Symmetry

Symmetry is key to organizing many complicated phenomena. In the last ~100 years, notion of "symmetry" has been interpreted mathematically.

$\boxed{\text{Symmetry} \equiv \text{Invariance under a group of transformations}}$

There are two basic types,

Discrete - Physical quantities transform by finite amounts, e.g., C, P, & T

Continuous - Physical quantities transform by any amount, including infinitesimal, e.g., rotational

Focus on this first

N.B. Continuous symmetries can be discussed in terms of infinitesimal transformations

→ This is relation between

Lie groups ⟷ Lie Algebras

You are already familiar with a number of continuous symmetries, including

- Rotations in 2 and 3 spatial dimensions
- Lorentz transformations in 3+1 spacetime dimensions
- Global phase transformation of Dirac spinors

$$\psi \to e^{i\theta}\psi$$

Each of these classes of symmetry transformations share the mathematical properties of a group

Groups

A group G is a set $\{g_i\}$ with an operation "group multiplication"

$$G \times G \to G$$

such that $\forall\ g_j, g_k, g_l \in G$

(1) Closure: $g_j g_k \in G$

(2) Associativity: $g_j(g_k g_l) = (g_j g_k) g_l$

(3) Identity: $\exists\ g_0 \in G$ such that $g_0 g_i = g_i$

(4) Inverse: $\exists\ g_j^{-1} \in G$ such that $g_i^{-1} g_i = g_0$

Examples

(a) $G = \{\pm 1, \pm i\}$ under ordinary multiplication is a group. Let's check, make group multiplication table

$\times$	$+1$	-1	$+i$	$-i$
$+1$	$+1$	-1	$+i$	$-i$
-1	-1	$+1$	$-i$	$+i$
$+i$	$+i$	$-i$	-1	$+1$
$-i$	$-i$	$+i$	$+1$	-1

identity element

Each row and column has every element of G $\Rightarrow$ closure

Ordinary multiplication is associative, and the inverse elements are $(\pm 1)^{-1} = \pm 1$, $(\pm i)^{-1} = \mp i$, which are elements of G $\Rightarrow$ G is a group!

N.B. this is an example of a Discrete group.

(b) $G = \{ e^{i\alpha} \mid \alpha \in \mathbb{R} \}$ under ordinary multiplication is a group. Let's check,

Let $\beta, \gamma \in \mathbb{R}$, then

- closure: $e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)} = e^{i\gamma} \in G$
- Associativity: $e^{i\alpha}(e^{i\beta} e^{i\gamma}) = (e^{i\alpha} e^{i\beta}) e^{i\gamma} \in G$
- Identity: if $\alpha = 0$, $e^{i0} = 1 \in G$
- Inverse: $e^{-i\alpha} \in G$, so $e^{-i\alpha} e^{i\alpha} = 1 \in G$

$\Rightarrow \{ e^{i\alpha} \}$ with $\alpha \in \mathbb{R}$ is a group!

N.B. this is an example of a continuous group.

Commutative (Abelian) Groups

If $g_j g_k = g_k g_j \;\; \forall \; g_j, g_k \in G$,
then the group is called commutative or Abelian

For example, $\{ e^{i\alpha} \}$ with $\alpha \in \mathbb{R}$ under ordinary multiplication is an Abelian group.

Non-commutative groups are simply called Non-Abelian

For example, rotations in 3 dimensions form a Non-Abelian group.

Continuous groups can also be defined as a smooth manifolds. Such groups are called Lie groups

Lie groups ≡ Continuous groups

They have (1) infinite number of elements

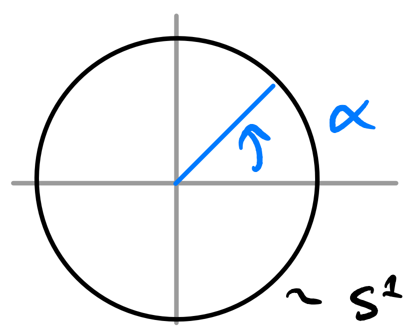
(2) the topological structure of manifold

Manifold = topological space that is locally Euclidean at each point.

Example

The group $G = \{ e^{i\alpha} \mid \alpha \in \mathbb{R} \}$ is a Lie group.

