

Spontaneous Symmetry Breaking

The procedure in constructing gauge theories has been successful for both electromagnetic and strong forces.

In attempting this construction for the weak interaction, we arrive at a problem in that the suspected gauge bosons, the W^\pm and Z^0 , are massive.

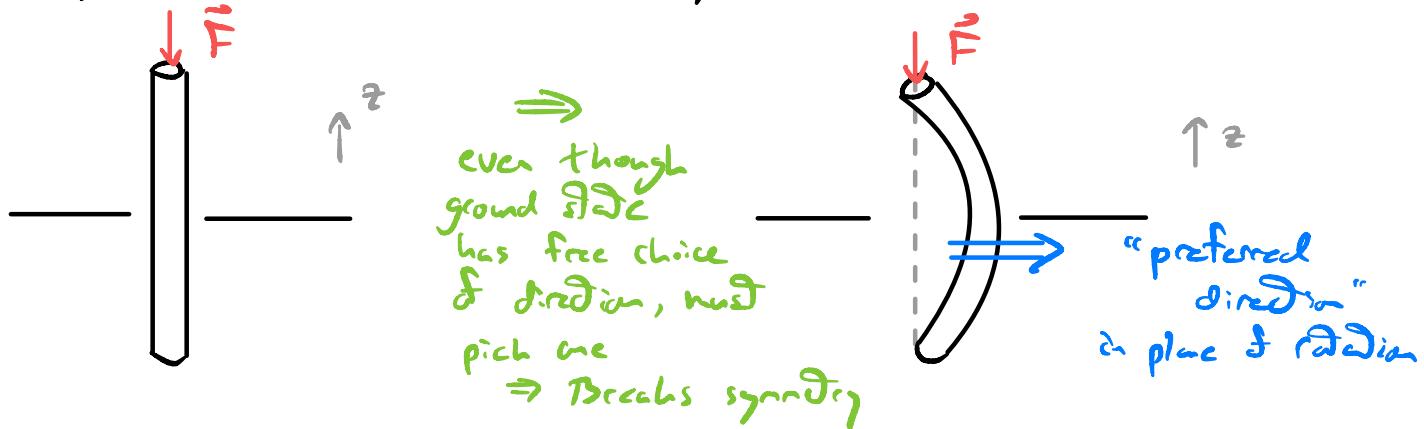
We know from QED & QCD that mass terms are not allowed as the break gauge covariance,

c.g.,

$$m_A^2 A_\mu A^\mu \xrightarrow[\text{Q(1)}]{} m_A^2 (A_\mu - \frac{1}{\xi} \partial_\mu \alpha)(A^\mu - \frac{1}{\xi} \partial^\mu \alpha) \neq m_A^2 A_\mu A^\mu.$$

There is a mechanism, however, to reduce mass terms as a consequence of interactions. The underlying feature is that the vacuum is not invariant under the symmetry.

Simple example - Compress a cylindrical stick



This is called "spontaneous symmetry breaking" or "hidden symmetry". Let's start with a simple field theory with discrete symmetry breaking.

Discrete Symmetry Breaking

Real classical scalar field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi \mp \frac{1}{2} \mu^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4 \quad , \lambda > 0$$

↳ two possible signs

- upper sign $\Rightarrow \mu = \text{mass}$

- lower sign \Rightarrow apparent instability

then,

$$\mathcal{H} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} (\vec{\nabla} \varphi)^2 + U(\varphi) \quad \text{"tachyonic" mass is}$$

with potential $U(\varphi) = \pm \frac{1}{2} \mu^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4$

