

Phenomenology II - QCD

Quantum Chromodynamics (QCD) is a non-abelian gauge theory describing the color interactions of quarks and gluons. The Lagrange density is

$$\mathcal{L} = \frac{1}{2} i \sum_f \bar{q}_f D^\mu q_f + \text{h.c.} - \sum_f m_f \bar{q}_f q_f - \frac{1}{2} \text{tr} [G_{\mu\nu} G^{\mu\nu}]$$

where $D_\mu = \partial_\mu + ig_s A_\mu$, $A_\mu = A_\mu^a T_a$

and

$$G_{\mu\nu} = \frac{1}{ig_s} [D_\mu, D_\nu]$$

$\hookrightarrow = \frac{\lambda_c}{2}$

There are 6 flavors, $f = \{u, d, s, c, b, t\}$

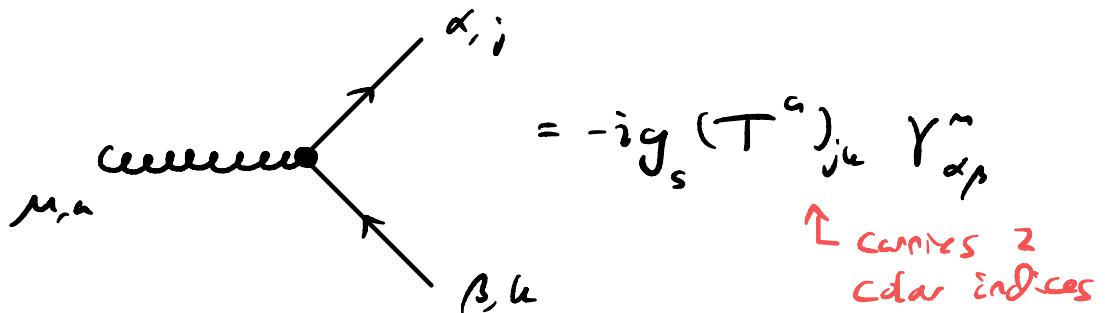
\hookrightarrow "decays" via weak
before hadronization

QCD, in principle, describes all
hadronic and nuclear phenomena we

observe in the universe. It depends on 6 quark masses,
and 1 coupling, g_s . Define

