



PHYS 303 – Classical Mechanics of Particles and Waves II

Problem Set 3

Due: Thursday, September 19 at 5:00pm

Term: Fall 2024

Instructor: Andrew W. Jackura

Readings

Read sections 14.1–14.8 of Taylor.

Problems

Problem 1. [10 pts.] – Free Particle Motion in Polar Coordinates

We saw that the trajectory of a free particle in polar coordinates is

$$r(\phi) = \frac{r_0}{\cos(\phi - \phi_0)},$$

where $r_0 = r(\phi_0)$. Show that this is an equation of a straight line by writing it in Cartesian coordinates (x, y) and casting it in standard form $ax + by = c$, where a , b , and c are constants to be determined.

Problem 2. [10 pts.] – Kepler's First Law

We showed in lecture that any Kepler orbit of an inverse-square force law can be written in the form

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi},$$

where $c > 0$ and $\epsilon > 0$. For the case $0 \leq \epsilon < 1$, rewrite this equation in Cartesian coordinates (x, y) and prove that this equation can be cast in the form

$$\frac{(x + d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a , b , and d are constants to be determined in terms of c and ϵ .

Problem 3. [15 pts.] – Halley's Comet

Halley's comet, which passed around the sun early in 1986, moves in a highly elliptical orbit of eccentricity 0.967 and a period of 76 years. Calculate its minimum and maximum distances from the sun.

Problem 4. [25 pts.] – Logarithmic Spiral

The relative motion of a two-body system with reduced mass μ is that of a logarithmic spiral given by $r = ke^{\alpha\phi}$, where k and α are constants.

- (a) [5 pts.] Find the central force law responsible for this relative motion in terms of k , α , μ , and the angular momentum ℓ . *Hint:* Recall that in cylindrical polar coordinates (r, ϕ) the equations of motion are

$$\mu r^2 \dot{\phi} = \ell, \quad \mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U(r)}{\partial r},$$

where $U(r)$ is the potential energy of the central force.

- (b) [10 pts.] Unlike inverse-square force laws, we can solve for the time dependence of both r and ϕ . If given that $\phi = 0$ at $t = 0$, determine $\phi(t)$ and $r(t)$.
- (c) [10 pts.] Determine the total energy E of the orbit, assuming we define $U(r \rightarrow \infty) = 0$.

Problem 5. [20 pts.] – Precession

A particle of mass m moves with angular momentum ℓ in the field of a fixed force center with

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3},$$

where k and λ are positive constants. Recall that the equations of motion for this system are

$$mr^2 \dot{\phi} = \ell, \quad m\ddot{r} = F(r) + \frac{\ell^2}{mr^3},$$

- (a) [10 pts.] Show that under a transformation $u = 1/r$ that the radial equation becomes

$$\frac{d^2 u}{d\phi^2} = -\alpha^2 u + \beta,$$

where α , and β are positive constants to be determined in terms of m , ℓ , k and λ .

- (b) [10 pts.] Assuming that $\phi = 0$ is defined to align with the periapsis, show that the solution to the radial equation gives an orbit

$$r(\phi) = \frac{c}{1 + \epsilon \cos(\alpha\phi)},$$

where c and ϵ are positive constants. Find c in terms of the given parameters. Describe the orbit for the case $0 < \epsilon < 1$.

- (c) [5 pts.] Are there any values of α for which the orbit is closed?