It has an infinite number of elements, labelled by the parameter $\alpha \in \mathbb{R}$, and a topological structure of a circle S^1



There is a useful result to note,

Every Lie group is isomorphic (1-1 correspondence) to a group of square matrices with group multiplication ≡ matrix multiplication.

N.B. Not necessarily compact!

A field F is also a set $\{f_i\}$ of "scalars" with two operations: $\begin{cases} \text{"scalar addition"} \\ \text{"scalar multiplication"} \end{cases}$

Such that

(1) F is Abelian group under addition with identity f_0

(2) F obeys group postulates under multiplication except f_0 has no inverse

(3) distributive:
$$f_j (f_k + f_\ell) = f_j f_k + f_j f_\ell$$
$$(f_j + f_k) f_\ell = f_j f_\ell + f_k f_\ell$$

e.g.,

- $\mathbb{R}$ $(f_0 \equiv 0)$ reals numbers under normal $+, \times$
- $\mathbb{C}$ $(f_0 \equiv 0)$ complex numbers under normal $+, \times$
- $\mathbb{Q}$ $(f_0 \equiv 0)$ rational numbers under normal $+, \times$

Some objects not fields, e.g., $\mathbb{Z}$ integers, $\mathbb{N}$ natural numbers

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A vector space V is a set $\{v_i\}$ of vectors and field F with extra operation "vector addition" $V \times V \to V$, and "extended scalar multiplication" $F \times V \to V$ such that

(1) V is Abelian group under vector addition

(2) extended scalar multiplication closes, associative, and has an identity

(3) bilinearity : $f_j(v_u + v_e) = f_j v_u + f_j v_e$

$$(f_j + f_u) v_e = f_j v_e + f_u v_e$$

e.g.,

- $\mathbb{R}^N$ as a vector space
- $\mathbb{C}^N$ as a vector space
- Set of $M \times N$ matrices (over F) under matrix addition

  [The "vectors" here are the matrices]

V is "M dimensional" if it can be spanned by M linearly independent vectors.

$$\boxed{\begin{array}{l}\text{Any such set is called a "basis" for } V \\ \text{Convention: denote basis elements by } \{x_i\}\end{array}}$$

An algebra A is a vector space V over a field F with extra operation "vector multiplication" $A \times A \to A$ such that

(1) Closure

(2) "bilinearity"
$$\begin{cases} (v_j + v_k) v_\ell = v_j v_\ell + v_k v_\ell \\ v_j (v_k + v_\ell) = v_j v_k + v_j v_\ell \\ (f_j v_k)(f_\ell v_m) = (f_j f_\ell)(v_k v_m) \end{cases}$$

Other possible combinations for special cases

- Commutative algebra : $v_j v_k = v_k v_j$
- associative algebra : $(v_j v_k) v_\ell = v_j (v_k v_\ell)$
- A with antisymmetry : $v_j v_k = - v_k v_j$
- A with identity : $v_0 v_j = v_j = v_j v_0$

  "unital"

But, many algebras don't have these properties

Example

$N \times N$ matrices under usual scalar multiplication, matrix addition (V); matrix multiplication (A)

This is a non-commutative, associative, unital algebra

The "vectors" are the matrices

It is N^2 dimensional if $\mathbb{F} = \mathbb{R}$

Example

$\mathbb{C}$ is a 2D algebra over $\mathbb{R}$ "vector over $\mathbb{R}$"

check: For $z \in \mathbb{C}$, can write $z = a + ib = (a, b)$

vector space:

$$V \times V \to V \quad : \quad (a,b) + (c,d) \longmapsto (a+c, b+d)$$

$$\mathbb{F} \times V \to V \quad : \quad r(a,b) \longmapsto (ra, rb), \ r \in \mathbb{R}$$

$$A \times A \to A \quad : \quad (a,b) \times (c,d) \longmapsto (ac - bd, ad + bc)$$

Required properties satisfied

This is a commutative, associative, unital algebra

Example (exercise)

$\mathbb{R}^3$ as vector space with vector multiplication = cross product

Check properties, find anticommutative algebra

A Lie algebra A is an algebra such that vector multiplication is anticommutative and obeys an identity "Jacobi Identity"

