WILLIAM & MARY
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PHYS 772 – The Standard Model of Particle Physics

Problem Set 10 – Solution

Due: Tuesday, April 29 at 4:00pm

Term: Spring 2025
Instructor: Andrew W. Jackura

1. Consider the Abelian Higgs model, with the symmetric Lagrange density given by

$$\mathcal{L} = (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{3!}(\varphi^* \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + iqA_\mu$. This theory has an unstable extremum at $\varphi = 0$, and is invariant under global U(1) transformations, $\varphi \to e^{i\alpha(x)}\varphi$ and $A_\mu \to A_\mu - q^{-1}\partial_\mu \alpha$.

(a) Let $\varphi(x) = \frac{1}{\sqrt{2}} r(x)\, e^{i\theta(x)}$ where $r(x)$, $\theta(x)$ are real scalar fields. Show that the Lagrange density in terms of these fields is

$$\mathcal{L} = \frac{1}{2}\partial_\mu r\, \partial^\mu r + \frac{1}{2} r^2 (\partial_\mu \theta + qA_\mu)^2 - \frac{\lambda}{4!}(r^2 - a^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \text{const.} \,,$$

where we have ignored non-dynamical constants. What is a in terms of the theory parameters μ, λ, and q?

Solution: Taking the covariant derivative on φ,

$$\begin{aligned} D_\mu \varphi &= \frac{1}{\sqrt{2}}(\partial_\mu + iqA_\mu) r(x) e^{i\theta(x)} \,, \\ &= \frac{1}{\sqrt{2}}\partial_\mu \left(r(x) e^{i\theta(x)} \right) + \frac{1}{\sqrt{2}} iqA_\mu r(x) e^{i\theta(x)} \,, \\ &= \frac{1}{\sqrt{2}}\partial_\mu r(x) e^{i\theta(x)} + \frac{i}{\sqrt{2}} r(x) \partial_\mu \theta(x)\, e^{i\theta(x)} + \frac{1}{\sqrt{2}} iqA_\mu r(x) e^{i\theta(x)} \,, \\ &= \frac{1}{\sqrt{2}} \left[\partial_\mu r(x) + i r(x) \left(\partial_\mu \theta(x) + qA_\mu \right) \right] e^{i\theta(x)} \,. \end{aligned}$$

Similarly, we also find

$$(D_\mu \varphi)^* = \frac{1}{\sqrt{2}} \left[\partial_\mu r(x) - i r(x) \left(\partial_\mu \theta(x) + qA_\mu \right) \right] e^{-i\theta(x)} \,,$$

thus the kinetic term is

$$\begin{aligned} (D_\mu \varphi)^* (D^\mu \varphi) &= \frac{1}{2} \left[\partial_\mu r(x) - i r(x) \left(\partial_\mu \theta(x) + qA_\mu \right) \right] \left[\partial^\mu r(x) + i r(x) \left(\partial^\mu \theta(x) + qA^\mu \right) \right] \,, \\ &= \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{1}{2} r^2 (\partial_\mu \theta + qA_\mu)^2 \,. \end{aligned}$$

Trivially, $\varphi^*\varphi = r^2/2$, so the Lagrange density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{1}{2}r^2(\partial_\mu\theta + qA_\mu)^2 + \frac{1}{2}\mu^2 r^2 - \frac{\lambda}{4!}r^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.$$

Completing the square on the potential terms, we find

$$\begin{aligned}\frac{1}{2}\mu^2 r^2 - \frac{\lambda}{4!}r^4 &= -\frac{\lambda}{4!}\left(r^4 - 2\frac{6\mu^2}{\lambda}r^2\right)\,,\\ &= -\frac{\lambda}{4!}\left(r^4 - 2\frac{6\mu^2}{\lambda}r^2 + \left(\frac{6\mu^2}{\lambda}\right)^2 - \left(\frac{6\mu^2}{\lambda}\right)^2\right)\,,\\ &= -\frac{\lambda}{4!}\left(r^2 - \left(\frac{6\mu^2}{\lambda}\right)^2\right)^2 - \frac{\lambda}{4!}\left(\frac{6\mu^2}{\lambda}\right)^2\,,\\ &\equiv -\frac{\lambda}{4!}\left(r^2 - a^2\right)^2 + \text{const.}\,,\end{aligned}$$

where in the last equality we identified $a = \sqrt{6\mu^2/\lambda}$, and discarded the irrelevant non-dynamical constant.
Combining this with before, we find the Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu r\,\partial^\mu r + \frac{1}{2}r^2(\partial_\mu\theta + qA_\mu)^2 - \frac{\lambda}{4!}(r^2 - a^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{const.}\,,$$

as desired.

(b) Expand the theory about the true vacuum of the theory, $r(x) \to a + \rho(x)$, $\theta(x) \to \theta(x)$, $A_\mu(x) \to A_\mu(x)$. Write each term of the new Lagrange density in terms of the fields ρ, A_μ, and θ. Determine the mass of each field in terms of the parameters of the theory μ, λ, and q, and the vacuum expectation value a.

