# CYCLIC COVERS: HODGE THEORY AND CATEGORICAL TORELLI THEOREMS

HANNAH DELL, AUGUSTINAS JACOVSKIS, AND FRANCO ROTA

Abstract. Let $Y$ admit a rectangular Lefschetz decomposition of its derived category, and consider a cyclic cover $X \to Y$ ramified over a divisor $Z$. In a setting not considered by Kuznetsov and Perry [KP17], we define a subcategory $\mathcal{A}_Z$ of the equivariant derived category of $X$ which contains, rather than is contained in, $\mathrm{D^b}(Z)$. We then show that the equivariant category of the Kuznetsov component of $X$ is decomposed into copies of $\mathcal{A}_Z$.

As an application, we relate $\mathcal{A}_Z$ with the cohomology of $Z$ under some numerical assumptions. In particular, we obtain a categorical Torelli theorem for the prime Fano threefolds of index 1 and genus 2.

## 1. Introduction

In [Kuz09], Kuznetsov systematically studies derived categories of Fano threefolds and subcategories – known as *Kuznetsov components* – arising as orthogonal complements to exceptional collections. These categories are linked to questions of rationality [Kuz10], stability conditions [BLMS23] and to hyperkähler geometry through moduli spaces [LPZ18], as well as questions in birational geometry and Hodge theory [Per22].

In this paper, we start from a variety $Y$ admitting a sheaf $\mathcal{O}_Y(1)$ and a rectangular Lefschetz decomposition of $\mathrm{D^b}(Y)$, i.e. an admissible subcategory $\mathcal{B}$ and a decomposition

$$\mathrm{D^b}(Y) = \langle \mathcal{B}, \dots, \mathcal{B}(m-1) \rangle,$$

with $\mathcal{B}(i)$ the image of $\mathcal{B}$ under the equivalence $- \otimes \mathcal{O}_Y(1)$. Examples of varieties with rectangular decompositions are projective space, Grassmannians, some homogeneous spaces (see [Kuz19, Section 4.1]). Another important case, also covered in this article, is when $Y$ is a Deligne–Mumford stack, for example a weighted projective space. Then, we consider an $n$-fold cover $X$ of $Y$, ramified along a divisor $Z \in |\mathcal{O}_Y(nd)|$. The derived category $\mathrm{D^b}(X)$ has a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \dots, \mathcal{B}_X(M-1) \rangle$$

where $\mathcal{B}_X$ is the pullback of $\mathcal{B}$ under the covering map, and $M := m-(n-1)d$. The action of $\mu_n$ on $X$ by the covering involution restricts to one on $\mathcal{A}_X$, giving rise to the Kuznetsov component $\mathcal{A}_X^{\mu_n} \subset \mathrm{D^b}(X)^{\mu_n}$. This is the same setting of [KP17], with the crucial difference that we work in the range $0 < M < d$, rather than $d \leq M$. This corresponds to allowing more positivity for $Z$ (see Section 3.3). In [KP17], $\mathrm{D^b}(Z)$ also admits a Kuznetsov component $\mathcal{A}_Z$, and $\mathcal{A}_X^{\mu_n}$ decomposes into copies of $\mathcal{A}_Z$. We define a subcategory $\mathcal{A}_Z \subset \mathrm{D^b}(X)^{\mu_n}$ which *contains*, rather than is contained in, $\mathrm{D^b}(Z)$ (see (3.2)), and in our first main result we exhibit a semiorthogonal decomposition for $\mathcal{A}_X^{\mu_n}$ with components $\mathcal{A}_Z$.

2020 *Mathematics Subject Classification.* Primary 14F08; secondary 14J45, 14C34.

*Key words and phrases.* Derived categories, cyclic covers, Fano threefolds, K-theory, categorical Torelli theorems.

**Theorem 1.1** (= Theorem 3.4)**.** *Assume that we are in the situation above, and that $0 < M < d$. Then there are fully faithful functors $\Phi_j\colon \mathcal{A}_Z \to \mathcal{A}_X^{\mu_n}$, with $j = 0, \dots, n-2$, and a semiorthogonal decomposition:*

$$\mathcal{A}_X^{\mu_n} = \langle \Phi_0(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z)\rangle \subset \mathrm{D}^{\mathrm{b}}(X)^{\mu_n}.$$

In the next part of the paper, we study the topological $\mathrm{K}$-theory of $\mathcal{A}_Z$, and its associated Hodge structure. For this, we use Blanc's topological $\mathrm{K}$-theory for categories [Bla16] and the non-commutative Hodge theory methods of [Per22] and [HLP20], slightly extending their results to admissible subcategories of global quotient DM stacks (Proposition 4.2).

We apply this to the case of very general three-dimensional double covers of weighted projective space. Here, we can make a direct comparison between $\mathrm{K}_0^{\mathrm{top}}(\mathcal{A}_Z)$ and $H^*(Z, \mathbf{Z})$, and show that primitive cohomology of $Z$ coincides with the part $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp$ of the topological $\mathrm{K}$-theory of $\mathcal{A}_X^{\mu_2}$ which is orthogonal to the algebraic classes.

**Proposition 1.2** (= Proposition 5.4)**.** *Suppose that $X$ is a very general prime double cover of weighted projective three-dimensional space, with $0 < M \leq d$. Then $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp$ is equipped with a Hodge structure such that the Chern character map induces a Hodge isometry*

$$\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp \simeq H^2_{\mathrm{prim}}(Z, \mathbf{Z}).$$

A landmark theorem by Bondal–Orlov [BO01] states that the derived category of a Fano variety $X$ determines $X$ up to isomorphism. This justifies the question of whether $\mathcal{A}_X$ suffices to determine $X$. When this is the case we say that a *categorical Torelli theorem* holds for $X$. Motivated by these questions, we apply Proposition 1.2 to reconstruct primitive cohomology from categorical equivalences:

**Theorem 1.3** (= Theorem 5.5)**.** *Let $X$ and $X'$ be very general prime double covers of a weighted projective three-dimensional space, with $d = d'$ and $0 < M \leq d$. A Fourier–Mukai type equivalence $\Phi^{\mu_2}\colon \mathcal{A}_X^{\mu_2} \to \mathcal{A}_{X'}^{\mu_2}$ induces a Hodge isometry*

$$H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \simeq H^2_{\mathrm{prim}}(Z', \mathbf{Z}).$$

Finally, an application of a Hodge-theoretic Torelli theorem [Don83] implies a categorical Torelli theorem the family of Fano threefolds of Picard rank 1, index 1, and genus 2 which settles an open case.

**Theorem 1.4** (= Theorem 6.4)**.** *Let $X, X'$ be very general Picard rank $1$, index $1$, genus $2$ Fano threefolds. Suppose there is a Fourier–Mukai type equivalence of Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then $X \simeq X'$.*

1.1. **Related works.** If we compare Theorem 3.4 and [KP17, Theorem 1.1] in the case where the base $Y$ is weighted projective space, we observe that the cases $0 < M < d$, $M = d$, and $M > d$ correspond to the branch divisor $Z$ being canonically polarized, $K$-trivial, or Fano (see Section 3.3). This is parallel to the trichotomy evidenced by Orlov [Orl09, Theorem 3.11] and suggests a relation between $\mathcal{A}_Z$ and categories of matrix factorizations in the hypersurface setting.

In the case of hypersurfaces, Hirano and Ouchi proved a result analogous to Theorem 3.4 in the language of matrix factorizations [HO23, Theorem 5.6]. From a similar perspective, the remarkable paper [BFK14] investigates Hodge theoretic applications of the language of matrix factorizations. The same philosophy was recently applied to prove categorical Torelli theorems for hypersurfaces in $\mathbf{P}^n$ [LZ23, Theorem 1.3]. A modification of this also applies to $X_2$, regarded as a sextic hypersurface in $\mathbf{P}(1,1,1,1,3)$ [LZ23, Theorem 6.1]. The upcoming work [LPS23] treats this case with independent methods.

Categorical Torelli theorems have been studied for many varieties. We direct the reader to [PS23] for a survey. One possible approach to proving such theorems is to recover the Fano threefold from Bridgeland moduli spaces, as in [BMMS12, APR22, PY22, BBF+22, FLZ23] (for prime Fano threefolds of index 2 and degree $\geq 2$) and [JLLZ21, JLZ22] (for prime Fano theefolds of index 1 and genus $\geq 6$). These methods use the fact that there is a unique $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$-orbit of Serre invariant stability conditions to show that any equivalence preserves moduli spaces of stable objects. However, for the Veronese double cone and lower genus index 1 Fano threefolds, it is not known (or even false, e.g. genus 4 [KP21, Corollary 1.9]) that such a unique orbit exists. Methods relating Hochschild cohomology and Hodge theory are developed in [HR19] and [Pir22] to prove categorical Torelli theorems for broad classes of hypersurfaces.

**Structure of the paper.** After recalling some preliminaries and fixing notations in Section 2, we dedicate Section 3 to the study of semiorthogonal decompositions for cyclic covers and the proof of Theorem 1.1. Section 4 is dedicated to topological K-theory and non-commutative Hodge theory. We apply these tools to the threefold setting in Section 5. The last section (Section 6) contains the categorical Torelli theorem.