$$\alpha_s = \frac{g_s^2}{4\pi}$$

This coupling governs the interactions of massless gluons with
quarks, e.g.,



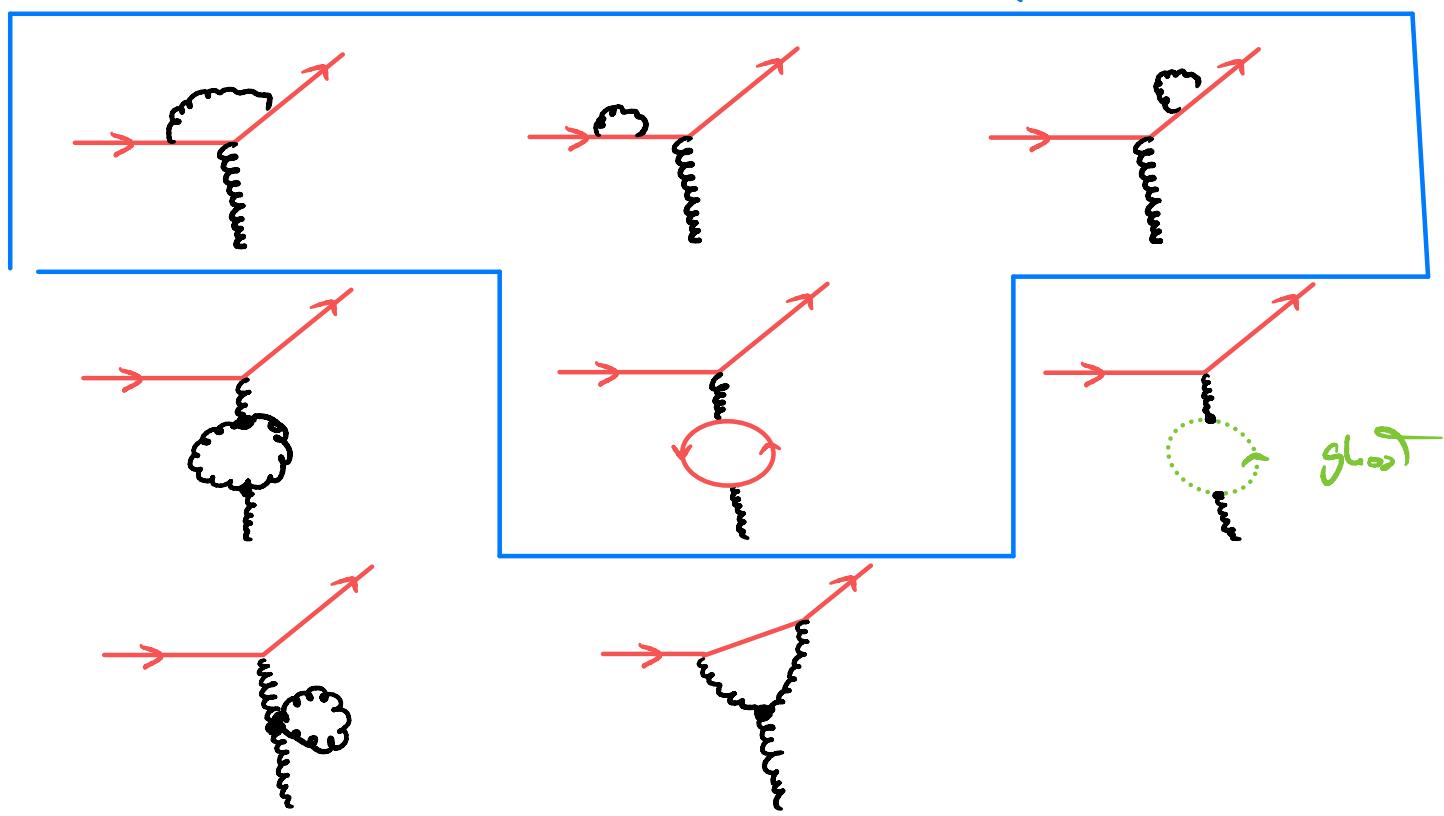
The same coupling also governs interactions of gluons with themselves,

$$= -ig_s f^{abc} (p_3^v g^{vp} + \text{perm.})$$

$$= -g_s^2 f^{abc} f^{ade} g^{np} g^{nr} + \text{perm.}$$

In perturbation theory, higher order corrections augment the coupling.

present in QED



Can show (using QFT II), that the coupling g_s runs with the momentum transfer Q^2 , and is described by a differential equation

$$\frac{dg_s}{d\ln Q} = \beta(g_s)$$

"beta function"

$$= -\frac{g_s^3}{16\pi^2} \left[11 - \frac{2}{3} N_f \right] + \mathcal{O}(g_s^5)$$

↓ LO β_0 (1-loop)

Notice that for $N_f < 16$, $\beta(g_s) < 0$

$\Rightarrow g_s$ decreases as Q increases. This is known as **Asymptotic freedom**

Conversely, as Q decreases, g_s increases, and eventually perturbation theory breaks down. This indicates that low-energy physics must be computed with non-perturbative techniques. This is the **confinement** region. Gluons are crucial for this, cf. QED with electrons and photons

$$\frac{de}{d\ln Q} = \frac{e^3}{16\pi} \cdot \frac{4}{3} + \mathcal{O}(e^5) > 0$$

Can write a general for α_s ,

$$\frac{d\alpha_s}{d\ln Q^2} = -\alpha_s \left[\left(\frac{\alpha_s}{4\pi} \right) \beta_0 + \left(\frac{\alpha_s}{4\pi} \right)^2 \beta_1 + \dots \right]$$