Solution: Expanding the theory about the true vacuum, $r(x) \to a + \rho(x)$, we find for the kinetic term,

$$\frac{1}{2}\partial_\mu r\,\partial^\mu r \to \frac{1}{2}\partial_\mu(a+\rho)\,\partial^\mu(a+\rho) = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho\,,$$

and

$$\begin{aligned}\frac{1}{2}r^2(\partial_\mu\theta + qA_\mu)^2 &\to \frac{1}{2}(a+\rho)^2(\partial_\mu\theta + qA_\mu)^2\,,\\ &= \frac{1}{2}a^2(\partial_\mu\theta + qA_\mu)^2 + a\rho(\partial_\mu\theta + qA_\mu)^2 + \frac{1}{2}\rho^2(\partial_\mu\theta + qA_\mu)^2\,,\end{aligned}$$

For the potential term, we find

$$
\begin{aligned}
-\frac{\lambda}{4!}(r^2 - a^2)^2 \to -\frac{\lambda}{4!}((a+\rho)^2 - a^2)^2\,, \\
&= -\frac{\lambda}{4!}(\rho^2 + 2a\rho)^2\,, \\
&= -\frac{\lambda}{4!}(\rho^4 + 4a\rho^3 + 4a^2\rho^2)\,, \\
&= -\frac{\lambda a^2}{6}\rho^2 - \frac{\lambda a}{6}\rho^3 - \frac{\lambda}{4!}\rho^4\,.
\end{aligned}
$$

Combining all these terms, we find after spontaneous symmetry breaking

$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}a^2(\partial_\mu\theta + qA_\mu)^2 + a\rho(\partial_\mu\theta + qA_\mu)^2 + \frac{1}{2}\rho^2(\partial_\mu\theta + qA_\mu)^2 \\
- \frac{\lambda a^2}{6}\rho^2 - \frac{\lambda a}{6}\rho^3 - \frac{\lambda}{4!}\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.
\end{aligned}
$$

The mass of the θ field is zero, while the mass of A_μ field is $m_A = aq$, and for the ρ field it is $m_\rho = \sqrt{2}\mu$.

(c) Eliminate the mixing term, $\partial_\mu\theta A^\mu$, by choosing the *unitary gauge*, $A_\mu \to A_\mu - q^{-1}\partial_\mu\theta$. Write down each term of the Lagrange density in this gauge.

Solution: Under the unitary gauge transformation, $A_\mu \to A_\mu - q^{-1}\partial_\mu\theta$, we find that the $F_{\mu\nu}F^{\mu\nu}$ term is invariant, while all terms with $(\partial_\mu\theta + qA_\mu)^2 \to (qA_\mu)^2 = q^2 A_\mu A^\mu$. Thus the Lagrange density becomes

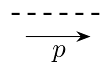
$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
+ \frac{1}{2}a^2q^2A_\mu A^\mu + a\rho q^2 A_\mu A^\mu + \frac{1}{2}\rho^2 q^2 A_\mu A^\mu \\
- \frac{\lambda a^2}{6}\rho^2 - \frac{\lambda a}{6}\rho^3 - \frac{\lambda}{4!}\rho^4\,.
\end{aligned}
$$

Let $m_A = qa$, $m_\rho = \sqrt{2}\mu$ where $\lambda a^2/6 = \mu^2$, we can write the theory in terms of the masses and the Higgs v.e.v. as

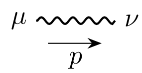
$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}m_\rho^2\rho^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2 A_\mu A^\mu \\
+ \frac{m_A^2}{a}\rho\,A_\mu A^\mu + \frac{m_A^2}{2a^2}\rho^2\,A_\mu A^\mu - \frac{m_\rho^2}{2a}\rho^3 - \frac{m_\rho^2}{8a^2}\rho^4\,.
\end{aligned}
$$

(d) Write the Feynman rules for the Abelian Higgs model in the unitary gauge.

Solution: We can read off the Feynman rules from the Lagrange density in part (c). The propagators are

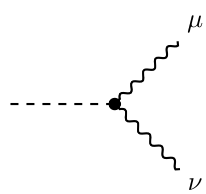


$$= \frac{i}{p^2 - m_\rho^2 + i\epsilon}\,;$$

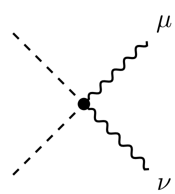


$$= \frac{-i}{p^2 - m_A^2 + i\epsilon}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_A^2}\right)\,;$$

The Higgs-gauge boson interactions are

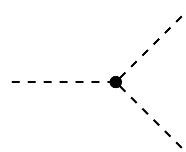


$$= i\frac{2m_A^2}{a}g^{\mu\nu}\,;$$

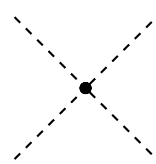


$$= -i\frac{2m_A^2}{a^2}g^{\mu\nu}\,;$$

The Higgs self-interactions are



$$= -i\frac{3m_\rho^2}{a}\,;$$



$$= -i\frac{3m_\rho^2}{a^2}\,;$$

2. Consider the leptonic decay of the negatively charged pion, $\pi^- \to \ell^- \bar{\nu}_\ell$ where $\ell = e, \mu$. The electroweak decay amplitude has the form

$$\mathcal{M}(\pi^- \to \ell \bar{\nu}_\ell) \propto f_\pi p^\mu \, \bar{u}_\ell \gamma_\mu (1 - \gamma_5) v_{\bar{\nu}_\ell}\,,$$

where f_π is the pion decay constant. Recall that the decay rate for an $a \to 1 + 2$ reaction in the rest frame of the decaying particle is

$$\Gamma = \frac{|\mathbf{p}|}{32\pi^2 m_a^2}\int \mathrm{d}\Omega\,\langle|\mathcal{M}|^2\rangle\,.$$

(a) Show that $\pi^- \to \tau^- \bar{\nu}_\tau$ is forbidden.