Convention: vector multiplication is denoted by $[\,,\,]$

$$A \times A \to A \ : \ v_j, v_k \longmapsto [v_j, v_k]$$

Lie Bracket

$\Rightarrow$ Lie algebra satisfies

(1) $[v_j, v_k] = -[v_k, v_j]$

(2) Jacobi : $\displaystyle\sum_{(jkl)} [[v_j, v_k], v_l] = 0$

cyclic sum

Useful Result (Ado theorem)

Every Lie algebra is isomorphic to algebra of square matrices with vector multiplication = commutator of matrix multiplication

i.e., $[v_j, v_k] \xrightarrow[\text{isomorphism}]{} v_j v_k - v_k v_j$

matrix multiplication

Given a basis $\{X_j\}$ for Lie algebra, can write

$$\boxed{[X_j, X_k] = C_{jk}{}^{\ell} X_\ell}$$

structure constants

The structure constants obey

$$\boxed{\sum_{(jk\ell)} C_{jk}{}^{m} C_{m\ell}{}^{n} = 0}$$

Proof

Recall the Jacobi identity,

$$\sum_{(jk\ell)} [[X_j, X_k], X_\ell] = 0$$

From Lie bracket $[X_j, X_k] = C_{jk}{}^{\ell} X_\ell$,

Find

$$\sum_{(jk\ell)} [[X_j, X_k], X_\ell] = \sum_{(jk\ell)} C_{jk}{}^{m} [X_m, X_\ell]$$

$$= \sum_{(jk\ell)} C_{jk}{}^{m} C_{m\ell}{}^{n} X_n$$

$$= 0$$

This is true for any basis set $\{X_j\}$, so

$$\sum_{(jk\ell)} C_{jk}{}^{m} C_{m\ell}{}^{n} = 0 \qquad \blacksquare$$

If $C_{jk}{}^{l} = 0$, Lie algebra is called "Abelian"

By a careful choice of canonical bases,
Lie algebra can be classified and partially enumerated.

Terminology

A mapping of abstract Lie algebra A
into $\begin{cases} \text{definite math structure} \equiv \text{"realization" of A} \\ N \times N \text{ matrices} \equiv \text{"N-dim representation" of A} \end{cases}$

└ Same terminology for group.

Warning

Don't confuse dim(A) with dim(rep)

Example:

dim of algebra

$\dim(su(2)) = 3$ "e.g., 3 pauli matrices"

$\dim(\sigma_j) = 2$ "2×2 matrix = 2-dim rep"

dim of rep

Connection between Lie groups and Lie algebras

Consider Lie group element $g(\alpha^i)$, identity @ $\alpha^i = 0$

think of as matrix

Expand in Taylor series about $\alpha^i = 0$

$$g(\alpha^i) = g(0) + \alpha^i X_i + \mathcal{O}(\alpha^2)$$

where $\boxed{X_j = \left.\frac{\partial g}{\partial \alpha_j}\right|_{\alpha_j = 0}}$ "infinitesimal group generator"

Inverse has the form $\left(g(\alpha^i)^{-1} g(\alpha^i) = \mathbb{1} \right)$

$$g(\alpha^i)^{-1} = g(0) - \alpha^i X_i + \mathcal{O}(\alpha^2)$$

Now, consider the group "commutator" of 2 elements

$$g(\beta^i)^{-1} g(\gamma^i)^{-1} g(\beta^i) g(\gamma^i) = g(\alpha^i)$$

true from group axioms

No sum on j, indices for parameters

Expand in Taylor series

$$(g(0) - \beta^j X_j)(g(0) - \gamma^k X_k)(g(0) + \beta^m X_m)(g(0) + \gamma^n X_n)$$
$$= g(0) + \alpha^l X_l$$

Keep $\mathcal{O}(\alpha)$, $\mathcal{O}(\beta)$, and $\mathcal{O}(\gamma)$ terms