**Notation and conventions.** We work over the field of complex numbers $\mathbf{C}$. Whenever we work with an arbitrary triangulated category $\mathcal{C}$, it will be assumed to be proper and $\mathbf{C}$-linear. For a projective variety (or a proper DM stack) $X$, we write $\mathrm{D}(X)$ (resp. $\mathrm{D}^{\mathrm{b}}(X)$, $\mathrm{D}_{\mathrm{perf}}(X)$) for the derived category of (bounded, perfect) complexes of coherent sheaves on $X$.

**Acknowledgements.** We thank Arend Bayer, Pieter Belmans, Wahei Hara, Alex Perry, and Sebastian Schlegel Mejia for helpful discussions. We thank Laura Pertusi, Paolo Stellari, and Shizhuo Zhang for informing us about their work on similar topics. HD and AJ were supported by ERC Consolidator Grant WallCrossAG, no. 819864. AJ was also supported by the Luxembourg National Research Fund (FNR–17113194). FR is supported by EPSRC grant EP/R034826/1 and acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2047/1 – 390685813.

## 2. Preliminaries on derived categories

For background on triangulated categories and derived categories, we recommend [Huy06]. Throughout this section, $\mathcal{C}$ is a triangulated category. For objects $E, F \in \mathcal{C}$, define

$$\mathrm{Hom}^\bullet_{\mathcal{C}}(E,F) := \bigoplus_{t\in\mathbf{Z}} \mathrm{Ext}^t_{\mathcal{C}}(E,F)[-t].$$

We will omit the Hom subscripts when the category we are working in is clear from context.

**Definition 2.1.** An object $E \in \mathcal{C}$ is called *exceptional* if $\mathrm{Hom}^\bullet(E,E) = \mathbf{C}$.

**Definition 2.2.** We say $\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a *semiorthogonal decomposition* of $\mathcal{C}$ if

(1) $\mathrm{Hom}(F,G) = 0$ for all $F \in \mathcal{A}_i, G \in \mathcal{A}_j$ if $i > j$;
(2) for any $F \in \mathcal{C}$, there exists a sequence of morphisms

$$0 = F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 = F$$

such that $\mathrm{Cone}(F_i \to F_{i-1}) \in \mathcal{A}_i$.

Let $i\colon \mathcal{A} \hookrightarrow \mathcal{C}$ be a full triangulated subcategory. If the inclusion $i$ has a left adjoint $i^*$, then $\mathcal{A}$ is called *left admissible*. If $i$ has a right adjoint $i^!$, then $\mathcal{A}$ is called *right admissible*. If both adjoints exist, then $\mathcal{A}$ is called *admissible*.

Define the *right orthogonal* $\mathcal{A}^\perp$ to $\mathcal{A}$ to be the subcategory

$$\mathcal{A}^\perp := \{F \in \mathcal{C} \mid \mathrm{Hom}(G, F) = 0 \,\text{for all}\, G \in \mathcal{A}\}.$$

Similarly, define the *left orthogonal* ${}^\perp\mathcal{A}$ to $\mathcal{A}$ to be the subcategory

$${}^\perp\mathcal{A} := \{F \in \mathcal{C} \mid \mathrm{Hom}(F, G) = 0 \,\text{for all}\, G \in \mathcal{A}\}.$$

Let $i\colon \mathcal{A} \hookrightarrow \mathcal{C}$ be an admissible subcategory. Let $i^*$ and $i^!$ be the left and right adjoint, respectively. For any $F \in \mathcal{C}$, we define the *left mutation functor* $\mathbf{L}_{\mathcal{A}}(F)$ by the triangle

$$ii^!(F) \to F \to \mathbf{L}_{\mathcal{A}}(F). \tag{2.1}$$

Similarly, we define the *right mutation functor* $\mathbf{R}_{\mathcal{A}}$(F) by the triangle

$$\mathbf{R}_{\mathcal{A}}(F) \to F \to ii^*(F). \tag{2.2}$$

In particular, when $\mathcal{A}$ is an exceptional object $E$, these triangles become

$$\mathrm{Hom}^\bullet(E, F) \otimes E \to F \to \mathbf{L}_E(F), \tag{2.3}$$

and

$$\mathbf{R}_E(F) \to F \to \mathrm{Hom}^\bullet(F, E)^\vee \otimes E, \tag{2.4}$$

respectively.

**Proposition 2.3.** *Let $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2\rangle$ be a semiorthogonal decomposition and let $\mathsf{S}_{\mathcal{C}}$ be the Serre functor of $\mathcal{C}$. Then*

$$\mathcal{C} \simeq \langle \mathsf{S}_{\mathcal{C}}(\mathcal{C}_2), \mathcal{C}_1\rangle \simeq \langle \mathcal{C}_2, \mathsf{S}_{\mathcal{C}}^{-1}(\mathcal{C}_1)\rangle$$

*are also semiorthogonal decompositions.*

**Proposition 2.4.** *Let $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2\rangle$. Then*

$$S_{\mathcal{C}_1}^{-1} = \mathbf{L}_{\mathcal{C}_2} \circ S_{\mathcal{C}}^{-1} \quad \textit{and} \quad \mathsf{S}_{\mathcal{C}_2} = \mathbf{R}_{\mathcal{C}_1} \circ \mathsf{S}_{\mathcal{C}}.$$

2.1. **Equivariant triangulated categories.** The definitions of a group action on a category and the corresponding equivariant category are due to Deligne [Del97]. The following background can be found in [KP17, Section 3], which we follow. Let $G$ be a finite group and $\mathcal{C}$ a triangulated category.

**Definition 2.5.** A *(right) action* of $G$ on $\mathcal{C}$ is given by the following data:

(1) for each $g \in G$, an autoequivalence $g^*\colon \mathcal{C} \to \mathcal{C}$;
(2) for each pair $g, h \in G$, a natural isomorphism $c_{g,h}\colon (gh)^* \xrightarrow{\sim} h^* \circ g^*$ such that

$$\begin{tikzcd}
(fgh)^* \arrow[r, "c_{fg,h}"] \arrow[d, "c_{f,gh}"'] & h^* \circ (fg)^* \arrow[d, "h^* c_{f,g}"] \\
(gh)^* \circ f^* \arrow[r, "c_{g,h} f^*"] & h^* \circ g^* \circ f^*
\end{tikzcd}$$

commutes for all $f, g, h \in G$.

**Definition 2.6.** A *$G$-equivariant object* of $\mathcal{C}$ is a pair $(F,\phi)$ consisting of an object $F\in\mathcal{C}$ and a collection of isomorphisms $\phi_g\colon F \xrightarrow{\sim} g^*(F)$ for all $g\in G$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\phi_h"] \arrow[drr, "\phi_{gh}"'] & h^*(F) \arrow[r, "h^*(\phi_g)"] & h^*(g^*(F)) \\ & & (gh)^*(F) \arrow[u, "c_{g,h}(F)"'] \end{tikzcd}$$

commutes for all $g,h\in G$. The isomorphisms $\phi=\{\phi_g\}_{g\in G}$ are called the *$G$-linearisation*. The *$G$-equivariant category* $\mathcal{C}^G$ of $\mathcal{C}$ is the category whose objects are the $G$-equivariant objects of $\mathcal{C}$, and morphisms are those between $G$-invariant objects of $\mathcal{C}$ that commute with the $G$-linearisations.

**Example 2.7** ([Ela15, p. 12])**.** Suppose $G$ is a finite abelian group. Let $\widehat{G}=\mathrm{Hom}(G,\mathbf{C}^\times)$ be the group of irreducible representations of $G$. Then there is an action of $\widehat{G}$ on $\mathcal{C}^G$ given as follows. For every $\rho\in\widehat{G}$, we have an autoequivalence

$$\rho^*(F,(\phi_h)) := (F,(\phi_h))\otimes\rho := (F,(\phi_h\cdot\rho(h))).$$

For $\rho_1$,$\rho_2\in\widehat{G}$, the equivariant objects $\rho_1^*\rho_2^*(F,(\phi_h))$ and $(\rho_2\circ\rho_1)^*(F,(\phi_h))$ are the same, hence we set the isomorphisms $c_{\rho_2,\rho_1}$ to be the identities.

**Theorem 2.8** ([Ela12, Theorem 6.3], [Ela15, Proposition 3.11], [KP17, Theorem 3.2])**.** *Let $X$ be a quasi-projective variety with an action of a finite group $G$. Suppose we have a semiorthogonal decomposition $\mathrm{D^b}(X)=\langle\mathcal{A}_1,\dots,\mathcal{A}_n\rangle$ where $g^*\mathcal{A}_i\subset\mathcal{A}_i$ for all $g\in G$. Then the $G$-equivariant category has the following semiorthogonal decomposition*

$$\mathrm{D^b}(X)^G=\langle\mathcal{A}_1^G,\dots,\mathcal{A}_n^G\rangle.$$

**Definition 2.9.** An action of $G$ on $\mathcal{C}$ is called *trivial* if for each $g\in G$ there is an isomorphism of functors, $\tau_g\colon \mathrm{id}\xrightarrow{\sim} g^*$, such that $c_{g,h}\circ\tau_{gh}=h^*\tau_g\circ\tau_h$ for all $g,h\in G$.