> **Solution**: Since $m_\tau > m_\pi$, then $\pi^- \to \tau^- \bar{\nu}_\tau$ is forbidden.

(b) Compute the ratio

$$\mathcal{R} \equiv \frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)}\,.$$

Compare the ratio to the measured value in the *Review of Particle Physics* by the PDG.

> **Solution**: Here we want to take the spin-averaged matrix element, $\langle |\mathcal{M}|^2 \rangle$. Let p be the momentum of the pion, k the momentum of the lepton, and q the momentum of the antineutrino. So,
>
> $$\langle |\mathcal{M}|^2 \rangle \propto f_\pi^2 \bar{u}(k) \not{p} (1-\gamma_5) v(q) \bar{v}(q) (1+\gamma_5) \not{p} u(k)\,,$$
>
> where we used $(\gamma^\mu(1-\gamma_5))^\dagger = (1-\gamma_5)^\dagger (\gamma^\mu)^\dagger = (1-\gamma_5)\gamma^0\gamma^\mu\gamma^0 = \gamma^0(1+\gamma_5)\gamma^\mu\gamma^0$. Since $p = k+q$, then $\not{p} u(k) = (\not{k} + \not{q}) u(k) = m_\ell u(k)$ and $\bar{u}(k)\not{p} = \bar{u}(k)(\not{k}+\not{q}) = m_\ell \bar{u}(k)$. So,
>
> $$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &\propto f_\pi^2 m_\ell^2 \,\mathrm{tr}\left[u(k)\bar{u}(k)(1-\gamma_5) v(q)\bar{v}(q)(1+\gamma_5)\right]\,, \\ &= f_\pi^2 m_\ell^2 \,\mathrm{tr}\left[(\not{k} + m_\ell)(1-\gamma_5)\not{q}(1+\gamma_5)\right]\,, \\ &= 8 f_\pi^2 m_\ell^2 (k\cdot q)\,, \\ &= 8 f_\pi^2 m_\ell^2 m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2\,, \end{aligned}$$
>
> where in the last line we used the fact that in the rest frame of the pion, $k\cdot q = E|\mathbf{q}| + |\mathbf{q}|^2 = m_\pi(1 - m_\ell^2/m_\pi^2)^2$ from kinematics. Therefore, the ratio of the partial widths is
>
> $$\mathcal{R} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \approx 1.28\times 10^{-4}\,.$$
>
> From the RPP, we find that $\mathcal{R} = 1.230(4)\times 10^{-4}$, which gives good agreement.

(c) Compute the ratio $\mathcal{R}$ for $K^- \to \ell^- \bar{\nu}_\ell$, and compare to the measured ratio reported in the RPP. The amplitude is identical to that for the pion with $f_\pi \to f_K$, f_K being the kaon decay constant.

> **Solution**: Since the amplitude is identical in structure, we can write down the ratio for $K^- \to \ell^- \bar{\nu}_\ell$ decays,
>
> $$\mathcal{R} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_K^2 - m_e^2}{m_K^2 - m_\mu^2}\right)^2 \approx 2.57\times 10^{-5}\,.$$
>
> The RPP gives $\mathcal{R} \approx 1.58\times 10^{-5}/0.636 \approx 2.49\times 10^{-5}$, which generally agrees.

(d) The weak interaction is known as a V-A interaction, or vector-axial vector. An alternative theory is the pseudoscalar interaction model, where the weak decay amplitude is given by

$$\mathcal{M} \propto g_\pi \bar{u}_\ell \gamma_5 v_{\bar{\nu}_\ell} \,,$$

where g_π is some coupling. Compute the ratio $\mathcal{R}$ for $\pi^- \to \ell^- \bar{\nu}_\ell$ in this theory. Compare the result to the experimentally measured ratio. What conclusions can you make about this theory?

Solution: For this pseudoscalar model, the spin-averaged matrix element is

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &\propto g_\pi^2 \left[\bar{v}(q)\gamma_5 u(k)\right]\left[\bar{u}(k)\gamma_5 v(q)\right], \\
&= g_\pi^2 \, \mathrm{tr}[\not{q}\gamma_5(\not{k}+m_\ell)\gamma_5]\,, \\
&= -g_\pi^2 \, \mathrm{tr}[\not{q}(\not{k}+m_\ell)]\,, \\
&= -4g_\pi^2\,(q\cdot k)\,.
\end{aligned}$$

In the π rest frame, $q\cdot k = m_\pi(1-m_\ell^2/m_\pi^2)^2$. Therefore, the ratio of e to μ decays is given by

$$\mathcal{R} = \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \approx 5.5\,.$$

Clearly this result is ruled out as it is not supported by the experimental data, $\mathcal{R}_{\text{exp.}} \approx 1.230(4)\times 10^{-4}$.