$\hookrightarrow 21 - 2N_f/3$

Solving the 1-loop equation, let $x = 1/\alpha_s$, $L = \ln Q^2$

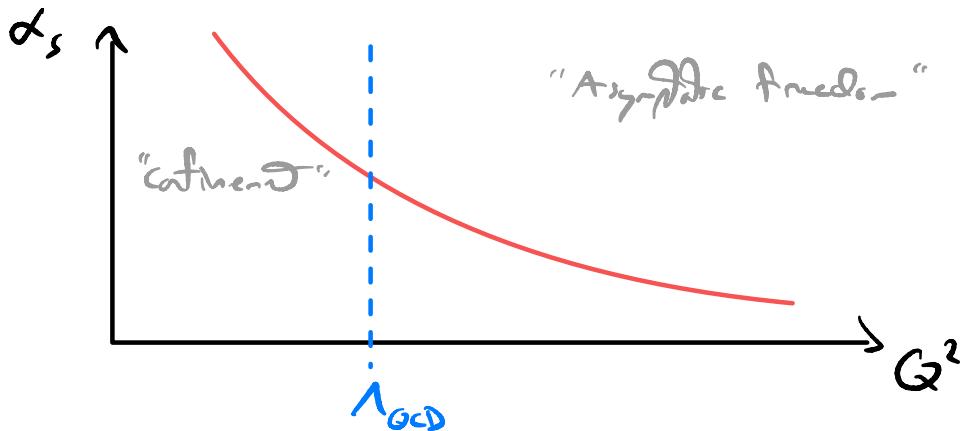
$$\Rightarrow \frac{dx}{dL} = -\frac{1}{\alpha_s^2} \frac{d\alpha_s}{dL} = \frac{\beta_0}{4\pi} \Rightarrow x = \frac{\beta_0}{4\pi} L + \text{const.}$$

$$\Rightarrow \alpha_s(Q^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln(Q^2/\Lambda_{QCD}^2)}$$

This is the running coupling.

\downarrow Integration const.

Λ_{QCD} is value of Q^2 where $\alpha_s = 1$. Consider massless QCD, with only the dimensionless coupling (1 parameter). But, the running produces a dimensional scale. This is known as dimensional transmutation. In effect, Λ_{QCD} sets the scale for strong interactions involving light quarks.



To study low-energy hadrons, we need a non-perturbative technique to access observables. There is another issue we need to deal with, and that is we do not observe quarks and gluons, but observe them only as confined inside hadrons.

What are the allowed quark & gluon combinations that can give a valid hadron? We have seen from the Quark Model

$q\bar{q}$ - Mesons

qqq - Baryons, $\bar{q}\bar{q}\bar{q}$ - Anti-baryons

In QCD, we need color singlet objects made from q_i, \bar{q}_i , and A_{jk}^{μ} (Field operators)

$\overbrace{\quad\quad\quad}^{\text{color indices}}$

For hadrons to exist, there must be a color singlet operator to create it from the QCD vacuum

e.g.,

- $\bar{q}_i \gamma_5 q_j'$ for $J^{PC}=0^{-+}$ mesons
- $\epsilon_{jkl} q_i q_k' q_l''$ for baryons

TB[†], also can have

- $G_{ij}^{\mu\nu} G_{\nu\rho,ij}$ for scalar glueball
- $\bar{q}_i [\gamma^\mu, \gamma^\nu] G_{\mu\nu,ji} q_i$ for $q\bar{q}g$ hybrid
- $\bar{q}_i \gamma^\mu q_j \bar{q}_k \gamma_\mu q_k$ for "tetragonal"
- $\epsilon_{ijk} q_j q_k \bar{q}_l \bar{q}_m$ for "pentagonal"
- ... as many (∞) more.

The problem is that there is no 1-to-1 correspondence between operators and states in QFT. Operators with the same quantum numbers can create the same state e.g.,

$$O_1 = \bar{u} \gamma_5 u + \bar{d} \gamma_5 d + \bar{s} \gamma_5 s \quad \text{looks like } \bar{q}q$$

$$O_2 = \epsilon_{\mu\nu\rho\sigma} G_{ij}^{\mu\nu} G_{ij}^{\rho\sigma} \quad \text{looks like pseudo scalar glueball}$$

Both can create η & η' , $\langle 0 | O_{1,2} | \eta' \rangle \neq 0$

From a QFT pov, asking whether η' is a $\bar{q}q$ state or a glueball is almost meaningless.