$$= \frac{\lambda}{4!} (\varphi^2 \pm a^2)^2 - \frac{\lambda}{4!} a^4$$

↳ $a = \sqrt{\frac{6\mu^2}{\lambda}}$

This theory has a discrete symmetry

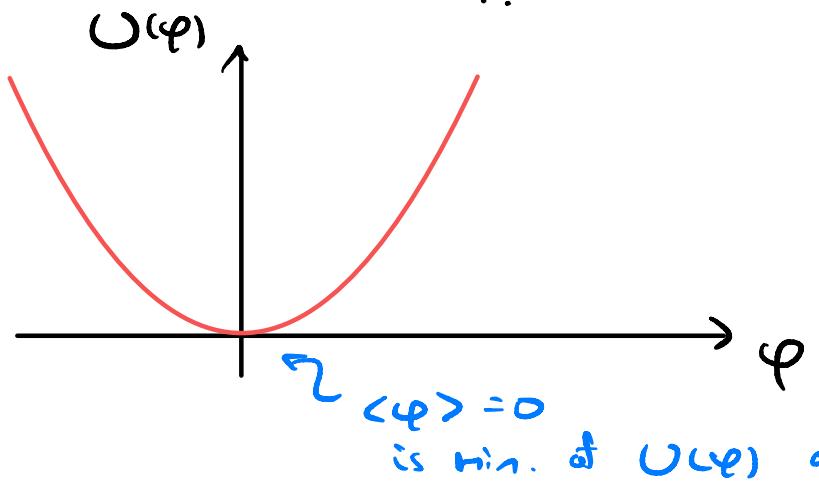
$$\varphi \rightarrow -\varphi$$

Lowest energy state, "vacuum", has $\varphi = \text{const.}$, denoted $\langle \varphi \rangle = \langle 0 | \varphi | 0 \rangle$

↳ vacuum expectation value

$\langle \varphi \rangle$ is determined by minimum of $U(\varphi)$.

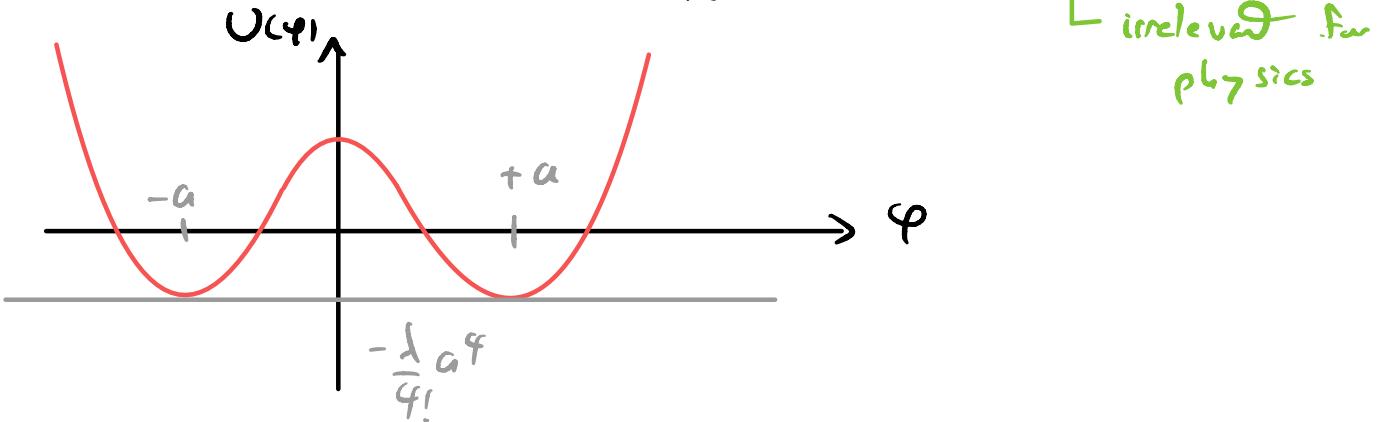
- For upper sign, $U(\varphi) = \frac{\lambda}{4!} (\varphi^2 + a^2)^2 - \frac{\lambda}{4!} a^4$



In this case, symmetry is evident in ground state
 \Rightarrow Quantization gives usual SHO states,

find particle & mass μ .