3. Consider $e^-e^+ \to \mu^-\mu^+$ within the Electroweak (EW) model of leptons. Assume the reaction occurs at a center-of-momentum (CM) energy $\sqrt{s} \gg m_e, m_\mu$.

(a) Within the EW model, the leading order diagrams which contribute to the $e^-e^+ \to \mu^-\mu^+$ amplitude involve e^-e^+ annihilation to virtual γ, Z^0, and H^0 exchange. Argue why the contribution from the H^0 is negligible compared with the γ and Z^0 exchanges.

Solution: Let us consider the reaction with the following kinematics

$$e^-(p,s) + e^+(k,r) \to \mu^-(p',s') + \mu^+(k',r')\,,$$

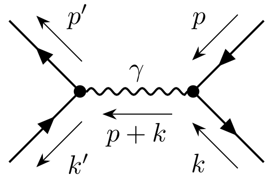
where in the CM frame $p = (E_e, \mathbf{p})$, $k = (E_e, -\mathbf{p})$, $p' = (E_\mu, \mathbf{p}')$, and $k' = (E_\mu, -\mathbf{p}')$. Since the electron and muon masses are negligible, we have $E_e = |\mathbf{p}| = E_\mu = |\mathbf{p}'| = \sqrt{s}/2$. The leading order scattering amplitude is given by three terms,

$$i\mathcal{M} = i\mathcal{M}_\gamma + i\mathcal{M}_Z + i\mathcal{M}_H\,,$$

where $\mathcal{M}_\gamma$ is the amplitude with photon exchange, $\mathcal{M}_Z$ is with Z^0 boson exchange, and $\mathcal{M}_H$ is with Higgs exchange.

Using the Feynman rules in the unitary gauge, the amplitude for γ exchange is

$$i\mathcal{M}_\gamma =$$



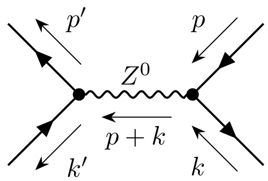
$$,$$

$$= \bar{u}_{s'}(p')(ie\gamma^\mu)v_{r'}(k')\,\frac{-ig_{\mu\nu}}{(p+k)^2}\,\bar{v}_r(k)(ie\gamma^\nu)u_s(p)\,,$$

$$= i\frac{4\pi\alpha}{s}\left[\bar{u}_{s'}(p')\gamma^\mu v_{r'}(k')\right]\left[\bar{v}_r(k)\gamma_\mu u_s(p)\right],$$

where $\alpha = e^2/4\pi$.
The amplitude for Z^0-boson exchange is

$$i\mathcal{M}_Z =$$



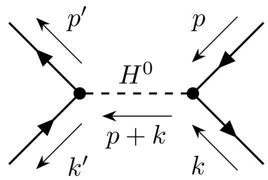
$$,$$

$$\begin{aligned}
&= \bar{u}_{s'}(p')\left(-i\frac{g}{4\cos\theta_W}\gamma^\mu(g_V-\gamma_5)\right)v_{r'}(k') \\
&\qquad \times \frac{-i}{(p+k)^2-m_Z^2}\left(g_{\mu\nu}-\frac{(p+k)_\mu(p+k)_\nu}{m_Z^2}\right) \\
&\qquad\qquad \times \bar{v}_r(k)\left(-i\frac{g}{4\cos\theta_W}\gamma^\nu(g_V-\gamma_5)\right)u_s(p)\,, \\
&= \frac{ig^2}{16\cos^2\theta_W(s-m_Z^2)}\Bigg[\bar{u}_{s'}(p')\gamma^\mu(g_V-\gamma_5)v_{r'}(k')\bar{v}_r(k)\gamma_\mu(g_V-\gamma_5)u_s(p) \\
&\qquad\qquad -\frac{1}{m_Z^2}\bar{u}_{s'}(p')(\not{p}+\not{k})(g_V-\gamma_5)v_{r'}(k')\bar{v}_r(k)(\not{p}+\not{k})(g_V-\gamma_5)u_s(p)\Bigg] \\
&= \frac{ig^2}{16\cos^2\theta_W(s-m_Z^2)}\left[\bar{u}_{s'}(p')\gamma^\mu(g_V-\gamma_5)v_{r'}(k')\right]\left[\bar{v}_r(k)\gamma_\mu(g_V-\gamma_5)u_s(p)\right],
\end{aligned}$$

where in going to the last line we used that $\not{p}u(p) = 0 = \not{k}v(k)$ since we are in the high-energy limit and the lepton masses are negligible. Here $g_V = 1 - 4\sin^2\theta_W$.