Moreover, states created by a given operator can consist of multiple hadrons, e.g.,

$$\bar{u} \gamma_5 d \bar{u} \gamma_5 d |0\rangle \sim |\pi^+ \pi^+\rangle$$

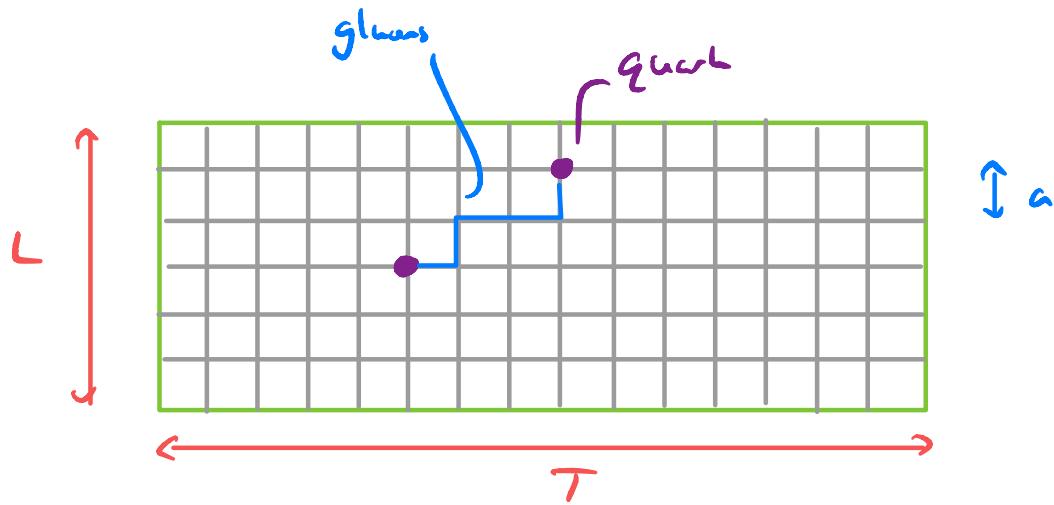
\Rightarrow Evidence for an operator Does Not imply existence of a new hadron.

How can we access such complicated non-perturbative objects from QCD? Lattice QCD is a numerical technique to stochastically compute the non-perturbative QCD path integral. Can now calculate light hadron masses w/ $\lesssim 1\%$ uncertainties and reaction amplitudes, involving a couple of hadrons. This is the only known method for making quantitative predictions for QCD in the low-energy region.

To compute with Lattice QCD, we rotate the action S_{QCD} to Euclidean time, $t \rightarrow -i\tau_E$

$$\Rightarrow e^{-iEt} \rightarrow e^{-E\tau_E}$$

and discretize spacetime (w/ spacing a) & use a finite size box $L^3 \times T$.



The QCD path integral is formulated as

$$Z_{QCD} = \int D\bar{q} D\bar{s} D\bar{u} e^{-S_{QCD}^{(E)}[\bar{q}, \bar{s}, \bar{u}]}$$

↓ $e^{i g_s a A_\mu \cdot n_\mu}$
 ↓ Aug. over all field "configurations"

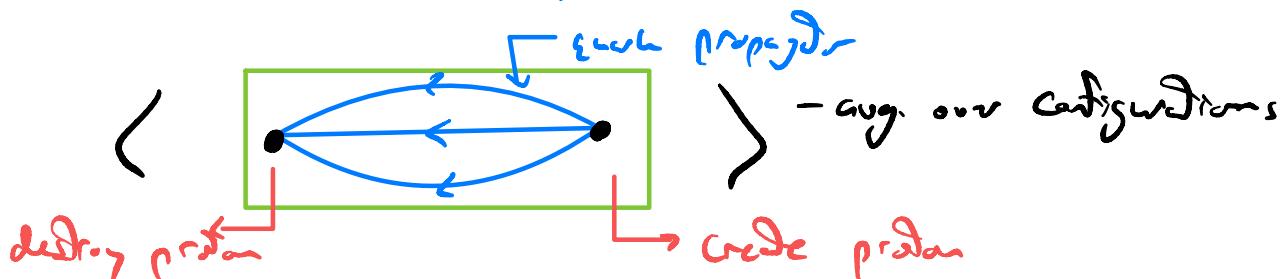
On a lattice, this is millions of degrees. We use Monte Carlo methods to stochastically estimate correlation functions. For example, to get proton mass, want

$$\langle O_p(\tau_c) O_p^+(\omega) \rangle \quad \text{where } O_p \propto \epsilon_{ijk} u_i u_{jk}$$

create proton
 $\Im T_E = 0$, and
 destroy $\Im \bar{T}_E$.