- For lower sign, $U(\varphi) = \frac{\lambda}{4!} (\varphi^2 - a^2)^2 + \text{const.}$

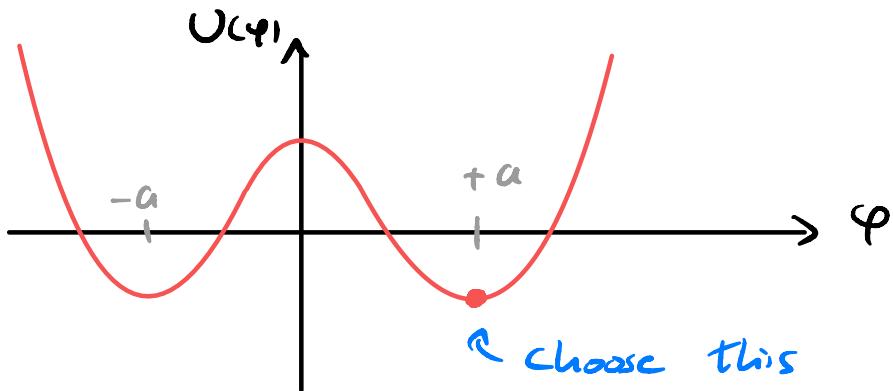


Now find 3 extrema : $\langle \varphi \rangle = 0, \pm a$

There are 2 minima. Due to symmetry, choice of $\langle \varphi \rangle = +a$ or $\langle \varphi \rangle = -a$ is irrelevant.

BS, must pick one!

\Rightarrow Spontaneously break discrete symmetry



To have QFT and perturbation theory, must choose stable minimum. Pick $\langle \varphi \rangle = +a$

Then, can write $\varphi(x) = a + \rho(x)$

$$\begin{aligned}\Rightarrow U(\varphi) &\rightarrow U(\rho) = \frac{\lambda}{4!} (2a\rho + \rho^2)^2 + \text{const.} \\ &= \frac{\lambda a^2}{6} \rho^2 + \frac{\lambda a}{6} \rho^3 + \frac{\lambda}{4!} \rho^4 \\ &= \underbrace{\mu^2}_{\mu^2} \rho^2 + \frac{\lambda a}{6} \rho^3 + \frac{\lambda}{4!} \rho^4\end{aligned}$$

So, in terms of perturbative field $\rho(x)$,
 we have particles with mass $\sqrt{2}\mu$

$$U(\rho) = \frac{1}{2} (2\mu^2) \rho^2 + \frac{\lambda a}{6} \rho^3 + \frac{\lambda}{4!} \rho^4$$

↑ ↑
 mass term cubic interaction

BD, also have a cubic interaction

⇒ Difficult to see discrete symmetry!

$$\begin{aligned} \varphi \rightarrow -\varphi &\Rightarrow a + \rho \rightarrow -a - \rho \\ &\Rightarrow a \rightarrow -a, \rho \rightarrow -\rho \end{aligned}$$

The 2 forms of \mathcal{L} are equivalent, but a simple field transformation seems to be changing physics!

Total physics is the same, but cannot solve full theory.

1st form of \mathcal{L}

- ⇒ Nonrenormalized perturbation theory
- ⇒ Correct picture of physics
- ⇒ Shifted field around true vacuum $a \neq 0$
 gives perturbation theory.

Continuous Symmetry Breaking

Consider complex scalar field.

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \mu^2 \varphi^* \varphi - \frac{\lambda}{3!} (\varphi^* \varphi)^2$$

↑ **No! wrong sign**

$$\begin{aligned} \text{Recall : } \varphi &= \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2) \\ \varphi^* &= \frac{1}{\sqrt{2}} (\varphi_1 - i\varphi_2) \end{aligned} \quad \left. \begin{array}{l} \varphi^* \varphi = \frac{1}{2} (\varphi_1^2 + \varphi_2^2) \end{array} \right\}$$

So,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \varphi_1 \partial^\mu \varphi_1 + \frac{1}{2} \partial_\mu \varphi_2 \partial^\mu \varphi_2 \\ &\quad + \frac{1}{2} \mu^2 (\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4!} (\varphi_1^2 + \varphi_2^2)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} U(\varphi_1, \varphi_2) &= -\frac{1}{2} \mu^2 (\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4!} (\varphi_1^2 + \varphi_2^2)^2 \\ &= \frac{\lambda}{4!} (\varphi_1^2 + \varphi_2^2 - a^2)^2 + \frac{\lambda}{4!} a^4 \end{aligned}$$