Finally, the amplitude for Higgs exchange is

$$i\mathcal{M}_H =$$



,

$$= \bar{u}_{s'}(p')\left(-\frac{ig}{2}\frac{m_\mu}{m_W}\right)v_{r'}(k')\,\frac{i}{(p+k)^2-m_H^2}\,\bar{v}_r(k)\left(-\frac{ig}{2}\frac{m_e}{m_W}\right)u_s(p)\,,$$

$$= -i\frac{g^2 m_\mu m_e}{4m_W^2}\left[\bar{u}_{s'}(p')v_{r'}(k')\right]\left[\bar{v}_r(k)u_s(p)\right].$$

Since we work in the high-energy limit, the Higgs exchange amplitude is negligible. Moreover, recall that $e = g\sin\theta_W$, so $\alpha \sim e^2 = g^2\sin^2\theta_W$. So, while both the γ and Z^0 exchange amplitudes are proportional to g^2, the Higgs amplitude is also proportional to the ratio $m_\mu m_e/m_W^2 \sim 10^{-8}$. So, the Higgs amplitude is negligible compared to the other two amplitudes, even in with massive leptons.

(b) Given that the weak mixing angle is $\sin^2\theta_W \approx 0.222$, and assuming that $s \ll m_Z^2$, show that the quantity

$$R(s) \equiv \frac{s}{(s-m_Z^2)\sin^2 2\theta_W} \approx -\frac{s}{m_Z^2\sin^2 2\theta_W} \ll 1\,.$$

For $\sqrt{s} = 35\,\text{GeV}$, verify this relation numerically.

Solution: Since $s \ll m_Z^2$, then $s - m_Z^2 \approx -m_Z^2$, so simply

$$R(s) \equiv \frac{s}{(s-m_Z^2)\sin^2 2\theta_W} \approx -\frac{s}{m_Z^2\sin^2 2\theta_W}\,.$$

Since $\sin^2\theta_W \approx 0.222$, then $\theta_W \approx 28.11^\circ$, therefore $\sin^2 2\theta_W \approx 0.691$. Also, $m_Z \approx 91.2\,\text{GeV}$, so $m_Z^2\sin^2 2\theta_W \approx 5\,733\,\text{GeV}^2$, and since $s \ll m_Z^2$, then we conclude $R(s) \ll 1$. Explicitly, for $\sqrt{s} = 35\,\text{GeV}$, then $R(s = (35\,\text{GeV})^2) \approx 0.214$, which is small compared to 1, but still $\sim 20\%$ of the value.

(c) Neglecting the H^0 contribution, and assuming that $s \ll m_Z^2$, show that the spin-averaged matrix element for this process is

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_\gamma + \mathcal{M}_Z|^2 \approx \frac{8e^4}{s^2}\left[1 + \frac{1}{2}(g_V^2+1)R(s)\right](p'\cdot k)(p\cdot k')$$
$$+ \frac{8e^4}{s^2}\left[1 + \frac{1}{2}(g_V^2-1)R(s)\right](p'\cdot p)(k'\cdot k)\,,$$

where p and k are the initial electron and positron momenta, respectively, and p' and k' are the final muon and anti-muon momenta, respectively, and we have neglected terms $R(s)^2$. Here we have introduced $g_V = 1 - 4\sin^2\theta_W$ for convenience.

Solution: The modulus squared of the amplitude is $|\mathcal{M}_\gamma + \mathcal{M}_Z|^2 = |\mathcal{M}_\gamma|^2 + |\mathcal{M}_Z|^2 + 2\mathrm{Re}(\mathcal{M}_\gamma^* \mathcal{M}_Z)$. Therefore the spin-averaged matrix element is

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_\gamma + \mathcal{M}_Z|^2 \,, \\
&= \frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_\gamma|^2 + \frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_Z|^2 + \frac{1}{2}\sum_{s,s'}\sum_{r,r'}\mathrm{Re}(\mathcal{M}_\gamma^*\mathcal{M}_Z)\,.
\end{aligned}$$

From Problem Set 6, we already have the spin-averaged matrix element,

$$\langle|\mathcal{M}_\gamma|^2\rangle = \frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_\gamma|^2 = \left(\frac{8e^4}{s^2}\right)\Big((p\cdot k')^2 + (p\cdot p')^2\Big)\,.$$

Note that $k'\cdot p = p'\cdot k$ and $p'\cdot p = k'\cdot k$ due to momentum conservation, thus

$$\langle|\mathcal{M}_\gamma|^2\rangle = \frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_\gamma|^2 = \left(\frac{8e^4}{s^2}\right)\Big((k'\cdot p)(p'\cdot k) + (p'\cdot p)(k'\cdot k)\Big)\,.$$