So,

$$(g(0) - \beta^j X_j)(g(0) - \gamma^k X_k)(g(0) + \beta^m X_m)(g(0) + \gamma^n X_n)$$

$$= (g(0) - \beta^j X_j - \gamma^k X_k + \beta^j \gamma^k X_j X_k)$$

$$\times (g(0) + \beta^m X_m + \gamma^n X_n + \beta^m \gamma^n X_m X_n)$$

$$= g(0) - \beta^j X_j - \gamma^k X_k + \beta^j \gamma^k X_j X_k$$

$$+ \beta^m X_m + \gamma^n X_n + \beta^m \gamma^n X_m X_n$$

$$+ (-\beta^j X_j - \gamma^k X_k + \beta^j \gamma^k X_j X_k)$$

$$\times (\beta^m X_m + \gamma^n X_n + \beta^m \gamma^n X_m X_n)$$

$$= g(0) + \beta^j \gamma^k X_j X_k + \beta^m \gamma^n X_m X_n$$

$$- \beta^j \gamma^n X_j X_n - \beta^m \gamma^k X_k X_m + \mathcal{O}(\beta^2, \gamma^2)$$

$$= g(0) + \beta^j \gamma^k (X_j X_k - X_k X_j) + \mathcal{O}(\beta^2, \gamma^2)$$

$$= g(0) + \beta^j \gamma^k [X_j, X_k] + \mathcal{O}(\beta^2, \gamma^2)$$

$$= g(0) + \alpha^l X_l + \mathcal{O}(\alpha^2)$$

So, conclude

$$[X_j, X_k] = C_{jk}{}^l X_l \qquad \text{where } \alpha^l = C_{jk}{}^l \beta^j \gamma^k$$

Commutator of matrix multiplication

Therefore,

$$\boxed{\begin{array}{c}\{X_j\} \text{ is a basis for Lie algebra} \\ \text{with structure constants } C_{jk}{}^{l}.\end{array}}$$

Convention: write $g(\alpha^j) = \mathrm{Exp}(\alpha^j X_j)$

generalized exponential
or exponential map

Result: For matrix representations of X_j

$$\mathrm{Exp}(\alpha^j X_j) = \exp(\alpha^j X_j)$$
$$= \mathbb{1} + \alpha^j X_j + \frac{1}{2}\alpha^j\alpha^k X_j X_k + \mathcal{O}(\alpha^3)$$

Suggestive argument:

$$g(\epsilon^j) \approx g(0) + \epsilon^j X_j = g(0) + \frac{\alpha^j}{N} X_j \quad \text{for large } N$$

then,

$$g(\alpha^j) = \left[\mathbb{1} + \frac{\alpha^j}{N} X_j\right]^N \longrightarrow \exp(\alpha^j X_j)$$

can multiply like this
because it is a group

Example: 2D rep of SO(2)

Consider $g(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$

These matrices form 2D rep of SO(2) group

$\underbrace{S}_{\substack{\text{special} \\ \det O = 1}} \underbrace{O}_{\substack{\text{orthogonal} \\ O^T O = \mathbb{1}}} \underbrace{(2)}_{2\times 2}$

This is an Abelian group. The associated algebra is called $\mathfrak{so}(2)$.

Associated generator X is obtained by Taylor expanding

$$g(\alpha) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{g(0)} + \underbrace{\alpha}_{\text{parameter}} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{generator}} + \mathcal{O}(\alpha^2)$$

$$\stackrel{?}{=} \exp\left[\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$$

$$\exp\left[\alpha\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \exp(-i\alpha\sigma^2)$$
$$= \cos\alpha - i\sigma^2 \sin\alpha$$
$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \checkmark$$

Algebra is Abelian: $[X, X] = 0$

Notice: There is one X and it is 2-Dimensional

algebra dim = 1 rep. dim = 2

Evidently, SO(2) useful for situations involving rotations. Generally, physical use of symmetry involves both group and a space on which it acts.

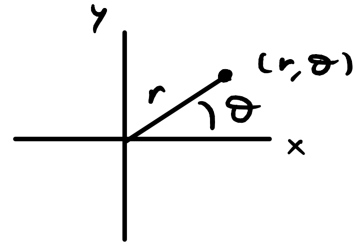
e.g., for rotations in plane

$$g(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \text{ acts on 2D vector } \begin{pmatrix} x \\ y \end{pmatrix}$$

This space is called "Basis for representation" or "representation"

Example of realization of SO(2)

Suppose physical space is represented by $f(r,\theta)$. What is realization of SO(2)?