$$= Z_p e^{-m_p T_c} + Z_{p'} e^{-m_{p'} T_E} + \dots$$

↑ ground state
 proton mass ↑ excited proton

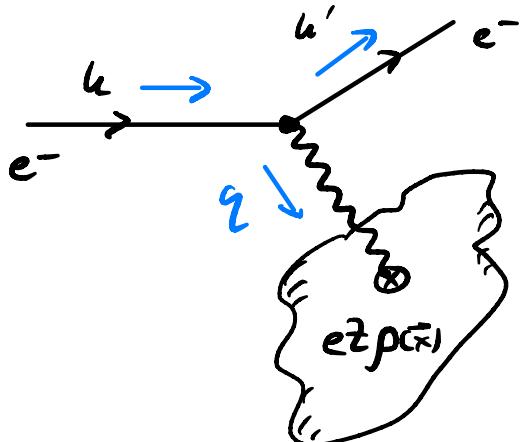


Deep Inelastic Scattering

If QCD is so complicated, how did we accept the theory as an explanation for hadrons? Consider the high-energy limit, $Q^2 \rightarrow \infty$, then the coupling is small, and QCD exhibits asymptotic freedom. That is, at high-energies quarks and gluons "behave" as "free" particles, before they hadronize.

There are many types experiments that show this, we will focus on a particular process called Deep Inelastic scattering (DIS).

Consider electron scattering off a static classical charge distribution,



The electrostatic potential is
 $A^{(n)}(\vec{x}) = (\varphi(\vec{q}\vec{x}), \vec{0})$
 $\vec{\nabla}^2 \varphi = -e z \rho$

$$\int d^3x \rho(\vec{x}) = 1$$

Fourier transform, $\tilde{A}^o(\vec{q})$

$$\begin{aligned}
 \tilde{A}^o(\vec{q}) &= 2\pi \delta(q^0) \int d^3\vec{x} e^{i\vec{q}\cdot\vec{x}} \varphi(\vec{x}) \\
 &= 2\pi \delta(q^0) \left(-\frac{1}{|\vec{q}|} \int d^3\vec{x} e^{i\vec{q}\cdot\vec{x}} \vec{\nabla}^2 \varphi(\vec{x}) \right) \\
 &= 2\pi \delta(q^0) \underbrace{\left[\frac{e^2}{|\vec{q}|} \int d^3\vec{x} e^{i\vec{q}\cdot\vec{x}} \rho(\vec{x}) \right]}_{= F(\vec{q}) \text{ "form-factor"}} \\
 &\quad Q^2 = -q^2
 \end{aligned}$$

Scattering amplitude is

$$\begin{aligned}
 iM &= \begin{array}{c} h' \\ \swarrow \downarrow \uparrow \\ h \\ \otimes \end{array} = ie \bar{u}(h') \gamma^\mu u(h) \tilde{A}_\mu(\vec{q}) \\
 &= 2\pi i e \delta(E-E') \frac{ze}{|\vec{q}|^2} F(\vec{q})
 \end{aligned}$$

So, cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{ze^2}{4E^2 s \sin^2 \frac{\theta}{2}} |F(\vec{q})|^2, \quad |\vec{q}|^2 = 2E^2(1-\cos\theta) \\
 = 4E^2 s \sin^2 \frac{\theta}{2}$$

$$= \frac{ze^2}{Q^2} |F(\vec{q})|^2$$

\hookrightarrow can learn about charge distribution

experimentally
measure

Let's Taylor expand the form-factor,

$$F(\vec{q}) = \int d^3\vec{x} \left[1 - i\vec{q} \cdot \vec{x} - \frac{(\vec{q} \cdot \vec{x})^2}{2} + \mathcal{O}(q^4) \right] \rho(\vec{x})$$

