} \cdot \left(\frac{\mathcal{P}}{2} - k \right) - \mathcal{P}^2 \left(\frac{\mathcal{P}}{2} - k \right)^2 \right] - 8n^2 \mathcal{P}^2 \\ &= -8 \left[2 \left(\frac{\mathcal{P}^2}{2} - k \cdot \mathcal{P} \right)^2 - 4n^2 \left(\frac{\mathcal{P}^2}{4} - k \cdot \mathcal{P} + k^2 \right) \right] - 32n^4 \\ &= -16 \left[(2n^2 - k \cdot \mathcal{P})^2 - 2n^2 (n^2 - k \cdot \mathcal{P} + k^2) \right] - 32n^4 \\ &= -16 \left[4n^4 + (k \cdot \mathcal{P})^2 - 4n^2 k \cdot \mathcal{P} - 2n^4 + 2n^2 k \cdot \mathcal{P} - 2n^2 k^2 + 2n^4 \right] \\ &= -16 \left[4n^4 + (k \cdot \mathcal{P})^2 - 2n^2 k \cdot \mathcal{P} - 2n^2 k^2 \right] \end{aligned}$$
- $$\begin{aligned} & \text{tr}[\gamma_\alpha [\mathcal{P}_{12} - k + n] \mathcal{P} [\mathcal{P}_{12} - k + n] \gamma^\alpha] \\ &= 4 \text{tr}[(\mathcal{P}_{12} - k + n) \mathcal{P} (\mathcal{P}_{12} - k + n)] \\ &= 4n \text{tr}[(\mathcal{P}_{12} - k) \mathcal{P}] + 4n \text{tr}[\mathcal{P} (\mathcal{P}_{12} - k)] \\ &= 32n (\mathcal{P}_{12}^2 - k \cdot \mathcal{P}) \\ &= 32n (2n^2 - k \cdot \mathcal{P}) \end{aligned}$$

So, the first trace is

$$\begin{aligned}
 T_1 &= m^2 \cdot 16(h^2 - P \cdot h - m^2) + 16(4m^4 - 2h^2 P \cdot h - 2h^2 m^2 + (P \cdot h)^2) \\
 &\quad - \frac{3}{2}m \cdot 32m(2m^2 - h \cdot P) \\
 &= -16m^2 h^2 + 16(P \cdot h)^2 - 48h^4 \\
 &= -16[3h^4 + m^2 h^2 - (h \cdot P)^2]
 \end{aligned}$$

The second trace needed is

$$T_2 = \text{tr}\left[\left(\frac{P}{2} + m\right)[\gamma_\mu, \gamma_\nu]\left(\frac{P}{2} + m\right)\frac{\partial N^\nu}{\partial g_\nu} \Big|_{g=0}\right]$$

with

$$\begin{aligned}
 \frac{\partial N^\nu}{\partial g_\nu} \Big|_{g=0} &= \frac{1}{2} \left\{ \gamma_\alpha \gamma^\nu \gamma^\mu (\not{P}_{1/2} - \not{k} + \not{m}) \gamma^\alpha \right. \\
 &\quad \left. - \gamma_\alpha (\not{P}_{1/2} - \not{k} + \not{m}) \gamma^\mu \gamma^\nu \gamma^\alpha \right\} \\
 &= \frac{1}{2} \left\{ -2 \left(\frac{P}{2} - k\right) \gamma^\mu \gamma^\nu + 4m g^{\mu\nu} \right. \\
 &\quad \left. + 2 \gamma^\nu \gamma^\mu \left(\frac{P}{2} - k\right) + 4m g^{\mu\nu} \right\}
 \end{aligned}$$