↳ $a^2 = \frac{6\mu^2}{\lambda}$

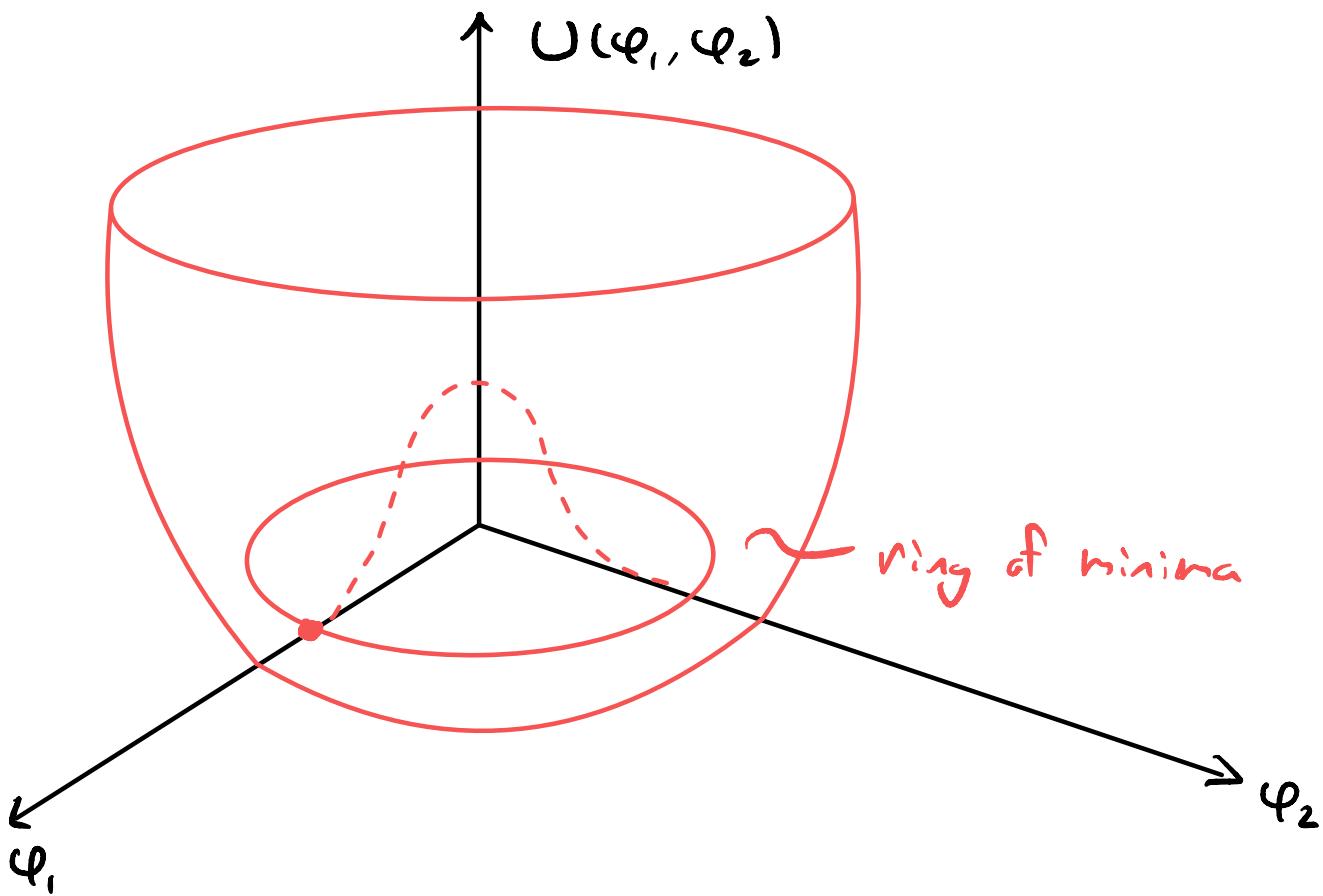
This theory has a global $U(1)$ invariance

$$\varphi \rightarrow \varphi' = e^{i\lambda} \varphi \quad , \quad \lambda \in \mathbb{R} \text{ const.}$$