→ assume spherically symmetric

$$= 1 - \frac{1}{6} |\vec{q}|^2 \int d^3\vec{x} |x|^2 \rho(x) + \mathcal{O}(q^4)$$

$$= 1 - \frac{1}{6} |\vec{q}|^2 \overline{|x|^2} + \mathcal{O}(q^4)$$

→ mean squared charge radius

$$\Rightarrow \langle r^2 \rangle = \langle |x|^2 \rangle = -6 \frac{dF}{dq^2} \Big|_{q^2=0}$$

For example, the proton is derived to have (to lowest approximation)

$$F(q) = \left(\frac{1}{1 - \frac{|\vec{q}|^2}{\Lambda^2}} \right)^2 \Rightarrow \rho(x) \propto e^{-\Lambda|x|}$$

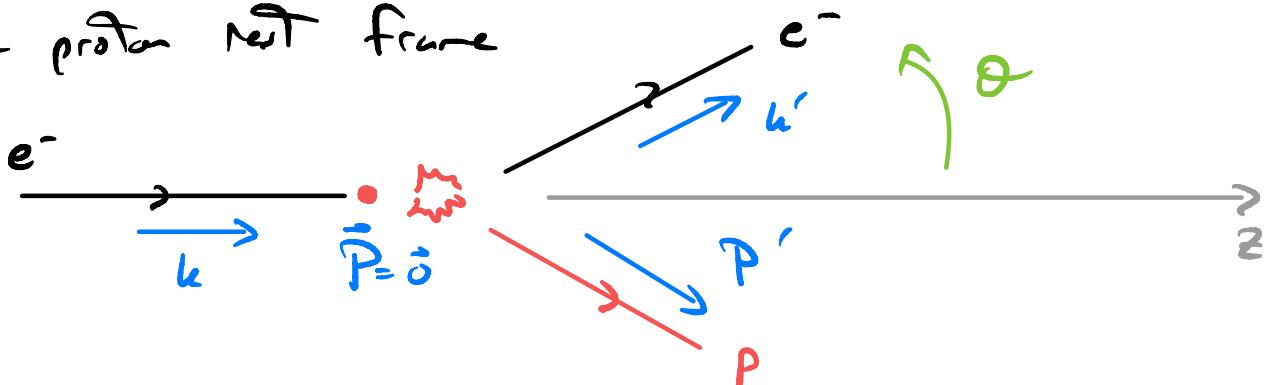
where $\Lambda \approx 1 \text{ fm}^{-1} = 1 \text{ GeV}$.

In a realistic experiment, the proton recoils.

Consider elastic $e^-p \rightarrow e^-p$ scattering, where the proton is a point particle. To leading order in α , we have

$$iM = \text{diagram with recoil} = \text{diagram without recoil} + O(\alpha^2)$$

In the proton rest frame



Cross-section is (exercise)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E'}{E} \left(\cos^2 \frac{\theta}{2} + \frac{Q^2}{2m_p c^2} \sin^2 \frac{\theta}{2} \right)$$

recoil factor

Kinematics: $k = (E, 0, 0, E)$ where $E \gg m_e$

$k' = (E', E' \sin \theta, 0, E' \cos \theta)$ $E' \gg m_e$

$p = (m_p, 0, 0, 0)$

and $Q^2 = (k - k')^2 = -2EE'(1 - \cos \theta) = -Q^2$

For a realistic proton, we need to account for it being a composite particle. As in QED, we parameterize in terms of form-factors,

$$\Rightarrow \Gamma^m = F_1(Q^2) \gamma^m + i \frac{\sigma^m}{2m_p} \epsilon_\nu F_2(Q^2)$$

In e^-n scattering & QED, $F_1(Q_1^2) - F_1(Q_2^2)$

for $e p$ scattering, it is observed that

$$\simeq -\frac{\alpha}{4\pi} \log\left(\frac{Q_1^2}{Q_2^2}\right)$$

a relatively mild change

$$F_1(Q^2) \simeq \left(\frac{1}{1 + \frac{Q^2}{\Lambda^2}} \right)^2 \quad \text{with } \Lambda \simeq 0.71 \text{ GeV}^2$$

This excludes non-perturbative QCD effects.