Consider function rotated by α

$$f(r,\theta+\alpha) = f(r,\theta) + \alpha\,\partial_\theta f(r,\theta) + \mathcal{O}(\alpha^2)$$

$$= \exp(\alpha\,\partial_\theta)\, f(r,\theta)$$

$$= g(\alpha)\, f(r,\theta)$$

$\Rightarrow \quad g(\alpha) = \exp(\alpha\,\partial_\theta) \longrightarrow X = \partial_\theta \quad , \quad [X, X] = 0$

↳ realization of SO(2)

Example: 1D rep of U(1)

Suppose physical space is represented by $z = re^{i\theta} \in \mathbb{C}$

To rotate, take $g(\alpha) = e^{i\alpha}$

↳ unitary → U(1)

↳ 1×1

generator is now $X = i$ (or $X = 1$)

$\Rightarrow [X, X] = 0$

So, $\mathfrak{u}(1) \cong \mathfrak{so}(2) \Rightarrow$ algebras are isomorphic

Some Matrix groups

- General linear groups

  - $GL(N,\mathbb{C})$ = group of invertible $N\times N$ matrices with complex entries

    det $\neq 0$

    Has $2N^2$ real parameters,

    generators are $2N^2$ matrices which are $N\times N$ with 1 or i as one non-zero entry.

  - $GL(N,\mathbb{R}) = GL(N,\mathbb{C})$ restricted to $F = \mathbb{R}$

    N^2 parameters, N^2 generators

    Notice: $GL(N,\mathbb{C}) \supset GL(N,\mathbb{R})$

- Special linear groups

  - $SL(N,\mathbb{C}) = GL(N,\mathbb{C})$ with det $= +1$

    $\Rightarrow$ $2(N^2-1)$ real parameters, generators (traceless)

  - $SL(N,\mathbb{R}) = SL(N,\mathbb{C})$ restricted to $F = \mathbb{R}$

e.g., $SL(2,\mathbb{C})$ is group of quantum Lorentz transformations

- Orthogonal groups

  - $O(N, \mathbb{C})$ = group of $N \times N$ complex orthogonal matrices
    $\hookrightarrow O^T O = 1$

    Notice: $\det O = \det O^T$

    $\Rightarrow (\det O)^2 = 1 \Rightarrow \det O = \pm 1$

    if $\det O = +1$
    $\Rightarrow SO(N, \mathbb{C})$
    "special"

    so, group is in (at least) two pieces.

    $N(N-1)$ real parameters,

    generators are $N \times N$ antisymmetric matrices.

    entries are ± 1, or $\pm i$.

  - $O(N, \mathbb{R}) = O(N, \mathbb{C})$ restricted to $F = \mathbb{R}$

    $\frac{1}{2} N(N-1)$ parameters, generators.

    Notice: if vector $x = \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix}$

    then $O(N, \mathbb{R})$ leaves invariant the quadratic form

$$x^T x = \sum_\alpha (x^\alpha)^2$$

$x \to O x$, $O \in O(N, \mathbb{R})$

then, $x^T \to x^T O^T$

and so $x^T x \longrightarrow x^T \underbrace{O^T O}_{= \mathbb{1}} x = x^T x$

- $O(N, M, \mathbb{R})$ = group of pseudo orthogonal $(N+M)\times(N+M)$ real matrices

  satisfy $O^T \eta O = \eta$

  with $$\eta = \begin{pmatrix} \underbrace{+ \\ & + \\ && + \\ &&& \ddots}_{M} \\ &&&& \underbrace{- \\ &&&&& - \\ &&&&&& - \\ &&&&&&& \ddots}_{N} \end{pmatrix}$$

  Leaves invariant

  $$x^T \eta x = \sum_{\alpha,\beta} x^\alpha \eta_{\alpha\beta} x^\beta$$

  $\frac{1}{2}(M+N)\times(M+N-1)$ parameters, generators

  e.g., $SO(3,1)$ is group of classical Lorentz trans.

- Unitary groups

  - $U(N)$ = group of $N\times N$ complex unitary matrices $\hookrightarrow U^\dagger U = \mathbb{1}$

    $U^\dagger = (U^T)^*$

    Leaves invariant quadratic form $z^\dagger z = \sum_\alpha z^*_\alpha z_\alpha$

    N^2 real parameters, generators

    generators are $N\times N$ antihermitian matrices

    $$z = \begin{pmatrix} z^1 \\ \vdots \\ z^N \end{pmatrix}$$

- $U(N,M) = U$ obeying $U^\dagger \eta U = \eta$

  Leaves invariant $z^\dagger \eta z$

- $SU(N,M) = U(N,M)$ restricted to $\det = +1$

  In particular, $SU(N,0) \equiv SU(N)$

  has N^2-1 real parameters

Note

$$\exp(\alpha X) \quad \text{vs.} \quad \exp(\alpha X)$$

First $\exp(\alpha X)$: α if real, X antihermitian.