Note,
 $[\gamma_\mu, \gamma_\nu] g^{\mu\nu} = [\gamma_\mu, \gamma^\nu] = 0$

$$\begin{aligned}
 \text{So, } T_2 &= -\text{tr}\left[\left(\frac{P}{2} + m\right) \underline{[\gamma_\mu, \gamma_\nu]} \left(\frac{P}{2} + m\right) \left(\frac{P}{2} - k\right) \gamma^\mu \gamma^\nu\right] \\
 &\quad + \text{tr}\left[\left(\frac{P}{2} + m\right) \underline{[\gamma_\mu, \gamma_\nu]} \left(\frac{P}{2} + m\right) \gamma^\nu \gamma^\mu \left(\frac{P}{2} - k\right)\right]
 \end{aligned}$$

$$\begin{aligned}
& \bullet \operatorname{tr} \left[\left(\frac{P}{2} + m \right) (Y_r Y_v - Y_v Y_r) \left(\frac{P}{2} + m \right) \left(\frac{P}{2} - k \right) Y^m Y^v \right] \\
& = m \operatorname{tr} \left[\frac{P}{2} (Y_r Y_v - Y_v Y_r) \left(\frac{P}{2} - k \right) Y^m Y^v \right] \\
& \quad + m \operatorname{tr} \left[(Y_r Y_v - Y_v Y_r) \frac{P}{2} \left(\frac{P}{2} - k \right) Y^m Y^v \right] \\
& = m \operatorname{tr} \left[\frac{P}{2} Y_r Y_v \left(\frac{P}{2} - k \right) Y^m Y^v \right] - m \operatorname{tr} \left[\frac{P}{2} Y_v Y_r \left(\frac{P}{2} - k \right) Y^m Y^v \right] \\
& \quad + m \operatorname{tr} \left[Y_r Y_v \frac{P}{2} \left(\frac{P}{2} - k \right) Y^m Y^v \right] - m \operatorname{tr} \left[Y_v Y_r \frac{P}{2} \left(\frac{P}{2} - k \right) Y^m Y^v \right] \\
& = 4m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - k \right) \right] + 2m \operatorname{tr} \left[\frac{P}{2} Y_v \left(\frac{P}{2} - k \right) Y^v \right] \\
& \quad - 8m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - k \right) \right] - 16m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - k \right) \right] \\
& = -20m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - k \right) \right] - 4m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - k \right) \right] \\
& = -24 \cdot 4m \frac{P}{2} \cdot \left(\frac{P}{2} - k \right) \\
& = -96m \left(\frac{P^2}{4} - k \cdot \frac{P}{2} \right) \\
& = -48m (2m^2 - k \cdot P)
\end{aligned}$$

$$\begin{aligned}
& \cdot \operatorname{tr} \left[\left(\frac{P}{2} + h \right) \left(Y_u Y_v - Y_v Y_u \right) \left(\frac{P}{2} + h \right) Y^u Y^v \left(\frac{P}{2} - h \right) \right] \\
& = m \operatorname{tr} \left[\frac{P}{2} \left(Y_u Y_v - Y_v Y_u \right) Y^u Y^v \left(\frac{P}{2} - h \right) \right] \\
& \quad + m \operatorname{tr} \left[\left(Y_u Y_v - Y_v Y_u \right) \frac{P}{2} Y^u Y^v \left(\frac{P}{2} - h \right) \right] \\
& = m \operatorname{tr} \left[\frac{P}{2} Y_u Y_v Y^u Y^v \left(\frac{P}{2} - h \right) \right] - m \operatorname{tr} \left[\frac{P}{2} Y_v Y_u Y^u Y^v \left(\frac{P}{2} - h \right) \right] \\
& \quad + m \operatorname{tr} \left[Y_u Y_v \frac{P}{2} Y^u Y^v \left(\frac{P}{2} - h \right) \right] - m \operatorname{tr} \left[Y_v Y_u \frac{P}{2} Y^u Y^v \left(\frac{P}{2} - h \right) \right] \\
& = 16m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - h \right) \right] + 8m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - h \right) \right] \\
& \quad + 4m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - h \right) \right] - 4m \operatorname{tr} \left[\frac{P}{2} \left(\frac{P}{2} - h \right) \right] \\
& = 24 \cdot 4m \frac{P}{2} \cdot \left(\frac{P}{2} - h \right) \\
& = 96m \left(\frac{P^2}{4} - \frac{Ph}{2} \right) \\
& = 48m \left(2m^2 - h \cdot P \right)
\end{aligned}$$