Can show, in proton rest frame

$$\begin{aligned} \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E'}{E} & \left[\left(F_1^2 + \frac{Q^2}{4m_p^2} F_2^2 \right) \cos^2 \frac{\theta}{2} \right. \\ & \left. + \frac{Q^2}{2m_p^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right] \end{aligned}$$

Convenient to define electric and magnetic form-factors

$$G_E = F_1 - \tau F_2 \quad , \quad \tau = \frac{Q^2}{4m_p} \\ G_M = F_1 + F_2$$

So,

Useful formula

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin \frac{\theta}{2}} \frac{E}{E'} \left[\frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right]$$

Experimentally, for proton $G_E(0) = 1$, $G_M(0) = 2.79$

neutron $G_E(0) = 0$, $G_M(0) = -1.91$

Recall for point particle, Dirac showed

$$g = 2G_M(0) = 2(1 + F_2(0)) = 2(1 + O(\alpha))$$

for the proton $F_2(0) = G_M(0) - 1 = 1.79 \neq O(\alpha)$

neutron $F_2(0) = G_M(0) - 1 = -2.91 \neq O(\alpha)$

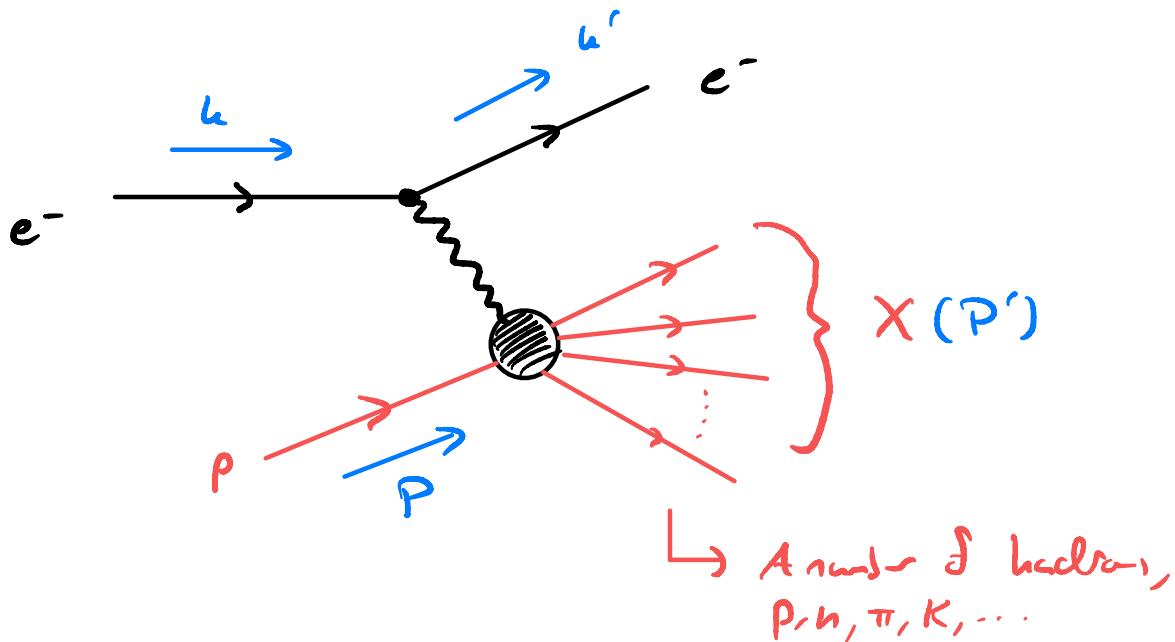
↓

proton & neutron are not
elementary fermions!

At higher virtualities $Q^2 \gg m_p^2$, then we can blow the proton apart in an inelastic collision

$$e^- p \rightarrow e^- X$$

\hookrightarrow A bunch of hadrons



In this process, we do not measure X (exclusive process). Can think of this as a high-energy probe probing the structure of the proton.