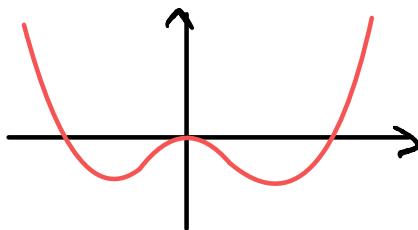
or,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \varphi'_1 \\ \varphi'_2 \end{pmatrix} = \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

The potential surface looks like



Any vertical section



As before, choice of vacuum is irrelevant to physics, but must be made.

Ring of minima at $\varphi_1^2 + \varphi_2^2 = a^2$

\Rightarrow choose $\langle \varphi_1 \rangle = a$, $\langle \varphi_2 \rangle = 0$ for definiteness

Spontaneously broken U(1) symmetry

Shift fields to minimum

$$\varphi_1(x) = a + \rho_1(x)$$

$$\varphi_2(x) = \rho_2(x)$$

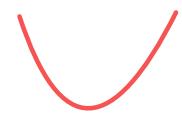
$$\begin{aligned} \Rightarrow U(\rho_1, \rho_2) &= \frac{\lambda}{4!} (za\rho_1 + \rho_1^2 + \rho_2^2)^2 \\ &= \frac{\lambda a^2}{6} \rho_1^2 + \frac{\lambda a}{6} \rho_1 (\rho_1^2 + \rho_2^2) + \frac{\lambda}{4!} (\rho_1^2 + \rho_2^2)^2 \end{aligned}$$

\hookrightarrow mass term, $\sqrt{2}\mu$, for ρ_1 .

Notice how hidden symmetry is in this case.

Notice also no mass term for ρ_2 !

Intuitively: "flat" direction in potential
 \Rightarrow zero mass mode



mass μ



mass $\epsilon\mu$



mass 0

The appearance of a massless mode is a characteristic of broken continuous symmetry.

To see in a different way, define

$$\varphi(x) = \frac{1}{\sqrt{2}} r(x) e^{i\theta(x)}$$

$$\text{or, } \varphi_1(x) = r(x) \cos \theta(x)$$

$$\varphi_2(x) = r(x) \sin \theta(x)$$

in terms of 2 other real fields $r(x), \theta(x)$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \partial_\mu r \partial^\mu r + \frac{1}{2} r^2 \partial_\mu \theta \partial^\mu \theta - U(r)$$

$U(1)$ covariance $\Leftrightarrow U(r, \theta) = U(r)$ only

Take vacuum at $\langle r \rangle = a$, $\langle \theta \rangle = 0$

Define shifted fields

$$r = a + \rho$$

$$\Theta = \theta$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} (a + \rho)^2 \partial_\mu \theta \partial^\mu \theta - U(a + \rho)$$

Can see that $\theta(x)$ particles are massless because

$U(a + \rho)$ independent of θ .

These massless modes are called "Goldstone bosons"
or "Nambu-Goldstone bosons".

Consider another example with more than one NG boson.

N real scalar fields $\varphi_j(x)$, $j=1, \dots, N$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi_j \partial^\mu \varphi_j + \frac{1}{2} \mu^2 \varphi_j \varphi_j - \frac{\lambda}{4!} (\varphi_i \varphi_j)^2$$

→ Notice!

$$\Rightarrow U(\varphi_i) = -\frac{1}{2} \mu^2 \varphi_i \varphi_i + \frac{\lambda}{4!} (\varphi_i \varphi_i)^2$$

$$= \frac{\lambda}{4!} (\varphi_i \varphi_i - a^2)^2 + \text{irrelevant const.}$$

$$\hookrightarrow a = \sqrt{\frac{6\mu^2}{\lambda}}$$

This theory is invariant under global $O(N)$ rotations

$$\varphi_j \rightarrow \varphi'_j = U_{jk} \varphi_k$$

$$\hookrightarrow U_{jk} \in O(N), \quad UU^T = \mathbb{1}$$

$O(N)$ has $\frac{1}{2}N(N-1)$ generators

\Rightarrow Corresponding \propto^a real const. parameters

Minima of $U(\varphi_i)$ lie on surface $\varphi_i \varphi_j = a^2$.