$$\text{So, } \tau_2 = 96m(2m^2 - h \cdot P)$$

Therefore, we find for the anomaly

$$\begin{aligned} \tilde{a}_e^{(1)} &= \frac{1}{12m^2} \text{tr} \left[\left(m^2 \gamma_\mu - \not{P}_\mu \not{P} - \frac{3}{2} m \not{P}_\mu \right) V_1^\mu \right. \\ &\quad \left. + \frac{m}{4} \left(\frac{\not{P}}{2} + m \right) [\gamma_\mu, \gamma_\nu] \left(\frac{\not{P}}{2} + m \right) \delta V_1^{\nu\mu} \right] \end{aligned}$$

$$= -i \frac{e^2}{12m^2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 [(P/2-k)^2 - m^2]^2} \cdot \left[\tau_1 + \frac{m}{4} \tau_2 \right]$$

Now,

$$\begin{aligned} \tau_1 + \frac{m}{4} \tau_2 &= -16 [3m^4 + m^2 k^2 - (k \cdot P)^2] \\ &\quad + 24 m^2 (2k^2 - k \cdot P) \\ &= -16 m^2 k^2 + 16 (k \cdot P)^2 - 24 m^2 k \cdot P \end{aligned}$$

Notice that the integrand is finite as $k \rightarrow 0$, since the numerator is proportional to k . Naively, we see that the integral could be UV divergent, however we will see that this integral is finite.

So,

$$a_e^{(1)} = -i e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 [(P/2-k)^2 - m^2]^2} \left(-\frac{4}{3} k^2 + \frac{4}{3} (k \cdot P)^2 - 2 k \cdot P \right)$$

We now introduce Feynman parameters

$$\begin{aligned}\frac{1}{A^2 B} &= 2 \int_0^1 dx_1 \int_0^1 dx_2 \delta(1-x_1-x_2) \frac{x_1}{[x_1 A + x_2 B]^3} \\ &= 2 \int_0^1 dx \frac{x}{[x A + (1-x) B]^3} \\ &= 2 \int_0^1 dx \frac{x}{[x(A-B)+B]^3}\end{aligned}$$

So, with

$$A = (P_{12} - k)^2 - m^2 = \frac{P^2}{4} + k^2 - k \cdot P - m^2$$

$$= k^2 - k \cdot P$$

and

$$B = k^2$$

$$\text{So, } \frac{1}{k^2 [(P_{12} - k)^2 - m^2]} = 2 \int_0^1 dx \frac{x}{[k^2 - k \cdot P x]^3}$$

So,

$$a_e^{(0)} = -2ie^2 \int_0^1 dx \times \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - k \cdot P_x)^3} \left(-\frac{4}{3} k^2 + \frac{4}{3m^2} (k \cdot P)^2 - 2k \cdot P \right)$$

To perform the momentum integral, let us perform this in d-dimensions to see the cancellation of the UV divergence.

Note the following integrals (No proof)

$$I(n) = \int d^d k \frac{1}{(k^2 + 2\vec{P} \cdot k - M^2)^n} = i\pi^{d/2} \frac{(-1)^n}{\Gamma(n)} \frac{\Gamma(n-d/2)}{(\vec{P}^2 + M^2)^{n-d/2}}$$

$$I_\mu(n) = \int d^d k \frac{k_\mu}{(k^2 + 2\vec{P} \cdot k - M^2)^n} = -\hat{P}_\mu I(n)$$

$$I_{\mu\nu}(n) = \int d^d k \frac{k_\mu k_\nu}{(k^2 + 2\vec{P} \cdot k - M^2)^n} = \left[\hat{P}_\mu \hat{P}_\nu - \frac{1}{2} g_{\mu\nu} \left(\frac{\vec{P}^2 + M^2}{n-1-d/2} \right) \right] I(n)$$