**Proposition 2.10** ([KP17, Proposition 3.3])**.** *Let $G$ be a finite group acting trivially on a triangulated category $\mathcal{C}$. Then $\mathcal{C}^G$ is also triangulated. Let $\rho_0,\dots,\rho_n$ be the irreducible representations of $G$. Then there is a completely orthogonal*[1] *decomposition*

$$\mathcal{C}^G=\langle\mathcal{C}\otimes\rho_0,\dots,\mathcal{C}\otimes\rho_n\rangle.$$

In the situation of Proposition 2.10, we define the functors

$$\iota_k\colon \mathcal{C}\to\mathcal{C}^G,\qquad F\mapsto F\otimes\rho_k; \tag{2.5}$$

$$\pi_k\colon \mathcal{C}^G\to\mathcal{C},\qquad F\mapsto F\otimes_{\mathbf{C}[G]}\rho_k^\vee. \tag{2.6}$$

The functors $\pi_k$ are both left and right adjoint to $\iota_k$.

## 3. Cyclic covers and semiorthogonal decompositions

3.1. **Cyclic covers and rectangular Lefschetz decompositions.** In this section we follow the notation of [KP17]. Let $Y$ be an algebraic variety (or a proper Deligne–Mumford (DM) stack). Consider $f\colon X\to Y$, the degree $n$ cyclic cover of $Y$ ramified over a divisor $Z$. Denote by $j\colon Z\hookrightarrow X$ the embedding of $Z$ as the ramification divisor.

Let $\mu_n$ denote the group of $n$-th roots of unity. This has dual group $\widehat{\mu_n}:=\mathrm{Hom}(\mu_n,\mathbf{C}^\times)\simeq\mathbf{Z}/n$. This isomorphism is given by the primitive character $\chi\colon\mu_n\to\mathbf{C}^\times$. Each irreducible representation $\rho_i$ of $\mu_n$ corresponds to $\chi^i$.

[1] Meaning the Homs vanish in both directions.

We consider the trivial action of $\mu_n$ on $Y$. Then $f$ is $\mu_n$-equivariant, and induces the following functors between equivariant derived categories:

$$f_k^*\colon \mathrm{D^b}(Y) \xrightarrow{\iota_k} \mathrm{D^b}(Y)^{\mu_n} \xrightarrow{f^*} \mathrm{D^b}(X)^{\mu_n},$$
$$f_{k*}\colon \mathrm{D^b}(X)^{\mu_n} \xrightarrow{f_*} \mathrm{D^b}(Y)^{\mu_n} \xrightarrow{\pi_k} \mathrm{D^b}(Y),$$
$$f_k^!\colon \mathrm{D^b}(Y) \xrightarrow{\iota_k} \mathrm{D^b}(Y)^{\mu_n} \xrightarrow{f^!} \mathrm{D^b}(X)^{\mu_n}.$$

Note that $f_k^*$ is left adjoint to $f_{k*}$, and $f_k^!$ is right adjoint to $f_{k*}$. Similarly, define the functors $j_k^*, j_{k*}, j_k^!$ as the compositions

$$j_k^*\colon \mathrm{D^b}(X)^{\mu_n} \xrightarrow{j^*} \mathrm{D^b}(Z)^{\mu_n} \xrightarrow{\pi_k} \mathrm{D^b}(Z),$$
$$j_{k*}\colon \mathrm{D^b}(Z) \xrightarrow{\iota_k} \mathrm{D^b}(Z)^{\mu_n} \xrightarrow{j_*} \mathrm{D^b}(X)^{\mu_n},$$
$$j_k^*\colon \mathrm{D^b}(X)^{\mu_n} \xrightarrow{j^!} \mathrm{D^b}(Z)^{\mu_n} \xrightarrow{\pi_k} \mathrm{D^b}(Z).$$

Also note that $j_k^*$ is left adjoint to $j_{k*}$, and $j_k^!$ is right adjoint to $j_{k*}$.

**Theorem 3.1** ([KP17, Theorem 4.1])**.** *For each $k \in \mathbf{Z}/n$, the functors $f_k^*$ and $j_{k*}$ are fully faithful. Moreover, there is a semiorthogonal decomposition*

$$\mathrm{D^b}(X)^{\mu_n} = \langle f_0^*\mathrm{D^b}(Y), j_{0*}\mathrm{D^b}(Z), \dots, j_{n-2*}\mathrm{D^b}(Z)\rangle.$$

Now additionally assume that $Y$ has a rectangular Lefschetz decomposition, i.e. assume that there is a line bundle $\mathcal{O}_Y(1)$ and an admissible subcategory $\mathcal{B} \subset \mathrm{D^b}(Y)$ such that

$$\mathrm{D^b}(Y) = \langle \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(m-1)\rangle$$

is a semiorthogonal decomposition, where $\mathcal{B}(t) := \mathcal{B} \otimes \mathcal{O}_Y(t)$. Assume moreover that $Z \in |\mathcal{O}_Y(nd)|$, where $n, d$ are positive integers satisfying $0 < m - (n-1)d =: M$. Then [KP17, Lemma 5.1] applies, hence $f^*$ is fully faithful and $\mathrm{D^b}(X)$ has the semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \dots, \mathcal{B}_X(M-1)\rangle, \tag{3.1}$$

where $\mathcal{B}_X := f^*\mathcal{B}$. We call $\mathcal{A}_X = \langle \mathcal{B}_X, \dots, \mathcal{B}_X(M-1)\rangle^\perp$ the *Kuznetsov component.*

For subsets $T \subset \mathbf{Z}$ and $S \subset \mathbf{Z}/n$, define

$$\mathcal{B}_X^S(T) := \langle \mathcal{B}_X(t) \otimes \rho_k \rangle_{t\in T, k\in S} \subset \mathrm{D^b}(Y)^{\mu_n}.$$

**Proposition 3.2.** *We have the following semiorthogonal decomposition,*

$$\mathrm{D^b}(X)^{\mu_n} = \langle \mathcal{A}_X^{\mu_n}, \mathcal{B}_X^{[0,n-1]}([0, M-1])\rangle.$$

*Proof.* Each piece of the semiorthogonal decomposition (3.1) is preserved by the action of $\mu_n$. Thus, by Theorem 2.8 the action of $\mu_n$ distributes through the semiorthogonal decomposition. Furthermore, since the action of $\mu_n$ is trivial on each piece $\mathcal{B}_X(i)$, we get $\mathcal{B}_X(i)^{\mu_n} = \mathcal{B}_X^{[0,n-1]}(i)$ by Proposition 2.10. The result follows. □

We also state a useful lemma which we will use in the next section.

**Lemma 3.3** ([KP17, Lemma 6.1])**.** *For any twist $t \in \mathbf{Z}$ and weight $k \in \mathbf{Z}/n$ we have*

$$\mathbf{R}_{j_{k*}\mathrm{D^b}(Z)}(\mathcal{B}_X^k(t)) = \mathcal{B}_X^{k+1}(t-d).$$

3.2. **A semiorthogonal decomposition of $\mathcal{A}_X^{\mu_n}$.** For this section, assume moreover that $0 < M < d$. Before stating the main theorem, let us set up some more notation. Define the functor $\Phi_k \colon \mathrm{D^b}(X)^{\mu_n} \to \mathrm{D^b}(X)^{\mu_n}$ for $0 \leq k \leq n-2$ as

$$\Phi_k(-) := \mathbf{L}_{\mathcal{B}_X^{[0,k]}([0,M-1])}(- \otimes \rho_k).$$

Furthermore, define the subcategory $\mathcal{A}_Z$ of $\mathrm{D^b}(X)^{\mu_n}$ as follows:

$$\mathcal{A}_Z := \langle j_{0*}\mathrm{D^b}(Z), \mathcal{B}_X^1([M-d,-1])\rangle. \tag{3.2}$$

**Theorem 3.4.** *Let $X$ and $Z$ be as in Section 3.1, with $0 < M < d$. We have a semiorthogonal decomposition,*

$$\mathcal{A}_X^{\mu_n} = \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z)\rangle.$$

**Remark 3.5.** The results of [KP17] apply to the numerical range $M \geq d$, while we work with $0 < M < d$. See Section 3.3 for a comparison between the results.