We now evaluate the Z-boson exchange spin-averaged matrix element

$$\begin{aligned}
\langle|\mathcal{M}_Z|^2\rangle = {} & \frac{g^4}{256\cos^4\theta_W(s-m_Z^2)^2} \\
& \times \frac{1}{4}\sum_{s,s'}\sum_{r,r'}[\bar{v}_{r'}(k')\gamma^\mu(g_V-\gamma_5)u_{s'}(p')]\,[\bar{u}_s(p)\gamma_\mu(g_V-\gamma_5)v_r(k)] \\
& \qquad \times [\bar{u}_{s'}(p')\gamma^\nu(g_V-\gamma_5)v_{r'}(k')]\,[\bar{v}_r(k)\gamma_\nu(g_V-\gamma_5)u_s(p)]\,, \\
= {} & \frac{g^4}{256\cos^4\theta_W(s-m_Z^2)^2} \\
& \times \frac{1}{4}\sum_{s',r'}[\bar{v}_{r'}(k')\gamma^\mu(g_V-\gamma_5)u_{s'}(p')][\bar{u}_{s'}(p')\gamma^\nu(g_V-\gamma_5)v_{r'}(k')] \\
& \qquad \times \sum_{s,r}[\bar{u}_s(p)\gamma_\mu(g_V-\gamma_5)v_r(k)]\,[\bar{v}_r(k)\gamma_\nu(g_V-\gamma_5)u_s(p)]\,, \\
= {} & \frac{g^4}{256\cos^4\theta_W(s-m_Z^2)^2} \\
& \times \frac{1}{4}\,\mathrm{tr}[\not{k}'\gamma^\mu(g_V-\gamma_5)\not{p}'\gamma^\nu(g_V-\gamma_5)]\,\mathrm{tr}[\not{p}\gamma_\mu(g_V-\gamma_5)\not{k}\gamma_\nu(g_V-\gamma_5)]\,, \\
= {} & \frac{g^4}{32\cos^4\theta_W(s-m_Z^2)^2}\left[(g_V^2-1)^2(k'\cdot k)(p'\cdot p) + (g_V^4+6g_V^2+1)(k'\cdot p)(p'\cdot k)\right]\,.
\end{aligned}$$

But, note that $e = g\sin\theta_W$, so $g^4 = e^4\sin^4\theta_W$, and $2\sin\theta_W\cos\theta_W = \sin 2\theta_W$. So,

$$\begin{aligned}
\langle|\mathcal{M}_Z|^2\rangle &= \frac{e^4}{2\sin^4 2\theta_W(s-m_Z^2)^2}\left[(g_V^2-1)^2(k'\cdot k)(p'\cdot p) + (g_V^4+6g_V^2+1)(k'\cdot p)(p'\cdot k)\right] , \\
&= \frac{e^4}{2s}R(s)^2\left[(g_V^2-1)^2(k'\cdot k)(p'\cdot p) + (g_V^4+6g_V^2+1)(k'\cdot p)(p'\cdot k)\right] .
\end{aligned}$$

Finally, for the interference term,

$$\begin{aligned}
\frac{1}{2}\sum_{s,s'}\sum_{r,r'}\mathrm{Re}(\mathcal{M}_\gamma^*\mathcal{M}_Z) &= \frac{e^2g^2}{16\cos^2\theta_W s(s-m_Z^2)} \\
&\quad\times\frac{1}{2}\sum_{s,s'}\sum_{r,r'}\mathrm{Re}\Bigg([\bar{v}_{r'}(k')\gamma^\mu u_{s'}(p')]\,[\bar{u}_s(p)\gamma_\mu v_r(k)] \\
&\qquad\times[\bar{u}_{s'}(p')\gamma^\nu(g_V-\gamma_5)v_{r'}(k')]\,[\bar{v}_r(k)\gamma_\nu(g_V-\gamma_5)u_s(p)]\Bigg) , \\
&= \frac{e^2g^2}{16\cos^2\theta_W s(s-m_Z^2)} \\
&\quad\times\frac{1}{2}\mathrm{Re}\Bigg(\sum_{s',r'}[\bar{v}_{r'}(k')\gamma^\mu u_{s'}(p')]\,[\bar{u}_{s'}(p')\gamma^\nu(g_V-\gamma_5)v_{r'}(k')] \\
&\qquad\times\sum_{s,r}[\bar{u}_s(p)\gamma_\mu v_r(k)]\,[\bar{v}_r(k)\gamma_\nu(g_V-\gamma_5)u_s(p)]\Bigg) , \\
&= \frac{e^2g^2}{16\cos^2\theta_W s(s-m_Z^2)} \\
&\quad\times\frac{1}{2}\mathrm{Re}\Bigg(\mathrm{tr}[\not{k}'\gamma^\mu\not{p}'\gamma^\nu(g_V-\gamma_5)]\,\mathrm{tr}[\not{p}\gamma_\mu\not{k}\gamma_\nu(g_V-\gamma_5)]\Bigg) , \\
&= \frac{e^2g^2}{\cos^2\theta_W s(s-m_Z^2)}\Bigg[(g_V^2-1)(k'\cdot k)(p'\cdot p) + (g_V^2+1)(k'\cdot p)(p'\cdot k)\Bigg] .
\end{aligned}$$

Using $e = g\sin\theta_W$, we find

$$\begin{aligned}
\frac{1}{2}\sum_{s,s'}\sum_{r,r'}\mathrm{Re}(\mathcal{M}_\gamma^*\mathcal{M}_Z) &= \frac{4e^4}{\sin^2 2\theta_W s(s-m_Z^2)}\Bigg[(g_V^2-1)(k'\cdot k)(p'\cdot p) + (g_V^2+1)(k'\cdot p)(p'\cdot k)\Bigg] , \\
&= \frac{4e^4}{s^2}R(s)\Bigg[(g_V^2-1)(k'\cdot k)(p'\cdot p) + (g_V^2+1)(k'\cdot p)(p'\cdot k)\Bigg] .
\end{aligned}$$