Second $\exp(\alpha X)$: α if complex ($\alpha \to i\alpha$), X hermitian.

choice between the two!

In Quantum theory, conserved quantity leads to symmetry.

Generators are observables associated with quantity

$\Rightarrow$ Observables must be Hermitian

So, for real parameter α^i

$$\Rightarrow \quad X_j = i\,Q_j$$

Q_j: Hermitian

$$\Rightarrow U = \exp(i\alpha^i Q_j)$$

U(1), SO(3) and SU(2)

Let us consider some specific groups important for the SM. The simplest is U(1)

- $U(1)$ = 1×1 unitary matrix (complex number)

so, $\boxed{U(1) = e^{i\alpha}, \quad \alpha \in \mathbb{R}}$

This is a simple phase rotation.

- $SO(3)$ = 3×3 real matrices obeying $O^T O = \mathbb{1}$, $\det O = +1$

  Generators are antisymmetric matrices L_1, L_2, L_3

  $N_{\text{Generators}} = \frac{1}{2}\, 3(3-1) = 3$

Can pick

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find Lie algebra $so(3)$ is

$$[L_j, L_k] = \epsilon_{jkl} L_l \quad , \quad \epsilon_{123} = +1 \quad , j,k,l \in \{1,2,3\}$$

The group has 3 parameters α^i

$\Rightarrow$ 3D rep of group is

$$O(\alpha^i) = \exp(\alpha^i L_j) \qquad \text{(Pset 4)}$$

$$= \mathbb{1}_3 + \frac{\alpha^j}{\alpha} L_j \sin\alpha + \left(\frac{\alpha^i}{\alpha} L_j\right)^2 (1 - \cos\alpha)$$

with $\alpha = |\vec{\alpha}|$

- $SU(2)$ = group of 2×2 complex matrices obeying

$$U^\dagger U = \mathbb{1}, \quad \det U = +1$$

$$N_{\text{generators}} = N^2 - 1 = 2^2 - 1 = 3 \text{ generators}$$

must be traceless, Hermitian matrices X_j

Recall: Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Hermitian

to get antihermitian

So, take $X_j = -\frac{1}{2} i \sigma_j$

convenient, normalizes algebra

$\Rightarrow$ algebra $su(2) \cong so(3)$ $\quad [X_j, X_k] = \epsilon_{jkl} X_l$

SU(2) has 3 parameters, α^j

$\Rightarrow$ 2D rep. is

$$U(\alpha^i) = \exp(\alpha^j X_j) \qquad \text{(PS 4)}$$
$$= \mathbb{1}_2 \cos\tfrac{1}{2}\alpha - i\frac{\alpha^j}{\alpha}\sigma_j \sin\tfrac{1}{2}\alpha$$

with $\alpha = |\vec{\alpha}|$

- Although $su(2) \cong so(3)$ as algebras, the groups SU(2) and SO(3) are different. In fact, SU(2) $\longrightarrow$ SO(3) is $2 \to 1$ map (double cover)

To see this, start at identity $\alpha^j = 0$, pick a direction $\hat{\alpha} = \frac{\vec{\alpha}}{\alpha}$ in group space, move away, and see what happens

Find:

$$O(\alpha) = +O(\alpha+2\pi) = +O(\alpha+4\pi)$$
$$U(\alpha) = -U(\alpha+2\pi) = +U(\alpha+4\pi)$$

So, SO(3): $\mathbb{1} \to \mathbb{1} \to \mathbb{1}$

SU(2): $\mathbb{1} \to -\mathbb{1} \to \mathbb{1}$

These groups are different!

For low dimensionalities, different groups may have same algebra,

$$\mathfrak{so}(3) \cong \mathfrak{su}(2) \quad - \quad SO(3) \cong SU(2)/\mathbb{Z}_2$$

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2) \quad - \quad SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$$

$$\vdots$$

Representations