*Proof.* We follow the strategy of the proof in Section 6 of [KP17]. From Theorem 3.1, we have the semiorthogonal decomposition

$$\mathrm{D^b}(X)^{\mu_n} = \langle \mathcal{B}_X^0([0,m-1]), j_{0*}\mathrm{D^b}(Z), \dots, j_{n-2*}\mathrm{D^b}(Z)\rangle. \tag{3.3}$$

Now rewrite the first component of the semiorthogonal decomposition above as

$$\begin{aligned}\mathcal{B}_X^0([0,m-1]) = \langle &\mathcal{B}_X^0([0,M-1]), \mathcal{B}_X^0([M,M+d-1]),\\ &\mathcal{B}_X^0([M+d,M+2d-1]), \dots, \mathcal{B}_X^0([m-d,m-1])\rangle.\end{aligned} \tag{3.4}$$

We next substitute the decomposition (3.4) into the semiorthogonal decomposition (3.3), and iteratively apply right mutations together with Lemma 3.3. Then, as in [KP17], we get

$$\begin{aligned}\mathrm{D^b}(X)^{\mu_n} = \langle &\mathcal{B}_X^0([0,M-1]), j_{0*}\mathrm{D^b}(Z), \mathcal{B}_X^1([M-d,M-1]), j_{1*}\mathrm{D^b}(Z),\\ &\mathcal{B}_X^2([M-d,M-1]), \dots, j_{n-2*}\mathrm{D^b}(Z), \mathcal{B}_X^{n-1}([M-d,M-1])\rangle.\end{aligned} \tag{3.5}$$

Now we deviate from the proof in *loc. cit.* and use $\mathcal{A}_Z$ as defined above. Since $M < d$,

$$\mathcal{B}_X^k([M-d,M-1]) = \langle \mathcal{B}_X^k([M-d,-1]), \mathcal{B}_X^k([0,M-1])\rangle$$

for any weight $k$. Also, since $j \colon Z \to X$ is equivariant, $j_{l*}(-) \otimes \rho_k = j_{l+k*}$. Therefore, (3.5) reads:

$$\begin{aligned}\mathrm{D^b}(X)^{\mu_n} = \langle &\mathcal{B}_X^0([0,M-1]),\\ &\mathcal{A}_Z, \mathcal{B}_X^1([0,M-1]),\\ &\mathcal{A}_Z \otimes \rho_1, \mathcal{B}_X^2([0,M-1]),\\ &\dots,\\ &\mathcal{A}_Z \otimes \rho_{n-2}, \mathcal{B}_X^{n-1}([0,M-1])\rangle.\end{aligned} \tag{3.6}$$

We next apply left mutations and regroup as follows.

$$\begin{aligned}
\mathrm{D^b}(X)^{\mu_n} &= \langle \mathbf{L}_{\mathcal{B}_X^0([0,M-1])}(\mathcal{A}_Z)), \\
&\qquad \mathbf{L}_{\mathcal{B}_X^0([0,M-1])}\mathbf{L}_{\mathcal{B}_X^1([0,M-1])}(\mathcal{A}_Z \otimes \rho_1), \\
&\qquad \dots, \\
&\qquad \mathbf{L}_{\mathcal{B}_X^0([0,M-1])}\mathbf{L}_{\mathcal{B}_X^1([0,M-1])} \cdots \mathbf{L}_{\mathcal{B}_X^{n-2}([0,M-1])}(\mathcal{A}_Z \otimes \rho_{n-2}), \\
&\qquad \mathcal{B}_X^0([0,M-1]), \mathcal{B}_X^1([0,M-1]), \dots, \mathcal{B}_X^{n-1}([0,M-1])\rangle \qquad (3.7)\\
&= \langle \mathbf{L}_{\mathcal{B}_X^0([0,M-1])}(\mathcal{A}_Z), \\
&\qquad \mathbf{L}_{\mathcal{B}_X^{[0,1]}([0,M-1])}(\mathcal{A}_Z \otimes \rho_1), \dots, \\
&\qquad \mathbf{L}_{\mathcal{B}_X^{[0,n-2]}([0,M-1])}(\mathcal{A}_Z \otimes \rho_{n-2}), \mathcal{B}_X^{[0,n-1]}([0,M-1])\rangle \\
&= \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z), \mathcal{B}_X^{[0,n-1]}([0,M-1])\rangle.
\end{aligned}$$

Since the right-hand sides of the semiorthogonal decomposition above and in Proposition 3.2 match up, the components to the left are equivalent. The statement of the theorem follows. □

**Corollary 3.6.** *Under the action of* $\widehat{\mu}_n \simeq \mathbf{Z}/n$ *on* $\mathrm{D^b}(X)^{\mu_n}$*, we have*

$$\mathcal{A}_X = \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z)\rangle^{\mathbf{Z}/n}.$$

*Proof.* By [Ela15, Theorem 4.2], $(\mathcal{A}_X^{\mu_n})^{\mathbf{Z}/n} \simeq \mathcal{A}_X$. The result then follows by Theorem 3.4. □

**Example 3.7.** Theorem 3.4 applies to the following classes of varieties:

- the Veronese double cone $Y_1$, defined as the double cover of $\mathbf{P}(1,1,1,2)$ branched over $Z \in |\mathcal{O}_{\mathbf{P}(1,1,1,2)}(6)|$. This is a smooth Fano threefold of Picard rank 1, index 2 and degree 1. In this case, $\mathcal{A}_Z = \langle \mathrm{D^b}(Z), \mathrm{D^b}(\mathrm{pt})\rangle$.
- the prime Fano threefold $X_2$ of index 1 and genus 2, which is realized as a double cover of $\mathbf{P}^3$ branched in a sextic. Then $\mathcal{A}_Z = \langle \mathrm{D^b}(Z), \mathrm{D^b}(\mathrm{pt}), \mathrm{D^b}(\mathrm{pt})\rangle$.

3.3. **Comparison to [KP17].** The article [KP17] works under the assumption $M \geq d$. If the inequality is strict, $\mathrm{D^b}(Z)$ admits a Kuznetsov component and a decomposition as follows. Let $\mathcal{B}_Z$ be the essential image of the restriction along the inclusion $Z \subset Y$. Then

$$\mathrm{D^b}(Z) = \langle \mathcal{A}_Z^{\mathsf{KP}}, \mathcal{B}_Z, \dots, \mathcal{B}_Z(M-d-1)\rangle,$$

where $\mathcal{A}_Z^{\mathsf{KP}}$ is the right orthogonal, i.e. the Kuznetsov component of $\mathrm{D^b}(Z)$. If $M = d$, then $\mathrm{D^b}(Z) = \mathcal{A}_Z^{\mathsf{KP}}$ (see [KP17, Lemma 5.5]).

In [KP17, Theorem 1.1] the authors show that for $M \geq d$ the $\mu_n$-equivariant Kuznetsov component, $\mathcal{A}_X^{\mu_n}$, has a semiorthogonal decomposition into $n-1$ copies of $\mathcal{A}_Z^{\mathsf{KP}}$.

Theorem 3.4 shows that, when $0 < M < d$, there exists a category $\mathcal{A}_Z$, defined as a subcategory of $\mathrm{D^b}(X)^{\mu_n}$ (see (3.2)), and a similar decomposition of $\mathcal{A}_X^{\mu_n}$. Observe that $\mathcal{A}_Z$ now *contains* $\mathrm{D^b}(Z)$ as a semiorthogonal component.

Suppose $Y$ is weighted projective space. Then $Y$ has a rectangular Lefschetz decomposition with $\mathcal{B} = \langle \mathcal{O}_Y\rangle$, and $\omega_Y \simeq \mathcal{O}_Y(-m)$. By adjunction, $\omega_Z \simeq \mathcal{O}_Z(nd-m)$,. Hence the sign of $nd - m = d - M$ determines whether $Z$ is Fano, K-trivial, or canonically polarized. The first two cases correspond to $M \geq d$ and are those considered in [KP17]. In summary,

$$\begin{aligned}
d \leq M \quad &\xRightarrow{\text{[KP17, Lemma 5.5]}} \quad \mathcal{A}_Z^{\mathsf{KP}} \hookrightarrow \mathrm{D^b}(Z), \\
0 < M < d \quad &\xRightarrow{\text{Section 3.2}} \quad \mathrm{D^b}(Z) \hookrightarrow \mathcal{A}_Z.
\end{aligned}$$

## 4. Equivariant equivalences and Hodge isometries

4.1. **Cohomology of varieties and Hodge theory.** Let $X$ be a smooth projective variety over $\mathbf{C}$ of dimension $n$. For all $k$, the singular cohomology group $H^k(X,\mathbf{Z})$ carries a Hodge structure. The complexification $H^k(X,\mathbf{C})$ decomposes as

$$H^k(X,\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) \simeq H^q(X,\Omega_X^p)$, for all $p+q=k\geq 0$. We have $H^{p,q}(X)=\overline{H^{q,p}(X)}$.

Singular cohomology has the structure of a commutative ring with cup product. The class of a hyperplane section $H$ of $X$ induces the operation $-\cup H\colon H^k(X,\mathbf{Z})\to H^{k+2}(X,\mathbf{Z})$. Then, *primitive cohomology* is defined as

$$H^{n-k}_{\mathrm{prim}}(X,\mathbf{Z}) := \ker(-\cup H^{k+1}\colon H^{n-k}(X,\mathbf{Z})\to H^{n+k+2}(X,\mathbf{Z}))$$

for all $k$. The Hodge structure on $H^{n-k}(X,\mathbf{Z})$ restricts to one on $H^{n-k}_{\mathrm{prim}}(X,\mathbf{Z})$ [Voi07, II.6-7].

4.2. **Topological K-theory.** Our main references for topological K-theory are [Wei97] and the papers [Bla16], [Per22, Section 5.1], and [HLP20, Section 2]. We collect the results needed for this work below.

There is a lax monoidal functor

$$\mathrm{K}^{\mathrm{top}}\colon \mathrm{Cat}_{\mathbf{C}}\to \mathrm{Sp}$$

from $\mathbf{C}$-linear categories to the $\infty$-category of spectra, which satisfies the following properties.