Let's define a few more useful kinematic variables,

$$v = E - E' = \frac{P \cdot q}{m_p} \quad \text{energy loss}$$

For DIS, $Q^2 \gg m_p^2 \Rightarrow v \gg m_p$

$$\hookrightarrow \Rightarrow \lambda \ll \frac{1}{m_p} \ll r_p$$

"probe probes internal structure"

"Bjorken x"

$$x = \frac{Q^2}{2m_p v} = \frac{Q^2}{2\rho \cdot \xi}$$

Since $P'^2 = M_x^2 \geq m_p^2$

\longrightarrow = for elastic scattering

$$\begin{aligned} &= (\xi + p)^2 \\ &= -Q^2 + 2\rho \cdot \xi + m_p^2 \end{aligned}$$

So, $x = \frac{Q^2}{Q^2 + M_x^2 - m_p^2} \Rightarrow 0 \leq x \leq 1$

$$\begin{aligned} x = 0 &\text{ if } Q^2 = 0 \\ x = 1 &\text{ if } M_x^2 = m_p^2 \end{aligned}$$

It is convenient to parametrize DIS by $x + Q^2$.

Another useful variable,

$$\gamma = \frac{E - E'}{E} \quad \text{relative energy loss}$$

$$= \frac{v}{E} = \frac{\rho \cdot \xi}{\rho \cdot k}$$

Can show that the cross-section for DIS is of the form

$$\frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{4\pi m_p Q^4} \frac{E'}{E} L^{uu} W_{uu}$$

↳ Max dof. since $M_x > m_p$

where $L^{uu} = \frac{1}{2} \text{tr}[t_L^\mu \gamma^\nu t_L^\nu]$ lepton tensor

\downarrow

$$W^{uu} = W_1 \left(-g^{uu} + \frac{g^u g^u}{Q^2} \right) + W_2 \left(\rho^u - \frac{\rho \cdot g}{Q^2} g^u \right) \left(\rho^u - \frac{\rho \cdot g}{Q^2} g^u \right)$$

↳ Ward identity enforced

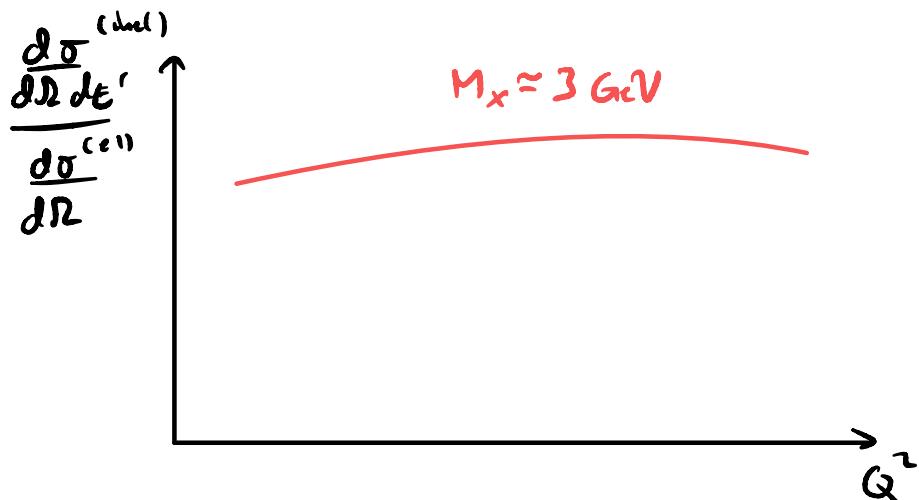
Hadronic tensor

& $W_{1,2} = W_{1,2}(x, Q^2)$ have been measured.

We find,

$$\begin{aligned} \frac{d\sigma}{d\Omega dE'} = & \frac{\alpha^2}{8\pi E^2 s \lambda \frac{Q^2}{2}} \left[\frac{m_p}{2} W_2(x, Q^2) \cos^2 \frac{\theta}{2} \right. \\ & \left. + \frac{1}{m_p} W_1(x, Q^2) \sin^2 \frac{\theta}{2} \right] \end{aligned}$$

It has been experimentally observed that



Result looks approx
constant \Rightarrow

looks like scattering
off point-like constituents.

Feynman called these
"Partons"