We ignore terms $R(s)^2$, which means we eliminate the contribution from $|\mathcal{M}_Z|^2$ as it is proportional to $R(s)^2$. Therefore, the spin-averaged matrix element is

$$\langle|\mathcal{M}|^2\rangle \approx \frac{8e^4}{s^2}\left[1+\frac{1}{2}(g_V^2+1)R(s)\right](p'\cdot k)(p\cdot k') + \frac{8e^4}{s^2}\left[1+\frac{1}{2}(g_V^2-1)R(s)\right](p'\cdot p)(k'\cdot k)\,,$$

as desired.

(d) From the amplitude in part (c), show that the unpolarized differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \frac{\alpha^2}{4s}\left[\left(1-\frac{g_V^2 s}{2m_Z^2\sin^2 2\theta_W}\right)(1+\cos^2\theta) - \frac{s}{m_Z^2\sin^2 2\theta_W}\cos\theta\right],$$

where $g_V = 1-4\sin^2\theta_W \ll 1$, and θ is the scattering angle in the CM frame.

Solution: In the CM frame, we have $p'\cdot k = k'\cdot p = s(1+\cos\theta)/4$, and $p'\cdot p = k'\cdot k = s(1-\cos\theta)/4$. Using $e^2 = 4\pi\alpha$, the spin-averaged matrix element is

$$\langle|\mathcal{M}|^2\rangle \approx \frac{(4\pi\alpha)^2}{2}\left[1+\frac{1}{2}(g_V^2+1)R(s)\right](1+\cos\theta)^2 + \frac{(4\pi\alpha)^2}{2}\left[1+\frac{1}{2}(g_V^2-1)R(s)\right](1-\cos\theta)^2\,.$$

Now, $(1\pm\cos\theta)^2 = 1\pm 2\cos\theta+\cos^2\theta$. Simplifying, we find

$$\langle|\mathcal{M}|^2\rangle \approx (4\pi\alpha)^2\left[\left(1+\frac{1}{2}g_V^2 R(s)\right)(1+\cos^2\theta) + R(s)\cos\theta\right].$$

The differential cross-section in the CM frame is

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle\,,\\ &\approx \frac{\alpha^2}{4s}\left[\left(1+\frac{1}{2}g_V^2 R(s)\right)(1+\cos^2\theta) + R(s)\cos\theta\right].\end{aligned}$$

Since

$$R(s) = \frac{s}{(s-m_Z^2)\sin^2 2\theta_W} \approx -\frac{s}{m_Z^2\sin^2 2\theta_W}\,,$$

we find the desired result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \frac{\alpha^2}{4s}\left[\left(1-\frac{g_V^2 s}{2m_Z^2\sin^2 2\theta_W}\right)(1+\cos^2\theta) - \frac{s}{m_Z^2\sin^2 2\theta_W}\cos\theta\right].$$

(e) The *forward-backward asymmetry* A_{FB} is defined as

$$A_{\mathrm{FB}} = \frac{\displaystyle\int_0^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} - \int_{-1}^0 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}{\displaystyle\int_{-1}^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}\,.$$

Determine A_{FB} given the leading order result in part (d).

Solution: The integral over $(1+\cos^2\theta)$ and $\cos\theta$ is

$$\int_a^b \mathrm{d}z\,(1+z^2) = \left(z + \frac{z^3}{3}\right)\Bigg|_a^b = (b-a) + \frac{1}{3}(b^3 - a^3)\,,$$

$$\int_a^b \mathrm{d}z\,z = \frac{z^2}{2}\Bigg|_a^b = \frac{1}{2}(b^2 - a^2)$$

For the denominator, $b = -a = 1$, so the term linear in $\cos\theta$ is identically zero, leaving the integral over $1+\cos^2\theta$ which gives a factor of $8/3$,

$$\int_{-1}^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\alpha^2}{3s}\left(1 - \frac{g_V^2 s}{2m_Z^2\sin^2 2\theta_W}\right)\,.$$

The numerator involves the difference of $b=1$, $a=0$ and $b=0$, $a=-1$. Since the integral over $1+\cos^2\theta$ is $4/3$ for both of these limits, the difference cancels. For the integral over $\cos\theta$, the first term is $+1/2$ while the second is $-1/2$. This gives

$$\int_0^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} - \int_{-1}^0 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = -\frac{\alpha^2}{4m_Z^2\sin^2 2\theta_W}\,.$$

Therefore, the forward-backward asymmetry is

$$\begin{aligned} A_{\mathrm{FB}} &= -\frac{\alpha^2}{4m_Z^2\sin^2 2\theta_W}\cdot\frac{3s}{2\alpha^2}\left(1 - \frac{g_V^2 s}{2m_Z^2\sin^2 2\theta_W}\right)^{-1}\,, \\ &= -\frac{3s}{8m_Z^2\sin^2 2\theta_W}\left(1 - \frac{g_V^2 s}{2m_Z^2\sin^2 2\theta_W}\right)^{-1}\,. \end{aligned}$$