**Theorem 4.1** ([Bla16]).

*(1) If $\mathcal{C}=\langle\mathcal{C}_1,\dots,\mathcal{C}_m\rangle$ is a $\mathbf{C}$-linear semiorthogonal decomposition, then there is an equivalence*

$$\mathrm{K}^{\mathrm{top}}(\mathcal{C})\simeq \mathrm{K}^{\mathrm{top}}(\mathcal{C}_1)\oplus\dots\oplus \mathrm{K}^{\mathrm{top}}(\mathcal{C}_m), \tag{4.1}$$

*where the map $\mathrm{K}^{\mathrm{top}}(\mathcal{C})\to \mathrm{K}^{\mathrm{top}}(\mathcal{C}_i)$ is induced by the projection functor to $\mathcal{C}_i$.*

*(2) Let $\mathrm{K}(\mathcal{C})$, $\mathrm{HN}(\mathcal{C})$, and $\mathrm{HP}(\mathcal{C})$ denote algebraic K-theory, negative cyclic homology, and periodic cyclic homology of $\mathcal{C}$ respectively. There is a functorial commutative square*

$$\begin{tikzcd} \mathrm{K}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{HN}(\mathcal{C}) \arrow[d] \\ \mathrm{K}^{\mathrm{top}}(\mathcal{C}) \arrow[r] & \mathrm{HP}(\mathcal{C}) \end{tikzcd}. \tag{4.2}$$

*(3) If $X$ is a proper complex variety, then there exists a functorial equivalence $\mathrm{K}^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X))\simeq \mathrm{K}^{\mathrm{top}}(X)$, the complex K-theory spectrum of $X$. Under this equivalence, the left vertical arrow in (4.2) recovers the usual inclusion of algebraic K-theory into topological K-theory, and the bottom arrow coincides with the usual Chern character under the identification of $\mathrm{HP}(\mathrm{D}_{\mathrm{perf}}(X))$ with 2-periodic de Rham cohomology.*

For $\mathcal{C}\in\mathrm{Cat}_{\mathbf{C}}$ and $t$ an integer, write $\mathrm{K}^{\mathrm{top}}_t(\mathcal{C})=\pi_t(\mathrm{K}^{\mathrm{top}}(\mathcal{C}))$. If in addition the category $\mathcal{C}$ is proper, then for each integer $t$ there is a bilinear Euler pairing $\chi^{\mathrm{top}}\colon \mathrm{K}^{\mathrm{top}}_t(\mathcal{C})\otimes \mathrm{K}^{\mathrm{top}}_t(\mathcal{C})\to\mathbf{Z}$ induced by the evaluation functor

$$\mathcal{H}\mathrm{om}(-,-)\colon \mathcal{C}^{\mathrm{op}}\otimes_{\mathrm{D}_{\mathrm{perf}}(\mathrm{Spec}(\mathbf{C}))}\mathcal{C}\to \mathrm{D}_{\mathrm{perf}}(\mathrm{Spec}(\mathbf{C})),$$

where $\mathrm{D}_{\mathrm{perf}}$ denotes the category of perfect complexes. The Euler pairing satisfies the following properties [Per22, Lemma 5.2]:

- The pairing is compatible with the inclusions of $\mathrm{K}_t^{\mathrm{top}}(\mathcal{C}_i)$, and it makes (4.1) into a semiorthogonal sum (i.e. $\chi^{\mathrm{top}}(v_i, v_j) = 0$ for $v_i \in \mathrm{K}_t^{\mathrm{top}}(\mathcal{C}_i)$, $v_j \in \mathrm{K}_t^{\mathrm{top}}(\mathcal{C}_j)$, $i > j$);
- when restricted to $\mathrm{K}_0(\mathcal{C})$ via the inclusion in (4.2), $\chi^{\mathrm{top}}$ coincides with the usual Euler pairing
$$\chi(E,F) = \sum_{i\in\mathbf{Z}}(-1)^i \dim \mathrm{Hom}^i_{\mathbf{C}}(E,F).$$
- (Riemann–Roch) If $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(X)$, with $X$ a proper complex variety, then for $v, w \in \mathrm{K}_t^{\mathrm{top}}(\mathcal{C})$ we have $\chi^{\mathrm{top}}(v,w) = p_*(v^\vee \otimes w) \in \mathrm{K}_{2t}^{\mathrm{top}}(\mathrm{Spec}(\mathbf{C})) \simeq \mathbf{Z}$, with $p\colon X \to \mathrm{Spec}(\mathbf{C})$ the structure morphism.

Now we suppose $\mathcal{C}$ is an admissible subcategory of the derived category of a proper smooth DM quotient stack, to leverage results on the non-commutative Hodge-to-de Rham spectral sequence. We say that $\mathrm{K}^{\mathrm{top}}(\mathcal{C})$ carries a *pure Hodge structure* if every homotopy group $\mathrm{K}_t^{\mathrm{top}}(\mathcal{C})$ carries a pure Hodge structure of weight $-t$.

**Proposition 4.2.** *Let $\mathcal{C}$ be an admissible subcategory of $\mathrm{D}_{\mathrm{perf}}(\mathcal{X})$, for $\mathcal{X} = [X/G]$ a proper smooth DM stack.*

*(1) $\mathrm{K}^{\mathrm{top}}(\mathcal{C})$ comes endowed with a canonical pure Hodge structure with graded pieces*
$$\mathrm{gr}^p(\mathrm{K}_t^{\mathrm{top}}(\mathcal{C})_{\mathbf{C}}) \simeq \mathrm{HH}_{t+2p}(\mathcal{C}), \tag{4.3}$$
*where $\mathrm{HH}_t(\mathcal{C})$ denotes the $t$-th Hochschild homology group of $\mathcal{C}$.*

*(2) For $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(\mathcal{X})$, the Chern character induces an isomorphism of Hodge structures*
$$\mathrm{K}_t^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathcal{X}))_{\mathbf{Q}} \simeq \bigoplus_{k\in\mathbf{Z}} H^{2k-t}(I_{\mathcal{X}}, \mathbf{Q})(k), \tag{4.4}$$
*where $I_{\mathcal{X}}$ denotes the (underived) inertia stack, and the summands on the right-hand side are its de Rham cohomology groups.*

*(3) If $\mathcal{X} = X$ is a smooth proper complex variety, the summands in (4.4) coincide with $H^{2k-t}(X,\mathbf{Q})(k)$, the $k$-th Tate twists of rational singular cohomology groups.*

*Proof.* To prove these statements, we claim that the map $\mathrm{K}^{\mathrm{top}}(\mathcal{C})_{\mathbf{C}} \to \mathrm{HP}(\mathcal{C})$ in (4.2) is an isomorphism (this is referred to as the *lattice property for* $\mathcal{C}$), and that the non-commutative Hodge-to-de Rham sequence degenerates for $\mathcal{C}$ (we say that $\mathcal{C}$ has the *degeneration property*). Granting the claims, the degeneration property gives a canonical filtration of $\mathrm{HP}_n(\mathcal{C})$ with $p$-th graded piece $\mathrm{HH}_{n+2p}(\mathcal{C})$, which together with the lattice property shows (1).

There are identifications $\mathrm{HP}_t(\mathrm{D}_{\mathrm{perf}}(\mathcal{X})) \simeq \oplus_{k\in\mathbf{Z}} H^{2k-t}(I_{\mathcal{X}}, \mathbf{Q})$ by [HLP20, Proposition 2.13], and $\mathrm{HP}_t(\mathrm{D}_{\mathrm{perf}}(X)) \simeq H^{2k-t}(X,\mathbf{Q})$ by [Wei97], under which the non-commutative Hodge filtration coincides with the usual Hodge filtrations. This proves parts (2) and (3).

To establish the claims, observe first that they are preserved under direct summands and arbitrary direct sums in the category of noncommutative motives (defined in [Tab08]). Indeed, this holds for the lattice property because of the functoriality of (4.2), and for the degeneration property by [HLP20, Lemma 1.22]. Therefore, it is sufficient to establish them in the case $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(\mathcal{X})$, for which they are proven, respectively, in [HLP20, Corollary 2.19] and [HLP20, Corollary 1.23]. ∎

4.3. **Equivalences of Fourier–Mukai type induce Hodge isometries.** Consider smooth proper DM stacks that are global quotients, $[X_j/G_j]$, for $j = 1, 2$. Fix admissible subcategories $i_1\colon \mathcal{K}_1 \to \mathrm{D}^{\mathrm{b}}(X_1)^{G_1}$ and $i_2\colon \mathcal{K}_2 \to \mathrm{D}^{\mathrm{b}}(X_2)^{G_2}$, and let $i_j^*$ denote the left adjoints to the inclusions for $j = 1, 2$. Recall that an equivalence $\phi\colon \mathcal{K}_1 \to \mathcal{K}_2$ is said to be of *Fourier–Mukai type* if the composition $\psi\colon \mathrm{D}^{\mathrm{b}}(X_1)^{G_1} \to \mathrm{D}^{\mathrm{b}}(X_2)^{G_2}$ given by
$$\mathrm{D}^{\mathrm{b}}(X_1)^{G_1} \xrightarrow{i_1^*} \mathcal{K}_1 \xrightarrow{\phi} \mathcal{K}_2 \xrightarrow{i_2} \mathrm{D}^{\mathrm{b}}(X_2)^{G_2} \tag{4.5}$$

is a Fourier–Mukai functor. Denote by $(\mathcal{E}_j, e_j)$ a dg-enhancement for each $\mathrm{D^b}(X_j)^{G_j}$, which in turn induce enhancements $(\mathcal{F}_j, f_j)$ for $\mathcal{K}_j$. In this setting:

**Lemma 4.3.** *An equivalence $\phi\colon \mathcal{K}_1 \to \mathcal{K}_2$ of Fourier–Mukai type lifts to an equivalence of dg-enhancements $(\mathcal{F}_1, f_1) \to (\mathcal{F}_2, f_2)$.*

*Proof.* The Fourier–Mukai assumption means that the composition $\psi$ in (4.5) is of Fourier–Mukai type, whence it lifts to a functor $\Psi\colon (\mathcal{E}_1, e_1) \to (\mathcal{E}_2, e_2)$ (this is [Toë07, Theorem 8.9] for schemes, and [Kün22, Corollary 3.7] for the equivariant version). Let $i_j^{\mathrm{dg}}$ and $i_j^{*\mathrm{dg}}$ denote the dg-lifts of the inclusions and their left adjoint functors. Define $\Phi := i_2^{*\mathrm{dg}} \circ \Psi \circ i_1^{\mathrm{dg}}$. The functor $\Phi\colon (\mathcal{F}_1, f_1) \to (\mathcal{F}_2, f_2)$ lifts $\phi$. In fact, taking cohomology, we have

$$H^0(\Phi) \simeq i_2^* \circ H^0(\Psi) \circ i_1 \simeq i_2^* \circ i_2 \circ \phi \circ i_1^* \circ i_1 \simeq \phi. \qquad \square$$

Observe that Proposition 4.2 applies to each $\mathcal{K}_j$, and endows each $\mathrm{K^{top}}(\mathcal{K}_j)$ with Hodge structures. Then we have the following Proposition.