For us, $\hat{P} = -\frac{P_x}{2}$, $M^2 = 0$, $n = 3$, and let $d = 4 - 2\varepsilon$

$$\text{so, } I(3) = -\frac{i\pi^2}{2\hat{P}^2} = -\frac{2i\pi^2}{P^2 x^2} = -\frac{i\pi^2}{2m^2 x^2}$$

↑
take $\varepsilon \rightarrow 0$
at end

and

$$I_\mu(3) = +\frac{x}{2} \hat{P}_\mu I(3)$$

$$\begin{aligned} I_{\mu\nu}(3) &= \left[\frac{x^2}{4} \hat{P}_\mu \hat{P}_\nu - \frac{1}{2} g_{\mu\nu} \frac{(P^2 x^2 / 4)}{2 - d/2} \right] I(3) \\ &= \frac{x^2}{4} \left[\hat{P}_\mu \hat{P}_\nu - \frac{2m^2}{\varepsilon} g_{\mu\nu} \right] I(3) \end{aligned}$$

Therefore,

$$\begin{aligned} \int d^4 k \frac{1}{(k^2 - k \cdot P_x)^3} &\left(-\frac{4}{3} k^2 + \frac{4}{3m^2} (k \cdot P)^2 - 2k \cdot P \right) \\ &= -\frac{4}{3} g^{\mu\nu} I_{\mu\nu}(3) + \frac{4}{3m^2} P^\mu P^\nu I_\mu(3) - 2P^\mu I_\mu(3) \end{aligned}$$

$$\begin{aligned}
& \int d^4k \frac{1}{(k^2 - k \cdot P_x)^3} \left(-\frac{4}{3} k^2 + \frac{4}{3m^2} (k \cdot P)^2 - 2k \cdot P \right) \\
&= -\frac{4}{3} g^{\mu\nu} I_{\mu\nu}(3) + \frac{4}{3m^2} P^\mu P^\nu I_{\mu\nu}(1) - 2P^\mu I_\mu(3) \\
&= -\frac{4}{3} g^{\mu\nu} \frac{x^2}{4} \left[P_\mu P_\nu - \frac{2m^2}{\epsilon} g_{\mu\nu} \right] I(3) \\
&\quad + \frac{4}{3m^2} P^\mu P^\nu \frac{x^2}{4} \left[P_\mu P_\nu - \frac{2m^2}{\epsilon} g_{\mu\nu} \right] I(3) \\
&\quad - 2P^\mu \frac{x}{2} P_\mu I(3) \\
&= \left\{ -\frac{4}{3} \frac{x^2}{4} \left[P^2 - \frac{2m^2 \cdot 4}{\epsilon} \right] + \frac{4}{3m^2} \frac{x^2}{4} \left[P^2 P^2 - \frac{2m^2}{\epsilon} P^2 \right] - x P^2 \right\} I(3) \\
&= \left\{ \frac{x^2}{3} \left[-P^2 + \frac{2m^2 \cdot 4}{\epsilon} \right] + \frac{P^2 P^2}{m^2} - \frac{2m^2}{\epsilon} \frac{P^2}{m^2} \right\} I(3) - x P^2 I(3) \\
&= \left\{ \frac{x^2}{3} \left[-4m^2 + 16m^2 \right] - x \cdot 4m^2 \right\} I(3) \quad \text{Naive UV divergence cancels.} \\
&= 4m^2 x (x-1) I(3) \\
&= -2 \frac{(x-1)}{x} i\pi^2
\end{aligned}$$