Since $g_V^2 \ll 1$, and $R(s) \ll 1$, then

$$\left(1 - \frac{g_V^2 s}{2m_Z^2\sin^2 2\theta_W}\right)^{-1} \approx 1 + \frac{g_V^2 s}{2m_Z^2\sin^2 2\theta_W}\,,$$

So the forward-backward asymmetry is approximately

$$A_{\mathrm{FB}} \approx -\frac{3s}{8m_Z^2\sin^2 2\theta_W}\,.$$

The numerical value for A_{FB} at $\sqrt{s} = 35\,\mathrm{GeV}$ is $A_{\mathrm{FB}} \approx -0.0801$, which agrees with the fit from Problem Set 6, $A_{\mathrm{FB}}^{(\mathrm{fit})} = -0.11(1)$.

(f) Plot the theoretical $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta \in [-1, 1]$ at a CM energy $\sqrt{s} = 35\,\mathrm{GeV}$ for $e^-e^+ \to \mu^-\mu^+$. Plot the y-axis in $\mathrm{nb} \cdot \mathrm{GeV}^2$, restricted to $(s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega)/\,(\mathrm{nb} \cdot \mathrm{GeV}^2) \in [0.0, 12.0]$. Plot the experimental data for each reaction, measured from the JADE experiment at PETRA, over the theoretical curves. Compare and comment on the quality of the theoretical description of the experimental data, and compare with the leading order QED result from Problem Set 6. **Note:** The data file presents the cross-section as $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$. The data files were obtained from the article by the JADE collaboration, https://link.springer.com/article/10.1007/BF01560255.

Solution: Plotting the cross-sections, we find the result shown in Fig. 1.

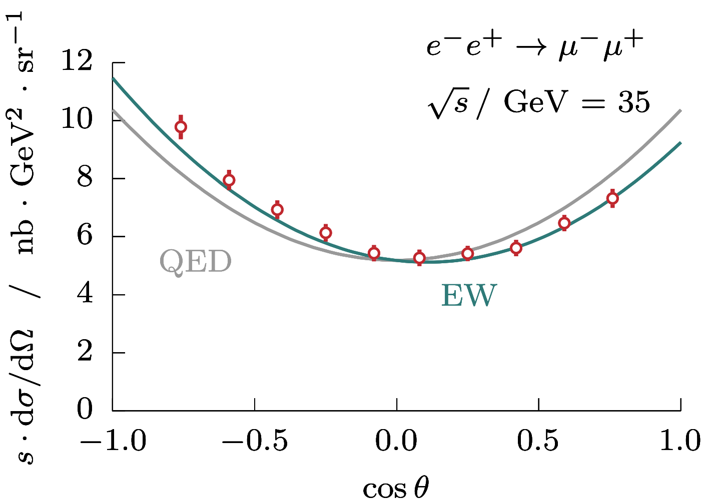


Figure 1: Plot of $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta$ for $\sqrt{s} = 35\,\mathrm{GeV}$ compared with the JADE data for the reaction $e^-e^+ \to \mu^-\mu^+$.

(g) Make a plot of the ratio of the EW theoretical differential cross-section to the leading order QED prediction from Problem Set 6 as a function of $\cos\theta \in [-1, 1]$ for the CM energy $\sqrt{s} = 35\,\mathrm{GeV}$. Also, plot the experimentally measured differential cross-section to compare with the theoretical ratio. Restrict the y axis between 0.5 and 1.5. Compare and comment on the quality of the theoretical description of the experimental data.

Solution: Plotting the ratios, we find the result shown in Fig. 2.

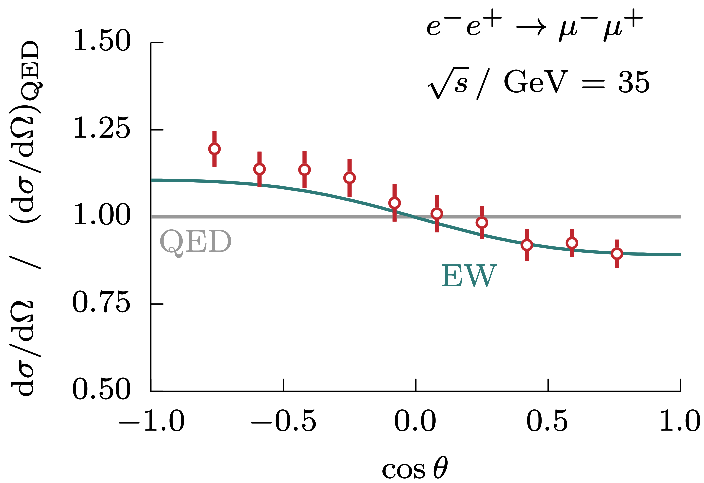


Figure 2: Plot of ratio of JADE experimental $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ to the QED theory result at $\mathcal{O}(\alpha^2)$ vs. $\cos\theta$ at $\sqrt{s} = 35\,\mathrm{GeV}$ for the reaction $e^-e^+ \to \mu^-\mu^+$.