**Proposition 4.4.** *A Fourier–Mukai type equivalence $\phi\colon \mathcal{K}_1 \to \mathcal{K}_2$ induces a Hodge isometry $\mathrm{K^{top}}(\mathcal{K}_1) \simeq \mathrm{K^{top}}(\mathcal{K}_2)$.*

*Proof.* By the functoriality of $\mathrm{K^{top}}$, and since the Euler pairing is defined through a canonical evaluation map, we immediately get isometries $\mathrm{K^{top}}(\mathcal{K}_1) \simeq \mathrm{K^{top}}(\mathcal{K}_2)$.

Since $\phi$ is of Fourier–Mukai type, it admits a lift $\Phi\colon (\mathcal{F}_1, f_1) \to (\mathcal{F}_2, f_2)$ to the dg-enhancements by Lemma 4.3. The construction of the Hodge filtration on $\mathrm{K^{top}}$ only depends on the noncommutative Hodge-to-de Rham spectral sequence, which is a motivic invariant of the dg-enhancements. Then $\mathrm{K^{top}}(\mathcal{K}_1) \simeq \mathrm{K^{top}}(\mathcal{K}_2)$ is a Hodge isometry. $\square$

## 5. Application: weighted double solids

In this section we focus on the three-dimensional case.

5.1. **Setup.** We collect here our assumptions and notation.

**Definition 5.1.** We say that $X$ is a *weighted double solid* (or, for brevity, just a double solid) if $X$ is a three-dimensional smooth DM stack equipped with a 2:1 map to weighted projective space (regarded as a smooth stack), branched over a divisor $Z \in |\mathcal{O}_Y(2d)|$. We say that $X$ is *prime* if it has Picard rank 1.

In the notation of 3.1, a double solid satisfies $n = 2$ and $M = m - d$. Here, $m$ is the sum of the weights of the coordinates of $Y$. We will sometimes need an additional generality assumption, which ensures that $Z$ has Picard rank 1 (this holds for $X$ prime and very general). We refer to $X$ in this case as a *very general* double solid.

**Remark 5.2.** A double solid with $0 < M$ is Fano. Indeed, denote by $\mathcal{O}_X(1)$ the (ample) pullback of $\mathcal{O}_Y(1)$. Then, by the Riemann–Hurwitz formula we have $K_X = f^*K_Y + \mathcal{O}_X(d) = \mathcal{O}_X(-m+d) = \mathcal{O}_X(-M)$, and $M > 0$ is the index of $X$. Similarly, $K_Z = \mathcal{O}_Z(-m+2d)$, so $Z$ is canonically polarized if $M < d$ and $K$-trivial if $M = d$.

For future reference we include the following result.

**Lemma 5.3.** *Let $X, X'$ be very general double solids. If there is an isomorphism $Z \simeq Z'$, then $X \simeq X'$.*

*Proof.* Denote the isomorphism of branch divisors $\phi\colon Z \simeq Z'$. Since $X$ and $X'$ are very general, $Z$ and $Z'$ are both Picard rank 1. This means that $\phi(h) \simeq h'$ where $h, h'$ are the ample generators of $\mathrm{Pic}(Z), \mathrm{Pic}(Z')$, respectively. Hencethe embeddings of $Z, Z'$ into $\mathbf{P}^3$ or $\mathbf{P}(w_0, \dots, w_3)$ coincide up to a projectivity. Hence, the covers $X \simeq X'$ are isomorphic. $\square$

In the rest of the paper, we will often assume that double solids satisfy $0 < M \leq d$. This willy imply that the results of Section 3 hold, including the limit case $\mathcal{A}_Z \simeq \mathrm{D^b}(Z)$.

5.2. **Hodge theory and Kuznetsov components.** Here we use the results of Section 4 to relate equivalences of Kuznetsov components to the cohomology of the ramification divisors. Fix a double solid $X$ with $0 < M \leq d$. By Theorem 3.4, we have the following semiorthogonal decomposition of the $\mu_n$-equivariant Kuzznetsov component.

$$\mathcal{A}_X^{\mu_2} = \langle \mathbf{L}_{\mathcal{B}} j_{0*}\mathrm{D^b}(Z), \mathcal{E}_1, \ldots, \mathcal{E}_{d-M} \rangle \subset \mathrm{D^b}(X)^{\mu_2},$$

where the $\mathcal{E}_i$ are exceptional objects, so that $\langle \mathcal{E}_i \rangle \simeq \mathrm{D^b}(\mathrm{pt})$. By the discussion in Section 4.2, we have an isomorphism of Hodge structures

$$\mathrm{K}_{-t}^{\mathrm{top}}(\mathcal{A}_X^{\mu_2}) \simeq \mathrm{K}_{-t}^{\mathrm{top}}(\mathrm{D^b}(Z)) \oplus \bigoplus_{i=1}^{d-M} \mathrm{K}_{-t}^{\mathrm{top}}(\mathrm{D^b}(\mathrm{pt})) \tag{5.1}$$

for all integers $t$.

If $\mathcal{C}$ is a proper category, we will write

$$\mathrm{K}_0(\mathcal{C})^{\perp} := \{ e \in \mathrm{K}_0^{\mathrm{top}}(\mathcal{C}) \mid \chi^{\mathrm{top}}(a, e) = 0 \text{ for all } a \in \mathrm{K}_0(\mathcal{C}) \}.$$

We denote by $\Lambda_Z$ the image of the Chern character map $\mathrm{K}_0^{\mathrm{top}}(\mathrm{D^b}(Z)) \to H^*(Z, \mathbf{Q})$. Observe that $\Lambda_Z \subset H^0(Z, \mathbf{Z}) \oplus H^2(Z, \mathbf{Z}) \oplus \frac{1}{2} H^4(Z, \mathbf{Z})$ since $Z$ is a smooth surface.

**Proposition 5.4.** *Suppose that $X$ is a very general prime double solid with $0 < M \leq d$. Then the Chern character induces an isometry*

$$\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^{\perp} \simeq H^2_{\mathrm{prim}}(Z, \mathbf{Z}).$$

*Proof.* First, observe that $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^{\perp} \simeq \mathrm{K}_0(\mathrm{D^b}(Z))^{\perp}$. Indeed, by definition of $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^{\perp}$ and through the decomposition (5.1), a class $e \in \mathrm{K}_0^{\mathrm{top}}(\mathcal{A}_X^{\mu_2})$ lies in $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^{\perp}$ if and only if $e \in \mathrm{K}_0(\mathrm{D^b}(Z))^{\perp}$ and $\chi^{\mathrm{top}}([\mathcal{E}_i], e) = 0$ for all $i$ (here we use that $[\mathcal{E}_i] \in \mathrm{K}_0(\mathcal{A}_X^{\mu_2})$). Since (5.1) is semiorthogonal, this is equivalent to saying that $e \in \mathrm{K}_0(\mathrm{D^b}(Z))^{\perp}$.

Now recall that the Chern character induces a functorial isometry $\mathrm{K}_0^{\mathrm{top}}(\mathrm{D^b}(Z))_{\mathbf{Q}} \simeq H^{\mathrm{even}}(Z, \mathbf{Q})$, by Proposition 4.2. More precisely, we make an integral statement, and claim that the Chern character, $\mathrm{ch}$, restricts to an isomorphism

$$\mathrm{K}_0(\mathrm{D^b}(Z))^{\perp} \simeq H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \subset \Lambda_Z.$$

In fact, the algebraic K-theory of $Z$ coincides with the algebraic part of its cohomology. By our assumptions on $X$, the Picard group $\operatorname{Pic} Z$ has rank 1, and is generated by an ample divisor $h$. Hence $\mathrm{K}_0(\mathrm{D^b}(Z))$ has rank 3, and we can choose as generators $[\mathcal{O}_Z]$, $[\mathcal{O}_h]$, $[\mathcal{O}_z]$ for some $z \in Z$.