Therefore, the simplified expression for $\alpha_e^{(1)}$ is

$$\begin{aligned}
\alpha_e^{(1)} &= -2 \frac{i e^2}{(2\pi)^4} \int_0^1 dx \times \left\{ \frac{2}{x} (1-x) i\pi^2 \right\} \\
&= + \frac{(2\pi)^2}{(2\pi)^4} e^2 \int_0^1 dx (1-x) = \frac{e^2}{4\pi^2} \left[x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2} \left(\frac{e^2}{4\pi^2} \right)
\end{aligned}$$

So,

$$\alpha_e^{(1)} = \frac{1}{2} \left(\frac{e^2}{4\pi^2} \right)$$

Recall the fine-structure constant , $\alpha = \frac{e^2}{4\pi^2}$

$$\Rightarrow \boxed{\alpha_e^{(1)} = \frac{\alpha}{2\pi}}$$

Schwinger's triumph!

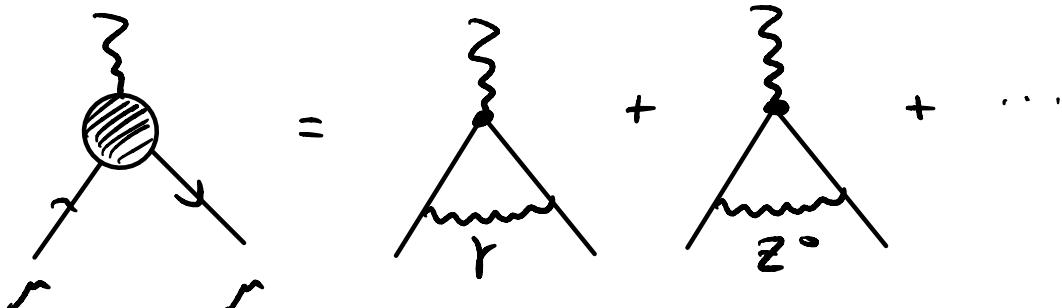
here, $\alpha_e^{(1)} = 1.16 \dots \times 10^{-3}$

The measured value is $(\alpha_e)^{\text{exp}} = 1.15965218073(28) \times 10^{-3}$
while the current theoretical prediction ($O(\alpha^{10})$!) is
 $(\alpha_e)^{\text{SM}} = 1.15965218188(78) \times 10^{-3}$

A remarkable prediction!

For muons, the leading order is identical , $\alpha_\mu^{(1)} = \frac{\alpha}{2\pi}$,
but since $\frac{m_\mu}{m_e} \sim 200$, the value of $\frac{g-2}{2}$ is susceptible

to searches for new physics.



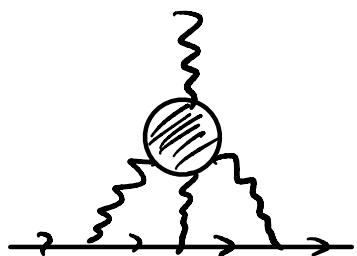
$$\text{For the } Z^0\text{-boson, } \alpha_\mu \sim \left(\frac{m_\mu}{m_Z}\right)^2 \cdot \frac{g_e^2 e}{16\pi} \cdot \log\left(\frac{m_Z^2}{m_\mu^2}\right)$$

↓ EW coupling.

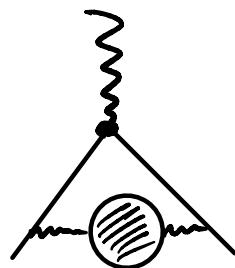
There is an intense search for new physics by looking at the muon anomaly. The current (as of Aug. 2023) experimental measurement is

$$(\alpha_\mu)^{\text{exp}} = 116592.059(22) \times 10^{-11}$$

The SM theory is a little bit muddled. There are discrepancies between lattice methods and data-driven analyses. There is an apparent 4.2σ discrepancy with experiment. \mathcal{BD} , two hadronic processes are the leading sources



and



Hadronic light-to-light

Hadronic vacuum polarization