This is a generalization of ring of minima in $U(1)$ example

where $N=2$ & minima formed S^1 .

Here, have $O(N)$ and minima form S^{N-1}

Pick a vacuum : $\langle \varphi_N \rangle = a$

$$\langle \varphi_j \rangle = 0 \quad \text{for } j=1, \dots, N-1$$

New feature in this example, vacuum is invariant
under a nontrivial continuous symmetry group $O(N-1)$

Shift as before

$$\varphi_N = a + \rho_N^{(\alpha)}$$

$$\varphi_j = \rho_j^{(\alpha)} \quad \text{for } j=1, \dots, N-1$$

So we find

$$\begin{aligned} U(p_i) &= \frac{\lambda}{4!} (2\alpha p_N + p_i p_i)^2 \\ &= \frac{\lambda \alpha^2}{6} p_N^2 + \frac{\lambda \alpha}{6} p_N (p_i p_i) + \frac{\lambda}{4!} (p_i p_i)^2 \\ &\quad \underbrace{\mu^2}_{\mu^2} \end{aligned}$$

We see that p_N has acquired mass $\sqrt{2}\mu$, but the other $N-1$ particles are NG modes.

Notice the

$$\text{Number of generators of } O(N) = \frac{1}{2}N(N-1)$$

$$\text{Number of generators of } O(N-1) = \frac{1}{2}(N-1)(N-2)$$

$$\Rightarrow \text{Difference} = N-1 \quad \leftarrow \text{Not a coincidence}$$

$$\text{Number of NG bosons} = N-1 \quad \leftarrow$$

Can understand this as follows:

1. Have theory with potential U invariant under some continuous group G with N generators

2. Vacuum solution is invariant under HCG
with M generators.
 $\Rightarrow N-M$ generators transform the vacuum nontrivially
3. Passing through $\langle\varphi\rangle$ is an $(N-M)$ -dimensional
surface of constant $\langle\varphi\rangle$.
4. \Rightarrow must be $N-M$ massless modes
 $= NG$ modes, one for each dimension.

The general result is called Goldstone's theorem

Goldstone's Theorem

Given a local, positive definite, & Lorentz invariant
field theory with a local conserved current j^μ
with an associated charge Q Not annihilating
the vacuum, then the theory contains massless modes (NG)

This naively looks useless as there are no massless bosons observed in nature. But, there are two interesting cases not covered by usual axioms.

- Chiral symmetry of QCD is spontaneously broken,
 $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$

So, expect $3^2 - 1 = 8$ Goldstones bosons.

But, this symmetry is approximate, so explicit breaking gives masses to the NG modes. These hadrons, the pseudoscalar octet, are unusually light compared with other hadrons \Rightarrow They play a special role in chiral symmetry breaking.

- Gauge theories are acceptable field theories that don't obey usual axioms \rightarrow Cannot make gauge choice with both positive definiteness & Lorentz invariance. Also, theorem says nothing of discreteness of the NG modes.

Higgs Mechanism

Next we consider SSB of a local symmetry.

Consider Scalar Electrodynamics with "wrong sign" mass term, which is known as the Abelian Higgs Model.

$$\mathcal{L} = (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{3!} (\varphi^* \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

↑ Notice

where $D_\mu \varphi = \partial_\mu \varphi + i q A_\mu \varphi$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This theory is locally invariant under U(1) gauge transformations

$$\varphi \rightarrow e^{i\alpha(x)} \varphi$$

$$D_\mu \varphi \rightarrow e^{i\alpha(x)} D_\mu \varphi$$

$$A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha$$