Suppose now that $[E] \in \mathrm{K}_0(\mathrm{D^b}(Z))^{\perp}$. The topological Euler pairing on $\mathrm{D^b}(Z)$ satisfies Hirzebruch–Riemann–Roch. Since $Z$ has Picard rank 1, we have

$$\mathrm{td}(Z) = \left(1, -\frac{a}{2}h, \mathrm{td}_2(Z)\right),$$

for some integer $a$.

Then, the orthogonality conditions satisfied by $[E]$ are:

$$\begin{aligned} 0 &= \chi^{\mathrm{top}}(\mathcal{O}_z, E) = \mathrm{ch}_0(E), \\ 0 &= \chi^{\mathrm{top}}(\mathcal{O}_h, E) = \int_Z \left(0, -h, -\frac{h^2}{2}\right) \cdot \mathrm{ch}(E) \cdot \mathrm{td}(Z) = -h \cdot \mathrm{ch}_1(E) \\ 0 &= \chi_{\mathrm{top}}(\mathcal{O}_Z, E) = \int_Z \mathrm{ch}(E) \cdot \mathrm{td}(Z) = \mathrm{ch}_2(E), \end{aligned} \tag{5.2}$$

which are precisely the conditions for $\mathrm{ch}(E) \in H^2_{\mathrm{prim}}(Z, \mathbf{Z})$. □

We now prove the main theorem of this section by combining the propositions above.

**Theorem 5.5.** *Let $X$ and $X'$ be very general prime double solids with $Y = Y'$ and $d = d'$. Suppose we have an equivalence $\Phi^{\mu_2} : \mathcal{A}_X^{\mu_2} \to \mathcal{A}_{X'}^{\mu_2}$ of Fourier–Mukai type. Then this induces a Hodge isometry*

$$H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \simeq H^2_{\mathrm{prim}}(Z', \mathbf{Z}).$$

*Proof.* We first show that $\Phi^{\mu_2}$ induces the required isomorphism of vector spaces. By the functorial diagram (4.2), and since the Euler pairing is defined through a canonical evaluation map, we immediately get an isometry $\gamma\colon \mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp \simeq \mathrm{K}_0(\mathcal{A}_{X'}^{\mu_2})^\perp$. This fits into the following commutative square

$$\begin{tikzcd} \mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp \arrow[r, "\gamma"] \arrow[d] & \mathrm{K}_0(\mathcal{A}_{X'}^{\mu_2})^\perp \arrow[d] \\ \mathrm{K}_0^{\mathrm{top}}(\mathcal{A}_X^{\mu_2}) \arrow[r, "\delta"] & \mathrm{K}_0^{\mathrm{top}}(\mathcal{A}_{X'}^{\mu_2}) \end{tikzcd}, \tag{5.3}$$

where $\delta$ is the Hodge isometry of Proposition 4.4, and the vertical maps are functorial inclusions. There is another commutative square

$$\begin{tikzcd} H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \arrow[r, "\alpha"] \arrow[d] & \mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp \arrow[d] \\ \Lambda_Z \arrow[r] & \mathrm{K}_0^{\mathrm{top}}(\mathcal{A}_X^{\mu_2}) \end{tikzcd}, \tag{5.4}$$

where the top arrow is the isometry of Proposition 5.4, the vertical maps are the natural inclusions, and the bottom horizontal arrow is the inverse of the Chern character $\Lambda_Z \to \mathrm{K}_0^{\mathrm{top}}(Z)$ followed by the inclusion of the first factor of (5.1). The commutativity of the diagram implies that $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp$ carries a Hodge structure which, on the one hand, makes $\alpha$ into a Hodge isometry, and on the other hand is induced by restricting the Hodge structure of $\mathrm{K}^{\mathrm{top}}(\mathcal{A}_X^{\mu_2})$. In other words, (5.3) is a commutative diagram of Hodge structures, where the horizontal arrows are isometries.

Therefore, the composition

$$H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \xrightarrow{\alpha} \mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp \xrightarrow{\gamma} \mathrm{K}_0(\mathcal{A}_{X'}^{\mu_2})^\perp \xrightarrow{(\alpha')^{-1}} H^2_{\mathrm{prim}}(Z', \mathbf{Z})$$

is the desired isometry of Hodge structures $H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \simeq H^2_{\mathrm{prim}}(Z', \mathbf{Z})$. □

**Remark 5.6.** It would be interesting to know whether the statement of Proposition 5.4 extends to higher dimensions, since then the argument of Theorem 5.5 would also carry through. We hope to explore this further in future work.

**Example 5.7.** Let $X, X'$ be very general Veronese double cones or sextic double covers of $\mathbf{P}^3$, as in Example 3.7. An equivalence $\mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}$ of Fourier–Mukai type induces a Hodge isometry $H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \simeq H^2_{\mathrm{prim}}(Z', \mathbf{Z})$, where $Z, Z'$ are the corresponding ramification divisors.

## 6. Categorical Torelli theorems

In this section, we apply Theorem 5.5 to prove a categorical Torelli theorem for the family of Fano threefolds $X_2$ (Theorem 6.4). Consider once again a double solid $X$ satisfying $0 < M$. Denote by $\tau\colon \mathcal{A}_X \to \mathcal{A}_X$ the categorical involution induced by the involution of the double cover $X$. Also define the *rotation functor* $\mathsf{R}\colon \mathcal{A}_X \to \mathcal{A}_X$ by $\mathsf{R}(-) := \mathbf{L}_{\mathcal{B}_X}(- \otimes \mathcal{O}_X(1))$. We have the following relationships between the Serre functor of $\mathcal{A}_X$, the rotation functor $\mathsf{R}$, and the categorical involution $\tau$:

**Proposition 6.1.** *There are isomorphisms of functors:*

- $\mathsf{R}^d \simeq \tau[1]$,
- $\mathsf{S}_{\mathcal{A}_X}^{-1} \simeq \mathsf{R}^M[-m+1]$.

*Proof.* The first bullet point is by [Kuz19, Corollary 3.18]. For the second bullet point, note that by Proposition 2.4 and Remark 5.2 we get

$$\begin{aligned}\mathsf{S}_{\mathcal{A}_X}^{-1}(-) &= \mathbf{L}_{\mathcal{B}}\mathbf{L}_{\mathcal{B}(1)}\cdots\mathbf{L}_{\mathcal{B}(M-1)}(-\otimes\mathcal{O}_X(M))[-m+1]\\ &= \mathsf{R}\mathbf{L}_{\mathcal{B}}\mathbf{L}_{\mathcal{B}(1)}\cdots\mathbf{L}_{\mathcal{B}(M-2)}(-\otimes\mathcal{O}_X(M-1))[-m+1]\\ &\cdots\\ &= \mathsf{R}^M[-m+1],\end{aligned}$$

by repeatedly using that $\mathbf{L}_F(-\otimes\mathcal{O}_X(1)) = \mathbf{L}_{F\otimes\mathcal{O}_X(-1)}(-)\otimes\mathcal{O}_X(1)$. □

Let $X'$ be another double solid with the same values of $m, d$ as $X$. We denote by $\tau \in \mathrm{Aut}(\mathcal{A}_X)$ and $\tau' \in \mathrm{Aut}(\mathcal{A}_{X'})$ the covering autoequivalences.

**Lemma 6.2.** *Suppose that an equivalence $\Phi\colon \mathcal{A}_X \simeq \mathcal{A}_{X'}$ commutes with the involutions, i.e. there is an isomorphism of functors $\Phi\circ\tau \simeq \tau'\circ\Phi$. Then $\Phi$ descends to an equivalence of equivariant categories $\Phi^{\mu_2}\colon \mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}$.*

*Proof.* All actions we discuss in this proof will be understood to be $\mathbf{Z}/2 \simeq \mu_2$-actions. We first observe that $\Phi$ preserves $1$-categorical actions (see Definition 2.5(1)). This is equivalent to the fact that $\Phi$ commutes with the involutions.

Next, we check that $\Phi$ intertwines the involutions, considered as $2$-categorical actions on $\mathcal{A}_X$ and $\mathcal{A}_{X'}$ (Definition 2.5(2)). The $1$-categorical actions $\tau$ and $\tau'$ lift to $2$-categorical actions because the functors $\tau\colon \mathcal{A}_X \to \mathcal{A}_X$ and $\tau'\colon \mathcal{A}_{X'} \to \mathcal{A}_{X'}$ are given by pulling back geometric involutions and pullbacks are functorial. Moreover, these lifts are unique because $H^2(B\mathbf{Z}/2, \mathbf{C}^\times) = 0$ (see [BP23, Corollary 3.4] for the lifting criterion, and [BP23, Example 3.12] for the vanishing). Thus $\Phi$ sends the $2$-categorical action $\tau$ to the unique $2$-categorical action $\tau'$. So $\Phi$ respects $2$-categorical actions, as required. □

**Lemma 6.3.** *Assume that $M$ divides an odd multiple of $d$. Then an equivalence $\Phi\colon \mathcal{A}_X \simeq \mathcal{A}_{X'}$ commutes with the covering involutions.*

*Proof.* Write $d(2b+1) = Ma$ for some integers $a, b$. Using Proposition 6.1, up to shifts we have

$$\tau = \tau^{2b+1} \simeq \mathsf{R}^{d(2b+1)} = \mathsf{R}^{Ma} \simeq \mathsf{S}_{\mathcal{A}_X}^{-a},$$

and $\tau' \simeq \mathsf{S}_{\mathcal{A}_{X'}}^{-a}$ up to the same shift. Then, because Serre functors and shifts commute with equivalences of categories, $\Phi$ commutes with the involution. □

We now consider the case where $Y = \mathbf{P}^3$. In this case, the constraint $0 < M = 4-d \leq d$ leaves us with the two families of sextic double solids for $d = 3$ (see Examples 3.7 and 5.7) and quartic double solids for $d = 2$ (which have been studied extensively, see Remark 6.5). We prove a Torelli theorem for $X_2$.

**Theorem 6.4.** *Let $X, X'$ be very general Fano threefolds of Picard rank $1$, index $1$, and genus $2$. Then an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ of Fourier–Mukai type implies $X \simeq X'$.*

*Proof.* In this case, $M = m - d = 1$ always divides $d$, so the equivalence $\Phi$ descends to the equivariant categories by 6.2 and 6.3.

Then, by Theorem 5.5 we get a Hodge isometry

$$H^{\dim X-1}_{\mathrm{prim}}(Z, \mathbf{Q}) \simeq H^{\dim X-1}_{\mathrm{prim}}(Z', \mathbf{Q}).$$

Next, we apply the Torelli theorem for generic hypersurfaces [Don83, p. 325] (see [Voi22, Theorem 0.2] for the statement in terms of a Hodge isometry), which implies $Z \simeq Z'$. Donagi's theorem applies to the present case, since its numerical assumptions are automatically satisfied as long as $d \neq 1, 2$. Now, by Lemma 5.3, we conclude that $X \simeq X'$. □

**Remark 6.5.** A quartic double solid $Y_2$ is an index 2 prime Fano threefolds of degree 2. The proof of Theorem 6.4 yields, with the same arguments, that an equivalence of Kuznetsov components $\mathcal{A}_{Y_2} \simeq \mathcal{A}_{Y_2'}$ yields a Hodge isometry $H^2_{\mathrm{prim}}(Z, \mathbf{Z}) \simeq H^2_{\mathrm{prim}}(Z', \mathbf{Z})$. However, the classical Torelli theorem in [Don83] does not apply to this case: the only double solid in our setting that satisfies the numerical conditions is $X_2$. Categorical Torelli theorems for $Y_2$ have been proven with several other methods, see the references in [PS23, Section 5.5] and the recent [BP23, Remark 7.5] and [FLZ23].

## References


[APR22] Matteo Altavilla, Marin Petković, and Franco Rota. Moduli spaces on the Kuznetsov component of Fano threefolds of index 2. *Épijournal Géom. Algébrique*, 6:Art. 13, 31, 2022.

[BBF+22] Arend Bayer, Sjoerd Beentjes, Soheyla Feyzbakhsh, Georg Hein, Diletta Martinelli, Fatemeh Rezaee, and Benjamin Schmidt. The desingularization of the theta divisor of a cubic threefold as a moduli space. *Geometry and Topology*, 2022.

[BFK14] Matthew Ballard, David Favero, and Ludmil Katzarkov. A category of kernels for equivariant factorizations and its implications for Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 120:1–111, 2014.

[Bla16] Anthony Blanc. Topological K-theory of complex noncommutative spaces. *Compositio Mathematica*, 152(3):489–555, 2016.

[BLMS23] Arend Bayer, Martí Lahoz, Emanuele Macrì, and Paolo Stellari. Stability conditions on Kuznetsov components. *Ann. Sci. Éc. Norm. Supér. (4)*, 56(2):517–570, 2023. With an appendix by Bayer, Lahoz, Macrì, Stellari and X. Zhao.

[BMMS12] Marcello Bernardara, Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. A categorical invariant for cubic threefolds. *Advances in Mathematics*, 229(2):770–803, 2012.

[BO01] Alexei Bondal and Dmitri Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Mathematica*, 125(3):327–344, 2001.

[BP23] Arend Bayer and Alexander Perry. Kuznetsov's Fano threefold conjecture via K3 categories and enhanced group actions. *J. Reine Angew. Math.*, 800:107–153, 2023.

[Del97] Pierre Deligne. Action du groupe des tresses sur une catégorie. *Inventiones mathematicae*, 128(1):159–175, 1997.

[Don83] Ron Donagi. Generic Torelli for projective hypersurfaces. *Compositio Mathematica*, 50(2-3):325–353, 1983.

[Ela12] Alexei D Elagin. Descent theory for semiorthogonal decompositions. *Sbornik: Mathematics*, 203(5):645, 2012.

[Ela15] Alexey Elagin. On equivariant triangulated categories. *arXiv preprint arXiv:1403.7027*, 2015.

[FLZ23] Soheyla Feyzbakhsh, Zhiyu Liu, and Shizhuo Zhang. New perspectives on categorical Torelli theorems for del Pezzo threefolds. *arXiv e-prints*, page arXiv:2304.01321, April 2023.

[HLP20] Daniel Halpern-Leistner and Daniel Pomerleano. Equivariant Hodge theory and noncommutative geometry. *Geom. Topol.*, 24(5):2361–2433, 2020.

[HO23] Yuki Hirano and Genki Ouchi. Derived factorization categories of non-Thom–Sebastiani-type sums of potentials. *Proceedings of the London Mathematical Society*, 126(1):1–75, 2023.

[HR19] Daniel Huybrechts and Jørgen Vold Rennemo. Hochschild cohomology versus the Jacobian ring and the Torelli theorem for cubic fourfolds. *Algebr. Geom.*, 6(1):76–99, 2019.

[Huy06] Daniel Huybrechts. *Fourier–Mukai transforms in algebraic geometry*. Clarendon Press, 2006.

[JLLZ21] Augustinas Jacovskis, Xun Lin, Zhiyu Liu, and Shizhuo Zhang. Categorical Torelli theorems for Gushel–Mukai threefolds. *arXiv preprint arXiv:2108.02946*, 2021.

[JLZ22] Augustinas Jacovskis, Zhiyu Liu, and Shizhuo Zhang. Brill–Noether theory for Kuznetsov components and refined categorical Torelli theorems for index one Fano threefolds. *arXiv preprint arXiv:2207.01021*, 2022.

[KP17] Alexander Kuznetsov and Alexander Perry. Derived categories of cyclic covers and their branch divisors. *Selecta Mathematica*, 23(1):389–423, 2017.

[KP21] Alexander Kuznetsov and Alexander Perry. Serre functors and dimensions of residual categories. *arXiv preprint arXiv:2109.02026*, 2021.

[Kün22] Felix Küng. Twisted Hodge diamonds give rise to non-Fourier–Mukai functors. *arXiv e-prints*, page arXiv:2207.03363, July 2022.

[Kuz09] Alexander Kuznetsov. Derived categories of Fano threefolds. *Proceedings of the Steklov Institute of Mathematics*, 264(1):110–122, 2009.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 219–243. Birkhäuser Boston, Boston, MA, 2010.

[Kuz19] Alexander Kuznetsov. Calabi–Yau and fractional Calabi–Yau categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(753):239–267, 2019.

[LPS23] Martí Lahoz, Laura Pertusi, and Paolo Stellari. Categorical Torelli theorems for weighted hypersurfaces. *upcoming*, 2023.

[LPZ18] Chunyi Li, Laura Pertusi, and Xiaolei Zhao. Twisted cubics on cubic fourfolds and stability conditions. *arXiv e-prints*, page arXiv:1802.01134, February 2018.

[LZ23] Xun Lin and Shizhuo Zhang. Serre algebra, matrix factorization and categorical Torelli theorem for hypersurfaces. *arXiv e-prints*, page arXiv:2310.09927, October 2023.

[Orl09] Dmitri Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 503–531. Birkhäuser Boston, Boston, MA, 2009.

[Per22] Alexander Perry. The integral Hodge conjecture for two-dimensional Calabi–Yau categories. *Compositio Mathematica*, 158(2):287–333, 2022.

[Pir22] Dmitrii Pirozhkov. Categorical Torelli theorem for hypersurfaces. *arXiv preprint arXiv:2208.13604*, 2022.

[PS23] Laura Pertusi and Paolo Stellari. Categorical Torelli theorems: results and open problems. *Rend. Circ. Mat. Palermo (2)*, 72(5):2949–3011, 2023.

[PY22] Laura Pertusi and Song Yang. Some remarks on fano three-folds of index two and stability conditions. *International Mathematics Research Notices*, 2022(17):13396–13446, 2022.

[Tab08] Gonçalo Tabuada. Higher $K$-theory via universal invariants. *Duke Math. J.*, 145(1):121–206, 2008.

[Toë07] Bertrand Toën. The homotopy theory of dg-categories and derived Morita theory. *Inventiones mathematicae*, 167(3):615–667, 2007.

[Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.

[Voi22] Claire Voisin. Schiffer variations and the generic Torelli theorem for hypersurfaces. *Compositio Mathematica*, 158(1):89–122, 2022.

[Wei97] Charles Weibel. The Hodge filtration and cyclic homology. *$K$-Theory*, 12(2):145–164, 1997.

HD: School of Mathematics and Maxwell Institute, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh, EH9 3FD,United Kingdom
*Email address*: h.dell@sms.ed.ac.uk

AJ: Department of Mathematics, Maison du Nombre, 6 avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg
*Email address*: augustinas.jacovskis@uni.lu

FR: School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QQ, United Kingdom
*Email address*: franco.rota@glasgow.ac.uk